The following pages constitute a special APPENDIX to the presentation by *Rudolf Gorenflo* to the

"International Symposium on Fractional PDEs: Theory, Numerics and Applications", June 3-5, 2013, Salve Regina University, New Port, RI, USA.

Berlin, July 27, 2013

In this informal essay (four pages) **Rudolf Gorenflo**, FU Berlin, Germany, presents the

Solution of an open problem concerning the distributed order fractional wave equation .

In [2] Gorenflo, Luchko and Stojanović have shown that the solution $u(x,t)$ to the Cauchy problem

(0) $$\int_{(1,2]} b(\beta) D_*^\beta u(x,t)d\beta = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad u(x,0) = \delta(x), \quad \frac{\partial}{\partial x} u(x,0) \equiv 0, \quad -\infty < x < \infty, \quad t > 0,$$

is a probability density in the spatial variable *x*, evolving in the time variable *t*, if we take D_*^β as the Caputo derivative with respect to *t*, $b(\beta)$ as a non-negative generalized function in the sense of Gelfan´d and Shilov with $0 < \int_{(1,2]} b(\beta)d\beta < \infty$. They raised the question for existence of a stochastic process so that $u(x,t)$ is the sojourn probability density for a wandering particle to be in point *x* at instant *t*. In the *International Symposium on Fractional PDEs: Theory, Numerics and Applications*,

Salve Regina University, Newport, RI, June 3-5, 2013, R. Gorenflo conveyed this question to the audience as a challenge and then had illuminating discussions on it with Mark M. Meerschaert who supposed that $\exp(-\sqrt{B(s)})$ is Laplace transform of an infinitely divisible distribution.

We use Laplace and Fourier transforms:

$$\tilde{f}(s) = \int_0^\infty f(t)\exp(-st)dt, \quad \hat{g}(\kappa) = \int_{-\infty}^\infty g(x)\exp(-\kappa x)dx.$$

Setting $B(s) = \int_{(1,2]} b(s) s^\beta d\beta$ we obtain $\hat{\tilde{u}}(\kappa, s) = \dfrac{B(s)/s}{B(s)+\kappa^2}$. From the spatial inversion

(1) $$\tilde{u}(x,s) = \frac{\sqrt{B(s)}}{2s}\exp(-|x|\sqrt{B(s)})$$

the authors of [2] have concluded, by aid of the basic theory of *completely monotone, Stieltjes, Bernstein and complete Bernstein functions* (see [5]), that $u(x,t)$ indeed is a probability density in *x* evolving in *t*.

We will see that $u(x,t)$ represents the spatial density of a stochastic process. We have $\sqrt{B(0)} = 0$ and in [2] it has been shown that $\sqrt{B(s)}$ is a complete Bernstein function. Hence, by Theorem1 in chapter XIII.7 of [1] (see also [5]) $\exp(-\sqrt{B(s)})$ is Laplace transform of an infinitely divisible distribution (on the positive half-line), thus suited as a subordinator. The inverse Laplace transform of $\exp(-x\sqrt{B(s)})$ is a density $r(t,x)$ (in $t \geq 0$, evolving in $x \geq 0$) of an increasing stochastic process $t = t(x)$ on the positive *t*-half-line starting in the origin $t = 0$ at instant $x = 0$. Note that here the roles of space and time are interchanged. Because of monotonicity we can ask for the inverse (also increasing) process $x = x(t)$ and its density $q(x,t)$, a density in $x \geq 0$, evolving in $t \geq 0$. These two processes are represented by the same graph if we visualize intervals of constancy and jumps by corresponding horizontal or vertical segments.

Now consider a fixed sample trajectory $t = t(x)$ and its likewise fixed inversion $x = x(t)$. For fixed $\bar{t}$ and fixed $\bar{x}$ we have, because of monotonicity, the equivalence $x(\bar{t}) \geq \bar{x} \Leftrightarrow \bar{t} \geq t(\bar{x})$. The corresponding probabilities coincide. By the replacements $x(\bar{t}) \Rightarrow x'$, $\bar{x} \Rightarrow x$, $\bar{t} \Rightarrow t$, $t(\bar{x}) \Rightarrow t'$ we now arrive for the densities *r* and *q* at the relation $\int_0^t r(t',x)dt' = \int_x^\infty q(x',t)dx'$, hence

$$q(x,t) = -\frac{\partial}{\partial x}\int_0^t r(t',x)dt',$$

$$\tilde{q}(x,s) = -\frac{\partial}{\partial x}\{\frac{1}{s}\tilde{r}(s,x)\} = -\frac{1}{s}\frac{\partial}{\partial x}\exp(-x\sqrt{B(s)})$$

(2) $\tilde{q}(x,s) = \dfrac{\sqrt{B(s)}}{s}\exp(-x\sqrt{B(s)})$ for $x \geq 0$.

Conclusion:

This **RESULT** allows us to interpret in equation (1) $u(x,t)$ as sojourn density (at *x*, evolving in *t*) of a randomly moving particle, starting at the origin $x(0) = 0$, deciding with probability 1/2 to move in positive or negative direction and then keeping to this decision. The motion in positive direction has density $q(x,t)$, that in negative direction has density $q(-x,t)$. Physically, when many particles are starting simultaneously, half (in probabilistic sense) of them move in positive, the other half in negative direction. Macroscopically we get two dispersing waves. This is the physical meaning of time-fractional wave. Simultaneously we have the character of wave and diffusion.

References:

[1] W. Feller: *Introduction to Probability Theory and its Applications*. Vol. II. Wiley, New York 1971.

[2] R. Gorenflo, Yu. Luchko, M. Stojanović: Fundamental solution of a distributed order fractional diffusion-wave equation as probability density. *FCAA* 16 (2013), 297-316.

[3] M. M. Meerschaert and H.-P. Scheffler: Triangular array limits for continuous time random walks. *Stochastic Processes and their Applications* **118** (2008), 1606-1633.

[4] F. Mainardi: *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, London 2010.

[5] R.L Schilling, R. Song and Z. Vondraček: *Bernstein Functions: Theory and Applications.* De Gruyter, Berlin 2010.

[6] A.N. Kochubei: General fractional calculus, evolution equations, and renewal processes. *Integral Equations and Operator Theory* **71** (2011), 583-600.

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Appendix: An alternative approach.

Let us first consider the Cauchy problem for the single-order time-fractional wave equation

(3) $D_*^{\beta} u(x,t) d\beta = \frac{\partial^2 u(x,t)}{\partial x^2},\ \ u(x,0) = \delta(x),\ \frac{\partial}{\partial x} u(x,0) \equiv 0,\ -\infty < x < \infty,\ t > 0,\ 0 < \beta \le 1$.

With $\nu = \frac{\beta}{2}$ we can split its Fourier-Laplace solution $\hat{\tilde{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + \kappa^2}$ into two terms:

(4) $$\hat{\tilde{u}}(\kappa, s) = \frac{1}{2}\left(\frac{s^{\nu-1}}{s^{\nu} + i\kappa} + \frac{s^{\nu-1}}{s^{\nu} - i\kappa}\right).$$

Here the second term $\frac{s^{\nu-1}}{s^{\nu} - i\kappa}$ represents the density (in *x,* evolving in *t*) of a positive-oriented fractional drift process starting in the origin $x = 0$ at $t = 0$ with monotonically increasing sample paths. The corresponding Cauchy problem is

(5) $$D_*^{\nu} q(x,t) = -\frac{\partial}{\partial x} q(x,t),\ \ q(x,0) = \delta(x).$$

To treat it conveniently by Laplace-Laplace analysis we introduce (in addition to the Laplace transform with respect to *t*) also the Laplace transform with respect to *x* as $\breve{g}(\kappa) = \int_0^{\infty} g(x) \exp(-\kappa x) dx$. From $s^{\nu} \tilde{\breve{q}}(\kappa, s) - s^{\nu-1} = -\kappa \tilde{\breve{q}}(\kappa, s)$ we obtain $\tilde{\breve{q}}(\kappa, s) = \frac{s^{\nu-1}}{s^{\nu} + \kappa}$, the Laplace-Laplace representation of the inverse ν- stable subordinator. Its relation to the positive-unilateral ν- stable density is visible in the partial inversion $\tilde{q}(x,s) = s^{\nu-1} \exp(-x s^{\nu})$. Its explicit form is given by Mainardi´s M-function (see [4]): $q(x,t) = t^{-\nu} M_{\nu}(x t^{-\nu})$. For convenience: $M_{\nu}(z) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\nu n) \sin(\pi \nu n)$.

Stochastic Interpretation: The fundamental solution to the fractional wave equation of single order $\beta \in (1,2]$ describes a process where a particle starting at $x = 0$ decides with probability 1/2 to wander in positive or negative direction (keeping this direction), thereby following the inverse $\frac{\beta}{2}$- stable subordinator (or its reflected process, respectively).

Analogously we now treat the case of **distributed orders** by splitting. We find

(5) $$\hat{\tilde{u}}(\kappa, s) = \frac{B(s)/s}{B(s) + \kappa^2} = \frac{1}{2}\left(\frac{\sqrt{B(s)}/s}{\sqrt{B(s)} + i\kappa} + \frac{\sqrt{B(s)}/s}{\sqrt{B(s)} - i\kappa}\right).$$

The analogy: $B(s)$ corresponds to s^{β}, $\sqrt{B(s)}$ to $s^{\beta/2} = s^{\nu}$. Again we have a positive-oriented and a reflected negative-oriented process, each chosen with probability 1/2 by a wandering particle, starting at the origin $x(0)=0$. Denoting the positive-oriented process again by $q(x,t)$, we can again work with Laplace-Laplace in place of Fourier-Laplace and obtain in the transform domain

(6) $\sqrt{B(s)}\,\breve{\tilde{q}}(\kappa,s)+\kappa\,\breve{\tilde{q}}(\kappa,s)=\dfrac{\sqrt{B(s)}}{s}$, $\breve{\tilde{q}}(\kappa,s)=\dfrac{\sqrt{B(s)}\,/\,s}{\sqrt{B(s)}+\kappa}$.

By partial inversion we again get equation (2), namely

$$\tilde{q}(x,s)=\frac{\sqrt{B(s)}}{s}\exp(-x\sqrt{B(s)}) .$$

Question: Can the operator A whose Laplace symbol is $\sqrt{B(s)}$ be analyzed with the theory developed by Kochubei in [6]?

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