International Symposium on Fractional PDEs
Theory, Numerics and Applications

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# ADVANCES ON FRACTIONAL DYNAMICS OF COMPLEX SYSTEMS


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# Presentation Outline

***1. Motivation***

***2. Fractional Variationl Principles***

***3. Fractional Field Theory***

***4. A Fractional Model for Simulation of Water Table Profile Between Two Parallel Subsurface Drains***

***5. Some Open Problems***

***1675: Leibnitz*** introduced the notion of a ***derivative of order n of a function***

The first published results are cited in a letter from L'Hopital to Leibnitz in 1695.

**1661–1704**
**Leibnitz**

***1695: L'Hopital*** wrote to Leibnitz asking him about a particular notation he had used in his publications for the nth-derivative of a linear function…

**1646–1716**
**L'Hopital**

- ***Leibniz, Euler, Lagrange, Fourier, Abel, Liouville,Riemann, Grünwald, Letnikov, Holmgren, Peacock ,Tarday, Cauchy, Hadamard, Hardy, Riesz, Weyl....***

# MOTIVATION

There are strong motivations today (the unelucidated nature of the dark matter and dark energy, the difficult **reconciliation** of Einstein's General Relativity (GR) and Quantum Theory) to consider alternative theories that modify, extend or replace GR.

Some of these theories presume a **higher dimensional space-time**, and part of them predict **violations** of the (Relativistic) Physics (both Special and General) fundamental principles: the Equivalence Principle and Lorentz symmetry could be broken, **the fundamental constants could vary**, **the space could be anisotropic**, and the physics could become **non-local**.

During the last decades, fractional differentiation has drawn increasing attention in the study of so-called "**anomalous**" social and physical behaviors, where scaling power law of fractional order appears universal as an empirical description of such complex phenomena.

It is worth noting that the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases where **power law is clearly observed**.

To accurately reflect the non-local, frequency- and history-dependent properties of power law phenomena, some **alternative** modeling tools have to be introduced such as fractional calculus.

Research in fractional differentiation is inherently multi-disciplinary and its application across various disciplines.

# Riemann-Liouville definitions

**Riemann-Liouville left-sided fractional integral of order α**

$$aD_x^{-\alpha}\Phi(x) = \frac{1}{\Gamma(x)}\int_a^x (x-t)^{\alpha-1}\Phi(t)dt \qquad ,x > a$$

**Riemann-Liouville right-sided fractional integral of order α**

$$xD_b^{-\alpha}\Phi(x) = \frac{1}{\Gamma(x)}\int_x^b (t-x)^{\alpha-1}\Phi(t)dt \qquad ,x < b$$

**where fractional order α > 0 and**

$$\Gamma(t) = \int_0^\infty s^{t-1}e^{-s}\, ds$$

$$\Gamma(t) = (t-1)\Gamma(t-1)$$

$$\Gamma(n) = (n-1)!$$

***Example:***

$$oD_x^{-\alpha}x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}x^{\mu+\alpha}$$

# Riemann-Liouville Fractional Derivatives

## Left Riemann-Liouville Fractional Derivative of order $\alpha$

$$aD_x^{\alpha}\Phi(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{\Phi(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a, n = [\Re(\alpha)] + 1,$$

- Right Riemann-Liouville Fractional Derivative of order $\alpha$

$$xD_b^{\alpha}\Phi(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b \frac{\Phi(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b, n = [\Re(\alpha)] + 1,$$

# Chain Rule:

$$For\ \Phi(x) = F\big(h(x)\big),$$

$$ {}_aD_x^{\alpha} F\big(h(x)\big) =$$

$$\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}\Phi(x) + \sum_{k=1}^{\infty}\binom{\alpha}{h}\frac{k!\,(x-a)^{k-\alpha}d^k}{\Gamma(k-\alpha+1)dx^k}F\big(h(x)\big)$$

with the help of ***Faa di Bruno formula***:

$$\frac{d^k}{dx^k}F\big(h(x)\big) = k!\sum_{m=1}^{k} F^m\big(h(x)\big)\sum\prod_{r=1}^{k}\frac{1}{a_r!}\left(\frac{h^r(x)}{r!}\right)^{a_r}$$

$$\sum_{r=1}^{k} r a_r = k\ ,\quad \sum_{r=1}^{m} a_r = m$$

# Power function:

$$ {}_aD_x^{\alpha}(x-a)^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}(x-a)^{k-\alpha}, k > -1 $$

so, for a constant A

$$ {}_aD_x^{\alpha}A = \frac{A}{\Gamma(1-\alpha)}(x-a)^{-\alpha} \neq 0 $$

# Leibnitz Rule:

$$aD_x^{\alpha}(\Phi(x)\Psi(x)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \Phi^{(k)}(x) aD_x^{\alpha-k}\Psi(x)$$

# Caputo definition

## The left Caputo fractional derivative

$$
{}_a^C D_x^\alpha \Phi(x) = {}_a D_x^\alpha \left( \Phi(x) - \sum_{k=0}^{n-1} \frac{\Phi^{(k)}(a)}{k!} (x-a)^k \right)
$$

$$
{}_a^C D_x^\alpha \Phi(x) = \frac{1}{\Gamma(N-\alpha)} \int_a^X \frac{\Phi^{(n)}(t)}{(x-t)^{\alpha+1-n}} dt
$$

## The right Caputo fractional derivative

$$
{}_x^C D_b^\beta \Phi(x) = \frac{(-1)^n}{\Gamma(N-\alpha)} \int_x^b \frac{\Phi^{(n)}(t)}{(t-x)^{\alpha+1-n}} dt
$$

where $0 \leq n-1 < \alpha < n$ and (x) has n+1 continuous and bounded derivatives in [a,b].

# Properties (Caputo definition):

$$ {}_{a}^{C}D_{x}^{\alpha}A = o \ , A = constant $$

$$ \lim_{x \to a} {}_{a}^{C}D_{x}^{\alpha}\Phi = 0 $$

In an infinite domain,

$$ {}_{-\infty}^{C}D_{x}^{\alpha}\Phi = {}_{-\infty}D_{x}^{\alpha}\Phi $$

$$ {}_{-\infty}^{C}D_{x}^{\alpha}\Phi = {}_{x}D_{\infty}^{\alpha}\Phi $$

As one could expect, since a fractional derivative is a generalization of an ordinary derivative, it is going to lose many of its basic properties.

For example, it loses its geometric or physical interpretation, the index law is only valid when working on very specific function spaces, the derivative of the product of two functions is difficult to obtain, and the chain rule is not straightforward to apply.

Fractional order differential equations, that is, those involving real or complex order derivatives, have assumed an important role in modelling the anomalous dynamics of numerous processes related to complex systems in the most diverse areas of Science and Engineering.

It is natural to ask, then:

***What properties fractional derivatives have that make them so suitable for modelling certain Complex Systems?***

We think the answer is based on the property exhibited by many of the aforementioned systems of non-local dynamics, that is, the processes dynamics have a certain degree of memory and fractional operators are non-local, while the ordinary derivative is clearly a local derivative.

The idea, that physical phenomena such as diffusion can be described by fractional differential equations, raises at least two **fundamental questions**:

**(1)** ***Are mathematical models with fractional space and/or time derivatives consistent with the fundamental laws and fundamental symmetries of nature ?***

**(2)** ***How can the fractional order of differentiation be observed experimentally or how does a fractional derivative emerge from models without fractional derivatives?***

# 2. FRACTIONAL VARIATIONAL PRINCIPLES

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**Article Title:** ***The Hamilton formalism with fractional derivatives***

**Authors:**
**Rabei, EM;Nawafleh, KI;Hijjawi, RS;Muslih, SI;Baleanu, D**
Journal: J MATH ANAL APPL
Volume: 327 , Issue: 2 , Page: 891-897 , Year: MAR 15 2007

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- **Why do you think your paper is highly cited?**
- The paper is highly cited because it describes a new method in the emerging field of fractional calculus. Also the field of the physical applications of fractional calculus is a hot subject.
- **Does it describe a new discovery, methodology, or synthesis of knowledge?**
- The method describes a new methodology to treat fractional dynamical systems.
- **Would you summarize the significance of your paper in layman's terms?**
- This new methodology is quite useful in discussing fractional dynamics and its applications. Some possible applications are the quantization of nonconservative systems using fractional calculus and the fractional quantization of constrained systems.

**How did you become involved in this research, and were there any problems along the way?**

We first became involved in this research by reading some old papers on variational principles involving fractional terms.

**Where do you see your research leading in the future?**

- The results obtained in the paper will provide a solid basis for understanding the fractional quantization method for discrete dynamical systems.

**DUMITRU BALEANU**

# Fractional Euler-Lagrange equations for discrete systems

- ***THEOREM:*** Let J[q] be a functional as follows

$$\int_a^b L(t, q^\rho, \; {}_aD_t^\alpha q^\rho, \; {}_tD_b^\beta q^\rho)dt$$

where ρ =1,…,n defined on the set of n functions which have continuous left Riemann-Liouville fractional derivative of order α and right Riemann-Liouville fractional derivative of order β in [a, b] and satisfy the boundary conditions $q^\rho(a) = q_a^\rho$ and $q^\rho(b) = q_b^\rho$.

A necessary condition for *J[qᵖ]* to admit an extremum for given functions *q(t)*, = *1*, · · · , n is that $q^\rho(t)$ satisfy Euler-Lagrange equations for

$$\frac{\partial L}{\partial q^\rho} + \; {}_tD_b^\beta \frac{\partial L}{\partial\, {}_aD_t^\alpha q^\rho} + \; {}_aD_t^\beta \frac{\partial L}{\partial\, {}_tD_b^\beta q^\rho} = 0$$



DUMITRU BALEANU

# Generalization

Assume that $\alpha_j$, (j = 1, · · · , $n_1$) and $\beta_k$, (k = 1, · · · , $n_2$) are two sets of real numbers all greater than 0, $\alpha_{max}$ =max($\alpha_1$, · · · , $\alpha_{n1}$, $\beta_k$, · · · , $\beta_{n2}$), and M is an integer such that $M - 1 \leq \alpha_{max} < M$. Let J[q] be a functional of the type

$$\int_a^b L(t, q^\rho, \ {}_aD_t^{\alpha_1} q^\rho, \ldots, \ {}_aD_t^{\alpha_{n1}} q^\rho, \ {}_tD_b^{\beta_1} q^\rho, \ldots, \ {}_tD_b^{\beta_{n2}} q^\rho) dt$$

defined on the set of n functions $q^\rho$, $\rho = 1, \cdots, n$ which have continuous left Riemann-Liouville fractional derivative of order $\alpha_j$, (j = 1, · · · , $n_1$) and right Riemann-Liouville fractional derivative of order $\beta_j$, (j = 1, · · · , $n_2$) in [a, b] and satisfy the boundary conditions $(q^\rho(a))^{(n)} = q^\rho_{aj}$ and $(q^\rho(b))^{(n)} = q^\rho_{bj}$ , j = 1, · · · ,M − 1.



DUMITRU BALEANU

# Generalization

A **necessary condition** for J[$q^\rho$] to admit an extremum for given functions $q^\rho(t)$, $\rho = 1, \cdots, n$ is that $q^\rho(t)$ satisfies Euler-Lagrange equations

$$\frac{\partial L}{\partial q^\rho} + \sum_{j=1}^{n_1} {}_aD_t^{\alpha_j} \frac{\partial L}{\partial\, {}_aD_t^{\alpha_j} q^\rho} + \sum_{j=1}^{n_2} {}_aD_t^{\beta_j} \frac{\partial L}{\partial\, {}_aD_t^{\beta_j} q^\rho} = 0$$

$${}_aD_t^{\alpha_j}$$

Here, if $\alpha_j$ is an integer, then ${}_aD_t^{\alpha_j}$ and ${}_aD_t^{\beta_j}$ must be replaced with the ordinary derivatives $(d/dt)^{\alpha j}$ and $(-d/dt)^{\alpha j}$ , respectively.[Agrawal OP,J. Math. Anal.Appl. 2002;272:368–379]

# Classical Euler-Lagrange equations

The Euler-Lagrange differential equation is the fundamental equation of calculus of variations.

It states that J if is defined by an integral of the form

$$I = \int f(t, y, \dot{y}) dt, \qquad where\ \dot{y} = \frac{dy}{dt}$$

then J has a stationary value if the Euler-Lagrange differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

***EXAMPLE***

$$L = \frac{1}{2}\dot{q}^2 - V(q)$$

$$-\frac{\partial V(q)}{\partial q} - \ddot{q} = 0$$

**DUMITRU BALEANU**

# Example

**I.)** $$L(x,y,z) = \dot{x}\dot{z} + \ddot{y}z^2$$

The classical solutions of Euler-Lagrange equations are

$$x(t) = at + b, \qquad z(t) = 0$$

y(t) has an undetermined evolution and a and b are constants to be determined from the initial conditions .

**II.)**One possible fractional generalization is as follows

$$L_f = ( \ {}_aD_t^\alpha x) \ \ {}_aD_t^\alpha z + yz^2$$

The Euler-Lagrange equations oare given by

$${}_tD_b^\alpha( \ {}_aD_t^\alpha z) = 0 \ , z^2 = 0 \, , \quad {}_tD_b^\alpha( \ {}_aD_t^\alpha x) + yz = 0$$

We notice that z = 0, y is not determined.

Besides,x fulfills the following equation

$${}_tD_b^\alpha( \ {}_aD_t^\alpha x) = 0$$

The solution of the above equation, under the assumption that $1 < \alpha < 2$ is given by

$$x(t) = A(t-a)^{\alpha-1} + B(t-a)^{\alpha-2} + C(t-a)^{\alpha}\ {}_2F_1\left(1, 1-\alpha, 1+\alpha, \frac{t-a}{b-a}\right)$$

$$+ D(t-a)^{\alpha}\ {}_2F_1\left(1, 2-\alpha, 1+\alpha, \frac{t-a}{b-a}\right)$$

Here $_2F_1$ represents Gauss hypergeometric function and A,B,C, and D are real constants.

It is easy to observe that if $\alpha \to 1^+$ and $a = 0$ the classical linear solution of one-dimensional space is obtain as

$$x(t) = A + Ct$$

# Lagrangian formulation of field systems with fractional derivatives

A covariant form of the action would involve a Lagrangian density $\boldsymbol{L}$ via

$$\boldsymbol{S} = \int \boldsymbol{L}\, d^4x = \int \boldsymbol{L}\, d^3x dt$$

where

$$\boldsymbol{L} = \boldsymbol{L}(\Phi, \partial_\mu \Phi)$$

The corresponding classical covariant Euler-Lagrange equation are

$$\frac{\partial \boldsymbol{L}}{\partial \Phi} - \frac{\partial_\mu \partial \boldsymbol{L}}{\partial(\partial_\mu \Phi)} = 0$$

where $\Phi$ is the field variable .

# Fractional generalization

$$S = \int L\left(\Phi(x), \left(D^{\alpha_k}_{a_k-}\right)\Phi(x), \left(D^{\alpha_k}_{a_k+}\right)\Phi(x), x\right) d^3x dt$$

Here $0 < \alpha_k \leq 1$ and $a_k$ correspond to $x_1$, $x_2$, $x_3$ and t respectively.

The fractional Euler-Lagrange equations are as follows

$$\frac{\partial S}{\partial \Phi} + \sum_{k}^{1,4} \left\{ \left(D^{\alpha_k}_{a_k+}\right) \frac{\partial S}{\partial \left(D^{\alpha_k}_{a_k-}\right)\Phi} + \left(D^{\alpha_k}_{a_k-}\right) \frac{\partial S}{\partial \left(D^{\alpha_k}_{a_k+}\right)\Phi} \right\} = 0$$

For $\alpha_k$ *->1*, the above equation becomes the usual Euler-Lagrange equations for classical fields.[**Dumitru Baleanu, Sami I. Muslih, PHYSICA SCRIPTA ,72 ( 2-3) 119-121 (2005)** ]

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# FRACTIONAL SCHRÖDINGER EQUATION

# *Fractional Dirac equation*

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**Sami I. Muslih ,Om P. Agrawal, Dumitru Baleanu**

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*Dumitru Baleanu, Sergiu Vacaru,*
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# 3.Fractional Field Theory

# A model of leptons

**Steven Weinberg**

**Laboratory for Nuclear Sciences and Physics Department**
**MIT, Cambridge,Massachusetts**
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# Gauge invariance in fractional field theories

Richard Herrmann

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# Fractional dynamics and the Standard Model for particle physics

**Ervin Goldfain**

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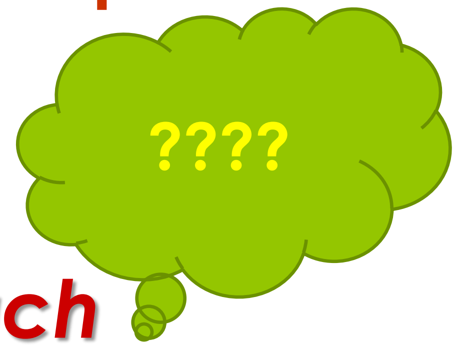
- **The mathematical modeling approach to problem solving consists of five steps:**

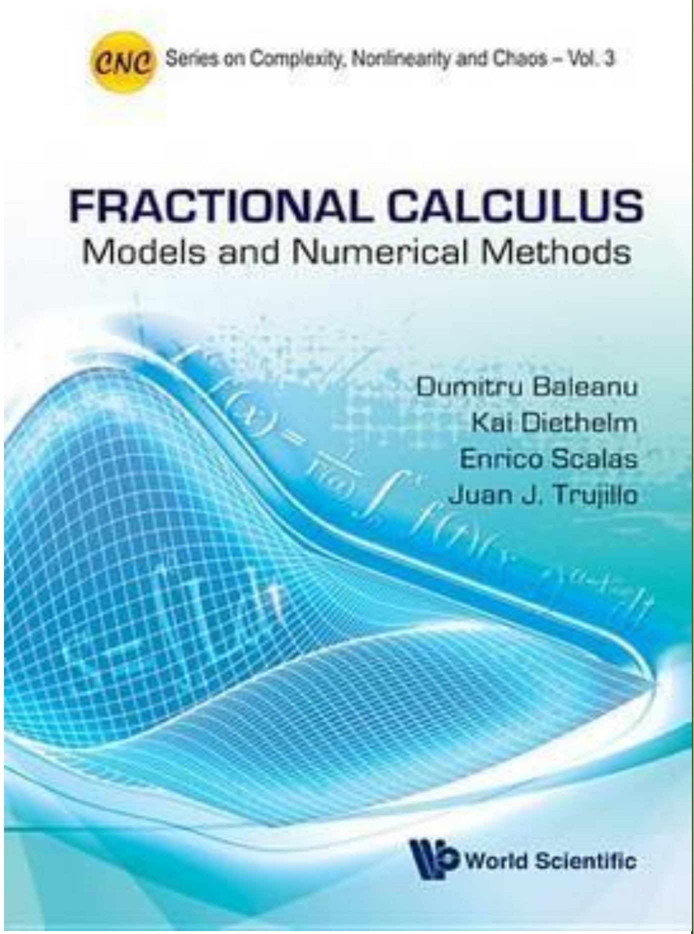



- ***Ask the question***
- ***Select the modelling approach***
- ***Formulate the model***
- ***Solve the model***
- ***Answer the question***

# Fractional Calculus Models and Numerical Methods

## Kai Diethelm, Dumitru Baleanu, Enrico Scalas, Juan J. Trujillo

This book will give readers the possibility of finding very important mathematical tools for working with fractional models and solving fractional differential equations, such as a generalization of Stirling numbers in the framework of fractional calculus and a set of efficient numerical methods. Moreover, we will introduce some applied topics, in particular fractional variational methods which are used in physics, engineering or economics. We will also discuss the relationship between semi-Markov continuous-time random walks and the space-time fractional diffusion equation, which generalizes the usual theory relating random walks to the diffusion equation. These methods can be applied in finance, to model tick-by-tick (log)-price fluctuations, in insurance theory, to study ruin, as well as in macroeconomics as prototypical growth models. All these topics are complementary to what is dealt with in existing books on fractional calculus and its applications currently existing in the market



# 4. A FRACTIONAL MODEL FOR SIMULATION OF WATER TABLE PROFILE BETWEEN TWO PARALLEL SUBSURFACE DRAINS

# A mathematical model for simulation of a water table profile between two parallel subsurface drains using fractional derivatives

**B. Mehdinejadiani, A. A. Naseri, H. Jafari, A. Ghanbarzadeh, D. Baleanu,**

**Computers and Mathematics with Applications, in press, 2013,doi:10.1016/j.camwa.2013.01.002**

# Introduction

- Predicting fluctuations of water table is very important from an agricultural and environmental perspective.
- Groundwater flow in an unconfined aquifer can be simulated using the Boussinesq equation.
- The **Boussinesq equation** is given by :

$$\frac{\partial}{\partial x}\left(K_x h \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y h \frac{\partial h}{\partial y}\right) + N = S_y \frac{\partial h}{\partial t} \qquad (1)$$

where

$\boldsymbol{K_x}$ is the saturated hydraulic conductivity in the x direction ( L/T) ,
$\boldsymbol{K_y}$ is the saturated hydraulic conductivity in the y direction ( L/T) ,
$\boldsymbol{h}$ is the hydraulic head ,
$\boldsymbol{S_y}$ is the specific yield (dimensionless), and
$\boldsymbol{N}$ is the recharge rate or discharge rate ( L/T)
*[B. Mehdinejadiani et al. 2013]*

- The fractional Taylor series

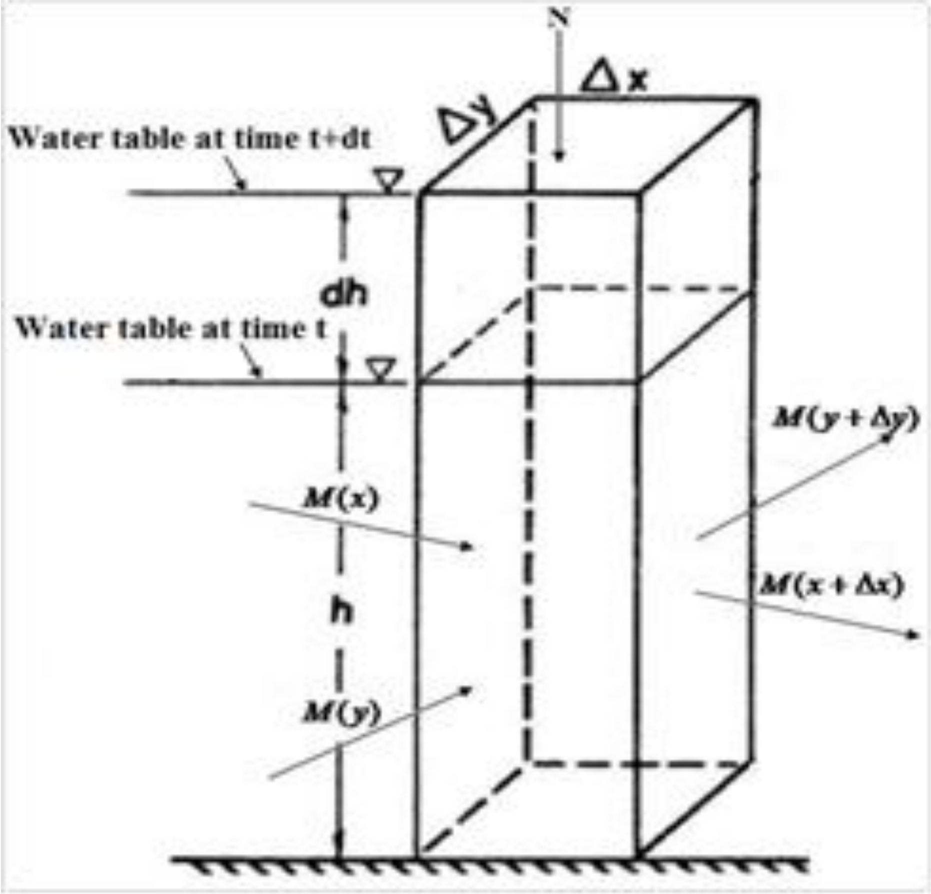
$$f(x+\Delta x)=f(x)+\frac{\partial^{\alpha} f(x)}{\partial x^{\alpha}}\cdot\frac{\Delta x^{\alpha}}{\Gamma(1+\alpha)}+\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial^{\alpha} f(x)}{\partial x^{\alpha}}\right)\cdot\frac{\Delta x^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots$$

$$0<\alpha\leq 1$$

## THE MODEL

- To develop the fractional Boussinesq equation, consider the fluid mass conservation for the control volume bounded by vertical surfaces at , $x, x+\Delta x$ , $y, y+\Delta y$ and as shown in Figure 1.
- [***B. Mehdinejadiani et al. 2013***]

Fig.1. Control volume in an unconfined aquifer.



- If the variation of  relative to the value of  is infinitesimal, one can consider that the average saturated thickness is equal to a constant value and derive a linear fractional Boussinesq equation:

$$\kappa_x \cdot D \frac{\partial^{\nu} h}{\partial x^{\nu}} + \kappa_y \cdot D \frac{\partial^{\mu} h}{\partial y^{\mu}} + N = S_y \frac{\partial h}{\partial t} \quad (2)$$

- Eq. (2)  is the linear fractional Boussinesq equation flow in a heterogeneous and an anisotropic aquifer.
- ***v*** is called the heterogeneity index in x direction and
- ***μ*** is called the *heterogeneity index in y direction*.

*[B. Mehdinejadiani et al. 2013]*

- Eq. (2) can be written as:
- $$C_\nu \cdot \frac{\partial^\nu h}{\partial x^\nu} + C_\mu \cdot \frac{\partial^\mu h}{\partial y^\mu} + N = \frac{\partial h}{\partial t} \qquad (3)$$
- where $C_\nu = \frac{\kappa_x \cdot D}{S_y}$ is the fractional hydraulic dispersion coefficient in the x direction $\left(L^\nu / T\right)$,
- and $C_\mu = \frac{\kappa_y \cdot D}{S_y}$ is the fractional hydraulic dispersion coefficient in the y direction $\left(L^\mu / T\right)$ .

- **The following assumptions are considered:**
- 1. *Flow toward subsurface drains is horizontal.*
- 2. *Unsaturated flow above the water table is neglected.*
- *3. The initial water table is flat.*
- *4. Recharge occurs instantaneously and the water table rises suddenly.*
- *5. There is a horizontal impermeable layer at a constant depth below drains.*
- *6. The subsurface drains have an equal spacing and lie in a parallel manner above the impermeable layer.*

*[B. Mehdinejadiani et al. 2013]*

- For one-dimensional transient flow, the linear fractional Boussinesq equation is in the following form:

$$\frac{\partial h(x,t)}{\partial t} = C_v \cdot \frac{\partial^v h(x,t)}{\partial x^v} \qquad (4)$$

- $C_v$ is equal to $\dfrac{\kappa_x \cdot d_e}{S_y}$
- As indicated in Fig. 2, the initial and boundary conditions for solving the one-dimensional linear fractional Boussinesq equation in the subsurface drains are as follows[B. Mehdinejadiani et al. 2013]

:

$$h(x,0) = h_0 \, , 0 \le x \le L \, ,$$

$$h(0,t) = 0 \, ,$$

$$h(L,t) = 0$$

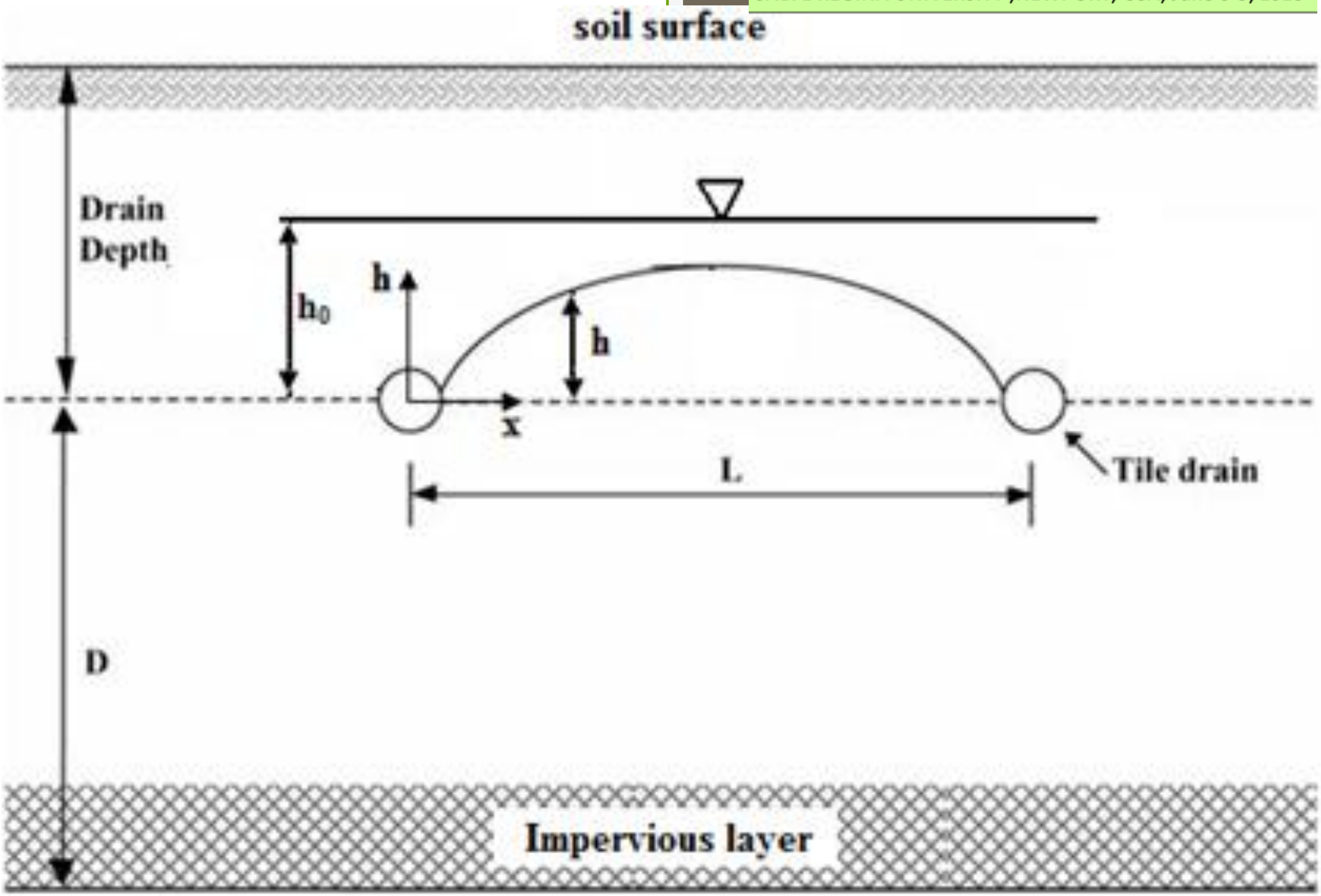



Fig.2. Considered subsurface drains in this manuscript.

- The analytical solution of Eq. (4) for the initial and boundary conditions is obtained using a spectral representation of the fractional derivative .
- To this end, the eigenvalues are $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n=1,2,3..$ and the corresponding eigenfunctions are nonzero constant multiples of $$h_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$
- Therefore, one can write $h(x,t)$ in the following form:

[B. Mehdinejadiani et al. 2013]

- $$h\,(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

- Eq. (4) satisfies the boundary conditions.
- For function *h(x,t)* , the operator $(-\Delta)^{\frac{\nu}{2}}$ is defined as
- $$(-\Delta)^{\frac{\nu}{2}} h(x,t) = -\frac{\partial^{\nu} h(x,t)}{\partial x^{\nu}} = \sum_{n=1}^{\infty} b_n(t) \cdot \lambda_n^{\frac{\nu}{2}} \sin\left(\frac{n\pi}{L} x\right). \qquad (5)$$
- Substituting Eqs. (4) and (5) into Eq. (2), we get:
- $$\sum_{n=1}^{\infty} \frac{db_n(t)}{dt} \cdot \sin\left(\frac{n\pi}{L} x\right) = -C_{\nu} \sum_{n=1}^{\infty} b_n(t) \lambda_n^{\frac{\nu}{2}} \sin\left(\frac{n\pi}{L} x\right). \qquad (6)$$
- Ordinary differential equation:
- $$\frac{db_n(t)}{dt} + C_{\nu} \lambda_n^{\frac{\nu}{2}} b_n(t) = 0 \qquad (7)$$

- Finally:
- $$h(x,t)=\frac{4h_0}{\pi}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n}\exp\left(-C_\nu\left(\frac{n\pi}{L}\right)^{\nu}t\right)\sin\left(\frac{n\pi}{L}x\right) \qquad (8)$$
- Eq. (8) is a new equation for predicting the water table profile between two parallel subsurface drains under unsteady state conditions.
- This equation is applicable for both homogeneous and heterogeneous soils.
- When $\nu \to 2$ , Eq. (8) reduces to Glover-Dumm's mathematical model which was developed by assuming the homogeneity of soil:
- $$h(x,t)=\frac{4h_0}{\pi}\sum_{n=1,3,5,\dots}^{\infty}\frac{1}{n}\exp\left(-C_2\left(\frac{n\pi}{L}\right)^{2}t\right)\sin\left(\frac{n\pi}{L}x\right) \qquad (9)$$

# Development of inverse models

- Eq. (8) has two parameters, the heterogeneity index (*v*) and the fractional hydraulic dispersion coefficient $(C_v)$, and Eq. (9) has a parameter, the hydraulic dispersion coefficient $(C_2)$ .
- The parameters of two mathematical models are estimated using inverse problem method [B. Mehdinejadiani et al. 2013].

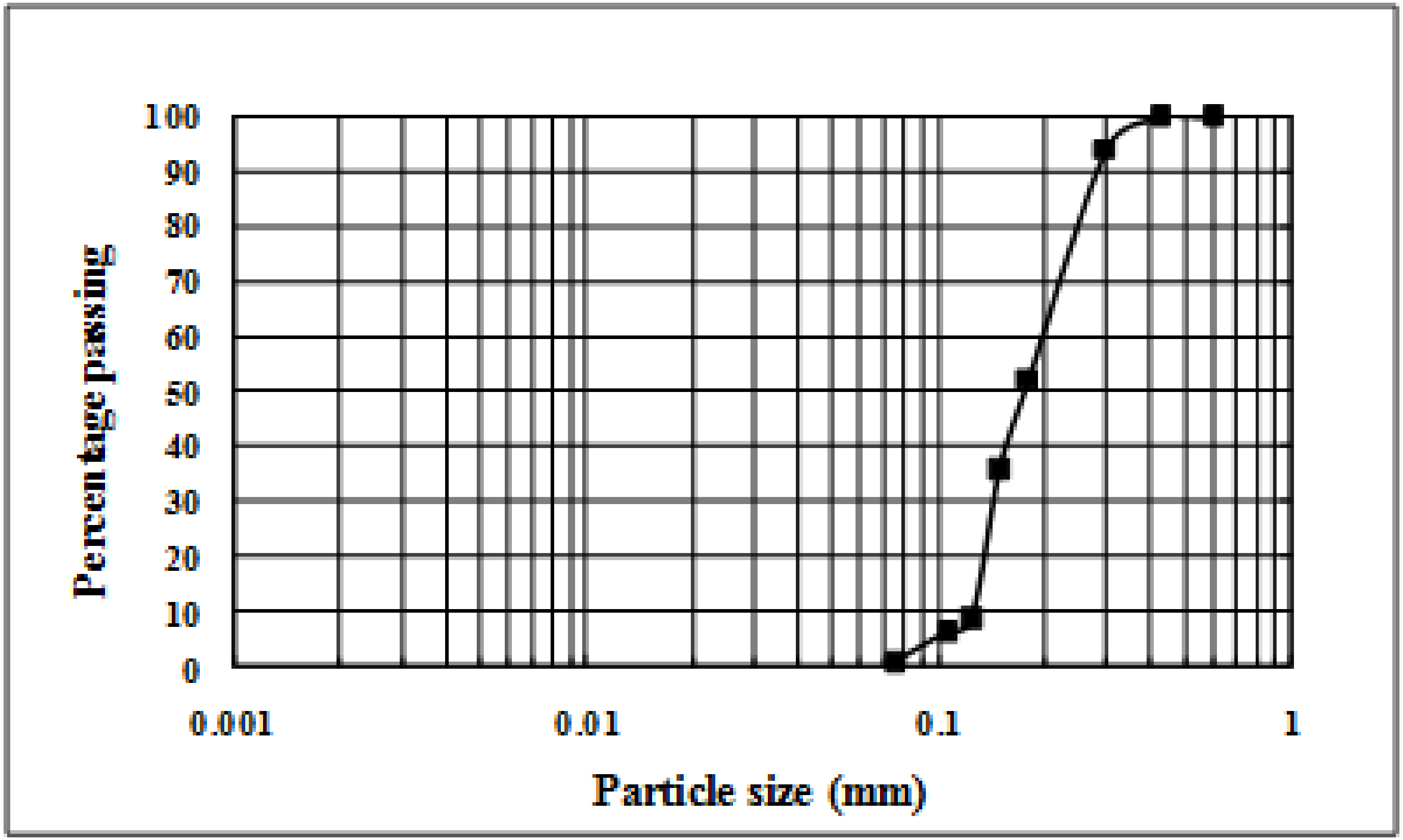
# Laboratory experiment

- The performance of developed mathematical model for simulation of the water table profile between two parallel subsurface drains in a homogenous soil is investigated. In this spirit, a sand tank having inside dimensions of 200 cm length 50 cm width 110 cm height was made of 3 mm thick steel sheets.
- A corrugated plastic drainpipe with inside diameter of 10 cm, lengthwise, was installed along one of the narrow ends of the sand tank at a depth of 80 cm below the top of the sand tank.
- The drainpipe was wrapped with synthetic envelope of PP450.
- A control valve was installed on the out of drainpipe to stop exit of water from drainpipe outlet during the soil saturation. In the front of wall of sand tank, a set of piezometers with diameter of 1 cm were installed at 7, 22, 37, 52, 67, 86, 105, 124, 143, 162 and 184 cm horizontal spacing from drainpipe.
- All the piezometers were inserted up to the middle of the sand tank to remove any seepage effect along the sand tank walls. There were three intake valves at the bottom of the sand tank.
- A variable head water supply tank fed from a water storage reservoir was connected to the intake valves*[B. Mehdinejadiani et al. 2013]*

# Fig. 3. shows the experimental setup

The sand tank was filled to 100 cm height with very uniform sand in an effort to minimize heterogeneity. [B. Mehdinejadiani et al. 2013]

The particle size distribution curve of sand used for laboratory experiment has been shown in Fig. 4.



# Field experiment

- To evaluate the performance of proposed mathematical model in the field conditions, the water table profiles between two parallel subsurface drains installed in an experimental field were measured.
- The area of this experimental field was 12 hectares.
- Brief information on the subsurface drainage system installed in the experimental field is reported in Table 1. *[B. Mehdinejadiani et al. 2013]*

Table 1
Drainage system characteristics at experimental field

| Parameter | Field data |
|---|---|
| Drain depth $(m)$ | 1.3 |
| Drain spacing $(m)$ | 30 |
| Depth to impervious layer $(m)$ | 1.5 |
| Radius of drain $(m)$ | 0.1 |
| Drain material | Corrugated PVC drainpipe |
| Envelope | PP450 |

- The observation wells, equal to depth of the drains, were installed to measure the water table height to determine the water table profiles.
- The water table, in addition to on the drain and at mid spacing, was also measured at 0.5, 1.5 and 5 m from the drain.
- The data of water table were measured for a period of 10 days at an interval of 1 day.
- The parameters of mathematical models were estimated using data of water table at times t=2, 4, and 6 days after beginning of drainage.

# Evaluation of mathematical models

- The water table profiles between two parallel subsurface drains at various times were simulated using the proposed mathematical model and the Glover-Dumm's mathematical model.
- To evaluate the performance of two mathematical models considered, the graphical displays and the statistical criteria were applied. In this paper, the two methods of graphical display were used for evaluation of two mathematical models considered (1) comparison of observed and predicted water table profiles;
- *(2)* comparison of matched simulated and observed integrated values. [B. Mehdinejadiani et al. 2013]

# Estimation of parameters of mathematical models

- The optimal values of the parameters of two mathematical models for two soil types (homogeneous soil and experimental field soil) are shown below.

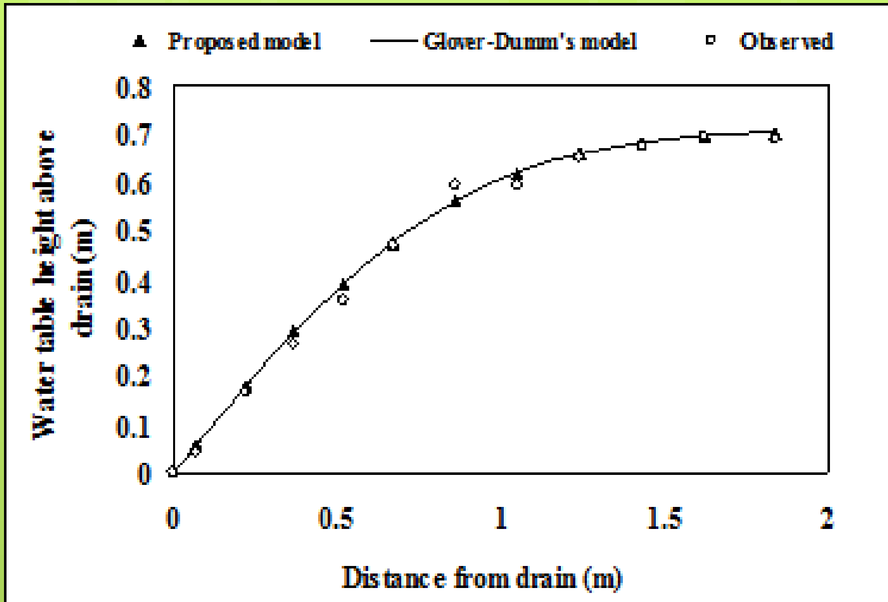
**Table 1.** Estimated parameters of two mathematical models

| Soil type | Glover-Dumm's mathematical model | Proposed mathematical model | |
|---|---|---|---|
| | $C_2\left(m^2/s\right)$ | $\nu$ | $C_\nu\left(m^\nu/s\right)$ |
| Homogenous soil | $1.96\times10^{-4}$ | 1.99 | $1.98\times10^{-4}$ |
| experimental field soil | $3.83\times10^{-6}$ | 1.04 | $3.28\times10^{-6}$ |

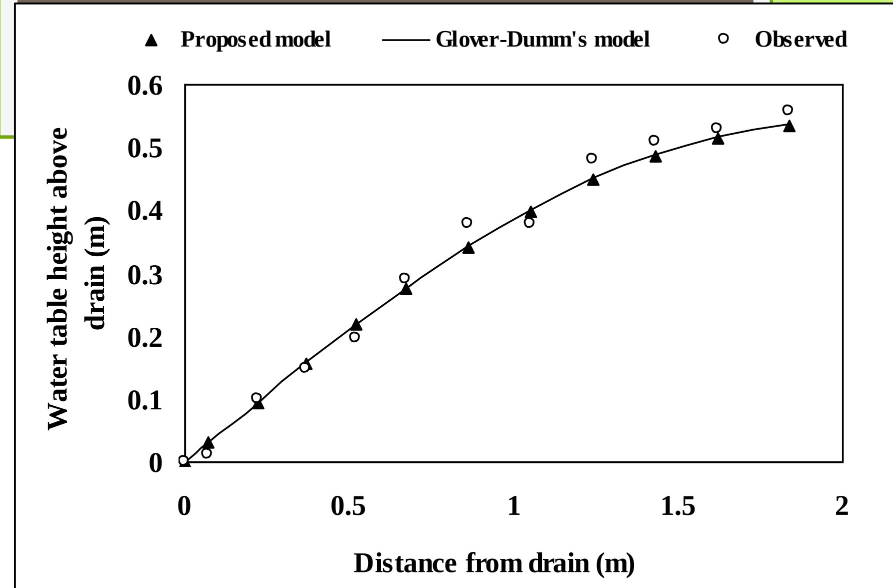
- The results of calibration of mathematical models for two soil types indicate that: (1) the heterogeneity index of homogenous soil is very close to 2; (2) the proposed mathematical model reduces to Glover-Dumm's mathematical model for the homogeneous soil. [B. Mehdinejadiani et al. 2013]

# Performance of mathematical models
## Homogeneous soil

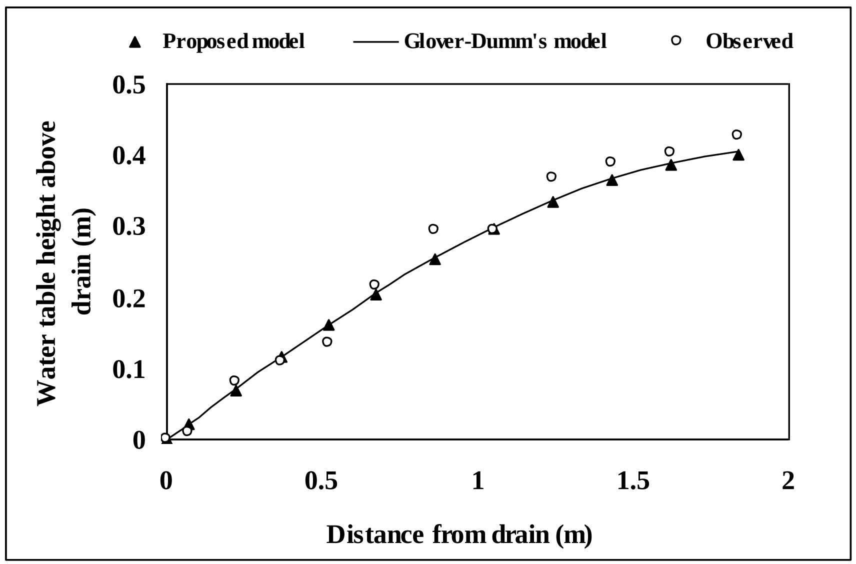
- The simulated results of two mathematical models considered in the homogeneous soil are shown in Figs. 5-6 and the corresponding values of statistical criteria are listed in Table 2.
- From Fig. 5, it can be found that the simulated results by both mathematical models are nearly coincident. *[B. Mehdinejadiani et al. 2013]*
- Fig. 6 indicates that the simulation results of two mathematical models are well correlated with the measured data.
- The statistical criteria corresponding to two mathematical models also indicate that both mathematical models have a similar performance and a minor error (Table1).




(a) t=20 minutes

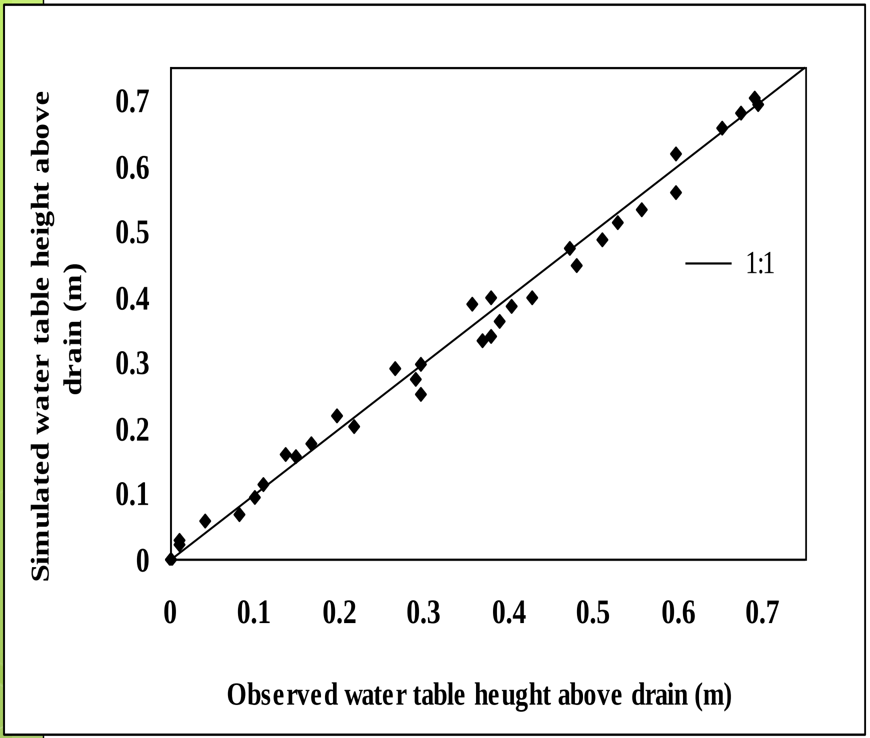


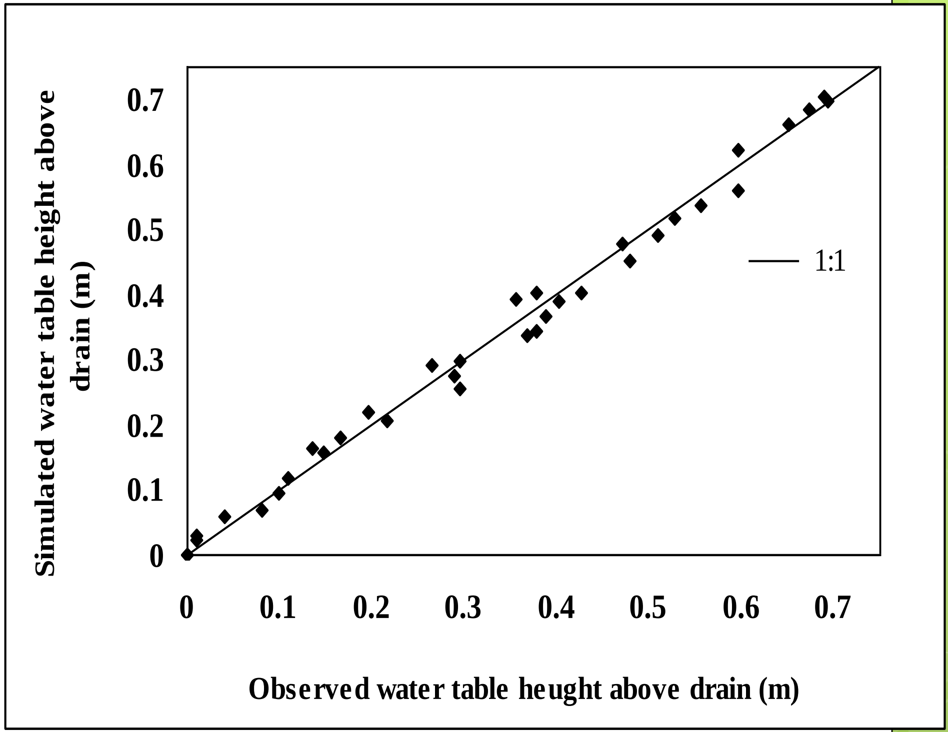

(b) t=70 minutes




(c) t=110 minutes

**Fig. 5. Comparison of water table profile between two drains simulated by proposed mathematical model and Glover-Dumm's mathematical model at times:**
(a) t=20 minutes; (b) t=70 minutes and (c) t=110 minutes after beginning of drainage.




(a) the proposed mathematical model    b) the Glover-Dumm's mathematical model.

**Fig. 6. Observed versus predicted water table height above drain.**
The water table height above drain was predicted by (a) the proposed mathematical model, and (b) the Glover-Dumm's mathematical model.
The line represents the potential 1:1 relationship between the data sets.

- The similar performance of two mathematical models is due to homogeneity of soil used in the sand tank. Indeed, the obtained results in the homogenous soil justify practically that the proposed mathematical model reduces to the Glover-Dumm's mathematical model in the homogenous soil.
- The satisfactory performance of two mathematical models considered come out from the validity of most assumptions applied to develop the mathematical models*[B. Mehdinejadiani et al. 2013]*

**Table 2.** The statistical criteria values of two mathematical models

| Glover-Dumm's mathematical model | | | | | Proposed mathematical model | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **ME** (cm) | **RMSE** (%) | **CRM** | **CD** | **EF** | **ME** (cm) | **RMSE** (%) | **CRM** | **CD** | **EF** |
| 7.24 | 5.09 | -0.016 | 1.02 | 0.99 | 7.19 | 5.11 | -0.012 | 1.02 | 0.99 |

# Experimental field soil

- Figs. 7 and 8 illustrate the simulated results by the **Glover-Dumm's mathematical model** and the proposed mathematical model in the experimental field soil. Moreover, the values of statistical criteria corresponding to two mathematical models are presented in Table 3.
- As shown in Figs. 7 and 8, compared to the Glover-Dumm's model, the simulation results of the proposed fractional model are very close to the observed results.
- The corresponding values of statistical criteria also indicate that the fractional model have a better performance than the Glover-Dumm's mathematical model (Table 3).
- The better performance of the proposed model is coming from the heterogeneity of the experimental field soil.
- The Glover-Dumm's model assumes that the soil is homogenous, while the proposed one considers the degree of heterogeneity of soil as a determinable parameter. In this study, this parameter was called the heterogeneity index (see subsection 2.1).
- Therefore, the fractional model has a less limitation and a better performance than the Glover-Dumm's model. [B. Mehdinejadiani et al. 2013]

- The graphical displays ( Figs. 7 and 8) and the values of statistical criteria (Table 3) show that both models predict the water table height above drain with an error.
- Fig 10 shows that the simulation results of both models are somewhat larger than measured results. This result can be also indicated by the negative CRM values of mathematical models (Table 3).
- We believe that the major source of this error is the assumptions used for derivation of the mathematical models, (for details, see subsection 2.2).
- *[B. Mehdinejadiani et al. 2013]*

**Table 3.** The statistical criteria values of two mathematical models

| Glover-Dumm's mathematical model | | | | | Proposed mathematical model | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **ME** (cm) | **RMSE** (%) | **CRM** | **CD** | **EF** | **ME** (cm) | **RMSE** (%) | **CRM** | **CD** | **EF** |
| 27.99 | 26.63 | -0.16 | 0.78 | 0.69 | 25.95 | 18.54 | -0.04 | 1.12 | 0.85 |

**Fig. 7. Comparison of water table profile between two drains simulated by proposed mathematical model and Glover-Dumm's mathematical model at times:**

**(a) t=3 days** and **(b) t=10 days** after beginning of drainage.

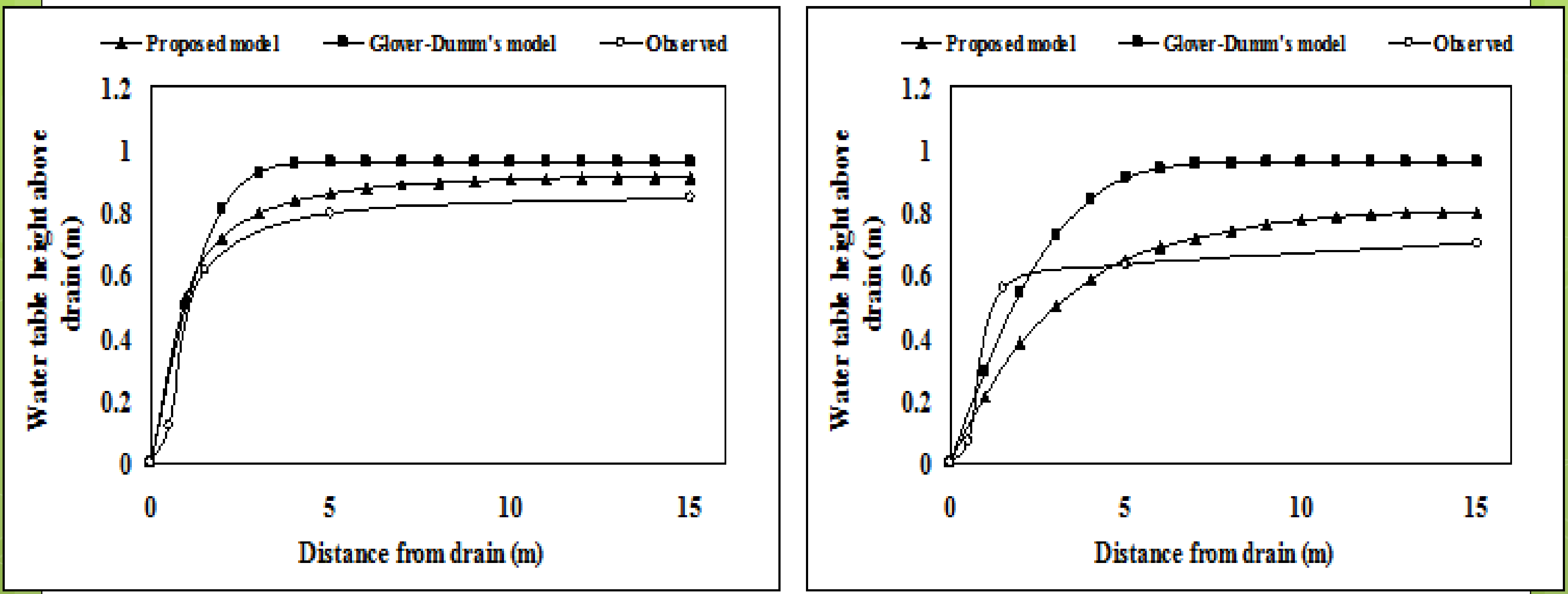
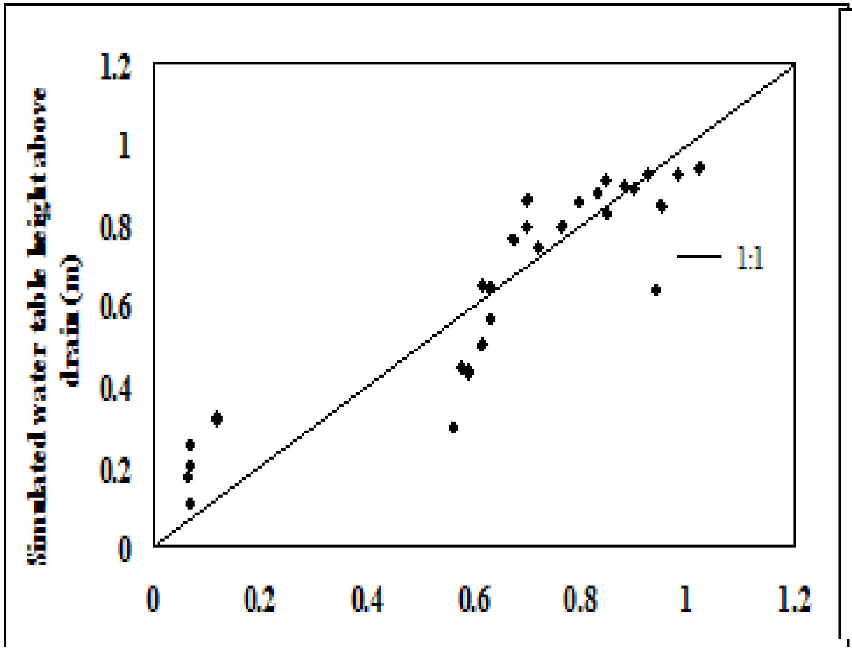


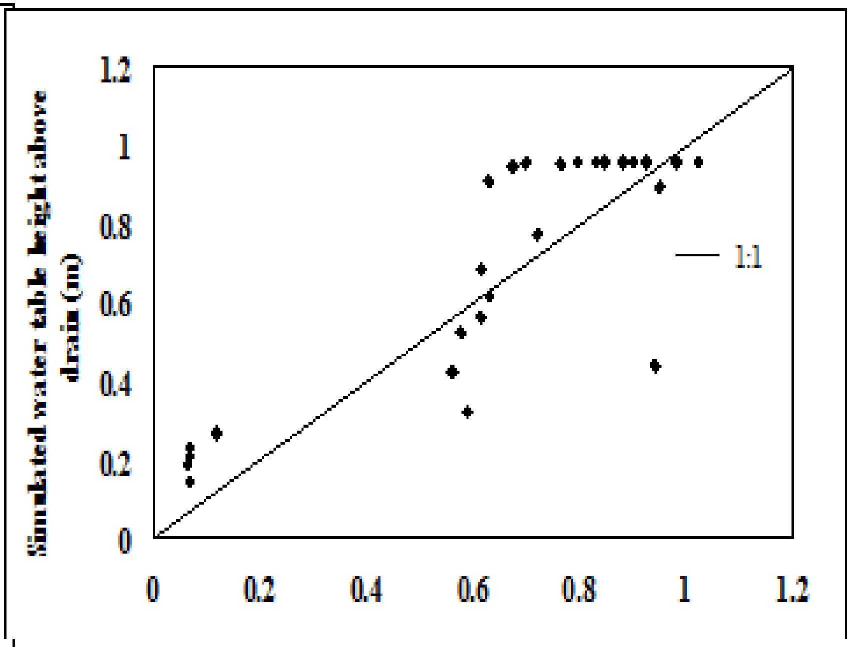
**Fig. 8. Observed versus predicted water table height above drain.** The water table height above drain was predicted by (a) the proposed mathematical model, and (b) the Glover-Dumm's mathematical model. The line represents the potential 1:1 relationship between the data sets.

**(a) the proposed mathematical model**,




Observed water table height above drain (m)

**(b) the Glover-Dumm's mathematical model**.




Observed water table height above drain (m)

PHILOSOPHICAL TRANSACTIONS:

**A Letter of Mr. Isaac Newton, Professor of the Mathematicks in the University of Cambridge; Containing His New Theory about Light and Colors: Sent by the Author to the Publisher from Cambridge, Febr. 6. 1671/72; In Order to be Communicated to the R. Society**

Isaac Newton

*Phil. Trans.* 1671 **6**, doi: 10.1098/rstl.1671.0072, published 1 January 1671

( 3075 ) Numb. 80.

PHILOSOPHICAL TRANSACTIONS.

February 19. 16 71/72.

The CONTENTS.

*A Letter of Mr.* Iſaac Newton, *Mathematick Profeſſor in the Univerſity of Cambridge; containing his New Theory about* Light *and* Colors: *Where* Light *is declared to be not Similar or Homogeneal, but conſiſting of difform rays, ſome of which are more refrangible than others: And* Colors *are affirm'd to be not Qualifications of Light, deriv'd from Refractions of natural Bodies, (as 'tis generally believed;) but Original and Connate properties, which in divers rays are divers: Where ſeveral Obſervations and Experiments are alledged to prove the ſaid Theory. An Accompt of ſome Books: I. A Deſcription of the EAST-INDIAN COASTS, MALABAR, COROMANDEL, CEYLON, &c. in* Dutch, *by* Phil. Baldæus. II. Antonii le Grand *INSTITUTIO PHILOSOPHIÆ,* ſecundùm principia Renati Des-Cartes; *novâ methodo adornata & explicata.* III. *An Eſſay to the Advancement of MUSICK; by* Thomas Salmon *M.A. Advertiſement about* Thæon Smyrnæus. *An* Index *for the Tracts of the Year* 1671.

*A Letter of Mr.* Iſaac Newton, *Profeſſor of the Mathematicks in the Univerſity of Cambridge; containing his New Theory about* Light *and* Colors: *ſent by the Author to the Publiſher from Cambridge, Febr. 6.* 16 71/72; *in order to be communicated to the* R. Society.

SIR,

TO perform my late promiſe to you, I ſhall without further ceremony acquaint you, that in the beginning of the Year 1666 (at which time I applyed my ſelf to the grinding of Optick glaſſes of other figures than *Spherical*,) I procured me a Triangular glaſs-Priſme, to try therewith the celebrated *Phænomena* of

G g g g *Colours,*

Godfrey Kneller's 1689 -portrait of **Isaac Newton** (age 46)

**Newton’s First Law:**

If the net force acting on a body is zero, then it is possible to find a set of reference frames in which that body has no accelaration.

**Newton’s Second Law:**

$$\sum \vec{F} = m\vec{a}$$

**Newton’s Third Law:**

When two bodies exert mutual forces on one another, the two forces are always equal in magnitude and opposite in direction.

**[R. Resnick, D. Halliday, K. S. Krane, Physics, 4th Edition, Volume 1, 1992]**

# *Newtonian law with memory*

**Baleanu D**, Golmankhaneh AK, Golmankhaneh AK, Nigmatullin RR

**NONLINEAR DYNAMICS**, **60**(**1-2)** Pages: **81-86** , **2010**

# *Is It Possible to Derive Newtonian Equations of Motion with Memory?*

Nigmatullin RR, **Baleanu D**

**INTERNATIONAL JOURNAL OF THEORETICAL PHYSICS 49**(**4** ), **701-708**, **2010**

# LOCAL FRACTIONAL INTEGRALS AND DERIVATIVES

# SOME RELATED REFERENCES

K. M. Kolwankar and A. D. Gangal, Local fractional Fokker-Planck equation, Physical Review Letters, vol. 80, no. 2, pp. 214–217, 1998

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W. Chen, H. Sun, X. Zhang and D. Korošak, Anomalous diffusion modeling by fractal and fractional derivatives, Computers & Mathematics with Applications, vol. 59, no.5, pp.1754–1758, 2010

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X. J. Yang and D. Baleanu, Fractal heat conduction problem solved by local fractional variation iteration method, Thermal Science, Thermal Science, vol. 17, no. 2, pp.625-628, 2013

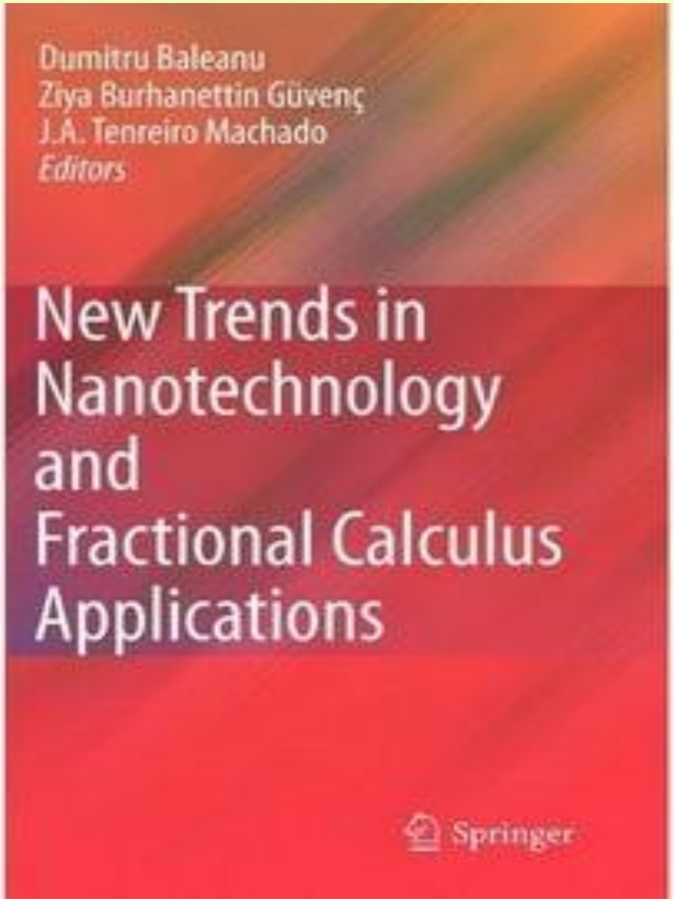
# Some books ...

# *Asymptotic Integration and Stability for Differential Equations of Fractional Order*

**Dumitru Baleanu,** Octavian Mustafa**,**
World Scientific,to appear, November **2013**

# New Trends in Nanotechnology and Fractional Calculus Applications

**Dumitru Baleanu,**
**Ziya B. Guvenc,**
**J. A. Tenreiro Machado**




In recent years fractional calculus has played an important role in various fields such as mechanics, electricity, , biology, economics, mode-ling, identification, control theory and signal processing.

The scope of this book is to present the state of the art in the study of fractional systems and the application of fractional differentiation.

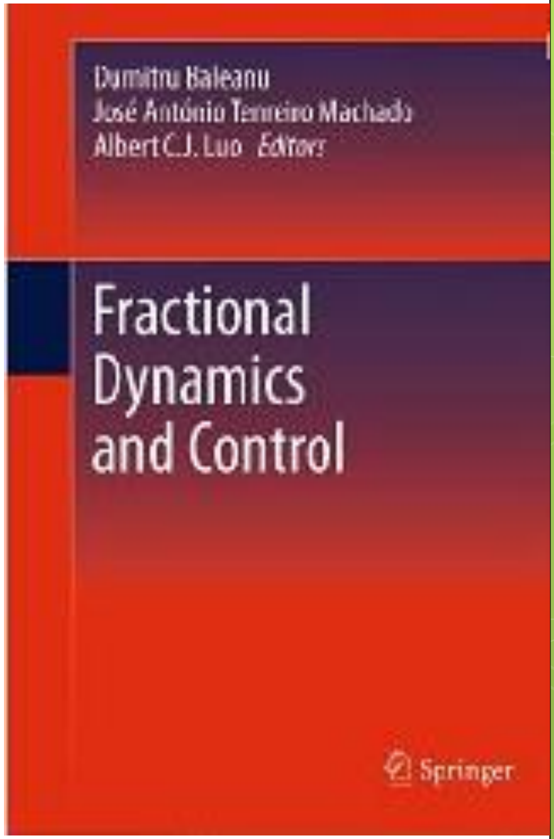
Furthermore, the manufacture of nanowires is important for the design of nanosensors and the development of high-yield thin films is vital in procuring clean solar energy.

# Fractional Dynamics and Control

- **Dumitru Baleanu**
- **José António Tenreiro Machado**
- **Albert C. J. Luo**

Fractional Dynamics and Control provides a comprehensive overview of recent advances in the areas of nonlinear dynamics, vibration and control with analytical, numerical, and experimental results.

This book provides an overview of recent discoveries in fractional control, delves into fractional variational principles and differential equations, and applies advanced techniques in fractional calculus to solving complicated mathematical and physical problems.Finally, this book also discusses the role that fractional order modeling can play in complex systems for engineering and science

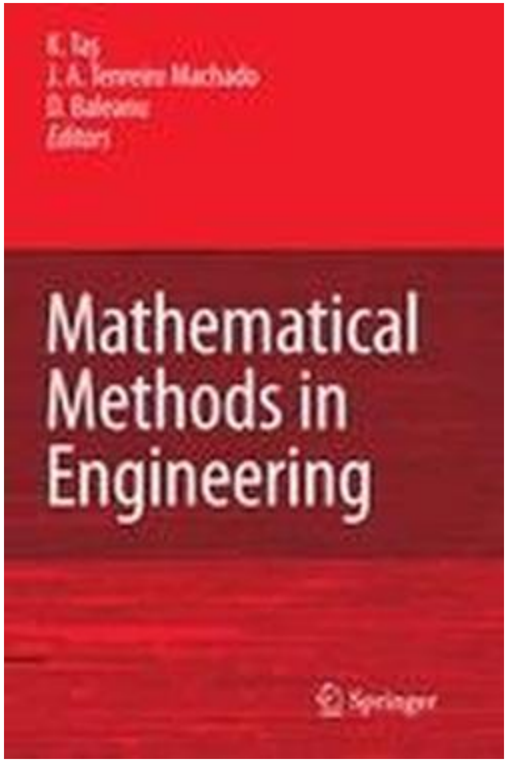


# Mathematical Methods in Engineering

**K. Tas**

**J.A. Tenreiro Machado**

**Dumitru Baleanu**




This book contains some of the contributions that have been carefully selected and peer-reviewed, which were presented at the International Symposium MME06 Mathematical Methods in Engineering, held in Cankaya University, Ankara, April 2006. The Symposium provided a setting for discussing recent developments in Fractional Mathematics, Neutrices and Generalized Functions, Boundary Value Problems, Applications of Wavelets, Dynamical Systems and Control Theory.

# Wavelet Transforms and Their Recent Applications in Biology and Geoscience

Edited by **Dumitru Baleanu**

ISBN 978-953-51-0212-0, 298 pages,
Publisher: InTech, Chapters published March 02, 2012


This book reports on recent applications in biology and geoscience. Among them we mention the application of wavelet transforms in the treatment of EEG signals, the dimensionality reduction of the gait recognition framework, the biometric identification and verification. The book also contains applications of the wavelet transforms in the analysis of data collected from sport and breast cancer. The denoting procedure is analyzed within wavelet transform and applied on data coming from real world applications.

The book ends with two important applications of the wavelet transforms in geoscience

# Some more papers ...

# ***Fractional radiative transfer equation within Chebyshev spectral approach***

Kadem, A; **Baleanu, D.,**

COMPUTERS & MATHEMATICS WITH APPLICATIONS ,
59(5) 1865-1873 **(2010)**

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**Dumitru Baleanu, Abdelouahab Kadem,Yury Luchko**

## *Spectral method for solution of the fractional transport equation*

**Reports on Mathematical Physics,**
**Vol.6,(2010),103-115.**

**D. Baleanu, O. Mustafa, Ravi P. Agarwal**

***On the asymptotic integration of a class of sublinear fractional differential equations***

JOURNAL OF MATHEMATICAL PHYSICS **50**, 123520 (2009)

**D. Baleanu, O. Mustafa, Ravi P. Agarwal**

***On the solution set for a class of sequential fractional differential equations***

J.Phys.A:Math.Theor.,2010 **43** 385209 doi: **10.1088/1751-8113/43/38/385209**

**D. Baleanu, O. Mustafa, Ravi P. Agarwal**

# *An existence result for a superlinear fractional differential equation*

Applied Mathematics Letters, **Volume 23, Number 9,** September 2010**,** 1129-1132

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- *Fractional Quantum Mechanics*
- *Fractional Field Theory*
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- *Fractional Differential Geometry*
- *Fractional Nonlinear Optics*
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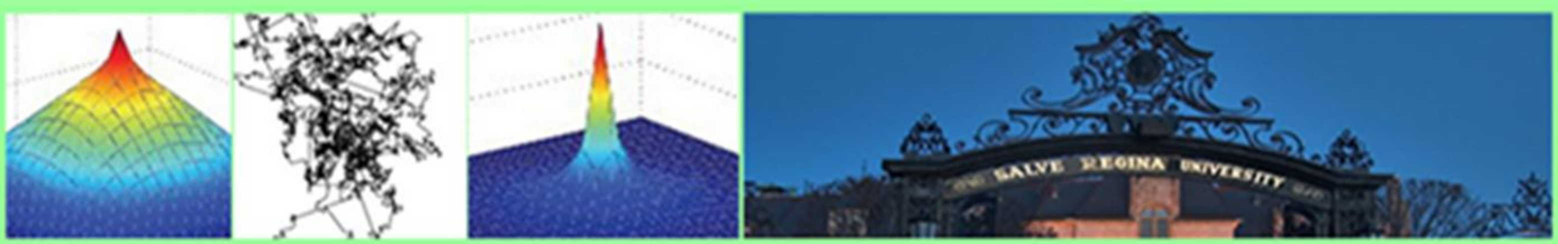


SALVE REGINA UNIVERSITY ,NEWPORT, USA, June 3-5, 2013

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