

Variable-coefficient space-fractional diffusion equations: mathematical analysis and fast numerical solution

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Fractional differential equations, which describe phenomena exhibiting anomalous diffusion that cannot be modeled accurately by second-order diffusion equations, raise mathematical and numerical difficulties that have not been encountered in the analysis and numerical simulation of second-order differential equations. The wellposedness of a Galerkin weak formulation to space-fractional diffusion equations with a constant diffusivity coefficient has been proved. Subsequently, error analysis for corresponding finite difference methods, finite element methods, discontinuous Galerkin methods, and spectral methods was derived, under the same assumption of a constant diffusivity coefficient.

In this talk we present a counterexample which shows that the Galerkin formulation loses coercivity in the context of variable-coefficient fractional differential equations. Hence, the previous results cannot be extended to variable-coefficient problems. We outline an alternative approach to prove the existence and uniqueness of the classical solution to a class of variable-coefficient conservative fractional differential equation and characterize the solution in terms of the classical solutions to second-order differential equations. We present a Petrov-Galerkin formulation to the problem and show that the formulation is weakly coercive and well posed.

In this talk we also address the issues on the development of faithful and efficient numerical methods for space-fractional partial differential equations. Because of the nonlocal property of fractional differential operators, the numerical methods for fractional diffusion equations often generate dense or even full coefficient matrices. Consequently, the numerical solution of these methods often require computational work of $O(N^3)$ per time step and memory of $O(N^2)$ for where N is the number of grid points. We present fast numerical methods that have computational cost of $O(N \log^2 N)$ per time step and memory of $O(N)$, while retaining the same accuracy and approximation property as the regular finite difference method with Gaussian elimination. Numerical experiments show that to achieve the same accuracy, the new method has a significant reduction of the CPU time from more than 2 months consumed by a traditional finite difference method to 5.74 seconds for a problem with 36,000 grid points on a work station with 128GB memory. This demonstrates the utility of the method.