1. Introduction

Reaction-diffusion models have found numerous applications in patten formation in biology, chemistry, physics and Engineering (see for example [?]). These systems show that diffusion can produce the spontaneous formation of spatial-temporal patterns. The simplest nonlinear reaction-diffusion model can be described by

$$\frac{\partial u}{\partial t} = K\Delta u + f(u), \tag{1}$$

where Δ is a Laplace operator, $0 < K$ is a diffusion constants and $f(u)$ is a nonlinear function representing the reaction kinetics. Here $f(u)$ is a source term that is assumed to satisfy the Lipschitz condition, i.e.,

$$\forall u_1, u_2, \quad |f(u_1, x, y, t) - f(u_2, x, y, t)| \leq L|u_1 - u_2|.$$

In the last decade, fractional models, because of their ability to model anomalous transport phenomena, have attracted considerable interest and have played a very important role in various fields of science and engineering. Recently, a growing number of works by many authors from various fields, such as system biology (see [?]), physics (see [?]), chemistry and biochemistry (see [?]), finance (see [?]), hydrology (see [?]) and thermodynamics (see [?]), deal with dynamical systems described by fractional differential equations. Fractional-order models provide an excellent instrument for describing the memory and hereditary properties of various processes. This is the main advantage of fractional models in comparison with their classical integer-order counterparts, in which such effects are neglected.

Ochoa-Tapia et al. [?] proposed a fractional Darcy's law to simulate shear stress phenomena in heterogeneous porous media. Valdes-Parada et al. [?] proposed a fractional Fick's law to simulate diffusion phenomena in disordered porous media. These anomalous diffusion behavior may be due to different mechanisms. Cushman et al. [?] have used the fractional Darcy's law or fractional Fick's law to derive fractional models to describe diffusion or transport behavior in complex heterogeneous structures or disordered porous media. Many fractional models have presented symmetric space fractional diffusion equations with a nonlinear reaction term to model many problems of practical interest. Construction of a mathematical model incorporating reactions is straightforward for superdiffusive systems where diffusion is characterized by a spatial non-locality but no memory is presented by Del-Castillo-Negrete in [?]. A random walk with a Lévy distribution of jump lengths is described on large scales by a diffusion equation with a Riesz fractional derivative. Some space Riesz fractional diffusion equations have been proposed and used in natural systems including heterogeneous soils, aquifers, and rivers, is typically observed to be non-Fickian, also called anomalous (see [?]). A one-component system is governed by the equation

$$\frac{\partial u}{\partial t} = -K(-\Delta)^{\frac{\alpha}{2}} u + f(u, t), \tag{2}$$

where $1 < \alpha \leq 2$.

In this paper, we consider two fractional Riesz space nonlinear reaction-diffusion models with zero-Dirichlet boundary conditions on finite domains, i.e., Model-1: one-dimensional variable order fractional Riesz space nonlinear reaction-diffusion model and Model-2: two-dimensional fractional Riesz space nonlinear reaction-diffusion model.

Model 1: One-dimensional variable order fractional Riesz space nonlinear reaction-diffusion model.

Model 2: A two-dimensional fractional Riesz space nonlinear reaction-diffusion model.

The remainder of this paper is arranged as follows. Some mathematical preliminaries are introduced in Section 2. In Section 3, we propose an implicit numerical method and an alternating direction method for Model-1, respectively. The Matrix transform techniques for Models 2 and 3 are introduced in Sections 4 and 5, respectively. Finally, some numerical results are given to assess the behaviours of these models with zero Dirichlet boundary conditions.

2. Preliminary knowledge

In this section, we outline some preliminary knowledge used throughout the remaining sections of this paper.

Definition 2.1. *Let $u(x, y, t)$ be defined on an infinite interval $\Omega : -\infty < x, y < +\infty$. The Riesz fractional operator $\frac{\partial^\alpha u}{\partial |x|^\alpha}$ is defined as [**?**]*

$$\frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} = -c_\alpha ({}_{-\infty}D_x^\alpha +_x D_{+\infty}^\alpha) u(x, y, t),$$

where $c_\alpha = \frac{1}{2\cos(\frac{\pi\alpha}{2})}$,

$$\begin{aligned}
{}_{-\infty}D_x^\alpha u(x, y, t) &= \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{x} \frac{u(\xi, y, t) d\xi}{(x-\xi)^{\alpha-1}}, \\
{}_x D_{+\infty}^\alpha u(x, y, t) &= \frac{(-1)^2}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{x}^{+\infty} \frac{u(\xi, y, t) d\xi}{(\xi - x)^{\alpha-1}}.
\end{aligned}$$

Similarly, we can define the Riesz fractional derivative $\frac{\partial^\alpha u(x,y,t)}{\partial |y|^\alpha}$ of order α $(1 < \alpha \le 2)$ with respect to y on an infinite interval.

Definition 2.2. *Let $u(x, y, t)$ be defined on a finite domain $\Omega : a < x < b; c < y < d$ and assume that $u(x, y, t) = 0$ on the boundary points and beyond the boundary points. The Riesz fractional operators $\frac{\partial^\alpha u}{\partial |x|^\alpha}$ and $\frac{\partial^\alpha u}{\partial |y|^\alpha}$ $(1 < \alpha \le 2)$ are defined as [**?**]*

$$\begin{aligned}
\frac{\partial^\alpha u(x, y, t)}{\partial |x|^\alpha} &= -\frac{1}{2\cos\frac{\pi\alpha}{2}} [{}_a D_x^\alpha + {}_x D_b^\alpha] u(x, y, t), \\
\frac{\partial^\alpha u(x, y, t)}{\partial |y|^\alpha} &= -\frac{1}{2\cos\frac{\pi\alpha}{2}} [{}_c D_y^\alpha + {}_y D_d^\alpha] u(x, y, t).
\end{aligned}$$

where

$$\begin{aligned}
{}_aD_x^{\alpha}u(x,y,t) &= \frac{\partial^{\alpha}u(x,y,t)}{\partial x^{\alpha}} = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\xi,y,t)d\xi}{(x-\xi)^{\alpha-1}},\\
{}_xD_b^{\alpha}u(x,y,t) &= \frac{\partial^{\alpha}u(x,y,t)}{\partial(-x)^{\alpha}} = \frac{(-1)^2}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,y,t)d\xi}{(\xi-x)^{\alpha-1}},\\
{}_cD_y^{\alpha}u(x,y,t) &= \frac{\partial^{\alpha}u(x,y,t)}{\partial y^{\alpha}} = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_c^y \frac{u(x,\zeta,t)d\zeta}{(y-\zeta)^{\alpha-1}},\\
{}_yD_d^{\alpha}u(x,y,t) &= \frac{\partial^{\alpha}u(x,y,t)}{\partial(-y)^{\alpha}} = \frac{(-1)^2}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial y^2}\int_y^d \frac{u(x,\zeta,t)d\zeta}{(\zeta-y)^{\alpha-1}}.
\end{aligned}$$

3. Model-1: A two-dimensional nonlinear space fractional model with the Riesz fractional derivative

In this paper, we consider the following two-dimensional nonlinear space fractional model with the Riesz fractional derivative:

$$\frac{\partial u}{\partial t} = K_x\frac{\partial^{\alpha}u}{\partial|x|^{\alpha}} + K_y\frac{\partial^{\alpha}u}{\partial|y|^{\alpha}} + f(u,x,y,t), \tag{3}$$

with initial conditions

$$u(x,y,0) = \psi(x,y), \ \ a \le x \le b, \ \ c \le y \le d, \tag{4}$$

and zero Dirichlet boundary conditions

$$u(a,y,t) = u(b,y,t) = u(x,c,t) = u(x,d,t) = 0, \tag{5}$$

where $1 < \alpha \le 2$ and $0 < K_x$, $0 < K_y$ are diffusion coefficients.

3.1. An implicit numerical method for Model 1

For the numerical simulation of equation (**??**), let $h_x = (b-a)/m_1$, $h_y = (d-c)/m_2$ and $\tau = T/N$ be the space and time grid sizes, respectively; $x_i = ih_x, i = 0,1,\cdots,m_1$; $y_j = jh_y, j = 0,1,\cdots,m_2$. Define $u_{i,j}^n$ as the numerical approximation to $u(x,y,t)$ at $x = x_i$, $y = y_j$, $t = t_n$. The initial conditions are set by $u_{i,j}^0 = \psi(x_i,y_j)$.

We use the shifted *Grünwald – Letnikov* schemes [**?**] to discretise the Riesz fractional derivatives as

$$\begin{aligned}
\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\Big|_{(x_i,y_j,t_n)} &= \frac{1}{(h_x)^{\alpha}}\sum_{l=0}^{i+1} g_{\alpha}^{(l)}u(x_{i-l+1},y_j,t_n) + O(h_x), &(6)\\
\frac{\partial^{\alpha}u}{\partial(-x)^{\alpha}}\Big|_{(x_i,y_j,t_n)} &= \frac{1}{(h_x)^{\alpha}}\sum_{l=0}^{m_1-i+1} g_{\alpha}^{(l)}u(x_{i+l-1},y_j,t_n) + O(h_x). &(7)
\end{aligned}$$

A similar result holds in the y-direction. Here $g_{\alpha_1}^{(0)} = 1$ and

$$g_{\alpha}^{(l)} = (-1)^l\begin{pmatrix}\alpha\\ l\end{pmatrix} = -g_{\alpha}^{(l-1)}\frac{\alpha-l+1}{l}. \tag{8}$$

Using these shifted *Grünwald − Letnikov* schemes, we obtain the discrete form of (**??**) as

$$\begin{aligned}
& u(x_i, y_j, t_n) = u(x_i, y_j, t_{n-1}) \\
+ \;& r^{(1)}_{i,j}\Big[\sum_{l=0}^{i+1} g^{(l)}_\alpha u(x_{i-l+1}, y_j, t_n) + \sum_{l=0}^{m_1-i+1} g^{(l)}_\alpha u(x_{i+l-1}, y_j, t_n)\Big] \\
+ \;& r^{(2)}_{i,j}\Big[\sum_{l=0}^{j+1} g^{(l)}_\alpha u(x_i, y_{j-l+1}, t_n) + \sum_{l=0}^{m_2-j+1} g^{(l)}_\alpha u(x_i, y_{j+l-1}, t_n)\Big] \\
+ \;& \tau f(u(x_i, y_j, t_{n-1}), x_i, y_j, t_{n-1}) + R_{i,j,n},
\end{aligned} \tag{9}$$

where $r^{(1)}_{i,j} = \frac{\tau K_x(x_i,y_j)c_\alpha}{(h_x)^\alpha}$, $r^{(2)}_{i,j} = \frac{\tau K_y(x_i,y_j)c_\alpha}{(h_y)^\alpha}$ and $|R_{i,j,n}| \le C(\tau^2 + \tau h_x + \tau h_y)$.

Thus, the implicit numerical method can be written in the following form:

$$\begin{aligned}
u^n_{i,j} = u^{n-1}_{i,j} + r^{(1)}_{i,j}\Big[\sum_{l=0}^{i+1} g^{(l)}_\alpha u^n_{i-l+1,j} + \sum_{l=0}^{m_1-i+1} g^{(l)}_\alpha u^n_{i+l-1,j}\Big] \\
+ \; r^{(2)}_{i,j}\Big[\sum_{l=0}^{j+1} g^{(l)}_\alpha u^n_{i,j-l+1} + \sum_{l=0}^{m_2-j+1} g^{(l)}_\alpha u^n_{i,j+l-1}\Big] + \tau f^{n-1}_{i,j},
\end{aligned} \tag{10}$$

with $(i = 0, 1, \cdots, m_1,\ j = 0, 1, \cdots, m_2)$

$$u^0_{i,j} = \psi_{i,j} = \psi(x_i, y_j), \tag{11}$$

$$u^n_{0,j} = u^n_{m_1,j} = u^n_{i,0} = u^n_{i,m_2} = 0. \tag{12}$$

A Gauss-Seidel iteration technique is used for solving the implicit difference scheme (**??**)-(**??**).

3.2. *Stability of the implicit numerical method*

We first prove the stability of (**??**)-(**??**). Let $\widetilde{u}^n_{i,j}$ be the approximate solution of (**??**)–(**??**) and $\varepsilon^n_{i,j} = u^n_{i,j} - \widetilde{u}^n_{i,j}$ denote the corresponding error with $\mathbf{E}^n = [\varepsilon^n_{1,1}, \varepsilon^n_{2,1}, \cdots, \varepsilon^n_{m_1-1,m_2-1}]^T$. Assuming $||\mathbf{E}^n||_\infty = \max_{1\le i\le m_1-1, 1\le j\le m_2-1} |\varepsilon^n_{i,j}|$, we have the following theorem:

Theorem 3.1. *The implicit numerical method (**??**)-(**??**) is unconditionally stable, and*

$$||\mathbf{E}^n||_\infty \le C||\mathbf{E}^0||_\infty, \quad n = 0, 1, 2, \cdots, N, \tag{13}$$

where C is a positive number independent of h_x, h_y and τ.

Proof. According to (**??**)–(**??**), the error $\varepsilon^n_{i,j}$ satisfies

$$\begin{aligned}
\varepsilon^n_{i,j} &= \varepsilon^{n-1}_{i,j} + r^{(1)}_{i,j}\Big[\sum_{l=0}^{i+1} g^{(l)}_\alpha \varepsilon^n_{i-l+1,j} + \sum_{l=0}^{m_1-i+1} g^{(l)}_\alpha \varepsilon^n_{i+l-1,j}\Big] \\
&+ r^{(2)}_{i,j}\Big[\sum_{l=0}^{j+1} g^{(l)}_\alpha \varepsilon^n_{i,j-l+1} + \sum_{l=0}^{m_2-j+1} g^{(l)}_\alpha \varepsilon^n_{i,j+l-1}\Big] \\
&+ \tau\Big[f(u^{n-1}_{i,j}, x_i, y_j, t_{n-1}) - f(\widetilde{u}^{n-1}_{i,j}, x_i, y_j, t_{n-1})\Big].
\end{aligned}$$

Using the properties of the coefficients $g_\alpha^{(l)}$ and the discrete Gronwall inequality [**?**], we can prove that

$$||\mathbf{E}^n||_\infty \leq (1+\tau L)^n ||\mathbf{E}^0||_\infty \leq e^{n\tau L}||\mathbf{E}^0||_\infty \leq e^{LT}||\mathbf{E}^0||_\infty, \quad n = 1, 2, \ldots, N,$$

i.e., $||\mathbf{E}^n||_\infty \leq e^{LT}||\mathbf{E}^0||_\infty = C||\mathbf{E}^0||_\infty$. □

3.3. Convergence of the implicit numerical method

Now we consider the convergence of the implicit scheme. We suppose that the continuous problem (**??**)–(**??**) has a smooth solution.

Theorem 3.2. *The implicit numerical method (**??**)-(**??**) is consistent for (**??**)-(**??**) with convergence order $O(\tau + h_x + h_y)$.*

Proof. Let $u(x_i, y_j, t_n)$ be the exact solution of (**??**)–(**??**) at mesh point (x_i, y_j, t_n). Define $\eta^n_{i,j} = u(x_i, y_j, t_n) - u^n_{i,j}$, $\mathbf{Y}^n = (\eta^n_{1,1}, \eta^n_{2,1}, \ldots, \eta^n_{m_1-1,m_2-1})^T$.

It can be shown that $\eta^n_{i,j} = u(x_i, y_j, t_n) - u^n_{i,j}$ satisfies

$$\begin{aligned} \eta^n_{i,j} &= \eta^{n-1}_{i,j} + r^{(1)}_{i,j}\Big[\sum_{l=0}^{i+1} g^{(l)}_\alpha \eta^n_{i-l+1,j} + \sum_{l=0}^{m_1-i+1} g^{(l)}_\alpha \eta^n_{i+l-1,j}\Big] \\ &+ r^{(2)}_{i,j}\Big[\sum_{l=0}^{j+1} g^{(l)}_\alpha \eta^n_{i,j-l+1} + \sum_{l=0}^{m_2-j+1} g^{(l)}_\alpha \eta^n_{i,j+l-1}\Big] \qquad (14) \\ &+ \tau\Big[f(u(x_i, y_j, t_{n-1}), x_i, y_j, t_{n-1}) - f(u^{n-1}_{i,j}, x_i, y_j, t_{n-1})\Big] + R_{i,j,n}. \end{aligned}$$

Using the properties of the coefficients $g_\alpha^{(l)}$ and the discrete Gronwall inequality [**?**], we can prove that

$$||\mathbf{Y}^n||_\infty \leq e^{n\tau L} n C^*(\tau^2 + \tau h_x + \tau h_y) \leq e^{LT} C^* T(\tau + h) = C(\tau + h_x + h_y).$$

Thus we see that for any x, y and t, as h_x, h_y and τ approach 0 in such a way that $(ih_x, jh_y, n\tau) \to (x, y, t)$, $u^n_{i,j}$ approaches $u(x, y, t)$. This proves that $u^n_{i,j}$ converges to $u(x_i, y_j, t_n)$ as h_x, h_y and τ tend to zero. □

4. Conclusion

References

[1] L. M. Pismen, Patterns and Interfaces in Dissipative Systems, Springer Verlag, Berlin, (2006), 163-169.

[2] S. B. Yuste, K. Lindenberg, Subdiffusion-limited A + A reactions, Phys. Rev. Lett. 87(11), 118301 (2001).

[3] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339, (2000), 1-77.

[4] S. B. Yuste, L. Acedo, K. Lindenberg, Reaction front in an A+B –¿ C reaction-subdiffusion process, Physical Review E 69(3), 036126 (2004)

[5] S. Picozzi and B. West, Fractional Langevin model of memory in financial markets, Phys. Rev. E 66, 046118 (2002).

[6] F. Liu , V. Anh , I. Turner, Numerical solution of space fractional Fokker-Planck equation J. Comp. and Appl. Math. 166, (2004), 209-219

[7] F. Liu , P. Zhuang , V. Anh , I. Turner , K. Burrage, tability and convergence of the difference methods for the space-time fractional advection-diffusion equation., Appl. Math. Comp. 191, (2007), 12-20.

[8] I. Podlubny, Fractional Differential Equations, (Academic Press, London and New York, 1999)

[9] J. A. Ochoa-Tapia, F. J. Valdes-Parada , J. Alvarez-Ramirez, A fractional-order Darcy's law, Physica A, 374, (2007), 1-14.

[10] F. J. Valdes-Parada , J. A. Ochoa-Tapia , J. Alvarez-Ramirez,Effective medium equations for fractional Fick's law in porous media, Physica A, 373, (2007), 339–353.

[11] J. H. Cushman , B. X. Hu , T. R. Ginn, Nonequilibrium statistical mechanics of preasymptotic dispersion, J. Stat. Phy. 75, (1994), 859–878.

[12] D. Del-Castillo-Negrete, Truncation effects in superdiffusive front propagation with L$\acute{e}$vy flights Phys. Rev. E 79, 031120 (2009).

[13] Y. Zhang, D. A. Benson , D. M. Reeves, Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications Advances in Water Resources 32, (2009), 561-581.

[14] V.B.L. Chaurasia and J. Singh, Unified fractional reaction-diffusion equations with solution, Appl. Math. Sci., 60(5), (2011), 2989-3000.

[15] P. Zhuang, F. Liu, V. Anh and I. Turner, Numerical methods for the variable-order fractional advection-diffusion equation with a nonlinear source term, SIAM J. Num. Anal., 47(3), (2009), 1160-1781.

[16] Meerschaert, M. M. and Tadjeran, C. (2004), Finite difference approximations for fractional advection-dispersion. *J. Comput. Appl. Math.* **172**, pp. 65–77.

[17] Podlubny, I. (1999), *Fractional Differential Euations*. (Acdemic Press, San Dieg).

[18] Liu, F., Zhuang, P., K. Burrage Numerical methods and analysis for a class of fractional advection-dispersion models, Comput. Math. Appl. ... (2012) ....