A Fractional Spectral Theory for Exponentially Accurate Spectral and Spectral Element Methods

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Abstract

We first consider a regular fractional Sturm-Liouville problem of two kinds RFSLP-I and RFSLP-II of order $\nu \in (0,2)$. The corresponding fractional differential operators in these problems are both of Riemann-Liouville and Caputo type, of the same fractional order $\mu = \nu/2 \in (0,1)$. We obtain the analytical eigensolutions to RFSLP-I &-II as non-polynomial functions, which we define as Jacobi *poly-fractonomials*. These eigenfunctions are orthogonal with respect to the weight function associated with RFSLP-I &-II. Subsequently, we extend the fractional operators to a new family of singular fractional Sturm-Liouville problems of two kinds, SFSLP-I and SFSLP-II. We show that the primary regular boundary-value problems RFSLP-I&-II are indeed asymptotic cases for the singular counterparts SFSLP-I&-II. Furthermore, we prove that the eigenvalues of the singular problems are real-valued and the corresponding eigenfunctions are orthogonal. In addition, we obtain the eigen-solutions to SFSLP-I &-II analytically, also as non-polynomial functions, hence completing the whole family of the Jacobi poly-fractonomials. Next, we develop efficient spectral and spectral element methods for fractional ordinary differential equations (FODEs) and fully time- and space-fractional partial differential equations (FPDEs), which are based upon the aforementioned fractional spectral theory. To this end, we examine our numerical schemes for solving FODEs of the form ${}_0\mathcal{D}_t^{\alpha} u(t) = f(t)$ denoted as initial-value problems, also ${}_t\mathcal{D}_T^{\alpha} u(t) = g(t)$ as fractional final-value problems, where $\alpha \in (0,1)$. Moreover, we extend the corresponding spectral and spectral element methods for time- and space-fractional advection and diffusion equations, where our numerical tests exhibit the theoretical exponential convergence in the p-refinements in addition to the algebraic convergence in the h-refinements.

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