Numerical Fractional Calculus Using Methods Based on Non-Uniform Step Sizes

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Traditionally, most numerical algorithms for handling problems in fractional calculus (such as, e. g., the computation of fractional derivatives or integrals or the approximate solution of fractional ordinary or partial differential equations) use uniform step sizes. Such algorithms can usually be implemented in a very easy way, and their convergence analysis is also quite simple. However, it has been known for quite some time that their computational cost may be far from optimal. We therefore look at various approaches using non-uniform step sizes. Our main emphasis in this context is placed on algorithms for the numerical solution of fractional ordinary differential equations and the application of these techniques to time-fractional partial differential equations.

We shall look at a number of different ideas for the construction of the underlying non-uniform grids. This includes, in particular, predetermined Shishkin-type methods that in the classical case of integer-order differential equations have proven to be valuable tools for differential equations whose solutions exhibit boundary layer phenomena, graded meshes, and ideas that can lead to adaptive meshes, i. e. to meshes whose nodes are not defined a priori but that are rather computed at the run time of the algorithm, based on the information collected so far.

Our goal is to provide a comparison of methods using such non-uniform grids with the standard methods based on uniform grids. The most important aspects of this comparison will be the computational effort and the resulting accuracy. As the computational complexity of numerical fractional calculus always tends to be relatively high due to the non-locality of the fractional differential and integral operators, the implementation of the algorithms on parallel computing systems is also an issue of interest.

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