Recent Progress in the Analytical and Numerical Treatment of Partial Differential Equations of Fractional Order

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Abstract

Analytical and numerical methods for the solution of fractional partial differential equations made enormous progress during the last 10 years because many complex physical and biological systems can be represented more accurately through fractional derivative formulation. In this talk we report on recent research work on the development of new analytical and numerical methods for the solution of partial differential equations of fractional order and explain their respective strengths and weaknesses. Several numerical examples are given to demonstrate the effectiveness and weaknesses of the present methods.

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Introduction

Some Recent Applications of fractional PDEs

- Fractional order PDEs, as generalization of classical order PDEs, are increasingly used to model problems in many fields of science and engineering.
- Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and process.
- Fractional calculus can be considered as a novel topic as well, since only from a little more than twenty years it has been object of specialized conferences and many papers.

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Example

- The laws of Hooke and Newton for elastic solids and viscous liquids, respectively, are

$$\begin{aligned} \sigma(t) &= ED^0\varepsilon(t), \\ \sigma(t) &= \eta D^1\varepsilon(t), \end{aligned}$$

where E is the modulus of elasticity and η is the viscosity of the material. It possible to model the relation between stress and strain for such a viscoelastic material via an equation of the form:

$$\sigma(t) = \upsilon_c D^k\varepsilon(t), \quad k \in (0,1),$$

where υ_c is a material constant.

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- Therefore, the development of robust and stable numerical and analytical methods for solving such equations has acquired an increasing interest in the last 10 years.
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Analytical methods are summarized as follows:

- Adomian Decomposition Method (ADM)
- Variational Iteration Method (VIM)
- Homotopy Perturbation Method (HPM)
- Homotopy Analysis Method (HAM)
- Generalized Two-dimensional Differential Transform Method (GDTM)
- Reproducing kernel Hilbet space method (RKHSM)

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- This method is based on the two-dimensional differential transform method (DTM) and generalized Taylor formula
- Comparison of the results obtained by using the GDTM with that obtained by other existing methods reveals that the present method is very effective and convenient for solving linear and nonlinear differential equations of fractional order.
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Application I: Analytical Method

Adomian Decomposition Method (ADM)

To use the decomposition method, we express the nonlinear fractional differential equation in terms of operator from as

$$D_{*t}{}^{\alpha}u(x,t) \quad = \quad D_{*x}{}^{\beta}u(x,t) + N_f(u(x,t)), \qquad (1)$$

where $n-1<\alpha\leq n, \quad m-1<\beta\leq m, \quad n,m\in N$

Application I: Analytical Method

Adomian Decomposition Method (ADM)

- where α and β are parameters describing the order of the fractional time- and space-derivatives in the Caputo sense, respectively, and N_f is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t.
- The function $u(x, t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses.

Application I: Analytical Method

Adomian Decomposition Method (ADM)

Applying the operator J^{α}, the inverse of the operator ${D_{*t}}^{\alpha}$, to both sides of equation (1) yields

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{n-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!} \\ &\quad + J^{\alpha}\left[D_* x^{\beta} u(x,t) + N_f(u(x,t))\right] \end{aligned} \tag{2}$$

- The ADM suggests the solution $u(x,t)$ be decomposed as

Application I: Analytical Method

Adomian Decomposition Method (ADM)

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t)$$

and the nonlinear function in equation (2) is decomposed as follows:

$$N_f(u(x,t)) = \sum_{i=0}^{\infty} A_i,$$

Application I: Analytical Method

Adomian Decomposition Method (ADM)

- Substitution of the decomposition series, the iterates are determined by the following recursive way

$$u_0(x,t) = \sum_{k=0}^{n-1} \frac{\partial^k u}{\partial t^k}(x,0^+)\frac{t^k}{k!}, \qquad (3)$$

$$u_{j+1}(x,t) = J^\alpha D_{*x}{}^\beta u_j + J^\alpha A_j, \quad j \geq 1 \qquad (4)$$

- The Adomian polynomial can be calculated for all forms of nonlinearity according to specific algorithms constructed by Adomian. Finally, we approximate the solution $u(x,t)$ by the truncated series

Application I: Analytical Method

Adomian Decomposition Method (ADM)

$$\phi_N(x,t) = \sum_{j=0}^{N} u_j(x,t) \ \ and \lim_{N\longrightarrow\infty} \phi_N(x,t) = u(x,t)$$

Application I: Analytical Method

Example

Consider the following space- and time-fractional diffusion-wave equation

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{\partial^\beta}{\partial t^\beta}u(x,t), \quad 0<\alpha\le 2, 1<\beta\le 2, 0<x<1, t>0$$

subject to the the initial conditions

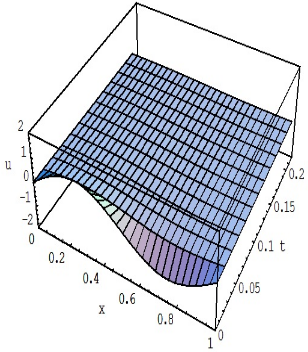
$$u(x,0) = \sin(2\pi x), \quad 0<\alpha\le 1,$$

$$u(x,0) = \sin(2\pi x), \quad \frac{\partial u(x,0)}{\partial t} = 2\sin(2\pi x), \quad 1<\alpha\le 2.$$

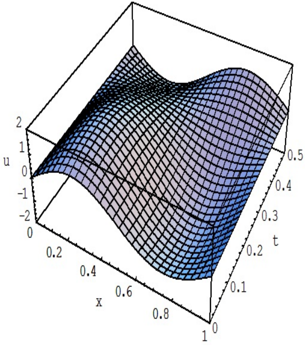
The evolution results for the above problem using ADM with different values α and β are shown in Figs. 1-4.

Application I: Analytical Method

- Evolution of the initial state using ADM: $\alpha = 1$, $\beta = 2$

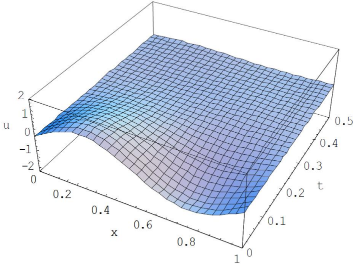


- Evolution of the initial state using ADM: $\alpha = 2$, $\beta = 2$

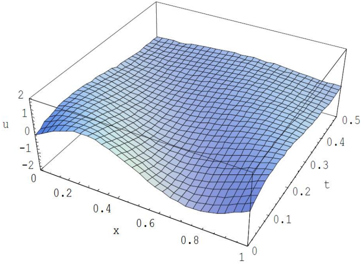


Application I: Analytical Method

- Evolution of the initial state using ADM: $\alpha = \frac{3}{2}$, $\beta = 2$



- Evolution of the initial state using ADM: $\alpha = 1$, $\beta = \frac{5}{4}$



Application I: Analytical Methods

Example

Consider the following nonlinear time-fractional KdV equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - (u^2)_x + [u(u)_{xx}]_x = 0, \quad 0 < \alpha \leq 1, 0 < x < 1, t > 0$$

subject to the the initial conditions

$$u(x,0) = \sinh^2(x/2),$$

The first three term approximate solutions for the above problem using ADM, VIM, HPM, and GDTM are given by:

Application I: Analytical Method

- $u_{ADM} = \sinh^2(x/2) - \frac{t^\alpha}{4\Gamma(\alpha+1)}\sinh(x) + \frac{t^{2\alpha}}{8\Gamma(2\alpha+1)}\cosh(x)$
- $u_{VIM} = \sinh^2(x/2) - \frac{1}{2}\cosh(x)t + \frac{t^{2-\alpha}}{4\Gamma(3-\alpha)}\sinh(x) + \frac{t^{\alpha+1}}{8\Gamma(\alpha+2)}\cosh(x)$
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- $u_{GDTM} = \frac{1}{2}[\cosh(x)\sum_{n=0}^{\infty}\frac{(t^\alpha/2)^{2n}}{\Gamma(2n\alpha+1)} - \sinh(x)\sum_{n=0}^{\infty}\frac{(t^\alpha/2)^{2n+1}}{\Gamma((2n+1)\alpha+1)} - 1]$

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Application I: Analytical Method

Table 1. Numerical values when $\alpha = 0.5$ and $\alpha = 1$ for Example 2.

t	x	$\alpha = 0.5$				$\alpha = 1$			
		u_{GDTM}	u_{ADM}	u_{VIM}	u_{HPM}	u_{GDTM}	u_{ADM}	u_{VIM}	u_{HPM}
0.2	0.25	0.00919	0.00962	0.00336	0.00962	0.00563	0.00565	0.00565	0.00565
	0.50	0.02474	0.02626	0.02995	0.02626	0.04053	0.04057	0.04057	0.04057
	0.75	0.07326	0.07596	0.08983	0.07596	0.10939	0.10946	0.10946	0.10946
	1.00	0.15779	0.16185	0.18676	0.16185	0.21654	0.21663	0.21663	0.21663
0.4	0.25	0.02174	0.02221	0.00173	0.02221	0.00062	0.00075	0.00075	0.00075
	0.50	0.02368	0.02722	0.01121	0.02722	0.02266	0.02297	0.02297	0.02279
	0.75	0.05852	0.06536	0.05279	0.06336	0.07755	0.07805	0.07805	0.07805
	1.00	0.12846	0.13902	0.12912	0.13902	0.16871	0.16945	0.16945	0.16945
0.6	0.25	0.03810	0.03786	0.00707	0.03786	0.00062	0.00102	0.00102	0.00102
	0.50	0.02909	0.03452	0.00231	0.03452	0.01003	0.01102	0.01102	0.01102
	0.75	0.05332	0.06475	0.02910	0.06475	0.05148	0.05312	0.05312	0.05312
	1.00	0.11232	0.13047	0.08913	0.13047	0.12758	0.12997	0.12997	0.12997

Some New Numerical Methods

Numerical methods are summarized as follows:

- Fractional Difference Method (FDM)
- Mickens Non-standard Discretization Method (MNSDM)

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Application II: Numerical Method

Mickens Non-standard Discretization Method

- The forward Euler method is one of the simplest discretization schemes. In this method the derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{h}$.
- However, in the Mickens schemes this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h)}$ where $\phi(h)$ is a continuous function of step size h.
- In addition to this replacement, if there are nonlinear terms such as $y^2(t)$ in the differential equation, these are replaced by $y(t)y(t+h)$ or $y(t-h)y(t)$.
- There is no appropriate general method for choosing the function $\phi(h)$ or for choosing which nonlinear terms are to be replaced, but some special techniques may be found in (Mickens, 1994) and (Erjaee and Momani, 2008).

Application II: Numerical Method

Mickens Non-standard Discretization Method

- Consider the single fractional differential equation

$$D_{*t}{}^{\alpha} y(t) \;=\; f(t, y), \quad y(t_0) = y_0, \tag{5}$$

$0 < \alpha \leq 1, \quad T \geq t \geq 0$, where $D_{*t}{}^{\alpha}$ denotes the fractional derivative in the Caputo sense. We have chosen to use the Grunwald-Letnikov method to enable us to apply Mickens scheme. This method approximates the one-dimensional fractional derivative as follows:

Application II: Numerical Method

Mickens Non-standard Discretization Method

$$D_{*t}^{\alpha} y(t) = \lim_{h \longrightarrow 0} h^{-\alpha} \sum_{j=0}^{[t/h]} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} y(t - jh),$$

where $[t]$ denotes the integer part of t and h is the step size. Thus equation (5) is discretized as

- $$\sum_{j=0}^{[t_n/h]} c_j^{\alpha} y(t_{n-j}) = f(t_n, y(t_n)), \tag{6}$$

Application II: Numerical Method

Mickens Non-standard Discretization Method

where $t_n = nh$ and c_j^α are the Grunwald-Letnikov coefficients defined as

$$c_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^\alpha, \tag{7}$$

and

$$c_0^\alpha = h^{-\alpha}, \quad j = 1, 2, 3, \ldots . \tag{8}$$

- We will now apply the Mickens discretization scheme to the fractional-order Rössler chaotic and hyperchaotic systems. The step size h is replaced by a function of $h, \phi(h)$, and by changing any nonlinear term to the corresponding one

The NSFD Scheme for Solving Fractional-Order Rössler Chaotic and Hyperchaotic Systems

- The fractional order Rössler chaotic system.

$$\begin{aligned} D^{\alpha_1} x(t) &= -y - z, \\ D^{\alpha_2} y(t) &= x + a_1 y, \qquad (9) \\ D^{\alpha_3} z(t) &= b_1 + z(x - c_1), \end{aligned}$$

- The fractional order Rössler hyberchaotic system.

$$\begin{aligned} D^{\alpha_1} x(t) &= -y - z, \\ D^{\alpha_2} y(t) &= x + a_2 y + w, \\ D^{\alpha_3} z(t) &= b_2 + xz, \quad (10) \\ D^{\alpha_4} w(t) &= -c_2 z + d_2 w, \end{aligned}$$

where α_i's are equal real numbers or rational numbers between 0 and 1 and $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the Caputo fractional derivative of order α_i, for $i = 1, 2, 3$. which is chaotic when $a_1 = 0.15$, and a hyperchaotic behavior when $a_2 = 0.25$, $b_2 = 3$, $c_2 = 0.5$ and $d_2 = 0.05$.

Mickens Non-standard Discretization Method

Using the Mickens non-standard method, we have

1. The linear terms on the right-side of (9) have the form

$$\begin{aligned} -y &= -y(t_n), \\ -z &= -z(t_n). \end{aligned}$$

2. The linear terms on the right-side of (9) have the form

$$\begin{aligned} x &= x(t_{n+1}), \\ y &= 2y - y \to 2y(t_n) - y(t_{n+1}). \end{aligned}$$

3. The linear and nonlinear terms on the right-side of (9) have the form

$$\begin{aligned} xz &= 2xz - xz \to 2x(t_{n+1})z(t_n) - x(t_{n+1})z(t_{n+1}), \\ -z &= -z(t_n). \end{aligned}$$

The NSFD Scheme for Solving Fractional-Order Rössler Chaotic and Hyperchaotic Systems

- The functions ϕ_i $(i = 1, 2, 3, 4)$ are chosen according to the non-diagonal elements of the Jacobian matrix of the original continuous system of the Rössler hyperchaotic system (10)

$$J_{ij} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & a_2 & 0 & 1 \\ z & 0 & x & 0 \\ 0 & 0 & -c_2 & d_2 \end{bmatrix}. \tag{11}$$

- Since $J_{11} = 0$, then we choose $\phi_1(h) = h$. The others are

$$\phi_2(h) = \frac{1 - e^{-a_2 h}}{a_2}, \tag{12}$$

$$\phi_3(h) = \frac{1 - e^{-\bar{x} h}}{\bar{x}}, \tag{13}$$

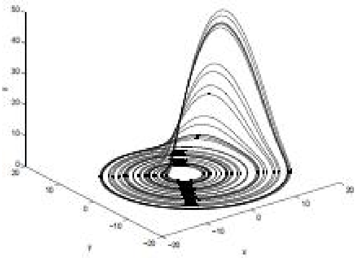
$$\phi_4(h) = \frac{1 - e^{-d_2 h}}{d_2}, \tag{14}$$

where $\bar{x}$ is a fixed point of the Rössler hyperchaotic system (10) as follows

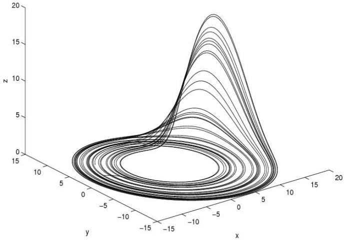
$$\bar{x} = \frac{\sqrt{b_2(c_2 - a_2 d_2)}}{\sqrt{d_2(c_2 - a_2 d_2)}}. \tag{15}$$

The NSFD Scheme for Solving Fractional-Order Rössler Chaotic and Hyperchaotic Systems

- Phase plot of chaotic attractor in the $x-y-z$ space, $\alpha_1=\alpha_2=\alpha_3=1$, $a=0.15$

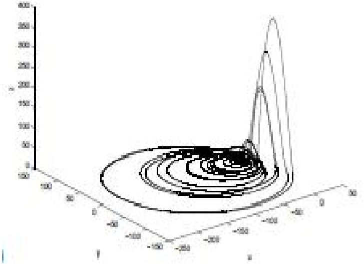


- Phase plot of chaotic attractor in the $x-y-z$ space, $x-y-z$ space, $\alpha_1=\alpha_2=\alpha_3=1$, $a=0.4$

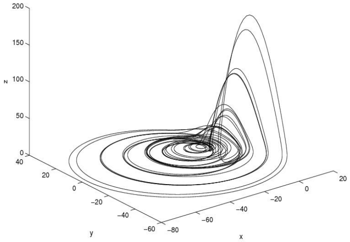


The NSFD Scheme for Solving Fractional-Order Rössler Chaotic and Hyperchaotic Systems

- Phase plot of hyperchaotic attractor in the $x - y - z$ space, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $a_2 = 0.25$



- Phase plot of hyperchaotic attractor in the $x - y - z$ space, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.95$, $a_2 = 0.32$



MSGDTM for Solving the fractional nonlinear Bloch system

Consider the fractional-order nonlinear Bloch equations:

$$\begin{aligned} D^{\alpha_1}x(t) &= \delta y + \gamma z(x\sin(c) - y\cos(c)) - \frac{x}{\Gamma_2}, \\ D^{\alpha_2}y(t) &= -\delta x - z + \gamma z(x\cos(c) + y\sin(c)) - \frac{y}{\Gamma_2}, \\ D^{\alpha_3}z(t) &= y - \gamma\sin(c)(x^2 + y^2) - \frac{z-2}{\Gamma_1}, \end{aligned} \tag{16}$$

subject to the initial conditions

$$x(0) = c_1, y(0) = c_2, z(0) = c_3$$

:

MSGDTM for Solving the fractional nonlinear Bloch system

The multi-step differential transform method is employed to study the behaviour of chaotic attractors for the fractional non-linear Bloch system under two sets of parameter values with different values of α_1, α_2 and α_3.

The first set of parameters is:

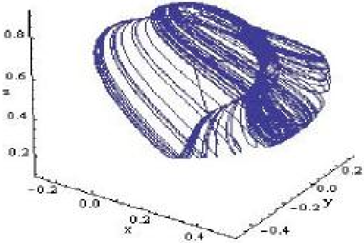
$$\gamma = 10, \delta = 1.26, c = 0.7764, \Gamma_1 = 0.5, \Gamma_2 = 0.25,$$

and the second set:

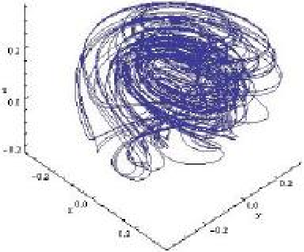
$$\gamma = 35, \delta = -1.26, c = 1.73, \Gamma_1 = 0.5, \Gamma_2 = 2.5.$$

MSGDTM for Solving the fractional nonlinear Bloch system

- Chaotic attractors of system (16) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$: First set of parameters

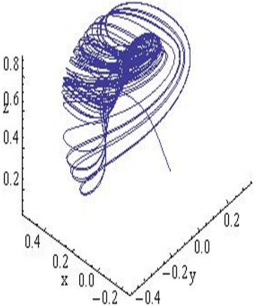


- Chaotic attractors of system (16) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$: Second set of parameters

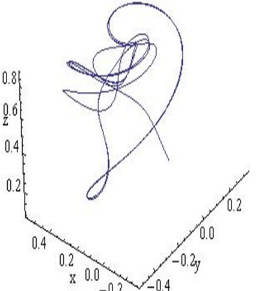


MSGDTM for Solving the fractional nonlinear Bloch system

- Chaotic attractors of system (16) when $\alpha_1 = 0.99$ and $\alpha_2 = \alpha_3 = 1$: First set of parameters

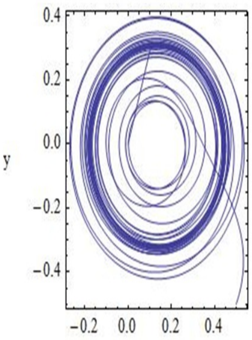


- Chaotic attractors of system (16) when $\alpha_1 = 0.97$ and $\alpha_2 = \alpha_3 = 1$: First set of parameters

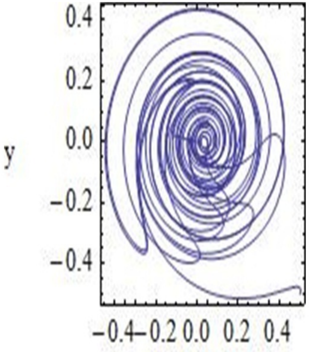


MSGDTM for Solving the fractional nonlinear Bloch system

- Phase portraits of x versus y of system (16) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\gamma = 5$: Second set of parameters



- Phase portraits of x versus y of system (16) when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\gamma = 20$: Second set of parameters



Conclusions

1. GDTM, HPM, ADM, VIM and HAM have been successfully applied to differential equations fractional order.
2. The main advantage of the five methods over mesh points methods is the fact that they do not require discretization of the variables, i.e. time and space, and thus they are not affected by computation round off errors and one is not faced with necessity of large computer memory and time.
3. The five methods provide the solutions in terms of convergent series with easily computable components.
4. The main disadvantage of these methods, they give a good approximation to the true solution in a small region.
5. Finally, the recent appearance of fractional differential equations as models in many fields of applied mathematics makes it necessary to develop new methods of solution for such equations.

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