High Order Numerical Methods for the Riesz Derivatives and the Space Riesz Fractional Differential Equation

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Outline

- Fractional calculus: partial but important
- Motivation
- Numerical methods for Riesz fractional derivatives
- Numerical methods for space Riesz fractional differential equation
- Conclusions
- Acknowledgements

Fractional calculus: partial but important

Calculus= integration + differentiation

Fractional calculus= fractional integration +
fractional differentiation

Fractional integration (or fractional integral)
Mainly one: **Riemann-Liouville (RL) integral**

Fractional derivatives
More than 6: **not mutually equivalent.**

RL and Caputo derivatives are mostly used.

Fractional calculus: partial but important

Definitions of RL and Caputo derivatives:

$$ {}_{RL}D_{0,t}^{\alpha}x(t) = \frac{d^m}{dt^m} D_{0,t}^{-(m-\alpha)}x(t), \quad m-1 < \alpha < m \in Z^+. $$

$$ {}_{C}D_{0,t}^{\alpha}x(t) = D_{0,t}^{-(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} x(t), \quad m-1 < \alpha < m \in Z^+. $$

$$ {}_{C}D_{0,t}^{\alpha}x(t) = {}_{RL}D_{0,t}^{\alpha}[x(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} x^{(k)}(0)], \qquad m-1 < \alpha < m \in Z^+. $$

The involved fractional integral is in the sense of RL.

$$ D_{0,t}^{-\alpha}x(t) = Y_{\alpha} * x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau, \qquad \alpha \in R^+. $$

Fractional calculus: partial but important

Once mentioned it, fractional calculus is taken for granted to be the mathematical generalization of calculus.

But this is not true!!!

For α-th $(m-1<\alpha<m\in Z^{+})$ Caputo derivative , fix t

$$\lim_{\alpha\to(m-1)^{+}} {}_{C}D_{0,t}^{\alpha}x(t)=x^{(m-1)}(t)-x^{(m-1)}(0), \qquad \lim_{\alpha\to m^{-}} {}_{C}D_{0,t}^{\alpha}x(t)=x^{(m)}(t).$$

Classical derivative is not the special case of the Caputo derivative.

Fractional calculus: partial but important

On the other hand, see an example below:

Investigate a function in $[0,1+\varepsilon]$,

$$x(t)=\begin{cases}1-t, & 0\le t\le 1,\\ t-1, & 1<t\le 1+\varepsilon, \varepsilon>0.\end{cases}$$

$${}_{RL}D_{0,t}^{\alpha}x(t)\ \exists \text{ in } (0,1+\varepsilon],\ \alpha\in(0,1).$$

$$x'(t)\ \exists \text{ in } [0,1)\cup(1,1+\varepsilon].$$

The existence intervals of two kind of derivatives for the same funtion is not the same, neither relation of inclusion.

RL derivative can not be regarded as the mathematical generalization of the classical derivative.

Fractional calculus: partial but important

Therefore, fractional calculus, closely related to classical calculus, is not the direct generalization of classical calculus in the sense of rigorous mathematics.

For details, see the following review article:

[1] Changpin Li, Zhengang Zhao, Introduction to fractional integrability and differentiability, The European Physical Journal-Special Topics, 193(1), 5-26, 2011.

Fractional calculus: partial but important

Where is fractional calculus?

It is here.

Example

$x(t) = t^{-\alpha}, 0 < \alpha \le 1/2,$ is not a solution to

$$\begin{cases} \dfrac{dx}{dt} = f(x,t), & \text{arbitrarily given,} \\ x(0) = x_0, & \text{arbitrarily given.} \end{cases}$$

But it is a solution to

$$\begin{cases} {}_{RL}D_{0,t}^{\alpha} x(t) = f(x,t), & \text{for some } f(x,t), \\ {}_{RL}D_{0,t}^{\alpha-1} x(t)\,|_{t=0} = x_0, & \text{for some } x_0. \end{cases}$$

Another Example

$$x(t)=\begin{cases}1, & t=\dfrac{q}{p}\in(0,1],\ (p,q)=1,\\ 0, & \text{others in } [0,1],\end{cases}$$

is not a solution to

$$\begin{cases}\dfrac{dx}{dt}=f(x,t), & \text{arbitrarily given},\\ x(0)=x_0, & \text{arbitrarily given}.\end{cases}$$

But it is a solution to

$$(*)\quad \begin{cases}{}_{RL}D_{0,t}^{\alpha}x(t)=0, \quad \alpha\in(0,1),\\ {}_{RL}D_{0,t}^{\alpha-1}x(t)\,|_{x=0}=0.\end{cases}$$

Actually, $x=0$ a.e. solves eqn (*).

Attention: fractional is very possibly a powerful tool for nonsmooth functions.

Motivation

For **Riemann-Liouvile derivative**, high order methods were very possibly proposed by Lubich (SIAM J. Math. Anal., 1986)

$$ {}_{RL}\mathrm{D}_{a,x_n}^{\alpha} f(x_n) = h^{-\alpha}\sum_{j=0}^{n}\varpi_{n-j}^{\alpha} f(x_j) + h^{-\alpha}\sum_{j=0}^{s}\varpi_{n,j} f(x_j) + O(h^p). $$

For **Caputo derivative**, high order schemes were constructed by Li, Chen, Ye (J. Comput. Phys., 2011)

In this talk, we focus on high order algorithms for **Riesz fractional derivatives** and **space Riesz fractional differential equation.**

Motivation

The Riesz fractional derivative is defined on (a, b) as follows,

$$\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} = -\Psi_{\alpha}\left({}_{RL}D_{a,x}^{\alpha} + {}_{RL}D_{x,b}^{\alpha} \right) u(x,t),$$

in which,

$$\Psi_{\alpha} = \frac{1}{2}\sec\left(\frac{\pi\alpha}{2}\right),\ n-1 < \alpha < n \in Z^{+},$$

$${}_{RL}D_{a,x}^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^{n}}{\partial x^{n}} \cdot \int_{a}^{x} \frac{u(\xi,t)}{(x-\xi)^{\alpha-n+1}} d\xi,$$

$${}_{RL}D_{x,b}^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^{n}}{\partial x^{n}} \int_{x}^{b} \frac{u(\xi,t)}{(\xi-x)^{\alpha-n+1}} d\xi.$$

Motivation

The space Riesz fractional differential equation reads as

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = K\dfrac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), 1<\alpha<2, a<x<b, 0<t\le T \\ {}_{RL}D_{a,x;t}^{\alpha-2}u(x,t)\,|_{x=a} = \phi(t), \quad 0\le t\le T, \\ {}_{RL}D_{x,b;t}^{\alpha-2}u(x,t)\,|_{x=b} = \varphi(t), \quad 0\le t\le T, \\ u(x,0)=\psi(x), \quad a\le x\le b. \end{cases} \tag{1}$$

The above boundary value conditions are of Dirichlet type.

Neumann or Rubin boundary value conditions can be similarly proposed.

Unsuitable initial or boundary value conditions: ill-posed.

Numerical methods for Riesz fractional derivatives

Already existed (typical) numerical schemes:

Let h be the step size with $x_m = a + mh$, $m = 0, 1, \cdots, M$, and $t_n = n\tau$, $n = 0, 1, \cdots, N$, where $h = (b-a)/M$, $\tau = T/N$.

1) The first order scheme

$$\frac{\partial^\alpha u(x_m,t)}{\partial |x|^\alpha} = -\frac{\Psi_\alpha}{h^\alpha}\left(\sum_{k=0}^{m} \varpi_k^{(\alpha)} u(x_{m-k},t) + \sum_{k=0}^{M-m} \varpi_k^{(\alpha)} u(x_{m+k},t)\right) + O(h),$$

in which $\varpi_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \dfrac{(-1)^k \Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}$.

Numerical methods for Riesz fractional derivatives

2) The shifted first order scheme

$$\frac{\partial^{\alpha} u(x_m,t)}{\partial |x|^{\alpha}} = -\frac{\Psi_{\alpha}}{h^{\alpha}}\left(\sum_{k=0}^{m+1}\varpi_k^{(\alpha)} u\left(x_{m-k+1},t\right) + \sum_{k=0}^{M-m+1}\varpi_k^{(\alpha)} u(x_{m+k-1},t)\right) + O(h),$$

in which $\varpi_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \dfrac{(-1)^k \Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}$.

3) The L2 numerical scheme

If $1<\alpha<2$, then one has

$$\frac{\partial^{\alpha}u(x_m,t)}{\partial|x|^{\alpha}}=-\frac{\Psi_{\alpha}}{\Gamma(3-\alpha)h^{\alpha}}\left\{\frac{(1-\alpha)(2-\alpha)u(x_0,t)}{m^{\alpha}}+\frac{(2-\alpha)\left[u(x_1,t)-u(x_0,t)\right]}{m^{\alpha-1}}\right.$$

$$+\sum_{k=0}^{m-1}d_k^{(\alpha)}\left[u(x_{m-k+1},t)-2u(x_{m-k},t)+u(x_{m-k-1},t)\right]$$

$$+\frac{(1-\alpha)(2-\alpha)u(x_M,t)}{(M-m)^{\alpha}}+\frac{(2-\alpha)\left[u(x_M,t)-u(x_{M-1},t)\right]}{m^{\alpha-1}}$$

$$\left.+\sum_{k=0}^{M-m-1}d_k^{(\alpha)}\left[u(x_{m+k-1},t)-2u(x_{m+k},t)+u(x_{m+k+1},t)\right]\right\}+O(h),$$

in which $d_k^{(\alpha)}=(k+1)^{2-\alpha}-k^{2-\alpha},\ k=0,1,\cdots,\max(m-1,\ M-m-1).$

4) The second order numerical scheme based on the spline interpolation method

$$\frac{\partial^{\alpha} u(x_m,t)}{\partial |x|^{\alpha}} = \frac{-\Psi_{\alpha}}{\Gamma(4-\alpha)h^{\alpha}} \sum_{k=0}^{M} z_{m,k}^{(\alpha)} u(x_k,t) + O(h^2),$$

in which $z_{m,k}^{(\alpha)}$ is defined below,

$$z_{m,k}^{(\alpha)} = \begin{cases} \overline{z}_{m,k}^{(\alpha)}, & k < m-1, \\ \overline{z}_{m,m-1}^{(\alpha)} + \tilde{z}_{m,m-1}^{(\alpha)}, & k = m-1, \\ \overline{z}_{m,m}^{(\alpha)} + \tilde{z}_{m,m}^{(\alpha)}, & k = m, \\ \overline{z}_{m,m+1}^{(\alpha)} + \tilde{z}_{m,m+1}^{(\alpha)}, & k = m+1, \\ \tilde{z}_{m,k}^{(\alpha)}, & k > m+1. \end{cases}$$

Numerical methods for Riesz fractional derivatives

$$\overline{z}_{m,k}^{(\alpha)} = \begin{cases} \overline{c}_{m-1,k} - 2\overline{c}_{m,k} + \overline{c}_{m+1,k}\,, & k \le m-1, \\ -2\overline{c}_{m,k} + \overline{c}_{m+1,k}\,, & k = m, \\ \overline{c}_{m+1,k}\,, & k = m+1, \\ 0\,, & k > m+1, \end{cases}$$

in which

$$\overline{c}_{j,k} = \begin{cases} (j-1)^{3-\alpha} - j^{2-\alpha}(j-3+\alpha), & k=0, \\ (j-k+1)^{3-\alpha} - 2(j-k)^{3-\alpha} + (j-k-1)^{3-\alpha}\,, & 1 \le k \le j-1, \\ 1\,, & k = j; \end{cases}$$

Numerical methods for Riesz fractional derivatives

$$\tilde{z}_{m,k}^{(\alpha)} = \begin{cases} 0\,, \quad k < m-1, \\ \tilde{c}_{m-1,m-1}\,, \quad k = m-1, \\ -2\tilde{c}_{m,m} + \tilde{c}_{m-1,m}\,, \quad k = m, \\ \tilde{c}_{m-1,k} - 2\tilde{c}_{m,k} + \tilde{c}_{m+1,k}\,, \quad m+1 \le k \le M, \end{cases}$$

in which

$$\tilde{c}_{j,k} = \begin{cases} 1, \quad k = j \\ (k-j+1)^{3-\alpha} - 2(k-j)^{3-\alpha} + (k-j-1)^{3-\alpha}\,, \; j+1 \le k \le M-1 \\ (3-\alpha-M+j)(M-j)^{2-\alpha} + (M-j-1)^{3-\alpha}\,, k = M \end{cases}$$

with $j = m-1, m, m+1.$

5) The second order scheme based on the fractional centered difference method

$$\frac{\partial^{\alpha} u(x_m,t)}{\partial |x|^{\alpha}} = -\frac{1}{h^{\alpha}} \Delta_h^{\alpha} u(x_m,t) + O(h^2).$$

The so called fractional centered difference operator is defined by

$$\Delta_h^{\alpha} u(x,t) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k+1\right)} u(x-kh,t).$$

6) The second order scheme based on the weighted and shifted Grünwald-Letnikov (GL) formula

$$\frac{\partial^{\alpha} u(x_m,t)}{\partial |x|^{\alpha}} = -\frac{\Psi_{\alpha}}{h^{\alpha}}\left(v_1 \sum_{k=0}^{m+\ell_1} \varpi_k^{(\alpha)} u\left(x_{m-k+\ell_1},t\right) + v_2 \sum_{k=0}^{m+\ell_2} \varpi_k^{(\alpha)} u\left(x_{m-k+\ell_2},t\right) \right.$$

$$\left. + v_1 \sum_{k=0}^{M-m+\ell_1} \varpi_k^{(\alpha)} u\left(x_{m+k-\ell_1},t\right) + v_2 \sum_{k=0}^{M-m+\ell_2} \varpi_k^{(\alpha)} u\left(x_{m+k-\ell_2},t\right) \right)$$

$$+ O\left(h^2\right),$$

in which

$v_1 = \dfrac{\alpha - 2\ell_2}{2(\ell_1 - \ell_2)},\ v_2 = \dfrac{2\ell_1 - \alpha}{2(\ell_1 - \ell_2)},\ \ell_1, \ell_2$ are two integers.

7) The third order scheme based on the weighted and shifted GL formula

$$\frac{\partial^\alpha u(x_m,t)}{\partial|x|^\alpha}=\frac{\Psi_\alpha}{h^\alpha}\Bigg(\kappa_1\sum_{m=0}^{m+\ell_1}\varpi_k^{(\alpha)}u\left(x_{m-k+\ell_1},t\right)+\kappa_2\sum_{k=0}^{m+\ell_2}\varpi_k^{(\alpha)}u\left(x_{m-k+\ell_2},t\right)$$
$$+\kappa_3\sum_{k=0}^{m+\ell_3}\varpi_k^{(\alpha)}u\left(x_{m-k+\ell_3},t\right)+\kappa_1\sum_{m=0}^{M-m+\ell_1}\varpi_k^{(\alpha)}u\left(x_{m+k-\ell_1},t\right)$$
$$+\kappa_2\sum_{m=0}^{M-m+\ell_2}\varpi_k^{(\alpha)}u\left(x_{m+k-\ell_2},t\right)+\kappa_3\sum_{m=0}^{M-m+\ell_3}\varpi_k^{(\alpha)}u\left(x_{m+k-\ell_3},t\right)$$
$$+O\left(h^3\right),$$

in which, $\kappa_1=\dfrac{12\ell_2\ell_3-(6\ell_2+6\ell_3+1)\alpha+3\alpha^2}{12\left(\ell_2\ell_3-\ell_1\ell_2-\ell_1\ell_3+\ell_1^2\right)}$,

$\kappa_2=\dfrac{12\ell_1\ell_3-(6\ell_1+6\ell_3+1)\alpha+3\alpha^2}{12\left(\ell_1\ell_3-\ell_1\ell_2-\ell_2\ell_3+\ell_2^2\right)},\kappa_3=\dfrac{12\ell_1\ell_2-(6\ell_1+6\ell_2+1)\alpha+3\alpha^2}{12\left(\ell_1\ell_2-\ell_1\ell_3-\ell_2\ell_3+\ell_3^2\right)}$,

ℓ_1,ℓ_2,ℓ_3 are three integers.

Numerical methods for Riesz fractional derivatives

In our work, we construct new three kinds of second order schemes and a fourth order numerical scheme. We first derive second order schemes.

Lemma 1. Assume that $u(x,t)$ with respect to x sufficiently smooth. For arbitrarily different numbers p, q and s , we have

$$u(x_s,t)=\frac{(x_s-x_q)u(x_p,t)+(x_p-x_s)u(x_q,t)}{x_p-x_q}$$

$$+O\left(\left\|(x_q-x_s)(x_p-x_s)\right\|\right).$$

Numerical methods for Riesz fractional derivatives

Lemma 2. For any positive number α , we have

$$\sum_{k=0}^{\infty} \varpi_k^{(\alpha)} = 0 ,$$

where $\varpi_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} = \dfrac{(-1)^k \Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}$.

Let

$$\mu_h^{\alpha}\left(u\left(x-jh,t\right)\right) = \frac{u\left(x,t\right)+f\left(x-2jh,t\right)}{2^{\alpha}}.$$

Numerical methods for Riesz fractional derivatives

Define

$$ {}_{AC}\Delta_h^{\alpha} u(x,t) = \mu_h^{\alpha}\left({}_{C}\Delta_h^{\alpha} u(x,t)\right), $$

where

$$ {}_{C}\Delta_h^{\alpha} u(x,t) = \sum_{k=0}^{\infty} \varpi_k^{(\alpha)} u\left(x - \left(k - \frac{\alpha}{2} \right) h, t \right). $$

Then one has

$$ {}_{AC}\Delta_h^{\alpha} u(x,t) = \frac{1}{2^{\alpha}} \sum_{k=0}^{\infty} \varpi_k^{(\alpha)} u\left(x - (2k - \alpha) h, t \right). $$

Numerical methods for Riesz fractional derivatives

Theorem 1. Let $u(x,t)$ and the Fourier transform of the ${}_{RL}D^{\alpha+2}_{-\infty,x}\mathrm{u}(x,t)$ with respect to x both be in $L_1(R)$, then

$$\frac{{}_{AC}\Delta_h^{\alpha}u(x,t)}{h^{\alpha}} = {}_{RL}D^{\alpha}_{-\infty,x}u(x,t)+O(h^2),\ \text{i.e.,}$$

$${}_{RL}D^{\alpha}_{-\infty,x}u(x,t)=\frac{1}{(2h)^{\alpha}}\sum_{k=0}^{\infty}\varpi_k^{(\alpha)}u(x-(2k-\alpha)h,t)+O(h^2).$$

Numerical methods for Riesz fractional derivatives

By choices of x_s, x_p and x_q, one has new three kinds of second order schems:

$$\frac{\partial^\alpha u(x_m,t_n)}{\partial |x|^\alpha} = -\frac{\Psi_\alpha}{(2h)^\alpha}\left[\left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{\left[\frac{m}{2}\right]}\varpi_k^{(\alpha)}u(x_{m-2k},t) + \frac{\alpha}{2}\sum_{k=0}^{\left[\frac{m}{2}\right]}\varpi_k^{(\alpha)}u(x_{m-2(k-1)},t) + \left(1-\frac{\alpha}{2}\right)\sum_{k=0}^{\left[\frac{M-m}{2}\right]}\varpi_k^{(\alpha)}u(x_{m+2k},t) + \frac{\alpha}{2}\sum_{k=0}^{\left[\frac{M-m}{2}\right]}\varpi_k^{(\alpha)}u(x_{m+2(k-1)},t)\right] + O(h^2), \quad \text{(I)}$$

Numerical methods for Riesz fractional derivatives

$$\frac{\partial^\alpha u(x_m,t_n)}{\partial|x|^\alpha} = -\frac{\Psi_\alpha}{(2h)^\alpha}\Bigg[\left(1+\frac{\alpha}{2}\right)\sum_{k=0}^{\left[\frac{m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m-2k},t) - \frac{\alpha}{2}\sum_{k=0}^{\left[\frac{m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m-2(k+1)},t) + \left(1+\frac{\alpha}{2}\right)\sum_{k=0}^{\left[\frac{M-m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m+2k},t) - \frac{\alpha}{2}\sum_{k=0}^{\left[\frac{M-m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m+2(k+1)},t)\Bigg] + O(h^2), \quad \text{(II)}$$

$$\frac{\partial^\alpha u(x_m,t_n)}{\partial|x|^\alpha} = -\frac{\Psi_\alpha}{(2h)^\alpha}\Bigg[\left(\frac{1}{2}+\frac{\alpha}{4}\right)\sum_{k=0}^{\left[\frac{m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m-2(k-1)},t) + \left(\frac{1}{2}-\frac{\alpha}{4}\right)\sum_{k=0}^{\left[\frac{m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m-2(k+1)},t) + \left(\frac{1}{2}+\frac{\alpha}{4}\right)\sum_{k=0}^{\left[\frac{M-m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m+2(k-1)},t) + \left(\frac{1}{2}-\frac{\alpha}{4}\right)\sum_{k=0}^{\left[\frac{M-m}{2}-1\right]}\varpi_k^{(\alpha)}u(x_{m+2(k+1)},t)\Bigg] + O(h^2). \quad \text{(III)}$$

Numerical methods for Riesz fractional derivatives

New kinds of third order numerical schemes. Skipped...

Now directly derive a fourth order scheme.

Numerical methods for Riesz fractional derivatives

Theorem 2. Let $u(x,t)$ lie in $C^7(R)$ whose partial derivatives up to order seven with respect to x belong to $L_1(R)$. Set $L_\theta u(x,t)=\sum_{k=-\infty}^{\infty} g_k^{(\alpha)} u\big(x-(k+\theta)h,t\big),\ \theta=-1,0,1,$

in which $$g_k^{(\alpha)}=\frac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k-1\right)},$$

then one has

$$\frac{\partial^\alpha u(x,t)}{\partial|x|^\alpha}=\frac{1}{h^\alpha}\left[\frac{\alpha}{24}L_{-1}u(x,t)+\frac{\alpha}{24}L_1 u(x,t)-(1+\frac{\alpha}{12})L_0 u(x,t)\right]+O\left(h^4\right).$$

Numerical methods for Riesz fractional derivatives

Based on Therom 2, a fourth order scheme can be established as

$$\frac{\partial^\alpha u(x_m,t)}{\partial|x|^\alpha} = \frac{\alpha}{24h^\alpha}\sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u\left(x_{m-(k+1)},t\right)$$

$$+\frac{\alpha}{24h^\alpha}\sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u\left(x_{m-(k-1)},t\right)$$

$$-\left(1+\frac{\alpha}{12}\right)\frac{1}{h^\alpha}\sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u\left(x_{m-k},t\right)+O\left(h^4\right),$$

where

$$g_k^{(\alpha)} = \frac{(-1)^k\,\Gamma(\alpha+1)}{\Gamma\left(\frac{\alpha}{2}-k+1\right)\Gamma\left(\frac{\alpha}{2}+k-1\right)}.$$

Numerical method for space Riesz fractional differential equation

Recall Equ (1).

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = K\dfrac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), 1 < \alpha < 2, a < x < b, 0 < t \le T \\ {}_{RL}D_{a,x;t}^{\alpha-2} u(x,t)|_{x=a} = \phi(t), \quad 0 \le t \le T, \\ {}_{RL}D_{x,b;t}^{\alpha-2} u(x,t)|_{x=b} = \varphi(t), \quad 0 \le t \le T, \\ u(x,0) = \psi(x), \quad 0 \le x \le L. \end{cases} \qquad (1)$$

Numerical method for space Riesz fractional differential equation

Let u_m^n be the approximation solution of $u(x_m, t_n)$, one has the following finite difference scheme for Equ (1).

$$u_m^n - \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k-1}^n - \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k+1}^n + \mu_2 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k}^n$$

$$=$$

$$u_m^{n-1} + \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k-1}^{n-1} + \mu_1 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k+1}^{n-1} - \mu_2 \sum_{k=-M+m+1}^{m-1} g_k^{(\alpha)} u_{m-k}^{n-1} + \frac{\tau}{2} f_m^n + \frac{\tau}{2} f_m^{n-1}, \quad (2)$$

where $\mu_1 = \dfrac{\alpha \tau K}{48 h^\alpha}, \mu_2 = \dfrac{\tau K}{2h^\alpha}\left(1 + \dfrac{\alpha}{12}\right).$

Numerical method for space Riesz fractional differential equation

Set $U^n = \left(u_1^n, u_2^n, \cdots, u_{M-1}^n\right)^T$, $F^n = (f_1^n, f_2^n, \cdots, f_{M-1}^n)$.

Then system (2) can by written in a compact form:

$$\left(I + H\right)U^n = \left(I - H\right)U^{n-1} + \frac{\tau}{2}F^n + \frac{\tau}{2}F^{n-1}, \qquad (3)$$

where I is an identity matrix with order M -1,

$H = \mu_2 G - \mu_1 G^+ - \mu_1 G^-$. The expressions of G, G^+, G^- are omitted here.

Numerical method for space Riesz fractional differential equation

Theorem 3. (Stability)

Difference scheme (3) (or (2)) is unconditionally stable.

Theorem 4. (Convergence)

$$\left| u\left(x_m, t_k\right) - u_m^k \right| \le C\left(\tau^2 + h^4\right).$$

Numerical Examples

Example 1

Consider the function $u(x) = x^2(1-x)^2, x \in [0,1]$.

Its exact Riesz fractional derivative is

$$\frac{\partial^\alpha u(x)}{\partial |x|^\alpha} = -\sec\left(\frac{\pi}{2}\alpha\right)\left\{\frac{1}{\Gamma(3-\alpha)}\left[x^{2-\alpha} + (1-x)^{2-\alpha}\right]\right.$$

$$-\frac{6}{\Gamma(4-\alpha)}\left[x^{3-\alpha} + (1-x)^{3-\alpha}\right]$$

$$\left.+\frac{12}{\Gamma(5-\alpha)}\left[x^{4-\alpha} + (1-x)^{4-\alpha}\right]\right\}.$$

Numerical Examples

We use the second order numerical fomulas (I),(II)and (III) to compute the given function, the absolute errors and convergence orders at x=0.5 are shown in Tables 1-3.

Table 1: The absolute errors and convergence orders of the Example 1 by numerical formula (21)

α	h	the absolute error	the convergence order
1.1	$\frac{1}{40}$	2.442043e-003	—
	$\frac{1}{80}$	6.226615e-004	1.9716
	$\frac{1}{160}$	1.571291e-004	1.9865
	$\frac{1}{320}$	3.946182e-005	1.9934
1.2	$\frac{1}{40}$	2.881892e-003	—
	$\frac{1}{80}$	7.337452e-004	1.9737
	$\frac{1}{160}$	1.850316e-004	1.9875
	$\frac{1}{320}$	4.645336e-005	1.9939
1.3	$\frac{1}{40}$	3.349953e-003	—
	$\frac{1}{80}$	8.513631e-004	1.9763
	$\frac{1}{160}$	2.145035e-004	1.9888
	$\frac{1}{320}$	5.382931e-005	1.9945
1.4	$\frac{1}{40}$	3.832567e-003	—
	$\frac{1}{80}$	9.719132e-004	1.9794
	$\frac{1}{160}$	2.446224e-004	1.9903
	$\frac{1}{320}$	6.135642e-005	1.9953

Numerical Examples

1.5	$\frac{1}{40}$	4.308646e-003	—
	$\frac{1}{80}$	1.089963e-003	1.9830
	$\frac{1}{160}$	2.740122e-004	1.9920
	$\frac{1}{320}$	6.868848e-005	1.9961
1.6	$\frac{1}{40}$	4.747369e-003	—
	$\frac{1}{80}$	1.197722e-003	1.9868
	$\frac{1}{160}$	3.007168e-004	1.9938
	$\frac{1}{320}$	7.533553e-005	1.9970
1.7	$\frac{1}{40}$	5.105376e-003	—
	$\frac{1}{80}$	1.284420e-003	1.9909
	$\frac{1}{160}$	3.220534e-004	1.9957
	$\frac{1}{320}$	8.062824e-005	1.9979
1.8	$\frac{1}{40}$	5.323451e-003	—
	$\frac{1}{80}$	1.335583e-003	1.9949
	$\frac{1}{160}$	3.344451e-004	1.9976
	$\frac{1}{320}$	8.367752e-005	1.9989
1.9	$\frac{1}{40}$	5.322657e-003	—
	$\frac{1}{80}$	1.332229e-003	1.9983
	$\frac{1}{160}$	3.332366e-004	1.9992
	$\frac{1}{320}$	8.333056e-005	1.9996

Numerical Examples

Table 2: The absoulue errors and convergence orders of the Example 1 by numerical formula (22)

α	h	the absolute error	the convergence order
1.1	$\frac{1}{40}$	9.194887e-004	—
	$\frac{1}{80}$	6.020935e-004	0.6108
	$\frac{1}{160}$	2.534351e-004	1.2484
	$\frac{1}{320}$	7.615967e-005	1.7345
1.2	$\frac{1}{40}$	3.119945e-003	—
	$\frac{1}{80}$	1.341436e-003	1.2177
	$\frac{1}{160}$	4.040947e-004	1.7310
	$\frac{1}{320}$	1.095322e-004	1.8833
1.3	$\frac{1}{40}$	6.094224e-003	—
	$\frac{1}{80}$	2.012477e-003	1.5985
	$\frac{1}{160}$	5.623605e-004	1.8394
	$\frac{1}{320}$	1.478881e-004	1.9270
1.4	$\frac{1}{40}$	9.343188e-003	—
	$\frac{1}{80}$	2.798110e-003	1.7395
	$\frac{1}{160}$	7.549643e-004	1.8900
	$\frac{1}{320}$	1.955374e-004	1.9490

Numerical Examples

1.5	$\frac{1}{40}$	1.328761e-002	—
	$\frac{1}{80}$	3.769343e-003	1.8177
	$\frac{1}{160}$	9.954245e-004	1.9209
	$\frac{1}{320}$	2.553328e-004	1.9629
1.6	$\frac{1}{40}$	1.823233e-002	—
	$\frac{1}{80}$	4.986055e-003	1.8705
	$\frac{1}{160}$	4.986055e-003	1.9430
	$\frac{1}{320}$	3.302848e-004	1.9731
1.7	$\frac{1}{40}$	2.448831e-002	—
	$\frac{1}{80}$	6.512590e-003	1.9108
	$\frac{1}{160}$	1.673480e-003	1.9604
	$\frac{1}{320}$	4.238452e-004	1.9812
1.8	$\frac{1}{40}$	3.241026e-002	—
	$\frac{1}{80}$	8.422706e-003	1.9441
	$\frac{1}{160}$	2.142421e-003	1.9750
	$\frac{1}{320}$	5.400199e-004	1.9882
1.9	$\frac{1}{40}$	4.241598e-002	—
	$\frac{1}{80}$	1.080210e-002	1.9733
	$\frac{1}{160}$	2.722999e-003	1.9880
	$\frac{1}{320}$	6.834357e-004	1.9943

Numerical Examples

Table 3: The absolute errors and convergence orders of the Example 1 by numerical formula (23)

α	h	the absolute error	the convergence order
1.1	$\frac{1}{40}$	2.786229e-003	—
	$\frac{1}{80}$	1.089239e-003	1.3550
	$\frac{1}{160}$	3.207996e-004	1.7636
	$\frac{1}{320}$	8.622670e-005	1.8955
1.2	$\frac{1}{40}$	4.564263e-003	—
	$\frac{1}{80}$	1.386489e-003	1.7189
	$\frac{1}{160}$	3.765776e-004	1.8804
	$\frac{1}{320}$	9.784745e-005	1.9443
1.3	$\frac{1}{40}$	5.536914e-003	—
	$\frac{1}{80}$	1.579259e-003	1.8098
	$\frac{1}{160}$	4.183401e-004	1.9165
	$\frac{1}{320}$	1.074773e-004	1.9606
1.4	$\frac{1}{40}$	6.268413e-003	—
	$\frac{1}{80}$	1.733278e-003	1.8546
	$\frac{1}{160}$	4.531147e-004	1.9356
	$\frac{1}{320}$	1.156983e-004	1.9695

Numerical Examples

1.5	$\frac{1}{40}$	6.832686e-003	—
	$\frac{1}{80}$	1.851368e-003	1.8839
	$\frac{1}{160}$	4.796681e-004	1.9485
	$\frac{1}{320}$	1.219608e-004	1.9756
1.6	$\frac{1}{40}$	7.207767e-003	—
	$\frac{1}{80}$	1.922400e-003	1.9066
	$\frac{1}{160}$	4.945380e-004	1.9588
	$\frac{1}{320}$	1.253156e-004	1.9805
1.7	$\frac{1}{40}$	7.328525e-003	—
	$\frac{1}{80}$	1.927488e-003	1.9268
	$\frac{1}{160}$	4.927113e-004	1.9679
	$\frac{1}{320}$	1.244746e-004	1.9849
1.8	$\frac{1}{40}$	7.096086e-003	—
	$\frac{1}{80}$	1.840758e-003	1.9467
	$\frac{1}{160}$	4.676169e-004	1.9769
	$\frac{1}{320}$	1.177843e-004	1.9892
1.9	$\frac{1}{40}$	6.377157e-003	—
	$\frac{1}{80}$	1.628844e-003	1.9691
	$\frac{1}{160}$	4.109576e-004	1.9868
	$\frac{1}{320}$	1.031780e-004	1.9939

Numerical Examples

Example 2

Consider the following Riesz fractional differential equation

$$\frac{\partial u(x,t)}{\partial t} = K\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} + f(x,t), \;\; 0 < x < 1, 0 < t \le 1$$

where

$$f(t) = (2\alpha+1)t^{2\alpha}x^4(1-x)^4 + t^{2\alpha+1}\sec(\frac{\pi}{2}\alpha)\left\{\frac{12}{\Gamma(5-\alpha)}\left[x^{4-\alpha}+(1-x)^{4-\alpha}\right]\right.$$
$$-\frac{240}{\Gamma(6-\alpha)}\left[x^{5-\alpha}+(1-x)^{5-\alpha}\right]+\frac{2160}{\Gamma(7-\alpha)}\left[x^{6-\alpha}+(1-x)^{6-\alpha}\right]$$
$$\left.-\frac{10080}{\Gamma(8-\alpha)}\left[x^{7-\alpha}+(1-x)^{7-\alpha}\right]+\frac{20160}{\Gamma(9-\alpha)}\left[x^{8-\alpha}+(1-x)^{8-\alpha}\right]\right\}$$

together with the initial condition $u(x,0)=0$ and homogenerous boundary value conditions.

The exact solution is $u(x,t) = t^{2\alpha+1}x^4(1-x)^4$.

Numerical Examples

The following table shows the maximum error, time and space convergence orders which confirms with our theoretical analysis.

Table 4: The maximum errors, temporal and spatial convergence orders of the Example 2 by difference scheme (37).

α		the maximum errors	temporal convergence orders	spatial convergence orders
1.2	$h=\frac{1}{4},\tau=\frac{1}{2}$	3.070974e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{8}$	2.038855e-005	1.9564	3.9129
	$h=\frac{1}{16},\tau=\frac{1}{32}$	1.318397e-006	1.9755	3.9509
	$h=\frac{1}{32},\tau=\frac{1}{128}$	8.371766e-008	1.9886	3.9771
	$h=\frac{1}{64},\tau=\frac{1}{512}$	5.279883e-009	1.9935	3.9870
	$h=\frac{1}{128},\tau=\frac{1}{2048}$	3.486707e-010	1.9603	3.9206

Numerical Examples

1.4	$h=\frac{1}{4},\tau=\frac{1}{2}$	3.715656e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{8}$	2.465865e-005	1.9567	3.9135
	$h=\frac{1}{16},\tau=\frac{1}{32}$	1.603083e-006	1.9716	3.9432
	$h=\frac{1}{32},\tau=\frac{1}{128}$	1.022389e-007	1.9854	3.9708
	$h=\frac{1}{64},\tau=\frac{1}{512}$	6.472320e-009	1.9908	3.9815
	$h=\frac{1}{128},\tau=\frac{1}{2048}$	4.083361e-010	1.9932	3.9865
1.6	$h=\frac{1}{4},\tau=\frac{1}{2}$	4.193160e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{8}$	2.775395e-005	1.9586	3.9173
	$h=\frac{1}{16},\tau=\frac{1}{32}$	1.810618e-006	1.9691	3.9381
	$h=\frac{1}{32},\tau=\frac{1}{128}$	1.158839e-007	1.9829	3.9657
	$h=\frac{1}{64},\tau=\frac{1}{512}$	7.363046e-009	1.9881	3.9762
	$h=\frac{1}{128},\tau=\frac{1}{2048}$	4.662698e-010	1.9905	3.9811
1.8	$h=\frac{1}{4},\tau=\frac{1}{2}$	4.417615e-004	—	—
	$h=\frac{1}{8},\tau=\frac{1}{8}$	2.895409e-005	1.9657	3.9314
	$h=\frac{1}{16},\tau=\frac{1}{32}$	1.884230e-006	1.9709	3.9417
	$h=\frac{1}{32},\tau=\frac{1}{128}$	1.204909e-007	1.9835	3.9670
	$h=\frac{1}{64},\tau=\frac{1}{512}$	7.658088e-009	1.9879	3.9758
	$h=\frac{1}{128},\tau=\frac{1}{2048}$	4.855499e-010	1.9896	3.9793

Numerical Examples

Figs.6.1 and 6.2 show the comparison of the analytical and numerical solutions with $\alpha = 1.1\ at\ t = 0.2,\ \alpha = 1.9\ at\ t = 0.8$

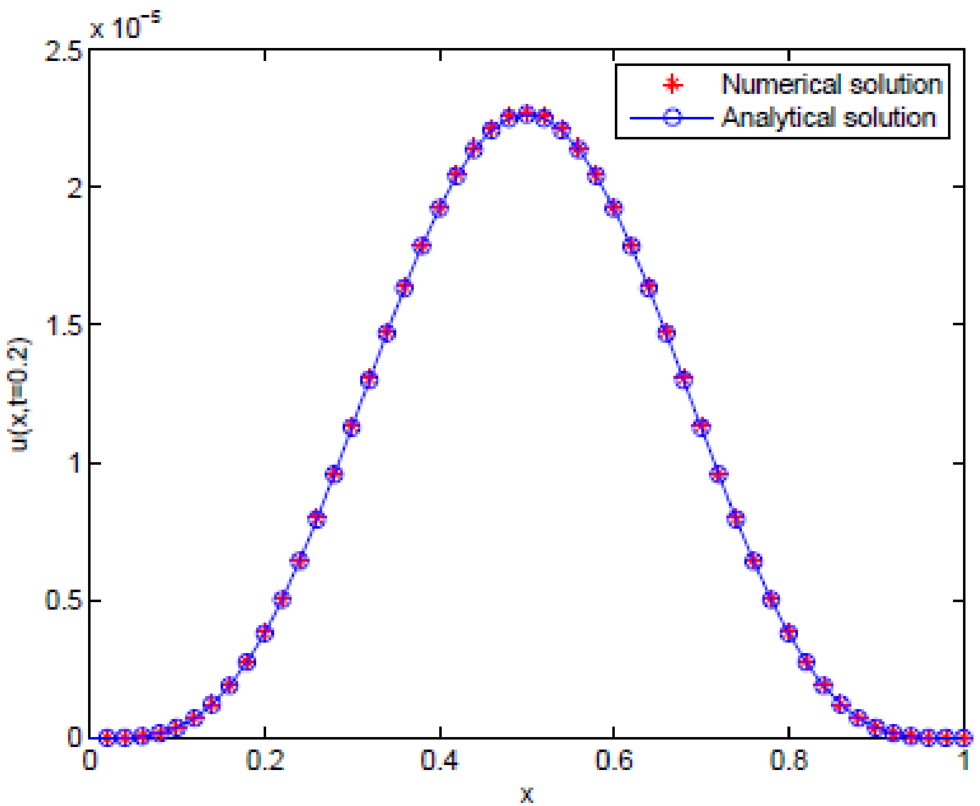

Figure 6.1: Comparison between the analytical solution and the numerical solution at $t = 0.2$ with $\alpha = 1.1$ in Example 2. $(\tau = \frac{1}{50}, h = \frac{1}{50})$

Numerical Examples

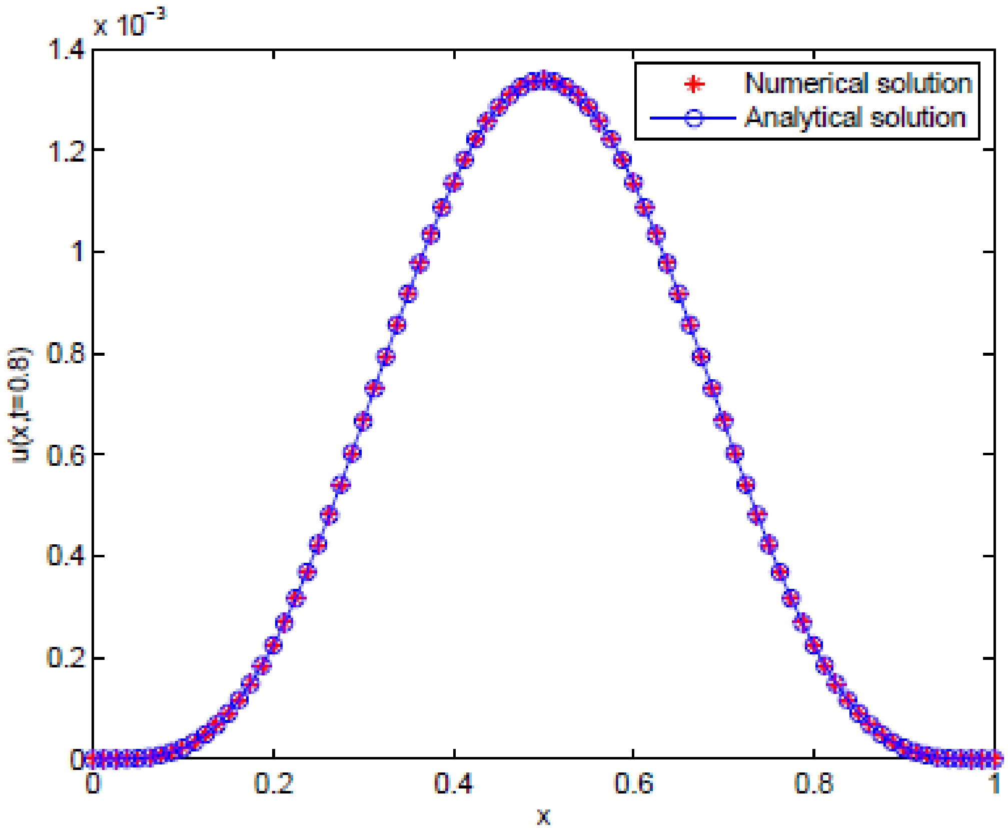

Figure 6.2: Comparison between the analytical solution and the numerical solution at $t = 0.8$ with $\alpha = 1.9$ in Example 2. $(\tau = \frac{1}{100}, h = \frac{1}{80})$

Numerical Examples

Figs. 6.3-6.6 display the numerical and analytical solution surface with $\alpha = 1.1$ and $\alpha = 1.8$

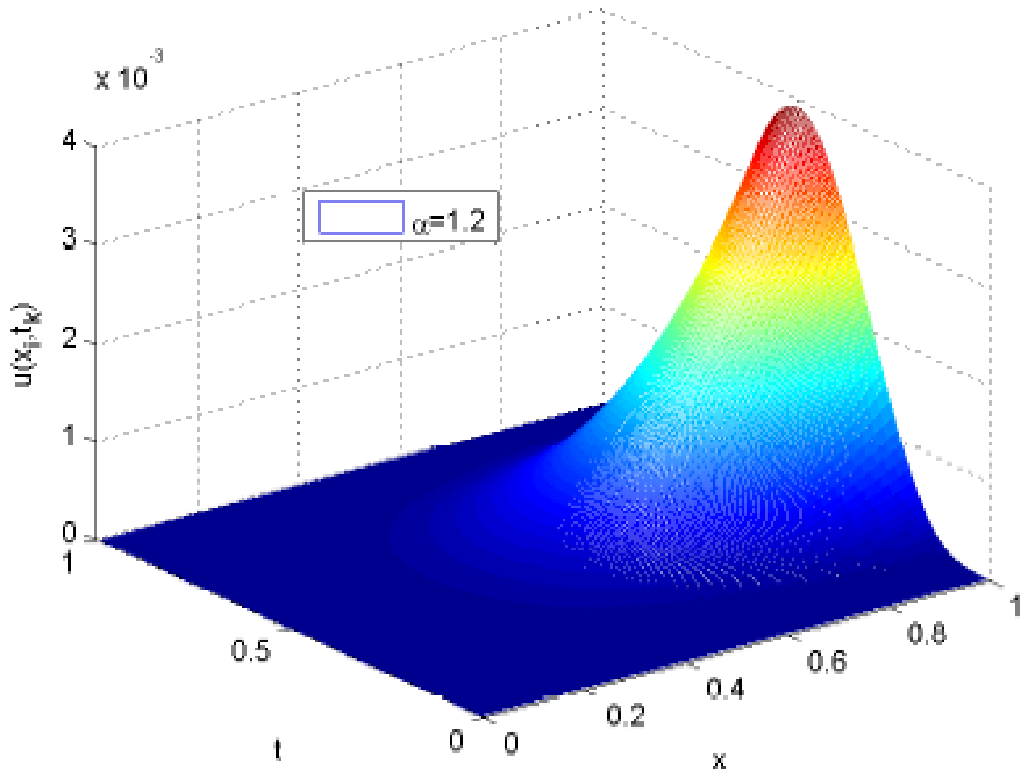

Figure 6.3: The numerical solution surface when $\alpha = 1.2$ in Example 2. ($\tau = \frac{1}{500}, h = \frac{1}{100}$)

Numerical Examples

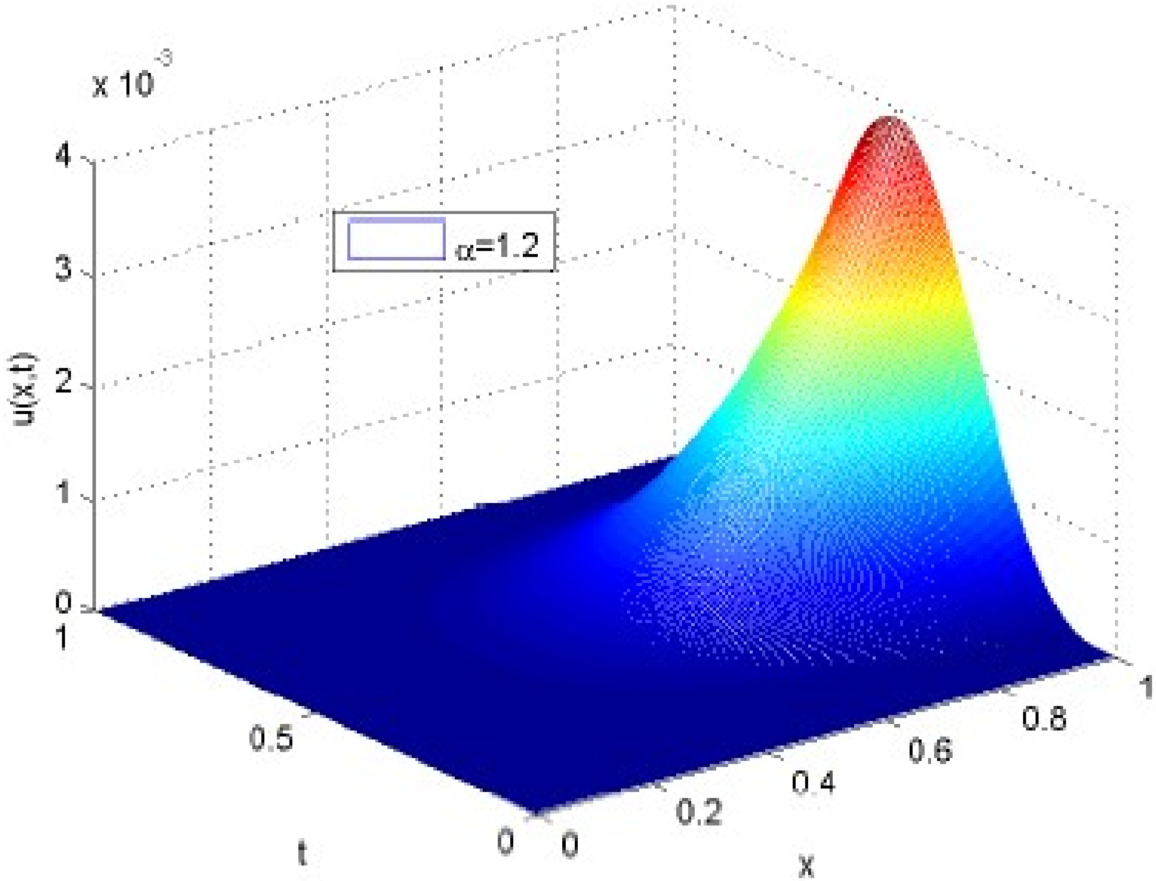

Figure 6.4: The analytical solution surface when $\alpha = 1.2$ in Example 2.

Numerical Examples

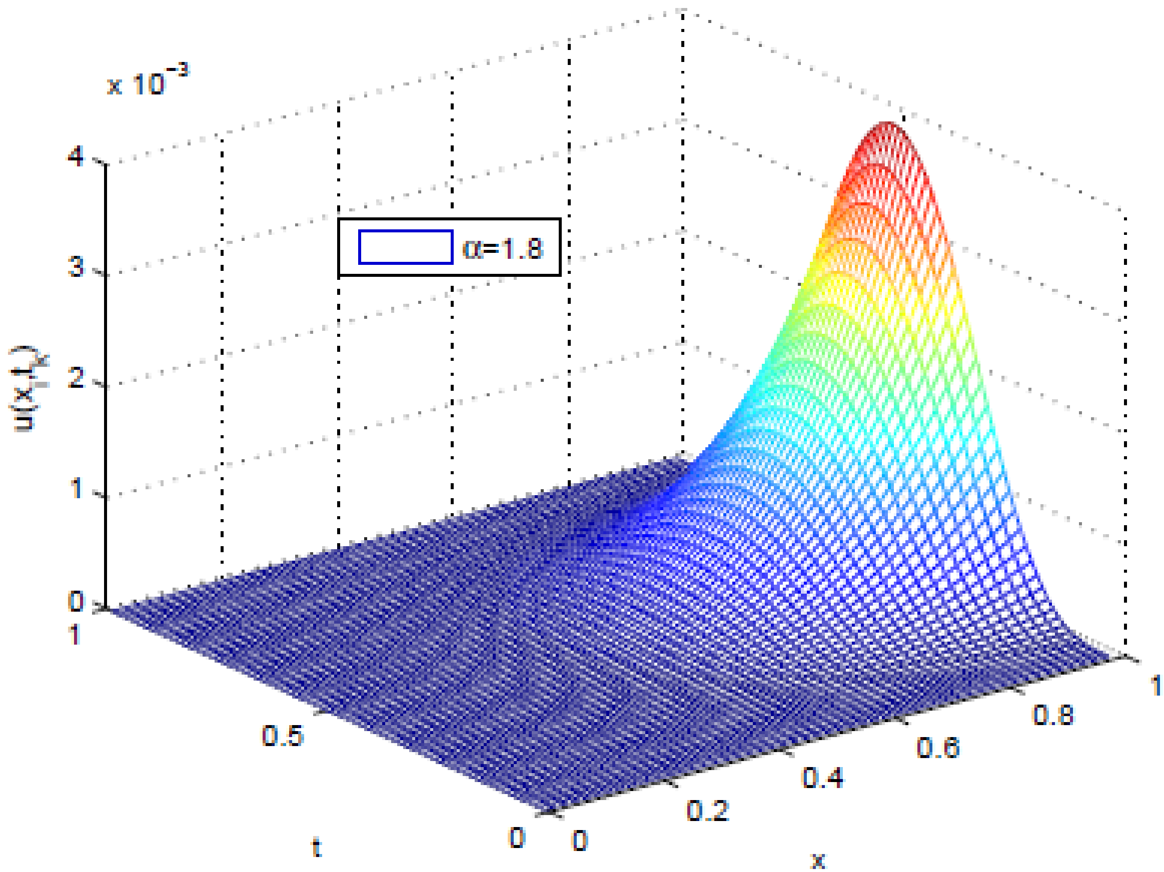

Figure 6.5: The numerical solution surface when $\alpha = 1.8$ in Example 2. ($\tau = \frac{1}{50}, h = \frac{1}{80}$)

Numerical Examples

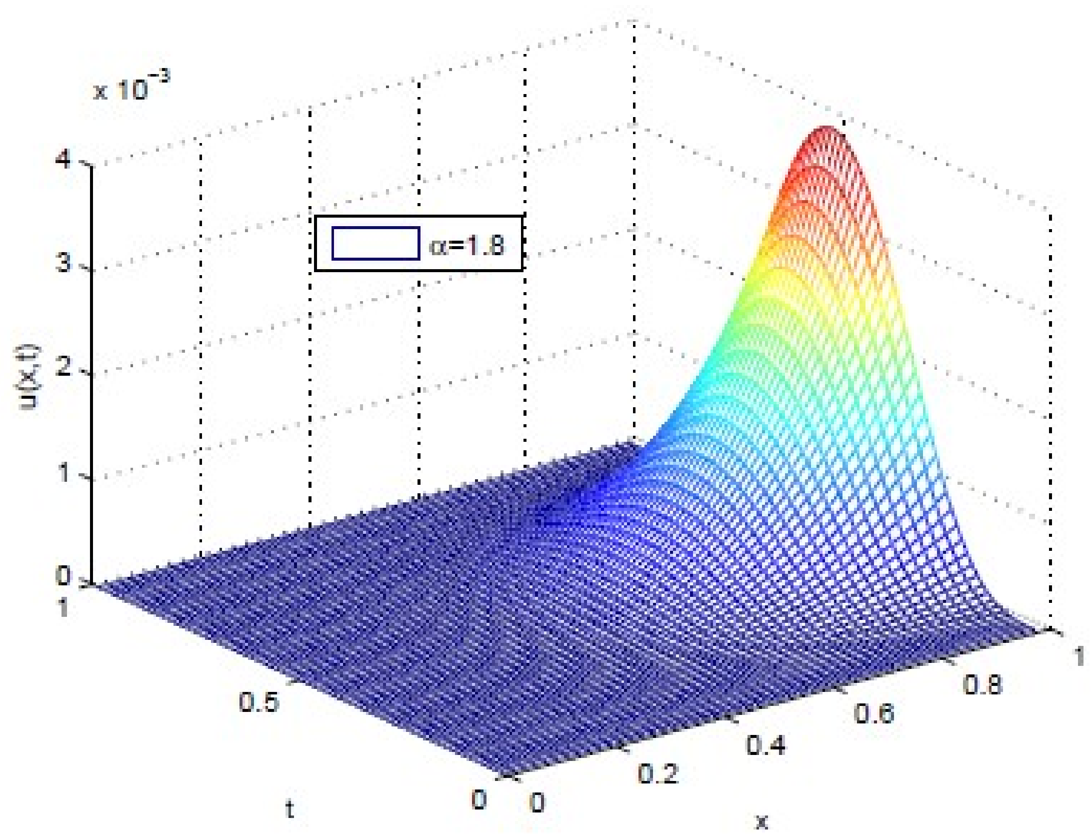

Figure 6.6: The analytical solution surface when $\alpha = 1.8$ in Example 2.

Comments

1) **Proper** and **exact** applications of fractional calculus

2) Long-term integrations at each step, induced by **historical dependencies** and/or **long-range interactions**, require more computational time and storage capacity.

   Therefore the effective and economical numerical methods are needed.

3) Fractional calculus + Stochastic process

   Stochastics and nonlocality may better characterize our complex world.

   So numerical fractional stochastic differential equations have been placed on the agenda.

   And more......

Conclusions?

Not yet, but Go Fractional with some lists.

Conclusions

Numerical calculations:

[1] Changpin Li, An Chen, Junjie Ye, J Comput Phys, 230(9), 3352-3368, 2011.
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[6] Hengfei Ding, Changpin Li, J Comput Phys 242(9), 103-123, 2013.

Conclusions

Fractional dynamics:

[1] Changpin Li, Ziqing Gong, Deliang Qian, Yangquan Chen, Chaos 20(1), 013127, 2010.

[2] Changpin Li, Yutian Ma, Nonlinear Dynamics 71(4), 621-633, 2013.

[3] Fengrong Zhang, Guanrong Chen, Changpin Li, Jürgen Kurths, Chaos synchronization in fractional differential systems, Phil Trans R Soc A 371 (1990), 20120155 (26 pages), 2013. **Review article**

Acknowledgements

Q & A!

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