On Stochastic Navier-Stokes Equation Driven by Stationary White Noise

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Abstract

We consider an unbiased approximation of stochastic Navier-Stokes equation driven by spatial white noise. This perturbation is unbiased in that the expectation of a solution of the perturbed equation solves the deterministic Navier-Stokes equation. The nonlinear term can be characterized as the highest stochastic order approximation of the original nonlinear term $u\nabla u$. We investigate the analytical properties and long time behavior of the solution. The perturbed equation is solved in the space of generalized stochastic processes using the Cameron-Martin version of the Wiener chaos expansion and generalized Malliavin calculus. We also study the accuracy of the Galerkin approximation of the solutions of the unbiased stochastic Navier-Stokes equations.

1 Introduction

Stochastic perturbations of the Navier-Stokes equation have received much attention over the past few decades. Among the early studies of the stochastic Navier-Stokes equations are those by Bensoussan and Temam [1], Foias et al. [3, 4, 5], Flandoli [6, 7], etc. Traditionally, the types of perturbations that were proposed includes stochastic forcing by a noise term such as a Gaussian random field or a cylindrical Wiener process, and are broadly accepted as a natural way to incorporate stochastic effects into the system. The stochastic Navier-Stokes equation

$$\begin{aligned}&u_t + u^i u_{x_i} + \nabla P = \nu\Delta u + f(t,x) + \left(\sigma^i(t,x)u_{x_i} + g(t,x)\right)\dot{W}(t,x),\\ &\operatorname{div} u \equiv 0,\\ &u(0,x) = w(x), \quad u|_{\partial D} = 0.\end{aligned} \tag{1}$$

is underpinned by a familiar physical basis, because it can be derived from Newton's Second Law via the the fluid flow map, using a particular assumption on the stochasticity of the governing SODE of the flow map, known as the Kraichnan turbulence. (See [11, 12] and the references therein.) However, due to the nonlinearity, stochastic Navier-Stokes equation (1) is a *biased* perturbation of the underlaying deterministic Navier-Stokes equation. That is, the mean of the solution of the stochastic equation does not coincide with the solution of the underlying deterministic Navier-Stokes equation. Of course, this observation is also true for other nonlinear equations such as the stochastic Burgers equation, Ginzburg-Landau equation, etc. In fact, the mean of (1) solves the famous Reynolds equation.

An unbiased version of stochastic Navier-Stokes equation (1)

$$\begin{aligned} &u_t + u^i \diamond u_{x_i} + \nabla P = \nu \Delta u + f(t,x) + \left(\sigma^i(t,x) u_{x_i} + g(t,x)\right) \dot{W}(t,x), \\ &\operatorname{div} u \equiv 0, \\ &u(0,x) = w(x), \quad u|_{\partial D} = 0. \end{aligned} \tag{2}$$

has been introduced and studied in [13]. The unbiased version of (1) differs from (1) by the nonlinear term: the product $u^i u_{x_i}$ is replaced by the Wick product $u^i \diamond u_{x_i}$. In fact, Wick product $u^i \diamond u_{x_i}$ can be interpreted as Malliavin integral of u_{x_i} with respect to u (see [10]). An important property of Wick product is that

$$\mathbb{E}[u^i \diamond u_{x_i}] = \mathbb{E}u^i \, \mathbb{E}u_{x_i}. \tag{3}$$

Due to this property, stochastic Navier-Stokes equation (2) with Wick nonlinearity is an unbiased perturbation of stochastic Navier-Stokes equation (1). In the future, we will reffer to unbiased perturbations of stochastic Navier-Stokes equation as *unbiased stochastic Navier-Stokes equation.*

In this paper we will study an unbiased stochastic Navier-Stokes equations on an open bounded smooth domain $D \in \mathbb{R}^d$, $d = 2, 3$, driven by purely spatial noise. In particular, we will study equation

$$\begin{aligned} &u_t + u^i \diamond u_{x_i} + \nabla P = \nu \Delta u + f(t,x) + \left(\sigma^i(x) u_{x_i} + g(t,x)\right) \diamond \dot{W}(x), \\ &\operatorname{div} u \equiv 0, \\ &u(0,x) = w(x), \quad u|_{\partial D} = 0. \end{aligned} \tag{4}$$

where the diffusivity constant is $\nu > 0$, and the functions f, g, σ are given deterministic $\mathbb{R}^d$-valued functions. Here, the driving noise $\dot{W}(x) = \sum_k \mathfrak{u}_l(x)\xi_l$ is a stationary Gaussian white noise on $L_2(D)$, and we assume that $\sup_l \|\mathfrak{u}_l\|_{L^\infty} < \infty$.

We will also study the stationary (elliptic) version of equation (4)

$$\begin{aligned} &\bar{u}^i \diamond \bar{u}_{x_i} + \nabla \bar{P} = \nu \Delta \bar{u} + \bar{f}(x) + \left(\bar{\sigma}^i(x) \bar{u}_{x_i} + \bar{g}(x)\right) \diamond \dot{W}(x), \\ &\operatorname{div} \bar{u} \equiv 0, \\ &\bar{u}|_{\partial D} = 0. \end{aligned} \tag{5}$$

where $\bar{f}(x), \bar{g}(x), \bar{\sigma}(x)$ are given deterministic $\mathbb{R}^d$-valued functions. It will be shown that $u\,(t,x) \to \bar{u}\,(x)$ as $t \to \infty$.

Solutions of equations (4) and (5) will be defined by their respective Wiener chaos expansions:

$$u\,(t,x) = \sum_\alpha u_\alpha\,(t,x)\,\xi_\alpha, \text{ and } \bar{u}\,(x) = \sum_\alpha \bar{u}_\alpha\,(x)\,\xi_\alpha, \tag{6}$$

where $\{\xi_\alpha, \alpha \in J\}$ is the Cameron-Martin basis generated by $\dot{W}(x)$, $v_\alpha := E(v\xi_\alpha)$, and J is the set of multiindices $\alpha = \{\alpha_k,\ k \geq 1\}$ such that for every k, $\alpha_k \in \mathbf{N}_0(\mathbf{N}_0 = \{0, 1, 2, \ldots\})$ and $|\alpha| = \sum_k \alpha_k < \infty$. It will be shown that Wiener chaos coefficients $u_\alpha(t, x)$ and $\bar{u}_\alpha(x)$ solve lower triangular systems of deterministic equations. We will reffer to these systems as *propagators* of $u_\alpha(t, x)$ and $\bar{u}_\alpha(x)$, respectively.

In fact, equations (4) and (5) could be viewed as the *highest stochastic order approximations* of similar equations with standard nonlinearities $u^i u_{x_i}$ and $\bar{u}^i \bar{u}_{x_i}$, respectively. Indeed, it was shown in [13] that under certain natural assumptions the following equality holds:

$$v\nabla v = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n v \Diamond \mathcal{D}^n \nabla v}{n!} \tag{7}$$

where $\mathcal{D}^n$ is the n^{th} power of Malliavin derivative $\mathcal{D} = \mathcal{D}_{\dot{W}}$. Taking into account expansion (7),

$$v\nabla v \approx v \Diamond \nabla v \tag{8}$$

This approximation is the *highest stochastic order* approximation of $v\nabla v$ in that $v\Diamond\nabla v$ contains the highest order Hermite polynomials of the driving noise, while the remaining terms of the right hand side of (7) include only lower order elements of the Cameron-Martin basis. This fact could be illustrated by the following simple fact:

$$\xi_\alpha \xi_\beta = \xi_{\alpha+\beta} + \sum_{\gamma < \alpha+\beta} k_l \xi_\gamma,$$

where k_γ are constants.

As a side note, we remark that in comparison, the usual stochastic Navier-Stokes equation has a propagator system that is a full system of equations which, comparatively, is a much tougher beast to tackle. Additionally, apart from the zero-th chaos mode which, being the mean, solves the deterministic Navier-Stokes equation, all higher modes in the propagator system solves a linearized Stokes equation. Thus, where a result is known for the deterministic Navier-Stokes equation, it is sometimes the case that an analogous result may be shown for the unbiased approximation of the stochastic Navier-Stokes equation. For instance, the existence of a unique stationary solution of (5) requires the same condition on the largeness of the viscosity ν as does the existence of a unique steady solution of the deterministic equation (13ba).

There is substantial theory on the steady solutions of the deterministic Stokes and Navier-Stokes equations, the long time convergence of a time-dependent solution to the steady solution, as well as other dynamical behavior of the solution. In the subsequent sections, we begin to study some of these same questions for the unbiased Navier-Stokes equation, focusing on the large viscosity case where the uniqueness of steady solutions and long time convergence has been established in the deterministic setting. We will study the existence of a unique stationary solution of (5) as well as the existence of a unique time-dependent solution of (4) on a finite time interval. The Wiener chaos expansion and the propagator system will be the central tool in obtaining a generalized solution, but to place the solution in a Kondratiev space involves a useful result invoking the Catalan numbers. The Catalan numbers arises naturally from the convolution of the Wiener chaos modes in the nonlinear term. It was used to study the Wick versions of the stochastic Burgers [9] and Navier-Stokes [13] equations.

2 Generalized random variables and functional analytic framework

To study equations (4) and (5), we will give the basic definitions for the generalized stochastic spaces that will be used. The definition of the generalized solution will be defined in the variational/weak sense such as described in [15, 14], and before stating those definitions, we first state some standard notation and facts about the vector spaces.

Let $d = 2, 3$ be the dimension. Denote the vector spaces $\mathbb{L}^2(D) = (L^2(D))^d$ with the norm $|\cdot|$, and $\mathbb{H}^m(D) = (H^m(D))^d$ with the norm $\|\cdot\|_{H^m}$. Denote the following spaces

$$\begin{aligned}
\mathcal{V} &:= \{v \in (C_0^\infty(D))^d : \operatorname{div} v = 0\} \\
V &:= \text{closure of } \mathcal{V} \text{ in the } \mathbb{H}_0^1(D) \text{ norm} \equiv \{u \in \mathbb{H}_0^1(D) : \operatorname{div} u = 0\} \\
H &:= \text{closure of } \mathcal{V} \text{ in the } \mathbb{L}^2(D) \text{ norm} \\
V' &:= \text{dual space of } V \text{ w.r.t. inner product in } H
\end{aligned}$$

Also denote the norms in V and V' by $\|\cdot\|_V$ and $\|\cdot\|_{V'}$, respectively. In particular, we have $\|\cdot\|_V := |\nabla \cdot|$.

The operator[1] $-\Delta$ on H, defined on the domain $\operatorname{dom}(-\Delta)$, is symmetric positive definite and thus defines a norm $|\cdot|_2$ via $|\cdot|_2 = |\Delta \cdot|$, which is equivalent to the norm $\|w\|_{H^2}$. For $m > 0$, the spaces $V_m := \operatorname{dom}((-\Delta)^{m/2})$ are closed subspaces of $\mathbb{H}^m(D)$ with the norms $|\cdot|_m = |(-\Delta)^{m/2} \cdot|$. In this paper, we will commonly use $m = 1/2, 3/2$ and 2. Note that $|\cdot|_1 = \|\cdot\|_V$, and the norms $|\cdot|_m$ and $\|\cdot\|_{H^m}$ are equivalent. We thus have a constant c_1 so that

$$c_1 \|w\|_{H^1}^2 \leq |w|_1^2 \leq \frac{1}{c_1} \|w\|_{H^1}^2, \quad \text{for all } w \in V.$$

Denote $\lambda_1 > 0$ to be the smallest eigenvalue of $-\Delta$, then we have a Poincare inequality,

$$\lambda_1 |v|^2 \leq \|v\|_V^2, \quad \text{for } v \in V. \tag{9}$$

Define the trilinear continuous form b on $V \times V \times V$ by

$$b(u, v, w) = \int_D u^k \partial_{x_k} v^j w^j \, dx,$$

[1]Technically, the correct operator is $Au := -P\Delta u$, where P is the orthogonal projection onto H. We abuse notation here and continue writing $-\Delta$.

and the mapping $B : V \times V \to V'$ by

$$\langle B(u,v), w\rangle = b(u,v,w).$$

It is easy to check that

$$b(u,v,w) = -b(u,w,v), \quad \text{and} \quad b(u,v,v) = 0$$

for all $u, v, w \in V$. B and b have many useful properties that follow from the following lemma.

Lemma 1 (Lemma 2.1 in [14])**.** *The form b is defined and is trilinear continuous on $H^{m_1} \times H^{m_2+1} \times H^{m_3}$, where $m_i \geq 0$ and*

$$\begin{array}{ll} m_1 + m_2 + m_3 \geq \frac{d}{2} & \text{if } m_i \neq \frac{d}{2},\ i = 1,2,3, \\ m_1 + m_2 + m_3 > \frac{d}{2} & \text{if } m_i = \frac{d}{2},\ \text{some } i. \end{array} \tag{10}$$

In view of Lemma 1, let c_b be the constant in

$$|b(u,v,w)| \leq c_b |u|_{m_1} |v|_{m_2+1} |w|_{m_3}$$

where m_i satisfies (10). Also let c_d, $d = 2,3$, be the constants in

$$\begin{array}{ll} |b(u,v,w)| \leq c_2 |u|^{1/2} \|u\|_V^{1/2} \|v\|_V^{1/2} |\Delta v|^{1/2} |w| & \text{if } d = 2 \\ |b(u,v,w)| \leq c_3 \|u\|_V \|v\|_V^{1/2} |\Delta v|^{1/2} |w| & \text{if } d = 3 \end{array}$$

for all $u \in V$, $v \in \mathrm{dom}(-\Delta)$, and $w \in H$ (equations (2.31-32) in [14]). Other useful consequences of Lemma 1 is that $B(\cdot,\cdot)$ is a bilinear continuous operator from $V \times H^2 \to L^2$, and also from $H^2 \times V \to L^2$.

Next, we introduce the basic notation that will be used to define the generalized stochastic spaces and the generalized solution. Let $(\Omega, \mathcal{F}, P)$ be a probability space where the σ-algebra $\mathcal{F}$ is generated by $\{\xi_k,\ k = 1,2,\dots\}$, where ξ_k are independent and identically distributed $N(0,1)$ random variables. Let $\mathcal{U} = L^2(D)$ and let $\{\mathfrak{u}_k(x),\ k = 1,2,\dots\}$ be a complete orthonormal basis for $\mathcal{U}$. Then the Gaussian white noise on $\mathcal{U}$ is

$$\dot{W}(x) = \sum_{k\geq 1} \mathfrak{u}_k(x)\xi_k.$$

Let $\mathcal{J} = \{\alpha = (\alpha_1, \alpha_2, \dots), \alpha_k \in \mathbb{N}_0\}$ be the set of multi-indices of finite length. Denote $|\alpha| - \sum_{k\geq 1} \alpha_k < \infty$, and ϵ_k is the unit multi-index with $|\alpha| = 1$, $\alpha_k = 1$. For $\alpha, \beta \in \mathcal{J}$,

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots), \quad \text{and} \quad \alpha! = \prod_{k\geq 1} \alpha_k!.$$

For a sequence $\rho = (\rho_1, \rho_2, \dots)$, set $\rho^\alpha = \prod \rho_k^{\alpha_k}$.

For each $\alpha \in \mathcal{J}$, let

$$\xi_\alpha = \prod_{k\geq 1} \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k}}$$

where H_n is the nth Hermite polynomial given by $H_n(x) = (-1)^n(\frac{d^n e^{-x^2/2}}{dx^n} e^{x^2/2})$. It is a well-known fact that the set $\Xi = \{\xi_\alpha,\, \alpha \in \mathcal{J}\}$ forms an orthonormal basis in $L^2(\Omega)$ [2]. Thus, for a Hilbert space X, if $f \in L^2(\Omega; X)$ and $f_\alpha = E[f\xi_\alpha]$, then the *Wiener chaos expansion* of f is $f = \sum_{k\leq 1} f_\alpha \xi_\alpha$, and moreover $E|f|_X^2 = \sum_{\alpha\in\mathcal{J}} |f_\alpha|_X^2$. The set Ξ is the Cameron-Martin basis of $L^2(\Omega)$.

For a Hilbert space X, define the (stochastic) test function and distribution spaces

$$\mathcal{D}(X) = \left\{ v = \sum_\alpha v_\alpha \xi_\alpha : v_\alpha \in X \text{ and only finitely many } v_\alpha \text{ are non-zero} \right\},$$

$$\mathcal{D}'(X') = \left\{ \text{All formal series } u = \sum_\alpha u_\alpha \xi_\alpha \text{ with } u_\alpha \in X' \right\}.$$

Random variables in $\mathcal{D}(X)$ serve as test functions for the distributions in $\mathcal{D}'(X')$. If $\langle\cdot,\cdot\rangle$ is the duality pairing between X', X, then the duality pairing between $u \in \mathcal{D}'(X')$ and $v \in \mathcal{D}(X)$ is

$$\langle\langle u, v\rangle\rangle = \sum_\alpha \langle u_\alpha, v_\alpha\rangle.$$

The space $\mathcal{D}'$ is a very large space. To quantify the asymptotic growth of the Wiener chaos coefficients, we introduce the Kondratiev spaces. For $q > 0$, Denote the sequence $(2\mathbb{N})^{-q} = ((2k)^{-q})_{k=1,2,\ldots}$, and let the weights $r_\alpha^2 = (2\mathbb{N})^{-q\alpha}/\alpha!$. The Kondratiev space $\mathcal{S}_{-1,-q}(X)$ is

$$\mathcal{S}_{-1,-q}(X) = \left\{ u = \sum_\alpha u_\alpha \xi_\alpha : u_\alpha \in X \text{ and } \sum_\alpha |u_\alpha|_X^2 r_\alpha^2 < \infty \right\}.$$

$\mathcal{S}_{-1,-q}(X)$ is a Hilbert space with the norm $\|u\|^2_{\mathcal{S}_{-1,-q}(X)} = \sum_\alpha |u_\alpha|_X^2 r_\alpha^2$.

Definition 2. *For $\alpha, \beta \in \mathcal{J}$, the* Wick product *is defined as*

$$\xi_\alpha \diamond \xi_\beta = \sqrt{\binom{\alpha+\beta}{\alpha}}\, \xi_{\alpha+\beta}.$$

Extending by linearity, for $u, v \in \mathcal{D}'(\mathbb{R})$, the Wick product $u \diamond v$ is a $\mathcal{D}'(\mathbb{R})$ element with

$$u \diamond v = \sum_\alpha \left(\sum_{0\leq\gamma\leq\alpha} \sqrt{\binom{\alpha}{\gamma}}\, u_\gamma v_{\alpha-\gamma} \right) \xi_\alpha.$$

In particular, for $G \in \mathcal{S}_{-1,-q}(L^2(D))$,

$$(G(x) \diamond \dot{W}(x))_\alpha = \sum_{k\geq 1} \sqrt{\alpha_k}\, G_{\alpha-\epsilon_k}(x)\mathfrak{u}_k(x).$$

We now proceed to define the weak solution of (4). Recall that for a smooth function p, $(\nabla p, v) = 0$ for all $v \in V$. This leads us to define the weak solution by taking the test function space V, so that the pressure term drops out.

Definition 3. *Let $T < \infty$. A generalized weak solution of* (4) *is a generalized random element $u \in \mathcal{D}'(L^2(0,T;V))$ such that*

$$\langle\!\langle u_t + u^i \diamond u_{x_i},\, \phi \rangle\!\rangle = \langle\!\langle \nu \Delta u + f + \big(\sigma^i u_{x_i} + g\big) \diamond \dot{W}(x),\, \phi \rangle\!\rangle \tag{11}$$

for all test functions $\phi \in \mathcal{D}(V)$.

The pressure term can be recovered from the generalized weak solution in the standard way.

Using the Wiener chaos expansion, we will study equations (4) and (5) through the analysis of the propagator system — an equivalent infinite system of deterministic PDE that gives the coefficients u_α of the solution, thereby equivalently characterizing the solution u. Recalling the definition of the Wick product, the propagator system of (4) is, for $\alpha = (0)$,

$$\begin{array}{l} \partial_t u_0 + B(u_0, u_0) = \nu \Delta u_0 + f \\ \operatorname{div} u_{(0)} = 0 \\ u_0(0,x) = w(x), \quad u_0|_{\partial D} = 0. \end{array}, \tag{12a}$$

and for $|\alpha| \geq 1$,

$$\begin{array}{l} \partial_t u_\alpha + B(u_\alpha, u_0) + B(u_0, u_\alpha) + \sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, B(u_\gamma, u_{\alpha-\gamma}) \\ \quad = \nu \Delta u_\alpha + \sum_l \sqrt{\alpha_l}\, \mathfrak{u}_l(x) \big(\sigma^i \partial_{x_i} u_{\alpha-\epsilon_l} + \mathbf{1}_{\alpha=\epsilon_l} g\big) \\ \operatorname{div} u_\alpha = 0 \\ u_\alpha(0,x) = 0, \quad u_\alpha|_{\partial D} = 0 \end{array} \tag{12b}$$

with equality holding in V'. Note that each equation in the propagator system involves only the divergence-free part; the pressure term P_α can be recovered from each equation by a standard technique (see e.g., [15]). Hereon, we will focus only on studying the velocity field u.

Similarly, the propagator system of (5) is

$$\begin{array}{l} B(\bar{u}_0, \bar{u}_0) = \nu \Delta \bar{u}_0 + \bar{f} \\ \operatorname{div} \bar{u}_0 = 0, \quad \bar{u}_0|_{\partial D} = 0 \end{array}, \tag{13a}$$

$$\begin{array}{l} B(\bar{u}_\alpha, \bar{u}_0) + B(\bar{u}_0, \bar{u}_\alpha) + \sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, B(\bar{u}_\gamma, \bar{u}_{\alpha-\gamma}) \\ \quad = \nu \Delta u_\alpha + \sum_l \sqrt{\alpha_l}\, \mathfrak{u}_l(x) \big(\bar{\sigma}^i \partial_{x_i} \bar{u}_{\alpha-\epsilon_l} + \bar{g}_{\alpha-\epsilon_l}\big) \\ \operatorname{div} \bar{u}_\alpha = 0, \quad \bar{u}_\alpha|_{\partial D} = 0 \end{array} \tag{13b}$$

with equality holding in V'.

The zeroth mode $u_0 = \mathbb{E}u$ is the mean of (4) and solves the the unperturbed Navier-Stokes equations (12ba).

3 The stationary unbiased stochastic Navier-Stokes equation

Given deterministic functions $\bar{f}, \bar{g}, \bar{\sigma} \in L_2(D)$, we seek a weak/variational solution $\bar{u} \in \mathcal{D}'(V)$ satisfying

$$-\nu\langle\langle \Delta\bar{u}, \varphi\rangle\rangle + \langle\langle \bar{u}^i \diamond \partial_{x_i}\bar{u},\, \varphi\rangle\rangle = \langle\langle \bar{f}, \varphi\rangle\rangle + \langle\langle \big(\bar{\sigma}^i\partial_{x_i}\bar{u} + \bar{g}\big) \diamond \dot{W}(x),\, \varphi\rangle\rangle$$

for all test random elements $\varphi \in \mathcal{D}(V)$.

We will first show the existence and uniqueness of a generalized strong solution.

Proposition 4. *Assume the dimension $d = 2, 3$. Assume $\bar{f}, \bar{g}, \bar{\sigma}$ are deterministic functions satisfying*

$$\bar{f}, \bar{g}, \bar{\sigma} \in H, \tag{A0}$$

$$\nu^2 > c_b\|\bar{f}\|_{V'}, \tag{A1}$$

$$\bar{g} \in \mathbb{H}^1(D), \quad \bar{\sigma} \in (W^{1,\infty}(D))^d. \tag{A2}$$

Then there exists a unique generalized strong solution $u \in \mathcal{D}'(\mathbb{H}^2(D)) \cap V$ of (5).

Remark. It is interesting to note that condition (A1) in Proposition 4, that ensures the existence of a generalized strong solution, is the same condition that ensures the uniqueness of the strong solution of the deterministic Navier-Stokes equation. Thus, Proposition 4 generalizes the analogous result in the deterministic Navier-Stokes theory, which is the subcase when $\bar{g} = \bar{\sigma} = 0$.

Proof.

Solution for $\alpha = (0)$. The equation for $\bar{u}_0$ is the deterministic stationary Navier-Stokes equation, for which the existence and uniqueness of weak solutions is well-known [15, 14]. From (A1), there exists a unique weak solution $\bar{u}_0 \in V$ of (13ba) satisfying

$$\|\bar{u}_0\|_V \leq \frac{1}{\nu}\|\bar{f}\|_{V'} < \frac{\nu}{c_b}. \tag{14}$$

Moreover, since $\bar{f} \in L_2(D)$, then $\bar{u}_0 \in \mathrm{dom}(-\Delta)$, with

$$|\Delta\bar{u}_0| \leq \frac{2}{\nu}|\bar{f}| + \frac{c_d^2}{\nu^5\lambda_1^{3/2}}|\bar{f}|^3.$$

The bilinear form $\bar{a}_0(\cdot,\cdot)$. Define the bilinear continuous form $\bar{a}_0$ on $V \times V$ by

$$\bar{a}_0(u, v) = \nu(\nabla u, \nabla v) + b(u, \bar{u}_0, v) + b(\bar{u}_0, u, v) \tag{15}$$

where $\bar{u}_0(x)$ is the solution of the stationary (deterministic) Navier-Stokes equation (13ba) just found. Also define the mapping $\bar{A}_0 : V \to V'$, by

$$\langle \bar{A}_0(u), v\rangle = \bar{a}_0(u, v), \quad \text{for all } v \in V.$$

Then (13bb) can be written as

$$\bar{A}_0(\bar{u}_\alpha) = -\sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, B(\bar{u}_\gamma, \bar{u}_{\alpha-\gamma}) + \sum_l \sqrt{\alpha_l}\mathfrak{u}_l(x)\big(\bar{\sigma}^i\partial_{x_i}\bar{u}_{\alpha-\epsilon_l} + \mathbf{1}_{\alpha=\epsilon_l}\bar{g}\big)$$

for $|\alpha| \geq 1$.

To obtain the existence and uniqueness of u_α, we intend to apply the Lax-Milgram lemma to the bilinear form $\bar{a}_0(\cdot,\cdot)$. To do this, we first check the coercivity of $\bar{a}_0(\cdot,\cdot)$ on V.

Lemma 5. *Assume (A1), and assume u_0 solves (13ba) with $f \in V'$. Then $\bar{a}_0(\cdot,\cdot)$ defined in* (15) *is coercive and bounded on V.*

Proof. Indeed, for any $v \in V$,

$$\begin{aligned}\bar{a}_0(v, v) &= \nu|\nabla v|^2 + b(v, \bar{u}_0, v) + b(\bar{u}_0, v, v)\\ &\geq \nu|\nabla v|^2 - c_b\|\bar{u}_0\|_V\|v\|_V^2\\ &= \big(\nu - c_b\|\bar{u}_0\|_V\big)\|v\|_V^2 = \bar{\beta}\|v\|_V^2,\end{aligned}$$

where $\bar{\beta} := \nu - c_b\|\bar{u}_0\|_V > 0$ by (14). Next, $\bar{a}_0(\cdot,\cdot)$ is bounded, because

$$\begin{aligned}|\bar{a}_0(v, w)| &\leq \nu\|v\|_V\|w\|_V + |b(v, \bar{u}_0, w)| + |b(\bar{u}_0, v, w)|\\ &\leq \big(\nu + c_b\|\bar{u}_0\|_V\big)\|v\|_V\|w\|_V\end{aligned}$$

for any $v, w \in V$. □

We continue with the proof of Proposition 4.

Solutions for $\alpha = \epsilon_l$. Equation (13bb) in variational form reduces to finding $\bar{u}_{\epsilon_l} \in V$ such that

$$\bar{a}_0(\bar{u}_{\epsilon_l}, v) = \langle \mathfrak{u}_l\big(\bar{\sigma}^i\partial_{x_i}\bar{u}_0 + \bar{g}\big),\, v\rangle =: \langle G_{\epsilon_l}, v\rangle$$

for all $v \in V$. To apply the Lax-Milgram lemma to (13bb), we check that the term

$$G_{\epsilon_l} := \mathfrak{u}_l\big(\bar{\sigma}^i\partial_{x_i}\bar{u}_0 + \bar{g}\big)$$

belongs to V'. In fact, we have that G_{ϵ_l} belongs to $\mathbb{L}^2(D)$. Indeed, due to assumption (A2), $|\bar{\sigma}^i\partial_{x_i}\bar{u}_0| \leq \|\bar{\sigma}\|_{L^\infty}\|\bar{u}_0\|_V$, and from (14),

$$\begin{aligned}|G_{\epsilon_l}| &\leq C\|\mathfrak{u}_l\|_{L^\infty}\Big(\|\bar{\sigma}\|_{W^{1,\infty}}\|\bar{u}_0\|_V + \|\bar{g}\|_{H^1}\Big)\\ &\leq C\|\mathfrak{u}_l\|_{L^\infty}\Big(\frac{\nu}{c_b}\|\bar{\sigma}\|_{W^{1,\infty}} + \|\bar{g}\|_{H^1}\Big)\end{aligned}$$

By the Lax-Milgram lemma, there exists a unique variational solution $\bar{u}_{\epsilon_l} \in V$ with the estimate

$$\|\bar{u}_{\epsilon_l}\|_V \le \frac{1}{\bar{\beta}} C \|\mathfrak{u}_l\|_{L^\infty} \Big(\frac{\nu}{c_b} \|\bar{\sigma}\|_{W^{1,\infty}} + \|\bar{g}\|_{H^1} \Big).$$

Additionally, by a standard technique in [15], there exists $\bar{P}_{\epsilon_l} \in L^2(D)$ such that (13bb) holds in V'.

Next, observe that by the continuity property of the bilinear form $B : V \times \mathbb{H}^2 \to \mathbb{L}^2$,

$$-\nu \Delta \bar{u}_{\epsilon_l} = G_{\epsilon_l} - B(\bar{u}_{\epsilon_l}, \bar{u}_0) - B(\bar{u}_0, \bar{u}_{\epsilon_l}) \in L^2(D)$$

Hence, $\bar{u}_{\epsilon_l} \in \operatorname{dom}(-\Delta)$, and we have the estimate

$$\begin{aligned}
|\Delta \bar{u}_{\epsilon_l}| &\le \frac{1}{\nu} \Big(|G_{\epsilon_l}| + |B(\bar{u}_{\epsilon_l}, \bar{u}_0)| + |B(\bar{u}_0, \bar{u}_{\epsilon_l})| \Big) \\
&\le \frac{1}{\nu} \Big(|G_{\epsilon_l}| + 2c_b |\Delta \bar{u}_0| \, \|\bar{u}_{\epsilon_l}\|_V \Big) \\
&\le \frac{C \sup_l \|\mathfrak{u}_l\|_{L^\infty}}{\nu \bar{\beta}} \Big(\frac{\nu}{c_b} \|\bar{\sigma}\|_{W^{1,\infty}} + \|\bar{g}\|_{H^1} \Big) \Big(1 + \frac{2c_b}{\bar{\beta}} |\Delta \bar{u}_0| \Big) \\
&= \bar{K}
\end{aligned}$$

and $\bar{K} = \bar{K}(\nu, \bar{f}, \bar{g}, \bar{\sigma})$ does not depend on l.

Solutions for $|\alpha| \ge 2$. Denote

$$\begin{aligned}
G_\alpha &:= \sum_l \sqrt{\alpha_l}\, \mathfrak{u}_l \big(\bar{\sigma}^i \partial_{x_i} \bar{u}_{\alpha - \epsilon_l} \big), \\
F_\alpha &:= - \sum_{0 < \gamma < \alpha} \sqrt{\tbinom{\alpha}{\gamma}}\; B(\bar{u}_\gamma, \bar{u}_{\alpha - \gamma})
\end{aligned}$$

We first find $\bar{u}_\alpha \in V$ such that

$$\bar{a}_0(\bar{u}_{\epsilon_l}, v) = \langle F_\alpha + G_\alpha, v \rangle$$

for all $v \in V$.

We prove by induction. Assume we have shown the existence of a unique solution $\bar{u}_\gamma \in \operatorname{dom}(-\Delta)$ for all $|\gamma| \le n - 1$. By a similar argument as above, we have $G_\alpha \in L^2(D)$ with

$$|G_\alpha| \le C \sum_l \sqrt{\alpha_l} \|\mathfrak{u}_l\|_{L^\infty} \|\bar{\sigma}\|_{W^{1,\infty}} \|\bar{u}_{\alpha - \epsilon_l}\|_V < \infty.$$

Also, since $B(\cdot, \cdot)$ is a bilinear continuous from $\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{L}^2$, we deduce that $F_\alpha \in \mathbb{L}^2(D)$ with

$$|F_\alpha| \le c_b \sum_{0 < \gamma < \alpha} \sqrt{\tbinom{\alpha}{\gamma}}\; |\Delta \bar{u}_\gamma| \, |\Delta \bar{u}_{\alpha - \gamma}| < \infty$$

Applying the Lax-Milgram lemma, there exists a unique solution $\bar{u}_\alpha \in V$ with the estimates

$$\|\bar{u}_\alpha\|_V \leq \frac{1}{\beta}(|G_\alpha| + |F_\alpha|).$$

Finally, since

$$-\nu\Delta\bar{u}_\alpha = F_\alpha + G_\alpha - B(\bar{u}_\alpha, \bar{u}_0) - B(\bar{u}_0, \bar{u}_\alpha) \in \mathbb{L}^2(D),$$

we deduce that $u_\alpha \in \mathrm{dom}(-\Delta)$, with

$$\begin{aligned} |\Delta\bar{u}_\alpha| &\leq \frac{1}{\nu}\left(|F_\alpha| + |G_\alpha| + |B(\bar{u}_\alpha, \bar{u}_0)| + |B(\bar{u}_0, \bar{u}_\alpha)|\right) \\ &\leq \frac{1}{\nu}\left(|F_\alpha| + |G_\alpha| + 2c_b\|\bar{u}_\alpha\|_V|\Delta\bar{u}_0|\right) \\ &\leq \frac{1}{\nu}(|F_\alpha| + |G_\alpha|)\Big(1 + \frac{2c_b}{\beta}|\Delta\bar{u}_0|\Big) < \infty \end{aligned}$$

Hence, we have found a solution $u \in \mathcal{D}'(\mathbb{H}^2(D) \cap V)$. □

Next, we find the appropriate Kondratiev space to which the solution u belongs. As described previously, the estimation of the Kondratiev norm makes use of the recursion properties of the Catalan numbers. The Catalan number rescaling technique used in our estimates has been described in [9], and is detailed in Appendix A.

Proposition 6. *Assume (A0-2) hold. Then there exists $q_0 > 2$, depending on ν, $\bar{f}$, $\bar{g}$, $\bar{\sigma}$ such that $\bar{u}$ belongs to the Kondratiev space $S_{-1,-q}(\mathbb{H}^2(D) \cap V)$, for $q > q_0$.*

Proof. For $|\alpha| \geq 1$, we have found in the proof of Proposition 4 estimates for $|\Delta\bar{u}_\alpha|$,

$|\Delta\bar{u}_{\epsilon_l}| \leq \bar{K}$

$$\frac{1}{\sqrt{\alpha!}}|\Delta\bar{u}_\alpha| \leq \bar{B}_0\left(\sum_{0<\gamma<\alpha}\frac{|\Delta\bar{u}_\gamma|}{\sqrt{\gamma!}}\,\frac{|\Delta\bar{u}_{\alpha-\gamma}|}{\sqrt{(\alpha-\gamma)!}} + \mathbf{1}_{\sigma\neq 0}\sum_l \mathbf{1}_{\alpha_l\neq 0}\frac{1}{\sqrt{(\alpha-\epsilon_l)!}}\|\bar{u}_{\alpha-\epsilon_l}\|_V\right).$$

where $\bar{B}_0$ depends on $\nu, \bar{f}, \bar{\sigma}$. Let $L_{\epsilon_l} = 1 + |\Delta\bar{u}_{\epsilon_l}|$, and $L_\alpha = \frac{1}{\sqrt{\alpha!}}|\Delta\bar{u}_\alpha|$ for $|\alpha| \geq 2$. Then by induction, we find that

$$L_\alpha \leq \bar{B}_0\sum_{0<\gamma<\alpha} L_{\alpha-\gamma}L_\gamma$$

and by the Catalan numbers method in Appendix A,

$$|\Delta\bar{u}_\alpha|^2 \leq \alpha!\,\mathcal{C}^2_{|\alpha|-1}\binom{|\alpha|}{\alpha}(2\mathbb{N})^\alpha\bar{B}_0^{2(|\alpha|-1)}\bar{K}^{2|\alpha|} \tag{16}$$

for $|\alpha| \geq 1$. The result holds with q_0 satisfying

$$\bar{B}_0^2\bar{K}^2 2^{5-q_0}\sum_{i=1}^{\infty} i^{1-q_0} = 1. \tag{17}$$

□

4 The time-dependent case

In this section, we consider for simplicity equation (4) with $\sigma(t,x) = 0$. We will consider the time-dependent solution $u(t)$ of (4) on a finite time interval $[0,T]$ if $d = 2,3$, and also study its uniform boundedness on $[0,\infty)$ for $d = 2$. The former result allows an arbitrarily large time interval, thereby ensuring a global-in-time solution. On the other hand, the latter result will become useful for showing the long-time convergence of the solution to a steady state solution.

For any $T < \infty$, it is known that a strong solution $u_0(t)$ of the deterministic Navier-Stokes equation (12ba) exists on the finite interval $[0,T]$ if $d = 2$, and exists on $[0,(T \wedge T_1)]$ for a specific $T_1 = T_1(u_0(0))$ depending on $u_0(0)$ if $d = 3$. Without further conditions, we have the following result for a generalized strong solution of the unbiased Navier-Stokes equation.

Lemma 7. *For $d = 2,3$, let $T < \infty$ if $d = 2$, or $T \leq T_1$ if $d = 3$. Assume the forcing terms f, g and initial condition $u(0)$ are deterministic functions satisfying*

$$f, g \in L^2(0,T;H), \qquad u(0) \in V.$$

Then there exists a unique generalized strong solution $u(t) \in \mathcal{D}'(\mathbb{H}^2(D) \cap V)$ for a.e. $t \in [0,T]$. Moreover, $u_\alpha \in C([0,T],V)$ for all α.

Proof. For $\alpha = (0)$, it is well-known that (12ba) has a unique solution u_0, and

$$u_0 \in L^2([0,T];\mathrm{dom}(-\Delta)), \quad u_0 \in C([0,T];V).$$

THE BILINEAR FORM $a_0(t)$. For $t \in [0,T]$, define the bilinear continuous form $a_0(t)$ on $V \times V$ by

$$a_0(u,v;t) = \nu(\nabla u, \nabla v) + b(u, u_0(t), v) + b(u_0(t), u, v)$$

where $u_0(t,x)$ is the solution of the time-dependent (deterministic) Navier-Stokes equations given in (12ba) just found. Also define the mapping $A_0(t) : V \to V'$, for $t \in [0,T]$, by

$$\langle A_0(t)u, v\rangle = a_0(u,v;t), \quad \text{for all } v \in V.$$

Then (12bb) can be written as

$$\partial_t u_\alpha + A_0(t)u_\alpha + \sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, B(u_\gamma, u_{\alpha-\gamma}) = \sum_l \sqrt{\alpha_l}\mathrm{u}_l(x)\big(\sigma^i \partial_{x_i} u_{\alpha-\epsilon_l} + \mathbf{1}_{\alpha=\epsilon_l} g\big)$$

This is a linear Stokes equation of the form

$$\begin{array}{l} \partial_t U + A_0(t)U = F \\ U|_{\partial D} = 0, \quad U(0) = w \end{array}.$$

Since $u_0 \in L^2(0,T;\mathrm{dom}(-\Delta))$, it can be shown by standard compactness techniques that if $F \in L^2(0,T;H)$ and $w \in V$, then there exists a unique strong solution $U \in L^2(0,T;\mathrm{dom}(-\Delta)) \cap C(0,T;V)$ and $U_t \in L^2(0,T;H)$ with the estimates

$$\sup_{t\le T} \|U(t)\|_V + \|U\|_{L^2(0,T;\mathrm{dom}(-\Delta))} + \|U_t\|_{L^2(0,T;H)} \le C\big(\|U(0)\|_V + \|F\|_{L^2(0,T;H)}\big), \tag{18}$$

where the constant C depends only on T,ν,D and $\|u_0\|_{L^2(0,T;\mathrm{dom}(-\Delta))}$.

We prove the lemma by induction. For $|\alpha| \ge 1$, assume that $u_\gamma \in L^2([0,T];\mathrm{dom}(-\Delta))$ for all $\gamma < \alpha$. We check for the RHS of (12bb),

$$-\sum_{0<\gamma<\alpha} \sqrt{\tbinom{\alpha}{\gamma}}\, B(u_\gamma, u_{\alpha-\gamma}) + \mathbf{1}_{\alpha=\epsilon_l} \mathrm{u}_l g \ \in L^2([0,T];H)$$

This follows from (A0$'$) and the fact that $|B(u_\gamma, u_{\alpha-\gamma})| \le c_b |\Delta u_\gamma|\,|\Delta u_{\alpha-\gamma}|$. It follows from (18) that there exists a unique solution u_α of (12bb) with

$$u_\alpha \in L^2([0,T];\mathrm{dom}(-\Delta)), \quad \partial_t u_\alpha \in L^2([0,T];H), \quad \text{and } u_\alpha \in C([0,T];V).$$

□

Remark. If $\sigma \ne 0$, then in addition to (A0$'$), we must require that $g \in L^2(0,T;\mathbb{H}^1(D))$ and $\sigma \in L^2(0,T;(W^{1,\infty}(D))^d)$. (Compare with (A2).)

Next, we study $\|u(t)\|_{-1,-q;\mathbb{H}^2}$ on a finite interval $[0,T]$ as well as the uniform boundedness of $\|u(t)\|_{-1,-q;V}$ for all time $t \in [0,\infty)$. We recall the following established result on the uniform bounds of u_0 in the V and $\mathbb{H}^2(D)$ norms.

Lemma 8. *(Lemma 11.1 in [14]; see also [8]) Assume for the initial condition that $u_0(0,\cdot) \in V$, and assume*

$$f \text{ is continuous and bounded from } [0,\infty) \text{ into } H$$

$$\frac{\partial f}{\partial t} \text{ is continuous and bounded from } [0,\infty) \text{ into } V'$$

Let $u_0(t)$ be the strong solution of the deterministic Navier-Stokes equations (12ba), defined on $[0,\infty)$ if $d=2$, or on $[0,T_1]$ if $d=3$. Then

$$\sup_{t\ge 0} \|u_0(t)\|_V \le c'(\|u_{(0)}(0,\cdot)\|_V, \nu, f, D). \tag{19a}$$

$$\sup_{t\ge \tau} |\Delta u_0(t)| \le c''(\tau, \|u_{(0)}(0,\cdot)\|_V, \nu, f, D). \tag{19b}$$

for any $\tau > 0$.

Proposition 9. *(i) For $d=2,3$, assume the same conditions as in Lemma 7. Then there exists some $q_1 > 2$ depending on ν, c', c_b and T, such that for $q > q_1$,*

$$u \in \mathcal{S}_{-1,-q}(\,L^2(0,T;dom(-\Delta))\,) \cap \mathcal{S}_{-1,-q}(L^\infty(0,T;V)).$$

(ii) For $d=2$, assume the hypothesis of Lemma 8, and assume g is bounded from $[0,\infty)$ into H. Also assume

$$\nu^4 > \frac{2^7 c_b^4 c'^4}{\lambda_1} \tag{A1$'$}$$

where $c' = c'(\|u(0,\cdot)\|_V, \nu, \bar{f}, D)$ *in* (19a).

Then there exists $q_2 > 2$ *depending on* ν, c' *and* c_b, *such that for* $q > q_2$,

$$\sup_{t\geq 0} \|u(t)\|_{-1,-q;V} < \infty \quad \textit{and} \quad \sup_{t\geq \tau} \|u(t)\|_{-1,-q;dom(-\Delta)} < \infty$$

with τ *in* (19b). *In fact,*

$$u \in \mathcal{S}_{-1,-q}(L^\infty([0,\infty);V)) \quad \textit{and} \quad u \in \mathcal{S}_{-1,-q}(L^\infty([\tau,\infty);V)).$$

Remark. Part (ii) of the equation asserts a uniform-in-time bound of the $\mathcal{S}_{-1,-q}(V)$ norm of the solution on the infinite time interval. Unfortunately, this result does not follow from part (i) because, under the present proof, the estimates for the $\mathcal{S}_{-1,-q}(\mathbb{H}^2(D)\cap V)$ norm of the solution on the finite time interval increases to infinity as the terminal time $T \to \infty$.

Proof. (i) The proof of this result is identical to the proof of Proposition 6, by using the estimates (18). For $\alpha = (0)$, (19a) and the usual deterministic theory implies that $u_0 \in L^2(0,T;\mathrm{dom}(-\Delta)) \cap L^\infty(0,T;V)$. Let $L_\alpha = \frac{1}{\sqrt{\alpha!}}\tilde{L}_\alpha$ for $|\alpha| \geq 1$. For $\alpha = \epsilon_l$, the estimates (18) yield

$$L_{\epsilon_l} \leq C \sup_l \|\mathfrak{u}_l\|_{L^\infty(D)}\, |g| =: K_1$$

where K_1 does not depend on l. For $|\alpha| \geq 2$,

$$\tilde{L}_\alpha \leq C \left(\sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, \|u_\gamma\|_{L^2(0,T;H^2)} \|u_{\alpha-\gamma}\|_{L^2(0,T;V)} \right) \leq C \sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}}\, \tilde{L}_\gamma \tilde{L}_{\alpha-\gamma}$$

Then for $L_\alpha := \frac{1}{\sqrt{\alpha!}}\tilde{L}_\alpha$,

$$L_\alpha \leq B_1 \sum_{0<\gamma<\alpha} L_\gamma L_{\alpha-\gamma}.$$

By the Catalan numbers method as per Appendix A,

$$\|u_\alpha\|_{L^\infty(0,T;V)} + \|\Delta u_\alpha\|_{L^2(0,T;H)} \leq \sqrt{\alpha!}\,\mathcal{C}_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_1^{|\alpha|-1} K_1^{|\alpha|}$$

and the statement of the Proposition holds with q_1 satisfying

$$B_1^2 K_1^2 2^{5-q_1} \sum_{i=1}^{\infty} i^{1-q_1} = 1.$$

(ii) We now show the uniform boundedness of each mode u_α on the infinite time interval. For $\alpha = (0)$, this is shown in the estimates of (19a),(19b). For

$|\alpha|=1$, $\alpha=\epsilon_l$, choose in (12bb) the test function $v=(-\Delta)u_\alpha$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_{\epsilon_l}\|_V^2+\nu|\Delta u_{\epsilon_l}|^2 &\le |b(u_{\epsilon_l},u_0,\Delta u_{\epsilon_l})|+|b(u_0,u_{\epsilon_l},\Delta u_{\epsilon_l})|+|\langle \mathfrak{u}_l g,\Delta u_{\epsilon_l}\rangle|\\ &\le 2c_b\|u_0\|_V\|u_{\epsilon_l}\|_V^{1/2}|\Delta u_{\epsilon_l}|^{3/2}+|\mathfrak{u}_l g|\,|\Delta u_{\epsilon_l}|\\ &\le \frac{\varepsilon}{2}|\Delta u_{\epsilon_l}|^2+\frac{1}{2\varepsilon}\big(2c_b\|u_0\|_V\|u_{\epsilon_l}\|_V^{1/2}|\Delta u_{\epsilon_l}|^{1/2}+|\mathfrak{u}_l g|\big)^2\\ &\le \frac{\varepsilon}{2}|\Delta u_{\epsilon_l}|^2+\frac{2c_b^2\|u_0\|_V^2}{2\varepsilon}\|u_{\epsilon_l}\|_V|\Delta u_{\epsilon_l}|+\frac{1}{\varepsilon}|\mathfrak{u}_l g|^2\\ &\le (\varepsilon)|\Delta u_{\epsilon_l}|^2+\frac{2^3c_b^4\|u_0\|_V^4}{\varepsilon^3}\|u_{\epsilon_l}\|_V^2+\frac{1}{\varepsilon}|\mathfrak{u}_l g|^2\end{aligned}$$

Taking $\varepsilon=\frac{\nu}{2}$,

$$\frac{d}{dt}\|u_{\epsilon_l}\|_V^2+\nu|\Delta u_{\epsilon_l}|^2\le \frac{2^7c_b^4}{\nu^3}\|u_0\|_V^2\|u_{\epsilon_l}\|_V^2+\frac{4}{\nu}|\mathfrak{u}_l g|^2$$

and from (9) and (19a),

$$\begin{aligned}\frac{d}{dt}\|u_{\epsilon_l}\|_V^2 &\le \Big(\frac{2^7c_b^4c'^4}{\nu^3}-\nu\lambda_1\Big)\|u_{\epsilon_l}\|_V^2+\frac{4}{\nu}|\mathfrak{u}_l g|^2\\ &\le -\beta\|u_{\epsilon_l}\|_V^2+\frac{4}{\nu}|\mathfrak{u}_l g|^2\end{aligned}$$

where $\beta:=-\big(\frac{2^7c_b^4c'^4}{\nu^3}-\nu\lambda_1\big)>0$ by (A1′). By Gronwall's inequality,

$$\|u_{\epsilon_l}(T)\|_V^2\le\int_0^T\frac{4}{\nu}|\mathfrak{u}_l g|^2e^{-\beta(T-s)}ds\le\frac{4}{\nu\beta}\|\mathfrak{u}_l\|_{L^\infty}^2\|g\|_{L^\infty(0,\infty;H)}^2\left(1-e^{-\beta T}\right)$$

for any $T>0$. Also,

$$|\Delta u_{\epsilon_l}(t)|^2\le\frac{2^7c_b^4c'^2}{\nu^4}\|u_{\epsilon_l}(t)\|_V^2+\frac{4}{\nu^2}|\mathfrak{u}_l g(t)|^2.$$

It follows that

$$L_{\epsilon_l}:=\sup_{t\ge0}\left(\|u_{\epsilon_l}(t)\|_V+|\Delta u_{\epsilon_l}(t)|\right)\le K_2,$$

for all l, where the constant K_2 is independent of l and t. For $|\alpha|\ge2$, let $L_\alpha:=\frac{1}{\sqrt{\alpha!}}\sup_{t\ge0}(\|u_\alpha(t)\|_V+|\Delta u_\alpha(t)|)$. Then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_V^2+\nu|\Delta u_\alpha|^2\\ &\le |b(u_\alpha,u_0,\Delta u_\alpha)|+|b(u_0,u_\alpha,\Delta u_\alpha)|+\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,|b(u_\gamma,u_{\alpha-\gamma},\Delta u_\alpha)|\\ &\le 2c_b\|u_0\|_V\|u_\alpha\|_V^{1/2}|\Delta u_\alpha|^{3/2}+\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,c_b\|u_\gamma\|_V\,|\Delta u_{\alpha-\gamma}|\,|\Delta u_\alpha|.\end{aligned}$$

By similar computations,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_\alpha\|_V^2 + \nu|\Delta u_\alpha|^2 \\
&\le \frac{2^7 c_b^4}{\nu^3}\|u_0\|_V^4\|u_\alpha\|_V^2 + \frac{4c_b^2}{\nu}\left(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,\|u_\gamma\|_V\,|\Delta u_{\alpha-\gamma}|\right)^2 \\
&\le \frac{2^7 c_b^4}{\nu^3}\|u_0\|_V^4\|u_\alpha\|_V^2 + \frac{4c_b^2}{\nu}\left(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\left(\sup_{s\ge 0}\|u_\gamma(s)\|_V\right)\left(\sup_{s\ge 0}|\Delta u_{\alpha-\gamma}(s)|\right)\right)^2
\end{aligned}
$$

and so

$$
\frac{d}{dt}\|u_\alpha\|_V^2 \le -\beta\|u_\alpha\|_V^2 + \frac{4c_b^2}{\nu}\left(\sum_{0<\gamma<\alpha}\sqrt{\alpha!}L_\gamma L_{\alpha-\gamma}\right)^2.
$$

By Gronwall's inequality and triangle inequality,

$$
\begin{aligned}
\|u_\alpha(T)\|_V^2 &\le \frac{4c_b^2}{\nu}\int_0^T\left(\sum_{0<\gamma<\alpha}\sqrt{\alpha!}L_\gamma L_{\alpha-\gamma}\,e^{-\beta(T-s)/2}\right)^2 ds \\
&\le \frac{4c_b^2}{\nu}\left(\sum_{0<\gamma<\alpha}\sqrt{\alpha!}L_\gamma L_{\alpha-\gamma}\Big(\int_0^T e^{-\beta(T-s)}ds\Big)^{1/2}\right)^2
\end{aligned}
$$

so

$$
\frac{1}{\sqrt{\alpha!}}\sup_{T\ge 0}\|u_\alpha(T)\|_V \le \frac{2c_b^2}{\sqrt{\nu\beta}}\sum_{0<\gamma<\alpha}L_\gamma L_{\alpha-\gamma}
$$

We have also,

$$
|\Delta u_\alpha(t)|^2 \le \frac{2^7 c_b^4 c'^4}{\nu^4}\|u_\alpha(t)\|_V^2 + \frac{4c_b^2}{\nu^2}\left(\sum_{0<\gamma<\alpha}\sqrt{\alpha!}L_\gamma L_{\alpha-\gamma}\right)^2
$$

for any $t \ge 0$. Hence, it follows that

$$
L_\alpha \le B_2\sum_{0<\gamma<\alpha}L_\gamma L_{\alpha-\gamma}
$$

where B_2 depends on ν, c' and c_b, but is independent of t. By the Catalan method in Appendix A,

$$
\sup_{t\ge 0}\left(\|u_\alpha(t)\|_V + |\Delta u_\alpha(t)|\right) \le \sqrt{\alpha!}\mathcal{C}_{|\alpha|-1}\binom{|\alpha|}{\alpha}B_2^{|\alpha|-1}K_2^{|\alpha|}
$$

for $|\alpha| \ge 1$, and the statement of the Proposition holds with q_2 satisfying

$$
B_2^2 K_2^2 2^{5-q_2}\sum_{i=1}^{\infty} i^{1-q_2} = 1.
$$

□

5 Long time convergence to the stationary solution

In this section, we study the solutions $u(t,x)$ of (4) and $\bar{u}(x)$ of (5) with $\sigma(t,x) = \bar{\sigma}(x) = 0$, and for simplicity consider the case with $f(t,x) = \bar{f}(x)$ and $g(t,x) = \bar{g}(x)$. We study the convergence of $u(t,x)$ to the stationary solution $\bar{u}(x)$ as $t \to \infty$, first in a weak sense (in a generalized space $\mathcal{D}'(H)$ with some exponential rate of convergence in each mode, then in a strong sense (in some Kondratiev space $\mathcal{S}_{-1,-q}(H)$) using a compact embedding argument. The latter proof, unfortunately, is does not provide a rate of convergence. For time-dependent f, g, similar results can be obtained under suitable assumptions, but the exponential convergence of each mode is not guaranteed.

Let $z(t) := u(t) - \bar{u}$. The propagator system for z is

$$z_{0,t} + B(u_0, u_0) - B(\bar{u}_0, \bar{u}_0) = \nu \Delta z_0 \tag{20a}$$

$$z_{\alpha,t} + A_0(t; u_\alpha) - \bar{A}_0(\bar{u}_\alpha) = - \sum_{0<\gamma<\alpha} \sqrt{\binom{\alpha}{\gamma}} \left(B(u_\gamma, u_{\alpha-\gamma}) - B(\bar{u}_\gamma, \bar{u}_{\alpha-\gamma}) \right) \tag{20b}$$

with $z_\alpha(0,x) = u_\alpha(0,x) - \bar{u}_\alpha(x)$, $z|_{\partial D} = 0$ and $\operatorname{div} z_\alpha \equiv 0$, for all α.

Proposition 10. *Let $d = 2$. Assume (A0), (A0′), (A1), and assume*

$$\nu \left(\frac{\lambda_1}{c_2'} \right)^{3/4} > \frac{2}{\nu} |\bar{f}| + \frac{c_2^2}{\nu^5 \lambda_1^{3/2}} |\bar{f}|^3 \tag{A3}$$

where c_2, c_2' are specific constants depending only on D.

Then the solution $u(t)$ of (4) converges in $\mathcal{D}'(H)$ to the solution $\bar{u}$ of (5),

$$u(t) \xrightarrow{\mathcal{D}'(H)} \bar{u}, \qquad \text{as } t \to \infty.$$

Remark. In the following proof, all computations follow through even when $d = 3$. So, a similar statement to Proposition 10 can be made for $d = 3$, provided a strong solution $u(t)$ exists in $\mathcal{D}'(\mathbb{H}^2 \cap V)$ for all $t > 0$, and the zero-th mode $u_0(t)$ satisfies the energy inequality (c.f. [14])

$$\frac{1}{2} \frac{d}{dt} |u_0(t)|^2 + \nu \|u_0(t)\|_V^2 \le \langle \bar{f}, u_0(t) \rangle.$$

Remark. If $f(t,x)$ and $g(t,x)$ depend on time, then an additional condition for the proposition to hold is that $f(t), g(t)$ converge to $\bar{f}, \bar{g}$ in H.

Proof. For $\alpha = (0)$, the convergence for the deterministic Navier-Stokes equation is well-known: if $u_0(t)$ is any weak solution of (12ba) with initial condition

$u_0(0) \in H$, then $u_0(t) \longrightarrow \bar{u}_{(0)}$ in H as $t \to \infty$, provided (A3) holds. Moreover, $|z_0(t)|$ decays exponentially,

$$|z_0(t)| \le |z_0(0)|\, e^{-\bar{\nu} t}, \tag{21}$$

where $\bar{\nu} := \nu\lambda_1 - \frac{c_2'}{\nu^{1/3}}|\Delta \bar{u}_0|^{4/3} > 0$. (See e.g., Theorem 10.2 in [14]; the positivity of $\bar{\nu}$ follows from the fact that $|\Delta \bar{u}_0|$ can be majorized by the RHS of (A3).)

For $\alpha = \epsilon_l$, choosing the test function $v = z_{\epsilon_l}$ in the weak formulation of (20bb),

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|z_{\epsilon_l}|^2 + \nu\|z_{\epsilon_l}\|_V^2 \\
&\le |b(z_{\epsilon_l}, \bar{u}_0, z_{\epsilon_l})| + |b(z_{\epsilon_l}, z_0, z_{\epsilon_l})| + |b(z_0, \bar{u}_{\epsilon_l}, z_{\epsilon_l})| + |b(\bar{u}_{\epsilon_l}, z_0, z_{\epsilon_l})| \\
&\le c_b\|\bar{u}_0\|_V\|z_{\epsilon_l}\|_V^2 + c_b\|z_0\|_{L^\infty}|z_{\epsilon_l}|\,\|z_{\epsilon_l}\|_V + 2c_b|\Delta\bar{u}_{\epsilon_l}|\,|z_0|\,\|z_{\epsilon_l}\|_V \\
&\le c_b\|\bar{u}_0\|_V\|z_{\epsilon_l}\|_V^2 + \frac{c_b^2}{2\varepsilon}\|z_0\|_{L^\infty}^2|z_{\epsilon_l}|^2 + \varepsilon\|z_{\epsilon_l}\|_V^2 + \frac{2c_b^2}{\varepsilon}|\Delta\bar{u}_{\epsilon_l}|^2|z_0|^2
\end{aligned}$$

where we have used the ε-inequality in the last line with any $0 < \varepsilon < \bar{\beta}$. So,

$$\frac{1}{2}\frac{d}{dt}|z_{\epsilon_l}|^2 + (\bar{\beta} - \varepsilon)\|z_{\epsilon_l}\|_V^2 \le \frac{c_b^2}{2\varepsilon}\|z_0\|_{L^\infty}^2|z_{\epsilon_l}|^2 + \frac{2c_b^2}{\varepsilon}|\Delta\bar{u}_{\epsilon_l}|^2|z_0|^2. \tag{22}$$

Using the Poincare inequality (9) and taking $\varepsilon = \frac{\bar{\beta}}{2}$,

$$\frac{d}{dt}|z_{\epsilon_l}|^2 + \bar{\beta}\lambda_1|z_{\epsilon_l}|^2 \le \frac{2c_b^2}{\bar{\beta}}\|z_0\|_{L^\infty}^2|z_{\epsilon_l}|^2 + \frac{8c_b^2}{\bar{\beta}}|\Delta\bar{u}_{\epsilon_l}|^2|z_0|^2$$

For some appropriately chosen $t_0 \in (0, \infty)$ to be discussed next, we apply Gronwall's inequality,

$$|z_{\epsilon_l}(T)|^2 \le e^{\int_{t_0}^T \varphi(t)dt}|z_{\epsilon_l}(t_0)|^2 + \int_{t_0}^T \psi_l(s)e^{\int_s^T \varphi(t)dt}ds$$

where

$$\begin{aligned}
\varphi(t) &= \frac{4c_b^2}{\bar{\beta}}\|z_0(t)\|_{L^\infty}^2 - \bar{\beta}\lambda_1, \\
\psi_l(t) &= \frac{8c_b^2}{\bar{\beta}}|\Delta\bar{u}_{\epsilon_l}|^2|z_0(t)|^2.
\end{aligned}$$

The t_0 is chosen large enough so that $\|z_0(t)\|_{L^\infty}^2 < \frac{\bar{\beta}^2\lambda_1}{4c_b^2}$ whenever $t \ge t_0$. Such t_0 exists, because by (19b) and the Sobolev embedding $z_0(t) \in C^{1/2}$ is Hölder continuous with exponent $\gamma < 1$ and $\sup_{t\ge\tau}\|z_0(t)\|_{C^\gamma} \le c''$ is uniformly in t. Then due to (21), we deduce that in fact $z_0(t, \cdot) \longrightarrow 0$ uniformly on D as $t \to \infty$. Consequently, we have that $\sup_{t\ge t_0}\varphi(t) < 0$. Set $\bar{\varphi} > 0$ satisfying

$$2\bar{\varphi} < \min\left\{-\sup_{t\ge t_0}\varphi(t),\, 2\bar{\nu}\right\}.$$

Obviously, $\exp\big\{\int_{t_0}^T \varphi(t)dt\big\} \le \exp\big\{-2\bar{\varphi}(T-t_0)\big\}$. Moreover, from (21),

$$|\psi_l(t)| \le \frac{8c_b^2}{\bar{\beta}}|\Delta\bar{u}_{\epsilon_l}|^2|z_0(t_0)|^2 e^{-2\bar{\nu}(t-t_0)} =: C_{\psi_l}e^{-2\bar{\nu}(t-t_0)} \longrightarrow 0$$

decays exponentially as $t \to \infty$. Combining these results,

$$\begin{aligned}|z_{\epsilon_l}(T)|^2 &\le e^{-2\bar{\varphi}(T-t_0)}|z_{\epsilon_l}(t_0)|^2 + \int_{t_0}^T C_{\psi_l}e^{-2\bar{\nu}(s-t_0)}e^{-2\bar{\varphi}(T-s)}ds \\ &\le e^{-2\bar{\varphi}(T-t_0)}|z_{\epsilon_l}(t_0)|^2 + \frac{C_{\psi_l}}{2(\bar{\nu}-\bar{\varphi})}\left(e^{-2\bar{\phi}(T-t_0)}e^{-2\bar{\nu}(T-t_0)}\right) \longrightarrow 0\end{aligned}$$

as $T \to \infty$. (In the first term, $|z_{\epsilon_l}(t_0)|^2$ has been shown to be finite for any finite t_0.) Since $\bar{\varphi} < \bar{\nu}$,

$$|z_{\epsilon_l}(T)|^2 \le \left(|z_{\epsilon_l}(t_0)|^2 + \frac{C_{\psi_l}}{2(\bar{\nu}-\bar{\varphi})}\right)e^{-2\bar{\varphi}(T-t_0)} =: K_{\epsilon_l}^2 e^{-2\bar{\varphi}(T-t_0)} \tag{23}$$

for $T \ge t_0$. K_{ϵ_l} does not depend on T. For $|\alpha| \ge 2$, we prove by induction. Fix α, and assume the induction hypothesis that: For each $0 < \gamma < \alpha$, for $T \ge t_0$,

$$|z_\gamma(T)| \le K_\gamma e^{-2^{1-|\gamma|}\bar{\varphi}(T-t_0)} \longrightarrow 0 \tag{24}$$

as $T \to \infty$, where K_γ does not depend on T. We want to show that (24) also holds for α. From (20bb) with test function $v = z_\alpha$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|z_\alpha|^2 + \nu|\nabla z_\alpha|^2 \\ &\le |b(z_\alpha,\bar{u}_0,z_\alpha)| + |b(z_\alpha,z_0,z_\alpha)| + |b(z_0,\bar{u}_\alpha,z_\alpha)| + |b(\bar{u}_\alpha,z_0,z_\alpha)| \\ &\quad + \sum_{0<\gamma<\alpha}\sqrt{\tbinom{\alpha}{\gamma}}\left(|b(z_\gamma,z_{\alpha-\gamma},z_\alpha)| + |b(z_\gamma,\bar{u}_{\alpha-\gamma},z_\alpha)| + |b(\bar{u}_\gamma,z_{\alpha-\gamma},z_\alpha)|\right)\end{aligned}$$

Similar to (22), using the ε-inequality with any $0 < \varepsilon < \bar{\beta}/2$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|z_\alpha|^2 + (\bar{\beta}-2\varepsilon)\|z_\alpha\|_V^2 &\le \frac{c_b^2}{2\varepsilon}\|z_0\|_{L^\infty}^2|z_\alpha|^2 + \frac{2c_b^2}{\varepsilon}|\Delta\bar{u}_\alpha|^2|z_0|^2 \\ &\quad + \frac{c_b^2}{4\varepsilon}\Big(\sum_{0<\gamma<\alpha}\sqrt{\tbinom{\alpha}{\gamma}}\left(\|z_{\alpha-\gamma}\|_V + 2\|\bar{u}_{\alpha-\gamma}\|_V\right)|z_\gamma|_{1/2}\Big)^2\end{aligned}$$

Using the Poincare inequality and taking $\varepsilon = \bar{\beta}/4$,

$$\begin{aligned}\frac{d}{dt}|z_\alpha(t)|^2 &\le \Big(\frac{4c_b^2}{\bar{\beta}}\|z_0\|_{L^\infty}^2 - \lambda_1\bar{\beta}\Big)|z_\alpha|^2 + \frac{16c_b^2}{\bar{\beta}}|\Delta\bar{u}_\alpha|^2|z_0|^2 \\ &\quad + \frac{2c_b^2}{\bar{\beta}}\Big(\sum_{0<\gamma<\alpha}\sqrt{\tbinom{\alpha}{\gamma}}\left(\|z_{\alpha-\gamma}\|_V + 2\|\bar{u}_{\alpha-\gamma}\|_V\right)|z_\gamma|^{1/2}\|z_\gamma\|_V^{1/2}\Big)^2 \\ &\le \varphi(t)|z_\alpha(t)|^2 + \psi_\alpha(t)\end{aligned}$$

where now

$$\psi_\alpha(t) = \frac{16c_b^2}{\bar{\beta}}|\Delta\bar{u}_\alpha|^2|z_0(t)|^2$$
$$+ \frac{2c_b^2}{\bar{\beta}}\left(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\left(\|z_{\alpha-\gamma}(t)\|_V + 2\|\bar{u}_{\alpha-\gamma}\|_V\right)^2\|z_\gamma(t)\|_V\right)\left(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,|z_\gamma(t)|\right)$$

From the hypothesis (24),

$$|\psi_\alpha(t)| \leq C_{\psi_\alpha}e^{-2\bar{\nu}(t-t_0)} + \tilde{C}_{\psi_\alpha}\Big(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,K_\gamma e^{-2^{-|\gamma|}2\bar{\varphi}(t-t_0)}\Big)$$

where

$$C_{\psi_\alpha} = \frac{16c_b^2}{\bar{\beta}}\|\bar{u}_\alpha\|_{H^2}^2|z_0(t_0)|^2,$$
$$\tilde{C}_{\psi_\alpha} = \frac{2c_b^2}{\bar{\beta}}\left(\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\left(\sup_{s\geq 0}\|z_{\alpha-\gamma}(s)\|_V + 2\|\bar{u}_{\alpha-\gamma}\|_V\right)^2\sup_{s\geq 0}\|z_\gamma(s)\|_V\right),$$

and $C_{\phi_\alpha}, \tilde{C}_{\phi_\alpha}$ do not depend on t. By Gronwall's inequality,

$$\begin{aligned}|z_\alpha(T)|^2 &\leq e^{-\bar{\varphi}(T-t_0)}|z_\alpha(t_0)|^2 + \int_{t_0}^T \psi_\alpha(s)e^{-\bar{\varphi}(T-s)}ds \\ &\leq e^{-\bar{\varphi}(T-t_0)}|z_\alpha(t_0)|^2 + \frac{C_{\psi_\alpha}}{2(\bar{\nu}-\bar{\varphi})}e^{-2\bar{\varphi}(T-t_0)} + \tilde{C}_{\psi_\alpha}\sum_{0<\gamma<\alpha}\sqrt{\binom{\alpha}{\gamma}}\,K_\gamma\frac{e^{-2^{1-|\gamma|}\bar{\varphi}(T-t_0)}}{1-2^{-|\gamma|}} \\ &\leq K_\alpha^2 e^{-2^{1-(|\alpha|-1)}\bar{\varphi}(T-t_0)}\end{aligned}$$

where K_α does not depend on T. Hence,

$$|z_\alpha(T)| \leq K_\alpha e^{-2^{1-|\alpha|}\bar{\varphi}(T-t_0)} \tag{25}$$

for all $T \geq t_0$. It follows that (24) holds also for α, and the result follows. □

We proceed to deduce the long time convergence of $u(t)$ in some Kondratiev space $\mathcal{S}_{-1,-q}(H)$. The manner of estimates in Proposition 10 is not directly suited for applying the Catalan numbers method. Instead, we will use a compact embedding type argument in the following lemma to show the result.

Lemma 11. *For $q > 0$, let the sequence $r = (2\mathbb{N})^{-q}$. Let $u^k \in \mathcal{S}_{-1,-q}(V)$ be a sequence satisfying*

$$\sum_\alpha \frac{r^\alpha}{\alpha!}\Big(\sup_k \|u_\alpha^k\|_V^2\Big) < \infty,$$

that is, satisfying $\{u^k\} \in \mathcal{S}_{-1,-q}(\ell^\infty(V))$.

Then there exists a subsequence $\tilde{k}_N$ such that $u^{\tilde{k}_N}$ converges in $\mathcal{D}'(H)$ to some $\bar{u} \in \mathcal{D}'(H)$. In fact, $\bar{u} \in \mathcal{S}_{-1,-q}(V)$ and the convergence is in $\mathcal{S}_{-1,-q}(H)$.

Proof. Since for every α'

$$\sum_\alpha \frac{r^\alpha}{\alpha!}\Big(\|u^k_\alpha\|^2_V\Big) = \sum_\alpha \Big(\|\frac{r^{\alpha/2}}{\sqrt{\alpha!}}u^k_\alpha\|^2_V\Big) \geq \|\frac{r^{\alpha'/2}}{\sqrt{\alpha'!}}u^k_{\alpha'}\|^2_V,$$

and

$$\sup_{\alpha,k}\|\frac{r^{\alpha/2}}{\sqrt{\alpha!}}u^k_\alpha\|^2_V \leq \sup_k \sum_\alpha \frac{r^\alpha}{\alpha!}\Big(\|u^k_\alpha\|^2_V\Big) < \infty.$$

Since $\mathcal{I}$ is countable and embedding $V \subseteq H$ is compact it follows that there are H-valued $\bar{u}_\alpha, \alpha \in \mathcal{I}$, and a subsequence $u^{k_N}_\alpha, \alpha \in \mathcal{I}$, so that for every α

$$|u^{k_N}_\alpha - \bar{u}_\alpha|_H \to 0 \text{ as } N \to \infty.$$

By Fatou,

$$\begin{aligned}\sum_\alpha \frac{r^\alpha}{\alpha!}\Big(\|\bar{u}_\alpha\|^2_H\Big) &\leq \sup_k \sum_\alpha \frac{r^\alpha}{\alpha!}\Big(|u^k_\alpha|^2_H\Big) \\ &\leq \sup_k \sum_\alpha \frac{r^\alpha}{\alpha!}\Big(\|u^k_\alpha\|^2_V\Big)\end{aligned}$$

and $\bar{u} = \sum_\alpha \bar{u}_\alpha \xi_\alpha \in \mathcal{S}_{-1,-q}(H)$. The convergence in $\mathcal{D}'(H)$ will follow easily from the fact that V is compactly embedded in H. Let $\mathcal{J}_N = \{\alpha \in \mathcal{J} : |\alpha| \leq N, \text{ and } \alpha_i = 0 \text{ for } i > N\}$. Since $\sup_k \|u^k_0\|_V < \infty$, there exists a subsequence $\{k^0_j\}^\infty_{j=1}$ such that $\|u^k_0 - \bar{u}_0\|_H \to 0$ for some $\bar{u}_0 \in H$. Iteratively, for each N, there exists further subsequences $\{k^N_j\}^\infty_{j=1} \subset \{k^{N-1}_j\}^\infty_{j=1}$ such that for every $\alpha \in \mathcal{J}_N$,

$$\|u^k_\alpha - \bar{u}_\alpha\|_H \to 0$$

for some $\bar{u}_\alpha \in H$. In particular, for each N, we can find j_N such that

$$\|u^{k^N_{j_N}}_\alpha - \bar{u}_\alpha\|_H \leq N^{-1}, \quad \text{for all } \alpha \in \mathcal{J}_N.$$

Consequently, choose the subsequence $\tilde{k}_N := k^N_{j_N}$ and we have found the limit $\bar{u} = \sum_\alpha \bar{u}_\alpha \xi_\alpha$. It follows that $u^{\tilde{k}_N} \to \bar{u}$ in $\mathcal{D}'(H)$. Now suppose $\bar{u} \in \mathcal{S}_{-1,-q}(H)$. Let $\varepsilon > 0$ be arbitrary. For any N,

$$\|u^{\tilde{k}_N} - \bar{u}\|^2_{-1,-q;H} = \sum_{\alpha \in \mathcal{J}_N} \frac{r^\alpha}{\alpha!}\|u^{\tilde{k}_N} - \bar{u}\|^2_H + \sum_{\alpha \notin \mathcal{J}_N} \frac{r^\alpha}{\alpha!}\|u^{\tilde{k}_N} - \bar{u}\|^2_H = (I) + (II)$$

By our special choice of $\tilde{k}_N$, there exists N_I such that

$$(I) \leq \sum_{\alpha \in \mathcal{J}_N} \frac{r^\alpha}{\alpha!}N^{-2} < \frac{\varepsilon}{2} \quad \text{whenever } N > N_I.$$

From the hypothesis of the lemma, there exists N_{II} such that

$$(II) \leq 2\sum_{\alpha \notin \mathcal{J}_N} \frac{r^\alpha}{\alpha!}\Big(\sup_k \|u^k\|^2_V\Big) + 2\sum_{\alpha \notin \mathcal{J}_N} \frac{r^\alpha}{\alpha!}\|\bar{u}\|^2_H < \frac{\varepsilon}{2} \quad \text{whenever } N > N_{II}.$$

Thus, $\|u^{\tilde{k}_N} - \bar{u}\|^2_{-1,-q;H} < \varepsilon$ whenever $N > \max\{N_I, N_{II}\}$. □

The hypothesis in Lemma 11 is stronger than requiring $u^k \in l^\infty(\mathcal{S}_{-1,-q}(V))$, thus it is a weaker statement of what might be construed as a compact embedding result for Kondratiev spaces. It is not shown whether $\mathcal{S}_{-1,-q}(V)$ is compactly embedded in $\mathcal{S}_{-1,-q}(H)$. Nonetheless, it is sufficient for our purposes.

Corollary 12. *Let $d = 2$. Assume the hypotheses of Propositions 6 and 9(ii). Then, for the solutions $u(t)$ and $\bar{u}$ of* (4), (5), *we have that*

$$u(t) \longrightarrow \bar{u} \quad \text{in } \mathcal{S}_{-1,-q}(H), \ \text{as } t \to \infty,$$

for $q > \max\{q_0, q_2\}$, where q_0, q_2 are the numbers from Propositions 6, 9.

Proof. In the proof of Proposition 9, we have in fact shown that $u(t)$ belongs to the space $\mathcal{S}_{-1,-q}(L^\infty([0,\infty); V))$ Taking any sequence of times, $t_k \to \infty$, the sequence $\{u(t_k)\}$ satisfies the hypothesis of Lemma 11. So, there exists a subsequence of $u(t_k)$ converging in $\mathcal{S}_{-1,-q}(H)$ to $\bar{u}$. This is true for any sequence $\{t_k\}$, hence $u(t) \longrightarrow \bar{u}$ in $\mathcal{S}_{-1,-q}(H)$ as $t \to \infty$. □

6 Finite Approximation by Wiener Chaos Expansions

In this section, we study the accuracy of the Galerkin approximation of the solutions of the unbiased stochastic Navier-Stokes equations. The goal is to quantify the convergence rate of approximate solutions obtained from a finite truncation of the Wiener chaos expansion, where the convergence is in a suitable Kondratiev space. In relation to being a numerical approximation, quantifying the truncation error is the first step towards understanding the error from the full discretization of the unbiased stochastic Navier-Stokes equation.

In what follows, we will consider the truncation error estimates for the steady solution $\bar{u}$. Recall the estimate (16) for $|\Delta \bar{u}|$: for $r_\alpha^2 = \frac{(2\mathbb{N})^{-q\alpha}}{\alpha!}$, with $q > q_0$, we have

$$r_\alpha^2 |\Delta \bar{u}_\alpha|^2 \le \mathcal{C}_{|\alpha|-1}^2 \binom{|\alpha|}{\alpha} (2\mathbb{N})^{(1-q)\alpha} B_0^{-2} (B_0 K)^{2|\alpha|}.$$

This estimate arose from the method of rescaling via Catalan numbers, and will be the estimate we use for the convergence analysis. For the time-dependent equation, similar analysis can be performed using the analogous Catalan rescaled estimate, and will not be shown.

Let $\mathcal{J}_{M,P} = \{\alpha : |\alpha| \le P, \dim(\alpha) \le M\}$, where M, P may take value ∞. The projection of $\bar{u}$ into $\text{span}\{\xi_\alpha, \alpha \in \mathcal{J}_{M,P}\}$ is $\bar{u}^{M,P} = \sum_{\alpha \in \mathcal{J}_{M,P}} \bar{u}_\alpha \xi_\alpha$.

Then the error $e = \bar{u} - \bar{u}^{M,P}$ can be written as

$$
\begin{aligned}
|\Delta e|^2 &= \sum_{\alpha \in \mathcal{J} \setminus \mathcal{J}_{M,P}} r_\alpha^2 |\Delta \bar{u}_\alpha|^2 \\
&= \sum_{|\alpha|=P+1}^{\infty} r_\alpha^2 |\Delta \bar{u}_\alpha|^2 + \sum_{\{|\alpha| \le P,\, |\alpha_{\le M}| < |\alpha|\}} r_\alpha^2 |\Delta \bar{u}_\alpha|^2 \\
&= \underbrace{\sum_{|\alpha|=P+1}^{\infty} r_\alpha^2 |\Delta \bar{u}_\alpha|^2}_{(IV)} + \underbrace{\sum_{|\alpha|=1}^{P} \underbrace{\sum_{i=0}^{|\alpha|-1} \underbrace{\sum_{|\alpha_{\le M}|=i} r_\alpha^2 |\Delta \bar{u}_\alpha|^2}_{(I)}}_{(II)}}_{(III)}
\end{aligned}
$$

We define the following values

$$
\hat{Q} := 2^{1-q} B_0^2 K^2 \sum_{i=1}^{\infty} i^{1-q},
$$

$$
\hat{Q}_{\le M} := 2^{1-q} B_0 K^2 \sum_{i=1}^{M} i^{1-q}, \qquad \hat{Q}_{>M} := 2^{1-q} B_0 K^2 \sum_{i=M+1}^{\infty} i^{1-q}.
$$

In particular, the term $\hat{Q}_{>M}$ decays on the order of M^{2-q}.

We proceed to estimate the terms (I)-(IV), by similar computations to Wan et al. For fixed $1 \le p \le P$, $|\alpha| = p$, and fixed $i < p$,

$$
\begin{aligned}
(I) &\le \mathcal{C}_{p-1}^2 B_0^{-2} \sum_{|\alpha_{\le M}|=i,\, |\alpha_{>M}|=p-i} \binom{|\alpha|}{\alpha} (2\mathbb{N})^{(1-q)\alpha} (B_0 K)^{2p} \\
&= \mathcal{C}_{p-1}^2 B_0^{-2} \binom{p}{i} \hat{Q}_{\le M}^i \hat{Q}_{>M}^{p-i}
\end{aligned}
$$

Then for fixed $1 \le p \le P$, $|\alpha| = p$,

$$
\begin{aligned}
(II) = \sum_{i=0}^{p-1} (I) &\le \mathcal{C}_{p-1}^2 B_0^{-2} \sum_{i=0}^{p-1} \binom{p}{i} \hat{Q}_{\le M}^i \hat{Q}_{>M}^{p-i} \\
&= \mathcal{C}_{p-1}^2 B_0^{-2} (\hat{Q}^p - \hat{Q}_{\le M}^p)
\end{aligned}
$$

And finally,

$$
\begin{aligned}
(III) = \sum_{|\alpha|=1}^{P} (II) &\le \sum_{p=1}^{P} \mathcal{C}_{p-1}^2 B_0^{-2} (\hat{Q}^p - \hat{Q}_{\le M}^p) \\
&\le \frac{1}{B_0^2} (\hat{Q} - \hat{Q}_{\le M}) + \frac{1}{16\pi B_0^2} \sum_{p=2}^{P} \frac{2^{4p}}{(p-1)^3} (\hat{Q}^p - \hat{Q}_{\le M}^p)
\end{aligned}
$$

Since $\hat{Q}^p - \hat{Q}^p_{\le M} \le p\hat{Q}^{p-1}(\hat{Q} - \hat{Q}_{\le M})$ by the mean value theorem for $x \mapsto x^p$,

$$
\begin{aligned}
(III) &\le \frac{1}{B_0^2}\hat{Q}_{>M} + \frac{1}{16\pi B_0^2}\hat{Q}_{>M}\sum_{p=2}^{P}\frac{p2^{4p}\hat{Q}^{p-1}}{(p-1)^3} \\
&\le \frac{1}{B_0^2}\hat{Q}_{>M} + \frac{1}{\pi B_0^2}\hat{Q}_{>M}\sum_{p=2}^{P}\frac{p(2^4\hat{Q})^{p-1}}{(p-1)^3} \\
&\le \frac{1}{B_0^2}\hat{Q}_{>M}\sum_{p=0}^{P-1}(2^4\hat{Q})^p
\end{aligned}
$$

To estimate Term (IV),

$$
\begin{aligned}
(IV) &\le \sum_{p=P+1}^{\infty}\sum_{|\alpha|=p}\mathcal{C}_{p-1}^2 B_0^{-2}(2^{1-q}B_0^2K^2)^p\binom{|\alpha|}{\alpha}(\mathbb{N})^{(1-q)\alpha} \\
&= B_0^{-2}\sum_{p=P+1}^{\infty}\mathcal{C}_{p-1}^2(2^{1-q}B_0^2K^2)^p\Big(\sum_{i\ge1}i^{1-q}\Big)^p \\
&\le B_0^{-2}\sum_{p=P+1}^{\infty}\frac{2^{4(p-1)}}{\pi(p-1)^3}\hat{Q}^p \le \frac{1}{16\pi B_0^2}\frac{(2^4\hat{Q})^{P+1}}{1-2^4\hat{Q}}
\end{aligned}
$$

Putting the estimates together,

$$
|\Delta e|^2 \le C\big((2^4\hat{Q})^{P+1} + M^{2-q}\big)
$$

Notice the condition $2^4\hat{Q} < 1$ in (17), which ensured summability of the weighted norm of the solution, is of course a required assumption for the convergence of the error estimate.

A The Catalan numbers method

The Catalan numbers method was used in the preceding sections to derive estimates for the norms in Kondratiev spaces. This method was previously described in [9, 13], but we restate it here just for the record.

Lemma 13. *Suppose L_α are a collection of positive real numbers indexed by $\alpha \in \mathcal{J}$, satisfying*

$$
L_\alpha \le B\sum_{0<\gamma<\alpha}L_\gamma L_{\alpha-\gamma}.
$$

Then

$$
L_\alpha \le \mathcal{C}_{|\alpha|-1}B^{|\alpha|-1}\binom{|\alpha|}{\alpha}\prod_i L_{\epsilon_i}^{\alpha_i}
$$

for all α, where $\mathcal{C}_n$ are the Catalan numbers.

Proof. The result is clearly true for $\alpha = \epsilon_i$. By induction, let $|\alpha| \geq 2$, and suppose the result is true for all $\gamma < \alpha$. Then

$$\begin{aligned}
L_\alpha &\leq \sum_{0<\gamma<\alpha} \mathcal{C}_{|\gamma|-1}\mathcal{C}_{|\alpha-\gamma|-1} B^{|\alpha|-1} \binom{|\gamma|}{\gamma}\binom{|\alpha-\gamma|}{\alpha-\gamma}\Big(\prod_i L_{\epsilon_i}^{\alpha_i}\Big) \\
&= \sum_{n=1}^{|\alpha|-1} \sum_{0<\gamma<\alpha|\gamma|=n} \mathcal{C}_{n-1}\mathcal{C}_{|\alpha|-n-1} \frac{n!}{\gamma!}\frac{(|\alpha|-n)!}{(\alpha-\gamma)!} B^{|\alpha|-1}\Big(\prod_i L_{\epsilon_i}^{\alpha_i}\Big) \\
&= \sum_{n=1}^{|\alpha|-1} \mathcal{C}_{n-1}\mathcal{C}_{|\alpha|-n-1} \underbrace{\sum_{0<\gamma<\alpha|\gamma|=n} \binom{|\alpha|}{n}^{-1}\binom{\alpha}{\gamma}}_{(*)} \frac{|\alpha|!}{\alpha!} B^{|\alpha|-1}\Big(\prod_i L_{\epsilon_i}^{\alpha_i}\Big)
\end{aligned}$$

We claim that $(*) = 1$, for any α and any $n < |\alpha|$. Indeed, let $K_\alpha = (k_1, \ldots, k_{|\alpha|})$ be the characteristic set of α. Each summand in $(*)$ is

$$\left(\frac{|\alpha|!}{\alpha!}\right)^{-1} \frac{n!}{\gamma!}\frac{(|\alpha|-n)!}{(\alpha-\gamma)!}$$

The term $\frac{|\alpha|!}{\alpha!}$ is the number of distinct permutations of K_α, whereas the term $\frac{n!}{\gamma!}\frac{(|\alpha|-n)!}{(\alpha-\gamma)!}$ is the number of distinct permutations of K_α where only $K_\gamma, K_{\alpha-\gamma}$ has been permuted within themselves. On the other hand, the latter term is the number of distinct permutations of K_α corresponding to a particular γ, where the correspondence of a permutation of K_α to a $\gamma \in \{\gamma : 0 < \gamma < \alpha, |\gamma| = n\}$ can be made by taking K_γ to be the first n entries of that permutation of K_α. Thus, each summand in $(*)$ is the relative frequency of γ over all distinct permutations of K_α, and hence their sum must equal 1. To complete the proof, using the recursion property of the Catalan numbers,

$$\begin{aligned}
L_\alpha &\leq \sum_{n=1}^{|\alpha|-1} \mathcal{C}_{n-1}\mathcal{C}_{|\alpha|-n-1}\left(\frac{|\alpha|!}{\alpha!}\right) B^{|\alpha|-1}\prod_i L_{\epsilon_i}^{\alpha_i} \\
&= \mathcal{C}_{|\alpha|-1}\left(\frac{|\alpha|!}{\alpha!}\right) B^{|\alpha|-1}\prod_i L_{\epsilon_i}^{\alpha_i}.
\end{aligned}$$

□

If L_α satisfies the hypothesis of Lemma 13, and if $L_{\epsilon_i} \leq K$ for all i, then for

$r = (2\mathbb{N})^{-q}$,

$$\begin{aligned}\sum_{|\alpha|=n} r^{\alpha} L_{\alpha}^{2} &\leq \sum_{|\alpha|=n} \mathcal{C}_{n-1}^{2} B^{2(|\alpha|-1)} K^{2|\alpha|} \binom{|\alpha|}{\alpha} (2\mathbb{N})^{(1-q)\alpha} \\ &= B^{-2} \mathcal{C}_{n-1}^{2} \big(B^{2} K^{2} 2^{1-q}\big)^{n} \sum_{|\alpha|=n} \binom{|\alpha|}{\alpha} \mathbb{N}^{(1-q)\alpha} \\ &= B^{-2} \mathcal{C}_{n-1}^{2} \big(B^{2} K^{2} 2^{1-q}\big)^{n} \Big(\sum_{i=1}^{\infty} i^{(1-q)}\Big)^{n}\end{aligned}$$

For large n, the Catalan numbers behave asymptotically like $\mathcal{C}_n \sim \frac{2^{2n}}{\sqrt{\pi} n^{3/2}}$. Hence, the sum $\sum_{n=0}^{\infty} \sum_{|\alpha|=n} r^{\alpha} L_{\alpha}^{2}$ converges for any $q > \max\{q_0, 2\}$, where q_0 satisfies

$$B^2 K^2 2^{5-q_0} \sum_{i=1}^{\infty} i^{(1-q_0)} = 1.$$

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