

THE FUNDAMENTAL SOLUTION OF THE DISTRIBUTED ORDER FRACTIONAL WAVE EQUATION IN ONE SPACE DIMENSION IS A PROBABILITY DENSITY

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June 2013

■ Fractional Wave Equation

$$D_t^\beta u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad 1 < \beta \leq 2, \quad -\infty < x < \infty, \quad t > 0 \quad (1)$$

$$u(x, 0) = \delta(x), \quad \frac{\partial u(x, t=0)}{\partial t} = 0,$$

D_t^β denote Caputo fractional differentiation.

■ The fundamental solution

$$\begin{aligned} u(x, t) &= g(x, t) \\ &= \frac{1}{2} t^{\frac{\beta}{2}} M_{\frac{\beta}{2}}(|x| t^{-\frac{\beta}{2}}) \end{aligned}$$

see [Mainardi \[3\]](#)

■ M-Wright Function

$$M_\nu(z) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n) \quad (2)$$

$t^\nu M_\beta(t_* t^{-\nu}) t^{-\nu}$ is density (in t_* , evolving with t) of inverse ν -stable subordination. Hence $g(x, t)$ is sojourn density of a randomly wandering particle,

monotonically rightwards with probability $\frac{1}{2}$,

monotonically leftwards with probability $\frac{1}{2}$.

- The process with density $M_{\frac{\beta}{2}}(x, t^{-\frac{\beta}{2}})$ is a fractional drift process, called “Mittag-Leffler process” by some people. It is distinct from Pillai’s ML-process.

■ Distributed Order Fractional Wave Equation

$\rho(\beta)$ generalized function, $\rho(\beta) \geq 0$,

$$0 < \int_{(1,2]} \rho(\beta) d\beta < \infty$$

Distributed order fractional Caputo

$$D_t^{\rho(\beta)} u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2}, \quad u(x, 0) = \delta(x), \quad u_t(x, t=0) \equiv 0 \quad (3)$$

Special Case:

$$\rho(\beta) = \delta(\beta - \beta_0).$$

■ Fourier-Laplace Solution

$$\nu(\hat{k}) = \int_{-\infty}^{\infty} e^{ikx} \nu(x) dx$$

$$\widetilde{w}(s) = \int_0^{\infty} e^{-st} w(t) dt$$

$$\widetilde{B}(s) = \int_{(1,2]} p(\beta) s^{\beta} d\beta$$

$$\hat{g}(k, s) = \frac{\frac{B(s)}{s}}{B(s) + k^2}$$

- Question: Is $g(x, t)$ a density in X , evolving in $t > 0$? Yes

$$\hat{g}(k=0, s) = \frac{1}{s}, \quad 1 = \hat{g}(k=0, t) = \int_{-\infty}^{\infty} g(x, t) dx$$

Remains to show that always and everywhere $g(x, t) \geq 0$.

- **Remark:** It is known (see[1]) that the fundamental solution of the distributed order fractional diffusion equation is a probability density evolving in time and that there exists a corresponding stochastic process. But for the fractional wave equation a different method of proof is required.

$$\int_{(0,1]} \rho(\beta) D_t^\beta u(x, t) d\beta = \frac{\partial^2 u(x, t)}{\partial x^2} \quad (4)$$

- Special types of functions
 - (a) **Completely monotone**
 - (b) **Stieltjes**
 - (c) **Bernstein**
 - (d) **Complete Bernstein** See the excellent book by Schilling et al.

- Consider function $\varphi : (0, \infty) \rightarrow \mathbb{R}$. (a): φ **Completely monotone** $\Leftrightarrow (-1)^n \varphi^{(n)}(\lambda) \geq 0$, for $n = 0, 1, 2, \dots$

Properties: Products of completely monotone functions are completely monotone
 positive linear combinations of completely monotone function are completely monotone.

- **Example:** $\lambda^{-\alpha}$, for $\alpha \geq 0$, $e^{-\lambda^\alpha}$ for $0 < \alpha \leq 1$

$$E_{\alpha, \beta}(-\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\alpha k + \beta)} \quad (\text{Mittag-Leffler})$$

- φ is **Stieltjes function** \Leftrightarrow
 φ is Laplace transform of a completely monotone function.
 The functions $\lambda^{-\alpha}$, for $0 < \alpha \leq 1$ are Stieltjes.
- φ is **Bernstein function** \Leftrightarrow
 $\varphi \in C^\infty$ and φ' is completely monotone.
- **Examples:** $\lambda^{-\alpha}$, for $0 < \alpha \leq 1$, $1 - e^{-\lambda}$
- **Properties:** Positive linear combinations and compositions are again Bernstein, likewise pointwise limits of sequences.
 φ completely monotone and ψ Bernstein $\Rightarrow \varphi(\psi)$ is completely monotone.
 φ Bernstein $\Leftrightarrow \frac{\varphi(\lambda)}{\lambda}$ is completely monotone.

- φ is Completely Bernstein $\Leftrightarrow \frac{\varphi(\lambda)}{\lambda}$ is Stieltjes.
- Properties: If φ is completely Bernstein and ψ is Stieltjes, then $\varphi(\psi)$ is Stieltjes.
- $\varphi \neq 0$ is complete Bernstein $\Leftrightarrow \frac{1}{\varphi}$ is Stieltjes.
The set of complete Bernstein function is convex cone.
- Examples: λ^α , for $0 < \alpha \leq 1$

■ Distributed order fractional diffusion equation

$$\int_{(0,1]} p(\beta) D_t^\beta u(x, t) d\beta = \frac{\partial^2 u(x, t)}{\partial x^2}$$

$$B(s) = \int_{(0,1]} p(\beta) s^\beta d\beta, \quad \hat{g}(k, s) = \frac{\frac{B(s)}{s}}{B(s) + k^2}$$

$$g(x, t) = \int_0^\infty \frac{1}{2\sqrt{\pi r}} e^{-\frac{x^2}{4r}} R(r, t) dr$$

$$\tilde{R}(r, s) = \frac{B(s)}{s} e^{-rB(s)}$$

- $B(s)$ is Bernstein, $\frac{B(s)}{s}$ completely monotone, $e^{-rB(s)}$ is completely monotone all in s . Hence \tilde{R} completely monotone in s , hence $R \geq 0$ and $g \geq 0$. That g is normalized, show as previously.

■ Distributed order fractional wave equation

$$\hat{g}(k, s) = \frac{\frac{B(s)}{s}}{B(s) + k^2}, \quad B(s) = \int_{(1,2]} p(\beta) s^\beta d\beta$$

By Fourier inversion

$$\tilde{g}(k, s) = \frac{1}{2} f_1(s) f_2(s)$$

$$f_2(s) = \frac{\sqrt{(B(s))}}{s}, \quad f_1(s) = \exp(-|x| \sqrt{(B(s))})$$

■ Plan: Show f_1 and f_2 completely monotone, hence

$$\hat{g}(k, s),$$

in s .

And consequently $g(x, t) \geq 0$.

■ Steps of proof

$$(f_2(s))^2 = \int_{(1,2]} p(\beta) s^{\beta-2} d\beta$$

$s^{\beta-2}$ is Stieltjes, hence also $(f_2(s))^2$

$sf_2(s) = \sqrt{(B(s))}$ is completely Bernstein, $f_2(s) = \frac{\sqrt{(B(s))}}{s}$ is completely monotone $f_1(s)$ is a completely monotone function, of a Bernstein function, hence completely monotone.

- Result: $\tilde{g}(x, s)$ is completely monotone in s , for all x , hence $g(x, t) \geq 0$.

■ Distributed order diffusion wave equation

$$D_t^{\rho(\beta)} u(x, t) d\beta = \frac{\partial^2 u(x, t)}{\partial x^2}$$

$$\rho(\beta) \geq 0, \quad \int_{(0,2]} \rho(\beta) d\beta > 0$$

Cases (a), (b), (c)

- (a): $\rho(\beta) \equiv 0$ in $(1,2]$ distributed order fractional diffusion, we know that $g(x, t)$ is a probability density in x .
- (b): $\rho(\beta) \equiv 0$ in $[0,1]$ distributed order wave, we have proved this.
- (c): $\rho(\beta) \neq 0$ in $[0,1]$, $\neq 0$ in $(1,2]$ distributed order diffusion wave equation.

- Case (c) is open .
A few very special cases have been discussed affirmatively in [4], e.g.

$$p(\beta) = \delta(\beta - 2\alpha) + 2\lambda\delta(\beta - \alpha), \quad 0 < \alpha \leq 1, \quad \lambda > 0$$

- Of course $0 < \alpha \leq \frac{1}{2}$ is d.o. fractional diffusion.

■ Stochastic Process?







Does there exist a stochastic process having $g(x, t)$ as density in x evolving in t ?

- Clearly yes in the special case

$$p(\beta) = \delta(\beta - \beta_0), \quad 0 < \beta_0 \leq 2.$$

- If $\beta_0 = 2$ the process is degenerated: the particle moves with constant velocity 1 monotonically rightwards with probability $\frac{1}{2}$, monotonically leftwards with probability $\frac{1}{2}$

References

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- THE SPEAKER APPRECIATES COLLABORATION WITH **YURI LUCHKO** AND **MIRA STOJANORIC** AND WORK BY **SEEMA S. NAIR** IN PRODUCING THE SLIDES.

REFERENCES



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