Tempered Fractional Calculus

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Abstract

Fractional Calculus has a close relation with Probability. Random walks with heavy tails converge to stable stochastic processes, whose probability densities solve space-fractional diffusion equations. Continuous time random walks, with heavy tailed waiting times between particle jumps, converge to non-Markovian stochastic limits, whose probability densities solve time-fractional diffusion equations. Time-fractional derivatives and integrals of Brownian motion produce fractional Brownian motion, a useful model in many applications. Fractional derivatives and integrals are convolutions with a power law. Including an exponential term leads to tempered fractional derivatives and integrals. Tempered stable processes are the limits of random walk models where the power law probability of long jumps is tempered by an exponential factor. These random walks converge to tempered stable stochastic process limits, whose probability densities solve tempered fractional diffusion equations. Tempered power law waiting times lead to tempered fractional time derivatives, which have proven useful in geophysics. Applying this idea to Brownian motion leads to tempered fractional Brownian motion, a new stochastic process that can exhibit semi-long range dependence. The increments of this process, called tempered fractional Gaussian noise, provide a useful new stochastic model for wind speed data.

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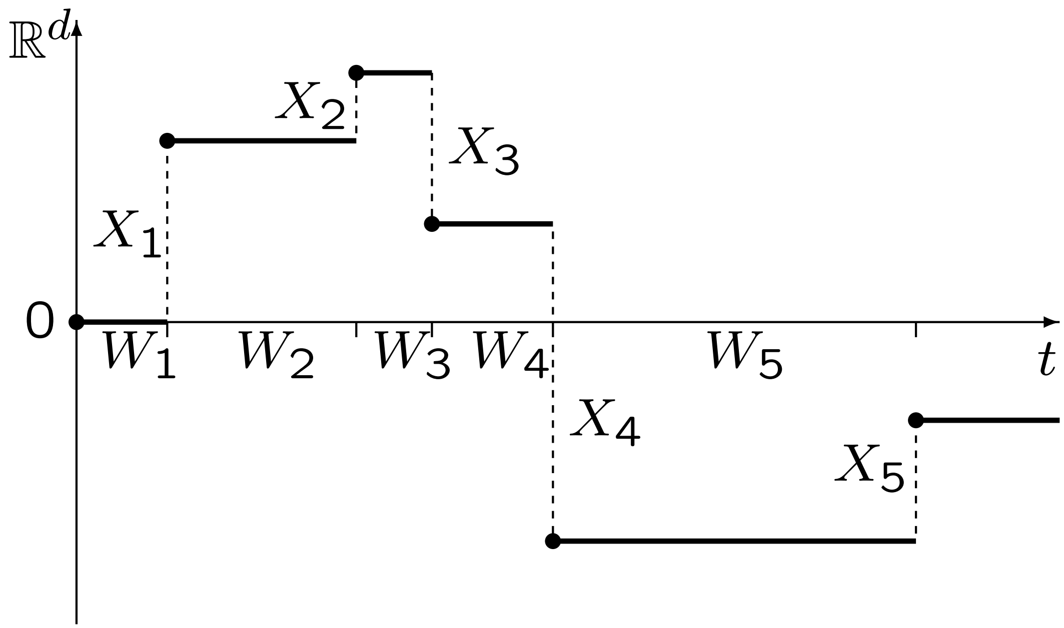
New Books

Stochastic Models for Fractional Calculus
Mark M. Meerschaert and Alla Sikorskii
De Gruyter Studies in Mathematics **43**, 2012
Advanced graduate textbook
ISBN 978-3-11-025869-1

Mathematical Modeling, 4th Edition
Mark M. Meerschaert
Academic Press, Elsevier, 2013
Advanced undergraduate / beginning graduate textbook
(new sections on particle tracking and anomalous diffusion)
ISBN 978-0-12-386912-8

Continuous time random walks



A random particle arrives at location $S(n) = X_1 + \cdots + X_n$ at time $T_n = W_1 + \cdots + W_n$. After $N_t = \max\{n \geq 0 : T_n \leq t\}$ jumps, particle location is $S(N_t)$.

CTRW limit theory

If $P(X_n > x) \approx x^{-\alpha}$ and $E(X_n) = 0$ then $n^{-1/\alpha}S(nu) \Rightarrow A(u)$. The α-stable limit $A(u)$ has pdf $p(x,u)$ with Fourier transform $\widehat{p}(k,u) = e^{u\psi_A(k)}$ and Fourier symbol $\psi_A(k) = (ik)^\alpha$ for $1 < \alpha \leq 2$.

If $P(W_n > t) \approx t^{-\beta}$ then $n^{-1/\beta}T_{nu} \Rightarrow D_u$. The β-stable limit has pdf $g(t,u)$ with Laplace transform $\tilde{g}(s,u) = e^{-u\psi_D(s)}$ and Laplace symbol $\psi_D(s) = s^\beta$ for $0 < \beta < 1$.

Inverse process: $n^{-\beta}N_{nt} \Rightarrow E_t = \inf\{u > 0 : D_u > t\}$, and then

$$n^{-\beta/\alpha}S(N_{nt}) = n^{-\beta/\alpha}S(n^\beta \cdot n^{-\beta}N_{nt}) \approx (n^\beta)^{-1/\alpha}S(n^\beta \cdot E_t) \Rightarrow A(E_t).$$

Since E_t has a pdf $h(u,t)$ with LT $\tilde{h}(u,s) = s^{-1}\psi_D(s)e^{-u\psi_D(s)}$, the CTRW limit $A(E_t)$ has pdf

$$q(x,t) = \int_0^\infty p(x,u)h(u,t)du \approx \sum_u P(A(u) = x | E_t = u)P(E_t = u).$$

Probability and fractional calculus

CTRW limit pdf has Fourier-Laplace transform

$$\begin{aligned}\bar{q}(k,s) &= \int_0^\infty \hat{p}(k,u)\tilde{h}(u,s)du \\ &= \int_0^\infty e^{u(ik)^\alpha} s^{\beta-1} e^{-us^\beta} du = \frac{s^{\beta-1}}{s^\beta - (ik)^\alpha}\end{aligned}$$

Then $s^\beta \bar{q}(k,s) - s^{\beta-1} = (ik)^\alpha \bar{q}(k,s)$ which inverts to

$$\partial_t^\beta q(x,t) = \mathbb{D}_x^\alpha q(x,t),$$

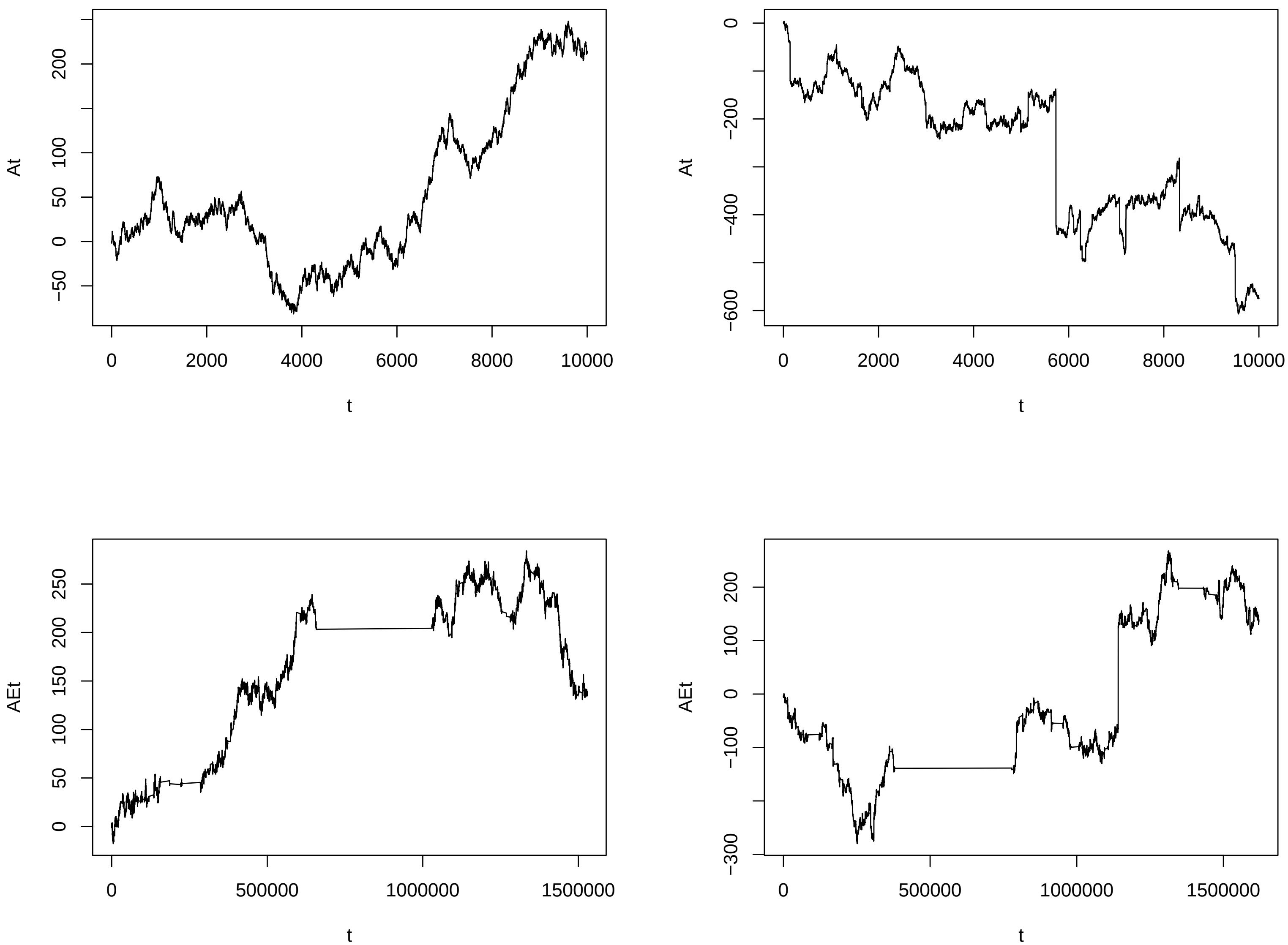
using Caputo on the LHS and Riemann-Liouville on the RHS.

Note ∂_t^β codes power law waiting times, $\mathbb{D}_x^\alpha$ power law jumps.

Reduces to traditional diffusion for $\alpha = 2$ and $\beta = 1$.

Anomalous diffusion processes

Here $\alpha = 2.0, 1.7$ (left/right) and $\beta = 1.0, 0.9$ (top/bottom).



Tempered fractional calculus

Tempered stable D_t has pdf with LT $\tilde{g}(t,u) = e^{-u\psi_D(s)}$ where the Laplace symbol $\psi_D(s) = (\lambda + s)^\beta - \lambda^\beta$ for some $\lambda > 0$ (small).

Tempered stable $A(u)$ has pdf with FT $\hat{p}(k,u) = e^{u\psi_A(k)}$ where the Fourier symbol $\psi_A(k) = (\lambda + ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}$ with $\lambda > 0$.

Again E_t has pdf with LT $\tilde{h}(u,s) = s^{-1}\psi_D(s)e^{-u\psi_D(s)}$, and again $A(E_t)$ has pdf $q(x,t) = \int_0^\infty p(x,u)h(u,t)du$ with FLT

$$\bar{q}(k,s) = \frac{s^{-1}\psi_D(s)}{\psi_D(s) - \psi_A(k)}$$

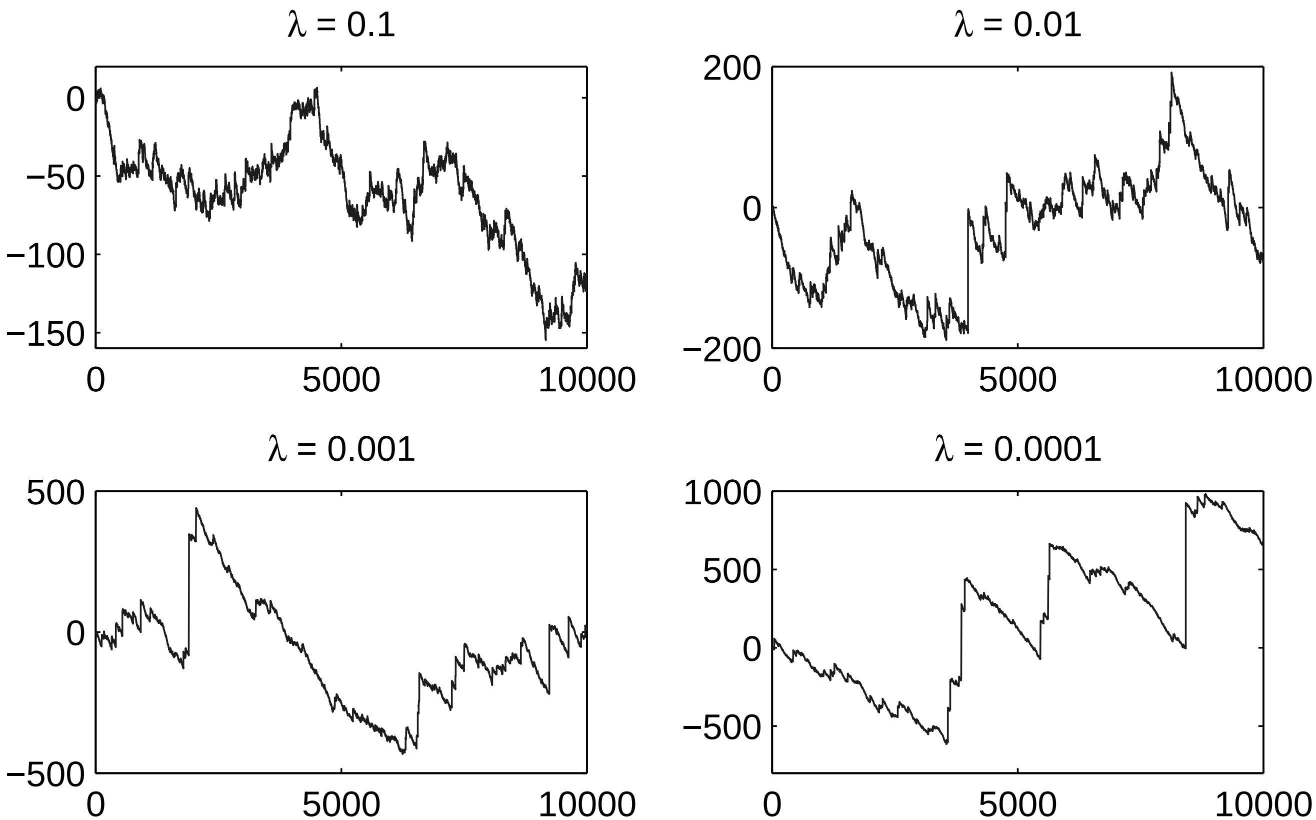
Then $\psi_D(s)\bar{q}(k,s) - s^{-1}\psi_D(s) = \psi_A(k)\bar{q}(k,s)$. Invert to get

$$\partial_t^{\beta,\lambda} q(x,t) = \mathbb{D}_x^{\alpha,\lambda} q(x,t).$$

Tempered jumps $P(X_n > x) \approx x^{-\alpha}e^{-\lambda x}$ and waiting times.

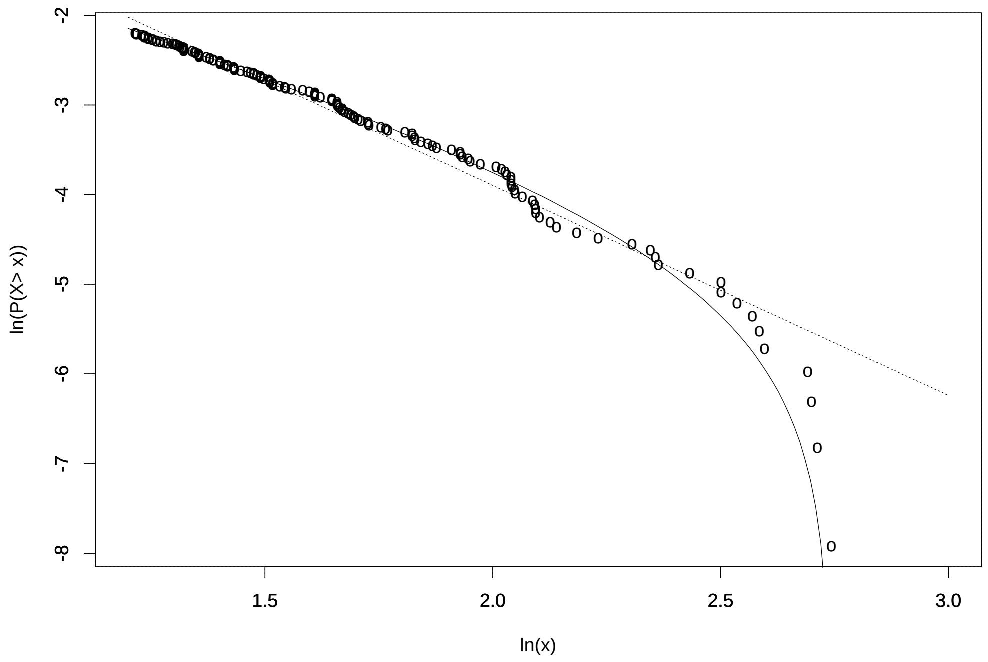
Space fractional tempered stable

Tempered stable Lévy motion with $\alpha = 1.2$



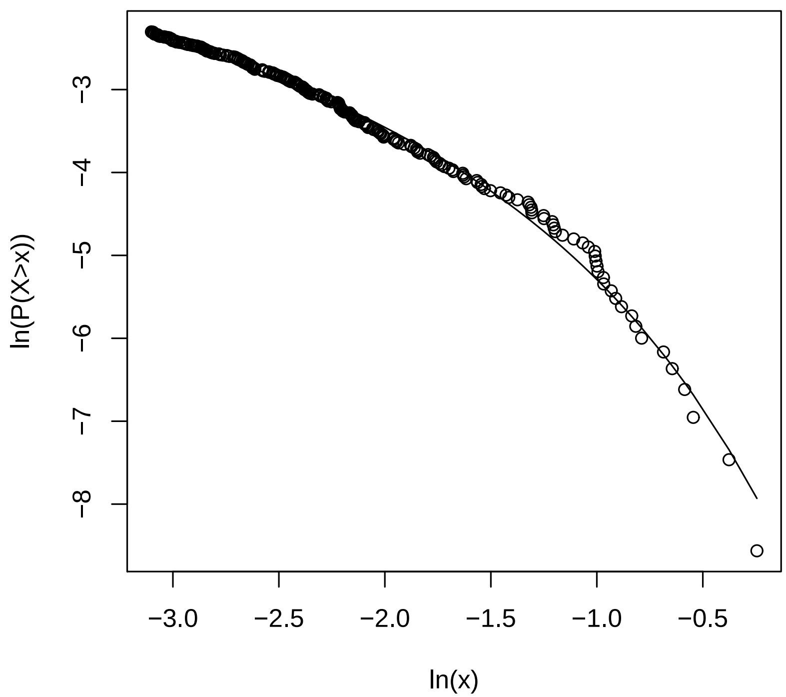
Tempered power laws in finance

AMZN stock price changes fit a tempered power law model $P(X > x) \approx x^{-0.6}e^{-0.3x}$ for x large



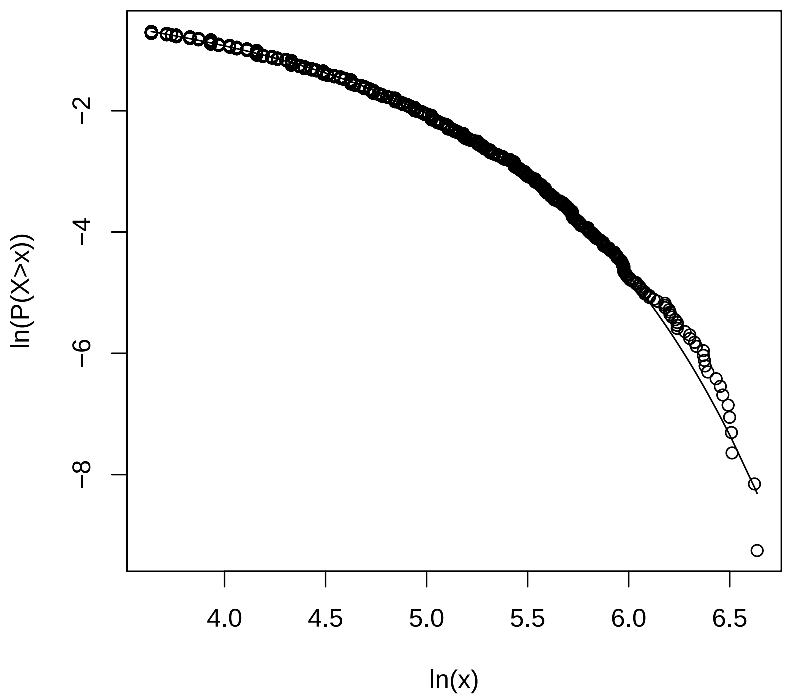
Tempered power laws in hydrology

Tempered power law model $P(X > x) \approx x^{-0.6}e^{-5.2x}$ for increments in hydraulic conductivity at the MADE site.

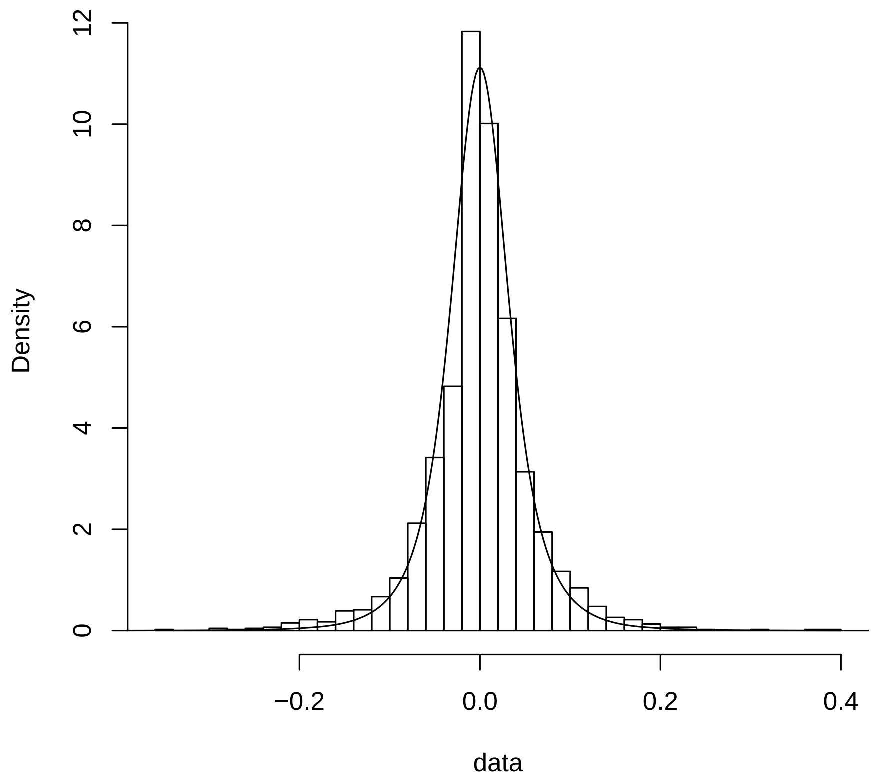


Tempered power laws in atmospheric science

Tempered power law model $P(X > x) \approx x^{-0.2} e^{-0.01x}$ for daily precipitation data at Tombstone AZ.

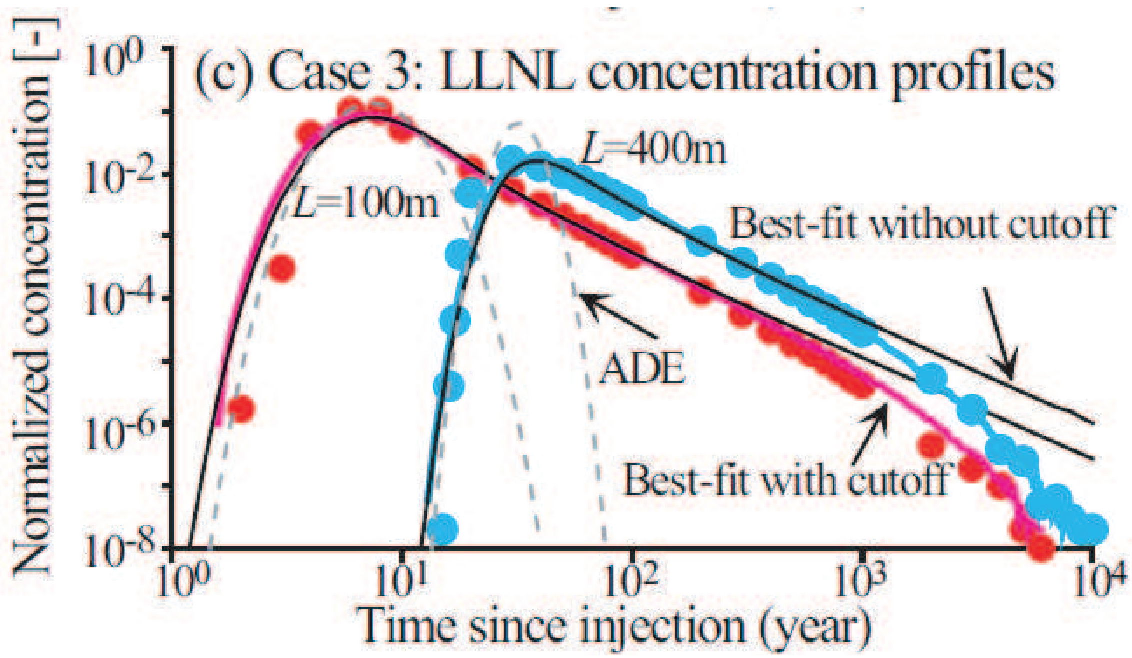


Tempered stable pdf in macroeconomics



One-step BARMA forecast errors for annual inflation rates, and a tempered stable with $\alpha = 1.1$ and $\lambda = 12$ (rough fit).

Tempered time-fractional diffusion model



Fitted concentration data from a 3-D supercomputer simulation. ADE fit uses $\alpha = 2, \beta = 1$. Without cutoff uses $\lambda = 0$.

Numerical codes for fractional diffusion

For $\alpha > 0$ we have $\mathbb{D}_x^\alpha f(x) = \lim_{h\to 0} h^{-\alpha}\Delta^\alpha f(x)$ where

$$\Delta^\alpha f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j f(x - jh), \qquad \binom{\alpha}{j} = \frac{\Gamma(\alpha+1)}{j!\Gamma(\alpha - j + 1)}$$

Use to construct explicit/implicit finite difference codes.

These codes are *mass-preserving* since $\displaystyle\sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j = 0.$

Change jh to $(j - \lfloor\alpha\rfloor)h$ for unconditional stability (implicit Euler).

Use operator splitting for 2-d, 3-d, or reaction term.

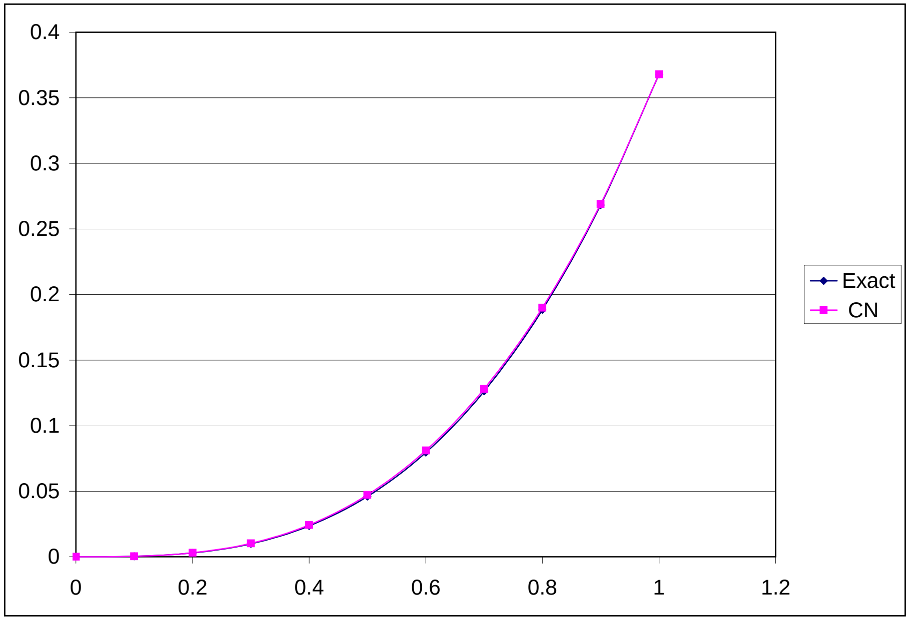
Finite element and finite volume methods also available.

Numerical example

Exact solution $p(x,t) = e^{-t}x^3$ and numerical approximation to

$$\frac{\partial p(x,t)}{\partial t} = d(x)\frac{\partial^{1.8} p(x,t)}{\partial x^{1.8}} + r(x,t)$$

on $0 < x < 1$ with $p(x,0) = x^3$, $p(0,t) = 0$, $p(1,t) = e^{-t}$, $r(x,t) = -(1+x)e^{-t}x^3$, $d(x) = \Gamma(2.2)x^{2.8}/6$.



Tempered fractional diffusion codes

For $0 < \alpha < 1$, $\lim_{h\to 0} h^{-\alpha}\Delta^{\alpha,\lambda} f(x) = \mathbb{D}_x^{\alpha,\lambda} f(x)$ where

$$\Delta^{\alpha,\lambda} f(x) = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} f(x - jh) - (1 - e^{-\lambda h})^{\alpha} f(x).$$

Codes are mass-preserving since (by the Binomial formula)

$$\sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j e^{-\lambda j h} = (1 - e^{-\lambda h})^{\alpha}$$

Stable, consistent implicit Euler codes: shift $jh \to (j - \lfloor \alpha \rfloor)h$.

Crank-Nicolson codes are $O(\Delta t^2 + \Delta x)$.

Apply Richardson extrapolation to get $O(\Delta t^2 + \Delta x^2)$.

For $1 < \alpha < 2$, $h^{-\alpha}\Delta^{\alpha,\lambda} f(x) \to \mathbb{D}_x^{\alpha,\lambda} f(x) + \alpha\lambda^{\alpha-1} f'(x)$.

Numerical example

Exact solution $u(x,t) = x^\beta e^{-\lambda x - t}/\Gamma(1+\beta)$ and Crank-Nicolson solution (extrapolated) to

$$\partial_t p(x,t) = c(x)\mathbb{D}_x^{\alpha,\lambda} p(x,t) + r(x,t)$$

on $0 < x < 1$ with $\alpha = 1,6$, $\lambda = 2$, $\beta = 2.8$, $p(x,0) = x^\beta e^{-\lambda x}/\Gamma(\beta+1)$, $p(0,t) = 0$, $p(1,t) = e^{-\lambda - t}/\Gamma(\beta+1)$, $c(x) = x^\alpha \Gamma(1+\beta-\alpha)/\Gamma(\beta+1)$, and $r(x,t) = r_1(x,t)e^{-\lambda x - t}\Gamma(1+\beta-\alpha)/\Gamma(\beta+1)$ where

$$r_1(x,t) = \frac{(1-\alpha)\lambda^\alpha x^{\alpha+\beta}}{\Gamma(\beta+1)} + \frac{\alpha\beta\lambda^{\alpha-1}x^{\alpha+\beta-1}}{\Gamma(\beta)} - \frac{2x^\beta}{\Gamma(1+\beta-\alpha)}.$$

Δt	Δx	CN Max Error	Rate	CNX Max Error	Rate
$1/10$	$1/50$	7.7738×10^{-5}	–	2.8514×10^{-6}	–
$1/20$	$1/100$	3.8353×10^{-5}	2.03	7.2120×10^{-7}	3.95
$1/40$	$1/200$	1.9055×10^{-5}	2.01	1.8157×10^{-7}	3.97
$1/80$	$1/400$	9.4976×10^{-6}	2.01	4.5555×10^{-8}	3.99

Fractional derivatives: Integral forms

In the simplest case $0 < \alpha < 1$ the *generator form* is

$$\mathbb{D}_x^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big) y^{-\alpha-1} dy.$$

Check: Since $f(x-y)$ has FT $e^{-iky}\widehat{f}(k)$, then the RHS has FT

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-iky}\right) \widehat{f}(k) y^{-\alpha-1} dy.$$

For $\lambda > 0$ it is not hard to compute [Prop. 3.10 in MS 2012]

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-(\lambda+ik)y}\right) y^{-\alpha-1} dy = (\lambda + ik)^\alpha$$

and then (let $\lambda \to 0$) $\mathbb{D}_x^\alpha f(x)$ has FT $(ik)^\alpha \widehat{f}(k)$. For $1 < \alpha < 2$

$$\mathbb{D}_x^\alpha f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \Big(f(x-y) - f(x) + yf'(x)\Big) y^{-\alpha-1} dy$$

and again the RHS has FT $(ik)^\alpha \widehat{f}(k)$.

Tempered fractional derivatives

In the simplest case $0 < \alpha < 1$ the generator form is

$$\mathbb{D}_x^{\alpha,\lambda} f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big) e^{-\lambda y} y^{-\alpha-1} dy.$$

Check: The integral on the RHS has Fourier transform

$$\frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left(1 - e^{-iky}\right) \hat{f}(k) e^{-\lambda y} y^{-\alpha-1} dy$$

$$= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \left((1 - e^{-(\lambda+ik)y}) - (1 - e^{-\lambda y})\right) \hat{f}(k) y^{-\alpha-1} dy$$

which reduces to $[(\lambda + ik)^\alpha - \lambda^\alpha]\hat{f}(k)$.

For $1 < \alpha < 2$ we have

$$\mathbb{D}_x^{\alpha,\lambda} f(x) = \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} \int_0^\infty \Big(f(x-y) - f(x) + y f'(x)\Big) e^{-\lambda y} y^{-\alpha-1} dy$$

and the RHS has FT $[(\lambda + ik)^\alpha - \lambda^\alpha - ik\alpha\lambda^{\alpha-1}]\hat{f}(k)$.

Laplace transform approach

Since $e^{\lambda t} f(t)$ has Laplace transform $\tilde{f}(s-\lambda)$ and since $\mathbb{D}_t^\alpha f(t)$ has LT $s^\alpha \tilde{f}(s)$, we see that

$$\int_0^\infty e^{-st} \mathbb{D}_t^\alpha \Big[e^{\lambda t} f(t) \Big] \, dt = s^\alpha \tilde{f}(s-\lambda).$$

Using the shift property one more time, we see that

$$\int_0^\infty e^{-st} e^{-\lambda t} \mathbb{D}_t^\alpha \Big[e^{\lambda t} f(t) \Big] \, dt = (s+\lambda)^\alpha \tilde{f}(s).$$

Then for $0 < \alpha \leq 1$ we have

$$\mathbb{D}_t^{\alpha,\lambda} f(t) = e^{-\lambda t} \, \mathbb{D}_t^\alpha \Big[e^{\lambda t} f(t) \Big] - \lambda^\alpha f(t)$$

and for $1 < \alpha \leq 2$ we have

$$\mathbb{D}_x^{\alpha,\lambda} f(x) = e^{-\lambda x} \, \mathbb{D}_x^\alpha \Big[e^{\lambda x} f(x) \Big] - \lambda^\alpha f(x) - \alpha \lambda^{\alpha-1} f'(x).$$

Fractional derivatives (other integral forms)

Recall that in the simplest case $0 < \alpha < 1$ the generator form

$$\mathbb{D}_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \Big(f(x) - f(x-y)\Big)\alpha y^{-\alpha-1} dy.$$

Integrate by parts $\int u dv = uv - \int v du$ with $u = f(x) - f(x-y)$ and $dv = \alpha y^{-\alpha-1} dy$ to get the Caputo form

$$\frac{1}{\Gamma(1-\alpha)} \int_0^\infty f'(x-y) y^{-\alpha} dy = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^\infty f'(u)(x-u)_+^{-\alpha} du$$

where $(x-u)_+ = (x-u)$ for $x > u$ and $(x-u)_+ = 0$ otherwise.

Move the derivative outside to get the Riemann-Liouville form

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^\infty f(x-y) y^{-\alpha} dy = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^\infty f(u)(x-u)_+^{-\alpha} dy.$$

Caputo is $\mathbb{I}_x^{1-\alpha}\mathbb{D}_x^1 f(x)$ and Riemann-Liouville is $\mathbb{D}_x^1\mathbb{I}_x^{1-\alpha} f(x)$ where

$$\mathbb{I}_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^\infty f(u)(x-u)_+^{\alpha-1} du.$$

Fractional Brownian motion

Given Z_n iid with mean zero and finite variance, the time series

$$X_t = \Delta^\alpha Z_n = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j Z_{t-j}$$

is long range dependent: $E(X_t X_{t+j}) \approx |j|^{2H-2}$ with $H = 1/2 - \alpha$.

Then $n^{-H}(X_1 + \cdots + X_{[nt]}) \Rightarrow B_H(t)$ fractional Brownian motion.

Here $B_H(t) = \partial_t^\alpha B(t) - \partial_t^\alpha B(0)$ where $B(t)$ is a Brownian motion:

$$B_H(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \left[(t-u)_+^{-\alpha} - (0-u)_+^{-\alpha} \right] B(du)$$

with $B(du) = B'(u)du$ in the distributional sense (Caputo).

Hence FBM is the fractional integral of the white noise $B(du)$.

Tempered fractional Brownian motion

Tempered fractional Brownian motion with $-1/2 < \alpha < 1/2$:

$$B_{\alpha,\lambda}(t) = \int_{-\infty}^{+\infty} \left[e^{-\lambda(t-u)_+}(t-u)_+^{-\alpha} - e^{-\lambda(0-u)_+}(0-u)_+^{-\alpha} \right] \, B(du).$$

TFBM is the tempered fractional integral of white noise $B(du)$:

$$\mathbb{I}_t^{\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} f(u) e^{-\lambda(t-u)_+}(t-u)_+^{\alpha-1} du.$$

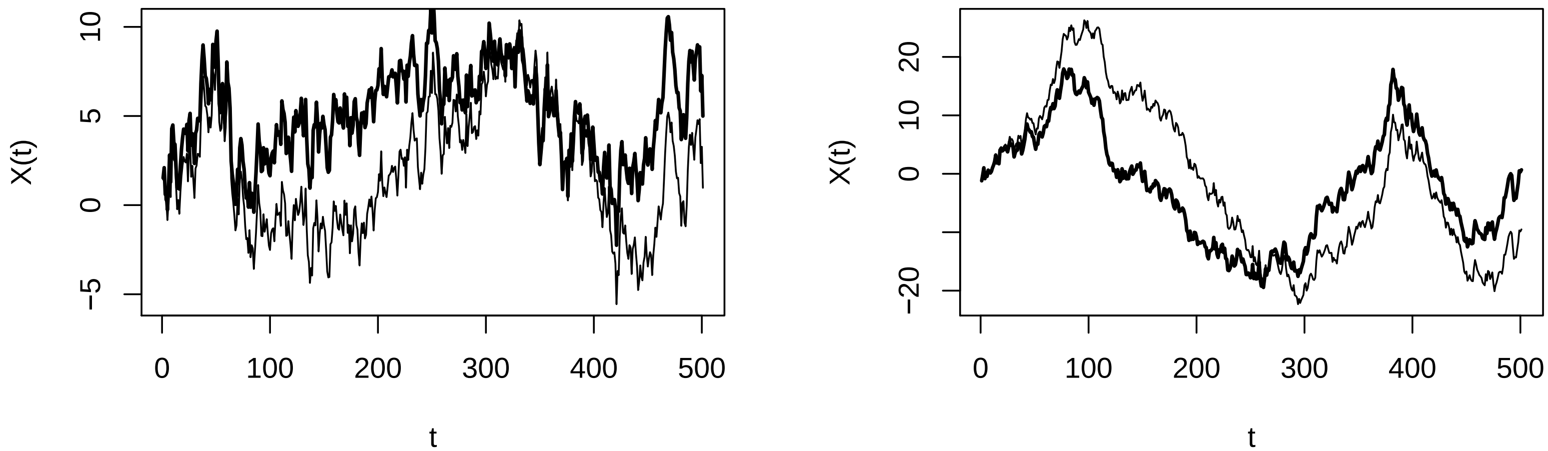
Tempered fractional Gaussian noise: $X_t = B_{\alpha,\lambda}(t) - B_{\alpha,\lambda}(t-1)$.

Semi-long range dependence: $E(X_t X_{t+j}) \approx |j|^{2H-2}$ for j small to moderate, then $E(X_t X_{t+j}) \approx |j|^{-2}$ as $j \to \infty$, where $H = 1/2 - \alpha$.

For any Z_n iid with mean zero and finite variance, the time series $X_t = \Delta^{\alpha,\lambda} Z_n$ also exhibits semi-long range dependence.

FBM and tempered FBM

FBM (thin line) and TFBM (thick line) with same $B(du)$ for $H = 0.3, \lambda = 0.03$ (left) and $H = 0.7, \lambda = 0.01$ (right).

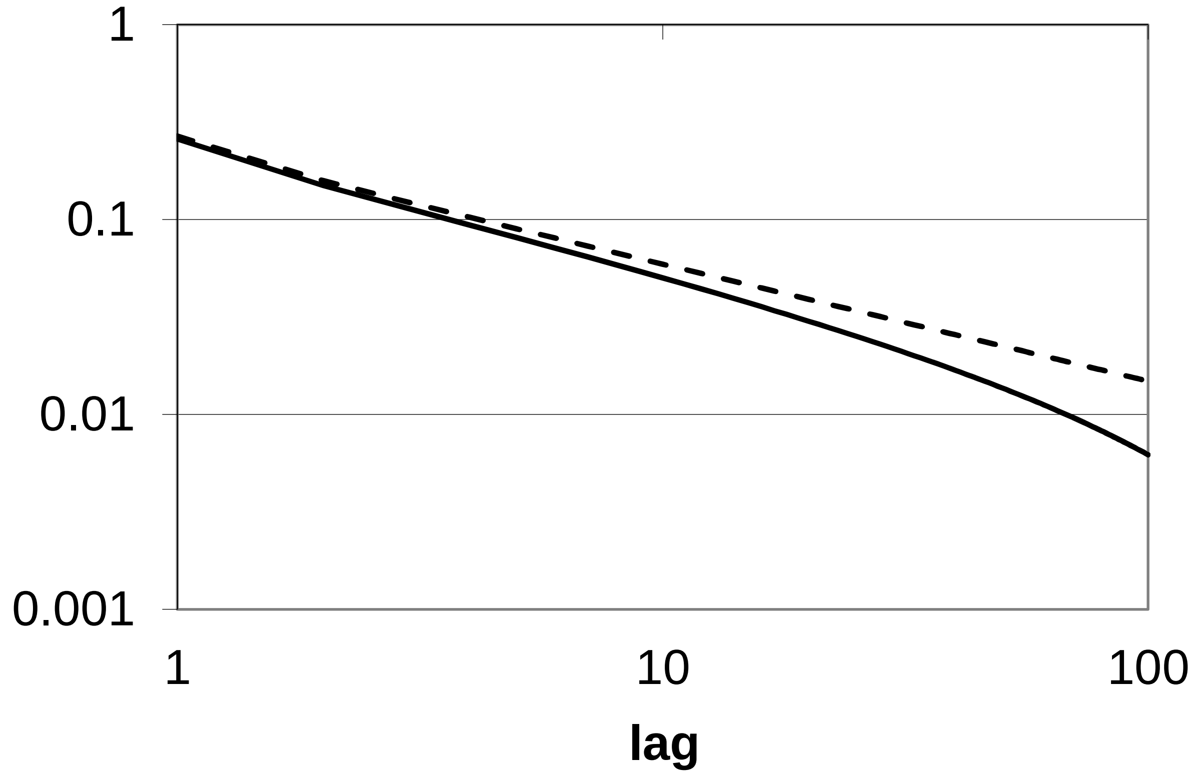


Sample paths are Hölder continuous of order H.

Scaling: $B_H(ct) \stackrel{d}{=} c^H B_H(t)$ and $B_{\alpha,\lambda}(ct) \stackrel{d}{=} c^H B_{\alpha,c\lambda}(t)$.

Semi-long range dependence

The covariance functions $E(X_t X_{t+j})$ for FGN and TFGN with $H = 0.7$ (and $\lambda = 0.001$) are quite similar until j gets large.



Spectral density

FGN spectral density blows up at low frequencies for $H > 1/2$:

$$f_H(\omega) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{i\omega j} E(X_t X_{t+j}) \approx |\omega|^{1-2H} \quad \text{as } \omega \to 0.$$

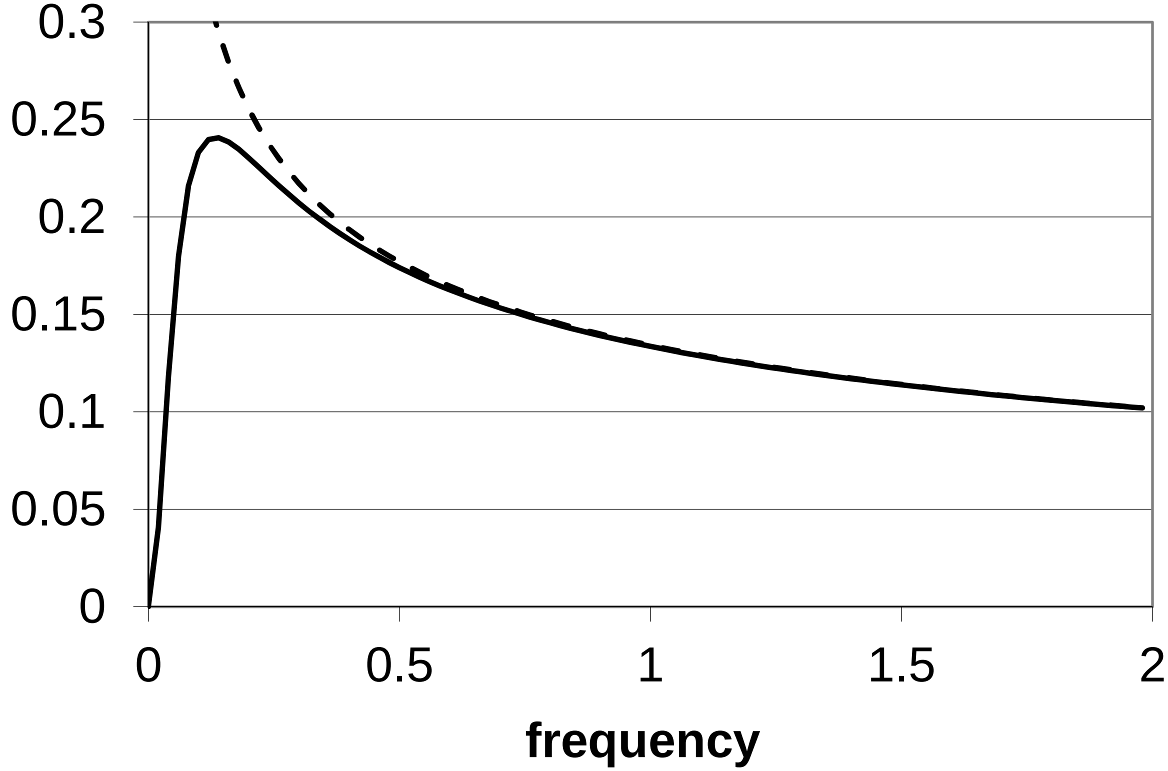
For TFGN with $\lambda \approx 0$ we get a similar result

$$f_{\alpha,\lambda}(\omega) \approx \frac{\omega^2}{[\lambda^2 + \omega^2]^{H+1/2}} \quad \text{as } \omega \to 0.$$

Hence $f_{\alpha,\lambda}(\omega) \approx |\omega|^{1-2H}$ for moderate frequencies, but remains bounded for very low frequencies (Davenport spectrum).

Spectral density comparison

Spectral density for FGN and TFGN with $H = 0.7, \lambda = 0.06$.



Davenport spectrum

Kolmogorov invented FBM to model turbulence in the inertial subrange. The Davenport spectrum $f(\omega) \approx \omega^2/[1+\omega^2]^{H+1/2}$ extends the model to include production and dissipation. Since TFBM has *the same* spectral density, it provides a comprehensive stochastic model for turbulence.

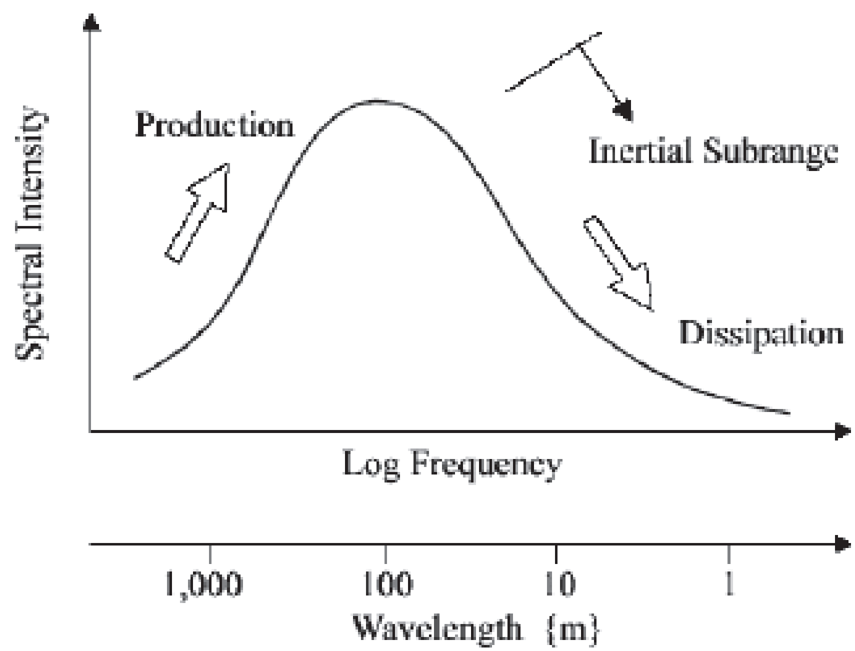
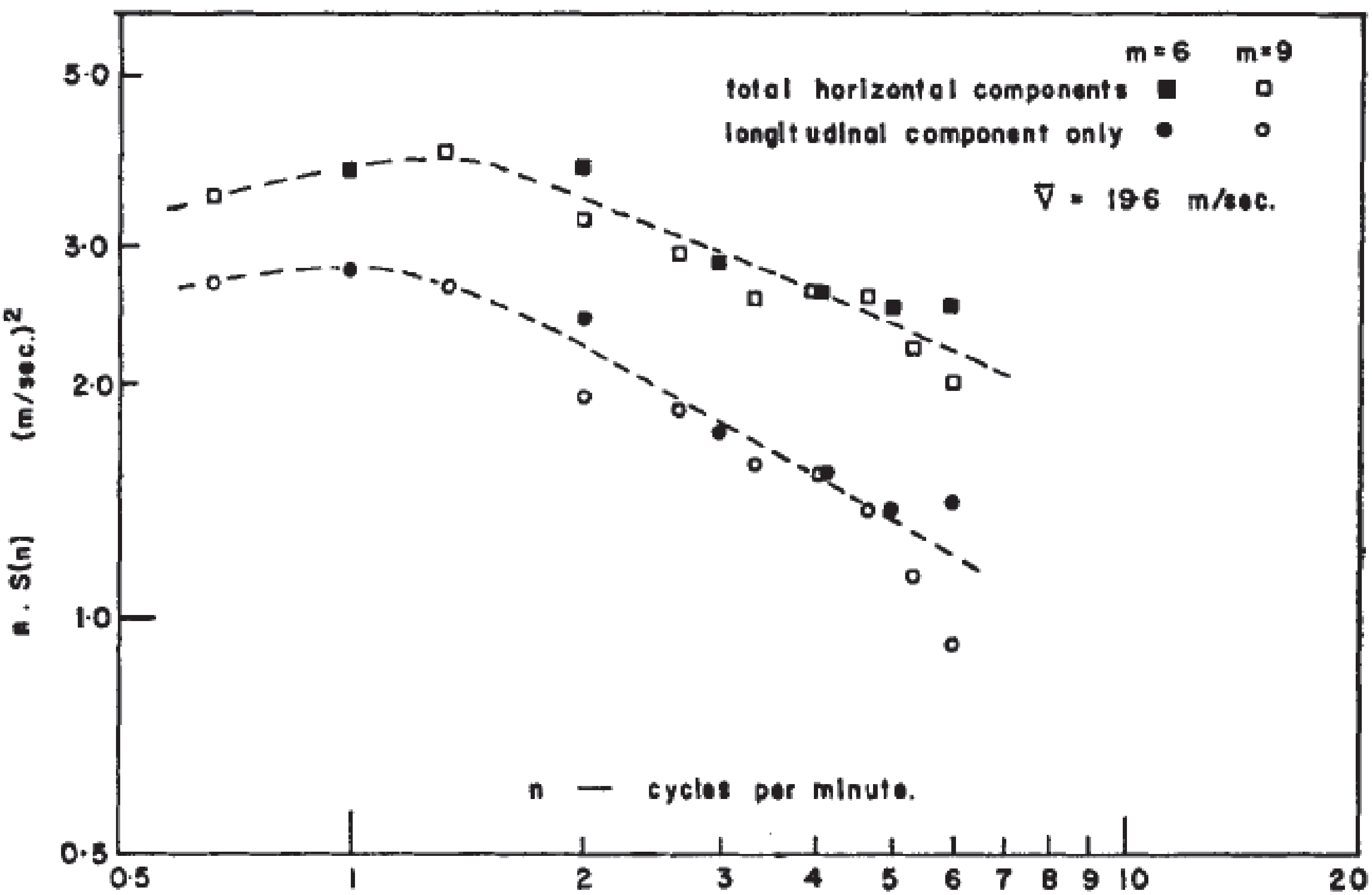


Figure reproduced from Beaupuits et al. (2004).

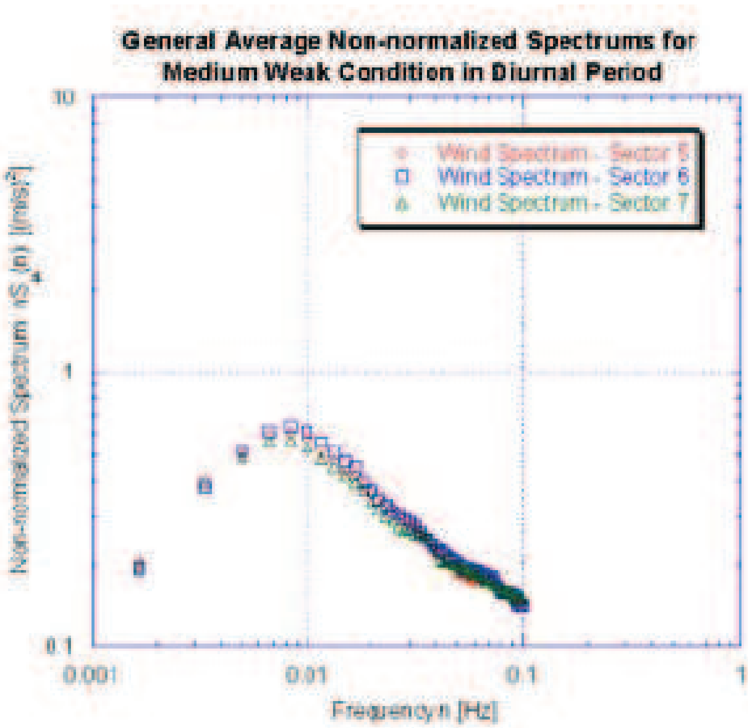
Davenport spectrum for wind gusts

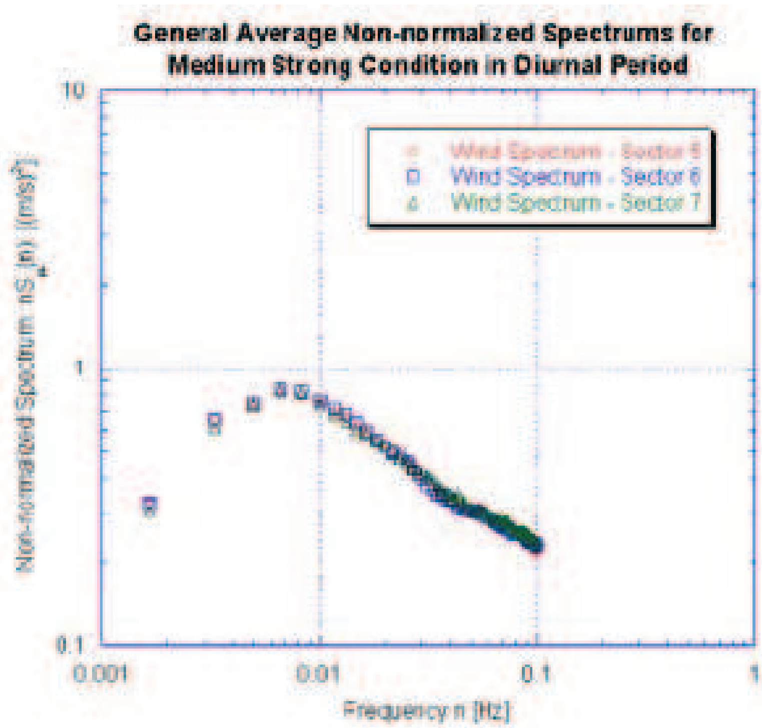
Spectral density of wind gustiness from Davenport (1961). TFBM provides a stochastic model for the Davenport spectrum.



Wind power study

Spectral density of wind speed at the Chajnantor radio telescope site in Chile.





Summary

- Tempered power laws
- Tempered fractional derivatives
- Numerical methods
- Tempered fractional Brownian motion
- Davenport model for wind speed

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Simulating tempered stable laws

Simulation codes for stable random variates are widely available.

If $X > 0$ has stable density density $f(x)$, TS density is

$$f_\lambda(x) = \frac{e^{-\lambda x} f(x)}{\int_0^\infty e^{-\lambda u} f(u)\, du}$$

Take $Y \sim \exp(\lambda)$ independent of X, (X_i, Y_i) IID with (X, Y).

Let $N = \min\{n : X_n \leq Y_n\}$. Then $X_N \sim f_\lambda(x)$.

Proof: Compute $P(X_N \leq x) = P(X \leq x | X \leq Y)$ by conditioning, then take d/dx to verify.

Triangular array scheme (SPL 2011)

Take $P(X > x) \approx Cx^{-\alpha}$ with $1 < \alpha < 2$. Triangular array limit

$$\sum_{k=1}^{[nt]} n^{-1/\alpha} X_k - b_t^{(n)} \Rightarrow A(t)$$

is stable. Define tempering variables:

$$P(Z > u) = u^\alpha \int_u^\infty r^{-\alpha-1} e^{-\lambda r} dr$$

Replace $n^{-1/\alpha} X_k$ by Z_k if $n^{-1/\alpha} X_k > Z_k$.

Triangular array limit is tempered stable.

Exponential tempering: sum of α and $\alpha - 1$ tempered stables.

Tail estimation (CIS 2012)

Hill-type estimator: Assume $P(X > x) \approx Cx^{-\alpha}e^{-\lambda x}$ for x large, use order statistics $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$.

Conditional MLE given $X_{(n-k+1)} > L \ge X_{(n-k)}$:

$$T_1 := \sum_{i=1}^{k} (\log X_{(n-i+1)} - \log L)$$
$$T_2 := \sum_{i=1}^{k} (X_{(n-i+1)} - L)$$
$$1 = \sum_{i=1}^{k} \frac{x_{(n-i+1)}}{kx_{(n-i+1)} + \widehat{\alpha}(T_2 - T_1 x_{(n-i+1)})}$$
$$\widehat{\lambda} = (k - \widehat{\alpha}T_1)/T_2$$
$$\widehat{C} = \frac{k}{n} L^{\widehat{\alpha}} e^{\widehat{\lambda} L}$$

R code available at `www.stt.msu.edu/users/mcubed/TempParetoR.zip`

Testing for pure power law tail (JASA 06)

Null hypthesis $H_0 : P(X > x) = Cx^{-\alpha}$ Pareto for $x > L$.

Test based on extreme value theory rejects H_0 if

$$X_{(1)} < \left(\frac{nC}{-\ln q} \right)^{1/\alpha}$$

where α, C can be estimated using Hill's estimator

$$\widehat{\alpha}_H = \left[k^{-1} \sum_{i=1}^{k} \{ \ln X_{(n-i+1)} - \ln X_{(n-k)} \} \right]^{-1}$$
$$\widehat{C}_H = (k/n)(X_{(n-k)})^{\widehat{\alpha}_H}$$

Simple p-value formula $p = \exp\{-n\, C\, X_{(n)}^{-\alpha}\}$.