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The Dual Role of Convection in 3D Navier-Stokes Equations

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Abstract

We investigate the dual role of convection on the large time behavior of the 3D incompressible Navier-Stokes equations. On the one hand, convection is responsible for generating small scales dynamically. On the other hand, convection may play a stabilizing role in potentially depleting nonlinear vortex stretching for certain flow geometry. Our study is centered around a 3D model that was recently proposed by Hou and Lei in [23] for axisymmetric 3D incompressible Navier-Stokes equations with swirl. This model is derived by neglecting the convection term from the reformulated Navier-Stokes equations and shares many properties with the 3D incompressible Navier-Stokes equations. In this paper, we review some of the recent progress in studying the singularity formation of this 3D model and how convection may destroy the mechanism that leads to singularity formation in the 3D model.

Key words: Finite time singularities, nonlinear nonlocal system, incompressible Navier-Stokes equations.

1.1 Introduction

Whether the 3D incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial data with finite energy is one of the seven Millennium problems posted by the Clay Mathematical Institute [16]. This

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problem is challenging because the vortex stretching nonlinearity is supercritical for the 3D Navier-Stokes equation. Conventional functional analysis based on energy type estimates fails to provide a definite answer to this problem. Global regularity results are obtained only under certain smallness assumptions on the initial data or the solution itself. Due to the incompressibility condition, the convection term seems to be neutrally stable if one tries to estimate the L^p $(1 < p \leq \infty)$ norm of the vorticity field. As a result, the main effort has been to use the diffusion term to control the nonlinear vortex stretching term by diffusion without making use of the convection term explicitly.

In [23], Hou and Lei investigated the role of convection by constructing a new 3D model for axisymmetric 3D incompressible Navier-Stokes equations with swirl. The 3D model is derived based on the reformulated Navier-Stokes equation given below

$$\partial_t u_1 + u^r (u_1)_r + u^z (u_1)_z = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)u_1 + 2\partial_z \psi_1 u_1, \tag{1.1}$$

$$\partial_t \omega_1 + u^r (\omega_1)_r + u^z (\omega_1)_z = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\omega_1 + \partial_z((u_1)^2), \tag{1.2}$$

$$-(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\psi_1 = \omega_1, \tag{1.3}$$

where $u_1 = u^\theta/r$, $\omega_1 = \omega^\theta/r$, $\psi_1 = \psi^\theta/r$. Here u^θ, ω^θ ψ^θ are the angular velocity, angular vorticity and angular stream-function, respectively. The radial velocity u^r and the axial velocity u^z are given by $u^r = -r(\psi_1)_z$ and $u^z = (r^2\psi_1)_r/r$. The 3D model of Hou-Lei is obtained by simply dropping the convection term in the reformulated Navier-Stokes equations (1.1)–(1.3), which is given by the following nonlinear nonlocal system

$$\partial_t u_1 = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)u_1 + 2\partial_z \psi_1 u_1, \tag{1.4}$$

$$\partial_t \omega_1 = \nu(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\omega_1 + \partial_z((u_1)^2), \tag{1.5}$$

$$-(\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2)\psi_1 = \omega_1. \tag{1.6}$$

Note that (1.4)–(1.6) is already a closed system. This model preserves almost all the properties of the full 3D Navier-Stokes equations, including the energy identity for smooth solutions of the 3D model, the non-blowup criterion of Beale-Kato-Majda type [1], the non-blowup criterion of Prodi-Serrin type [34, 35], and the partial regularity result [24] which is an analogue of the well-known Caffarelli-Kohn-Nirenberg theory [2] for the full Navier-Stokes equations.

One of the main findings of [23] is that the 3D model (1.4)–(1.6) has a very different behavior from that of the full Navier-Stokes equations although it shares many properties with those of the Navier-Stokes equations. In [23], the authors presented numerical evidence which supports the notion that the 3D model may develop a potential finite time singularity. However, the Navier-Stokes equations with the same initial data seem to have a completely different behavior.

In a recent paper [26], we rigorously proved the finite time singularity formation of this 3D model for a class of initial boundary value problems with smooth initial data of finite energy. The analysis of the finite time singularity for the 3D model was rather subtle. Currently, there is no systematic method of analysis available to study singularity formation of a nonlinear nonlocal system. In [26], we introduced an effective method of analysis to study singularity formation of this nonlinear nonlocal multi-dimensional system. The initial boundary value problem considered in [26] uses a mixed Dirichlet Robin boundary condition. The local well-posedness of this mixed initial boundary problem is nontrivial. In this paper, we provide a rigorous proof of the local well-posedness of the 3D model with this mixed Dirichlet Robin boundary problem.

We remark that formation of singularities for various model equations for the 3D Euler equations or the surface quasi-geostrophic equation has been investigated by Constantin-Lax-Majda [9], Constantin [5], DeGregorio [12, 13], Cordoba-Cordoba-Fontelos [8], Chae-Cordoba-Cordoba-Fontelos [4], and Li-Rodrigo [30]. In a recent paper related to the present one, Hou, Li, Shi, Wang and Yu [25] have proved the finite time singularity of a one-dimensional nonlinear nonlocal system:

$$u_t = 2uv, \quad v_t = H(u^2), \tag{1.7}$$

where H is the Hilbert transform. This is a simplified system of the original 3D model along the symmetry axis. Here v plays the same role as ψ_z. The singularity of this nonlocal system is remarkably similar to that of the 3D model.

The work of Hou and Lei [23] was motivated by the recent study of Hou and Li in [22], where the authors studied the stabilizing effect of convection via a new 1D model. They proved dynamic stability of this 1D model by exploiting the stabilizing effect of convection and constructing a Lyapunov function. A surprising result from their study is that there is a beautiful cancellation between the convection term and the nonlinear stretching term when one constructs an appropriate Lyapunov function. This Lyapunov estimate gives rise to a global pointwise estimate for the derivatives of the vorticity in their model.

We would like to emphasize that the study of [22, 23] is based on a reduced model for certain flow geometry. It is premature to conclude that the convection term could lead to depletion of singularity of the Navier-Stokes equations in general. Convection term may act as a destabilizing term for a different flow geometry. A main message from this line of study is that the convection term carries important physical information. We need to take the convection term into consideration in an essential way in our analysis of the Navier-Stokes equations.

The rest of the paper is organized as follows. In Section 2, we discuss the role of convection from the Lagrangian perspective and present some numerical evidence that the local geometric regularity of the vortex lines may deplete the nonlinear vortex stretching dynamically. In Section 3, we investigate the role of convection by studying the potential singular behavior of the 3D model which neglects convection in the reformulated Navier-Stokes equation. We present some theoretical results on finite time singularity formation of the 3D model in Section 4. Finally we present the analysis of the local well-posedness of the 3D model with the mixed Dirichlet Robin boundary condition in Section 5.

1.2 The role of convection from the Lagrangian perspective

Due to the supercritical nature of the nonlinearity of the 3D Navier-Stokes equations, the 3D Navier-Stokes equations with large initial data are convection dominated. Thus the understanding of whether the corresponding 3D Euler equations would develop a finite time blowup could shed useful light on the global regularity of the Navier-Stokes equations.

We consider the 3D Euler equations in the vorticity form. We note that we can rewrite the vorticity equation in a commutator form (or a Lie derivative) as follows:

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = 0. \tag{1.8}$$

Through this commutator formulation, we can see that the convection term may have the potential to dynamically cancel or weaken the vortex stretching term under certain geometric regularity conditions.

Another way to realize the importance of convection is to use the Lagrangian formulation of the vorticity equation. When we consider the two terms together, we preserve the Lagrangian structure of the solution [32]:

$$\omega(X(\alpha, t), t) = X_\alpha(\alpha, t)\omega_0(\alpha), \tag{1.9}$$

where $X_\alpha = \frac{\partial X}{\partial \alpha}$ and $X(\alpha, t)$ is the flow map:

$$\frac{dX}{dt}(\alpha, t) = \mathbf{u}(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha. \tag{1.10}$$

We believe that (1.9) is an important signature of the 3D incompressible Euler equation. An immediate consequence of (1.9) is that vorticity increases in time only through the dynamic deformation of the Lagrangian flow map, which is volume preserving, i.e. $\det(X_\alpha(\alpha, t)) \equiv 1$. Thus, as vorticity increases dynamically, the parallelepiped spanned by the three vectors, $(X_{\alpha_1}, X_{\alpha_2}, X_{\alpha_3})$, will experience severe deformation and become flattened dynamically. Such deformation tends to weaken the nonlinearity of vortex stretching dynamically.

1.2.1 A Brief Review

In this subsection, we give a brief review of some of the theoretical and computational studies of the 3D Euler equation. Due to the formal quadratic nonlinearity in vortex stretching, classical solutions of the 3D Euler equation are known to exist only for a short time [32]. One of the most well-known non-blowup results on the 3D Euler equations is due to Beale-Kato-Majda [1] who showed that the solution of the 3D Euler equations blows up at T if and only if $\int_0^T \|\boldsymbol{\omega}\|_\infty(t)\, dt = \infty$, where $\boldsymbol{\omega}$ is vorticity.

There have been some interesting recent theoretical developments. In particular, Constantin-Fefferman-Majda [6] showed that local geometric regularity of the unit vorticity vector can lead to depletion of the vortex stretching. Denote $\boldsymbol{\xi} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ as the unit vorticity vector and $\mathbf{u}$ the velocity field. Roughly speaking, Constantin-Fefferman-Majda proved that if (1) $\|\mathbf{u}\|_\infty$ is bounded in a $O(1)$ region containing the maximum vorticity, and (2) $\int_0^t \|\nabla\boldsymbol{\xi}\|_\infty^2 d\tau$ is uniformly bounded for $t < T$, then the solution of the 3D Euler equations remains regular up to $t = T$.

There has been considerable effort put into computing a finite time singularity of the 3D Euler equation. The finite time collapse of two anti-parallel vortex tubes by R. Kerr [28, 29] has received a lot of attention. With resolution of order $512\times 256\times 192$, his computations showed that the maximum vorticity blows up like $O((T-t)^{-1})$ with $T = 18.9$. In his subsequent paper [29], Kerr applied a high wave number filter to the data obtained in his original computations to "remove the noise that masked the structures in earlier graphics" presented in [28]. The singularity time was revised to $T = 18.7$. Kerr's blowup scenario is consistent with the Beale-Kato-Majda non-blowup criterion [1] and the Constantin-Fefferman-Majda non-blowup criterion [6]. It is worth noting that there is still a considerable gap between the predicted singularity time $T = 18.7$ and the final time $t = 17$ of Kerr's original computations which he used as the primary evidence for the finite time singularity.

1.2.2 The local non-blowup criteria of Deng-Hou-Yu [10, 11]

Motivated by the result of [6], Deng, Hou and Yu [10] have obtained a sharper non-blowup condition which uses a Lagrangian approach and the very localized information of the vortex lines. More specifically, they assume that at each time t there exists some vortex line segment L_t on which the local maximum vorticity is comparable to the global maximum vorticity. Further, they denote $L(t)$ as the arclength of L_t, $\mathbf{n}$ the unit normal vector of L_t, and κ the curvature of L_t. If (1) $\max_{L_t}(|\mathbf{u}\cdot\boldsymbol{\xi}| + |\mathbf{u}\cdot\mathbf{n}|) \le C_U(T-t)^{-A}$ with $A < 1$, and (2) $C_L(T-t)^B \le L(t) \le C_0/\max_{L_t}(|\kappa|,\ |\nabla\cdot\boldsymbol{\xi}|)$ for $0 \le t < T$, then they show that the solution of the 3D Euler equations remains regular up to $t = T$ provided that $A + B < 1$.

In Kerr's computations, the first condition of Deng-Hou-Yu's non-blowup criterion is satisfied with $A = 1/2$ if we use $\|\mathbf{u}\|_\infty \le C(T-t)^{-1/2}$ as alleged in [29]. Kerr's computations suggested that κ and $\nabla\cdot\boldsymbol{\xi}$ are bounded by $O((T-t)^{-1/2})$ in the inner region of size $(T-t)^{1/2}\times(T-t)^{1/2}\times(T-t)$ [29]. Moreover, the length of the vortex tube in the inner region is of order $(T-t)^{1/2}$. If we choose a vortex line segment of length $(T-t)^{1/2}$ (i.e. $B = 1/2$), then the second condition is satisfied. However, this would violate the condition $A + B < 1$. Thus Kerr's computations fall into the critical case of the non-blowup criterion of [10]. In a subsequent paper [11], Deng-Hou-Yu improved the non-blowup condition to include the critical case $A + B = 1$, with some additional constraint on the scaling constants.

We remark that in a recent paper [27], Hou and Shi introduced a different method of analysis to study the non-blowup criterion of the 3D Euler and the SQG model. By performing estimates on the integral of the absolute value of vorticity along a local vortex line segment, they established a relatively sharp dynamic growth estimate of maximum vorticity under some mild assumptions on the local geometric regularity of the vorticity vector. Under some additional assumption on the vorticity field, which seems to be consistent with the computational results of [19], they proved that the maximum vorticity can not grow faster than double exponential in time. This analysis extends to some extent the earlier results by Cordoba-Fefferman [7] and Deng-Hou-Yu [10, 11].

1.2.3 Computing potentially singular solutions using pseudo-spectral methods

It is an extremely challenging task to compute a potential Euler singularity numerically. First of all, it requires a tremendous amount of numerical resolution in order to capture the nearly singular behavior of the Euler equations. Secondly, one must perform a careful convergence study. It is risky to interpret the blowup of an under-resolved computation as evidence of finite time singularities for the 3D Euler equations. Thirdly, we need to validate the asymptotic blowup rate, i.e. is the blowup rate $\|\omega\|_{L^\infty} \approx \frac{C}{(T-t)^\alpha}$ asymptotically valid as $t \to T$? If a numerical solution is well resolved only up to T_0 and there is still an $O(1)$ gap between T_0 and the predicted singularity time T, then one can not apply the Beale-Kato-Majda criterion [1] to this extrapolated singularity since the most significant contribution to $\int_0^T \|\omega(t)\|_{L^\infty} dt$ comes from the time interval $[T_0,\ T]$. But ironically there is no accuracy in the extrapolated solution in this time interval if $(T - T_0) = O(1)$. Finally, the blowup rate of the numerical solution must be consistent with other non-blowup criteria [6, 10, 11]. Guidance from analysis is clearly needed.

In [19], Hou and Li performed high resolution computations of the 3D Euler equations using the two-antiparallel vortex tubes initial data. They used the same initial condition whose analytic formula was given by [28]. They used two different pseudo-spectral methods. The first pseudo-spectral method used the standard 2/3 de-aliasing rule to remove the aliasing error. For the second pseudo-spectral method, they used a novel 36th order Fourier smoothing to remove the aliasing error. In order to perform a careful resolution study, they used a sequence of resolutions: $768 \times 512 \times 1536$, $1024 \times 768 \times 2048$ and $1536 \times 1024 \times 3072$ in their computations. They computed the solution up to $t = 19$, beyond the alleged singularity time $T = 18.7$ by Kerr [29].

We first illustrate the dynamic evolution of the vortex tubes. Figure 1.2 describes the isosurface of the 3D vortex tubes at $t = 0$ and $t = 6$, respectively. As we can see, the two initial vortex tubes are very smooth and relatively symmetric. As time evolves, the two vortex tubes approach each other and become flattened dynamically. By time $t = 6$ there is already a significant flattening near the center of the tubes. In Figure 1.3 we plot the local 3D vortex structure of the upper vortex tube at $t = 17$. By this time the vortex tube has turned into a thin vortex sheet with rapidly decreasing thickness. We observe that the vortex lines become relatively straight and the vortex sheet rolls up near the left edge of the sheet.

We now perform a convergence study for the two numerical methods using a sequence of resolutions. For the Fourier smoothing method, we use the resolutions $768\times512\times1536$, $1024\times768\times2048$, and $1536\times1024\times3072$, respectively, whereas the 2/3 de-aliasing method uses the resolutions $512 \times 384 \times 1024$, $768 \times 512 \times 1536$ and $1024 \times 768 \times 2048$, respectively.

In Figure 1.1 we compare the Fourier spectra of the energy obtained by using the 2/3 de-aliasing method with those obtained by the Fourier smoothing method. For a fixed resolution $1024\times768\times2048$, the Fourier spectra obtained

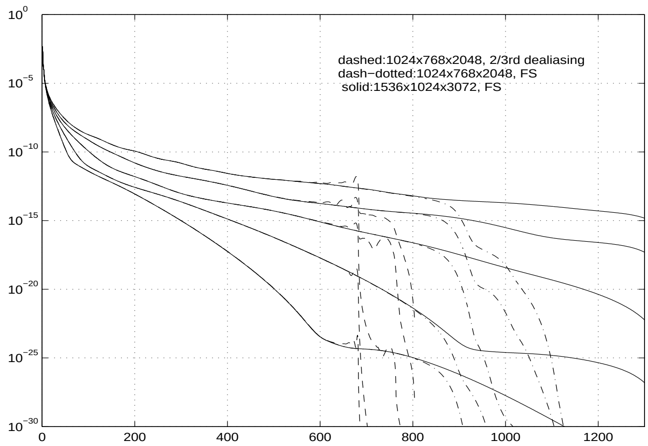


Figure 1.1 The energy spectra versus wave numbers. The dashed lines and dashed-dotted lines are the energy spectra with the resolution $1024 \times 768 \times 2048$ using the 2/3 de-aliasing rule and Fourier smoothing, respectively. The times for the spectra lines are at $t = 15, 16, 17, 18, 19$, respectively.

by the Fourier smoothing method retain more effective Fourier modes than those obtained by the $2/3$ de-aliasing method and does not give the spurious oscillations in the Fourier spectra. In comparison, the Fourier spectra obtained by the $2/3$ de-aliasing method produce some spurious oscillations near the $2/3$ cut-off point. It is important to emphasize that the Fourier smoothing method conserves the total energy extremely well. More studies including the convergence of the enstrophy spectra can be found in [19, 20, 21].

To gain more understanding of the nature of the dynamic growth in vorticity, we examine the degree of nonlinearity in the vortex stretching term. In Figure 1.4 we plot the quantity $\|\xi \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}\|_\infty$ as a function of time. If the maximum vorticity indeed blew up like $O((T - t)^{-1})$, as alleged in [28], this quantity should have grown quadratically as a function of maximum vorticity. We find that there is tremendous cancellation in this vortex stretching term. Its growth rate is bounded by $C\|\vec{\omega}\|_\infty \log(\|\vec{\omega}\|_\infty)$, see Figure 1.4. It is easy to show that if $\|\xi \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}\|_\infty \leq C\|\vec{\omega}\|_\infty \log(\|\vec{\omega}\|_\infty)$, then the maximum vorticity can not grow faster than doubly exponential in time.

In the right plot of Figure 1.4, we plot the double logarithm of the maximum vorticity as a function of time. We observe that the maximum vorticity indeed does not grow faster than doubly exponential in time. We have also examined the growth rate of maximum vorticity by extracting the data from Kerr's paper [28]. We find that $\log(\log(\|\omega\|_\infty))$ basically scales linearly with respect to t

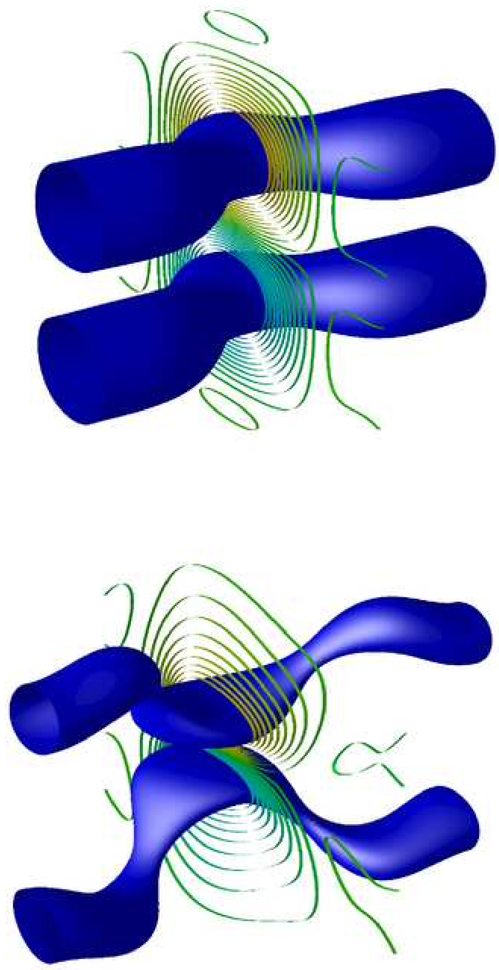


Figure 1.2 The 3D view of the vortex tube for $t = 0$ and $t = 6$. The tube is the isosurface at 60% of the maximum vorticity. The ribbons on the symmetry plane are the contours at other different values.

from $14 \leq t \leq 17.5$ when his computations are still reasonably resolved. This implies that the maximum vorticity up to $t = 17.5$ in Kerr's computations does not grow faster than doubly exponential in time, which is consistent with our conclusion.

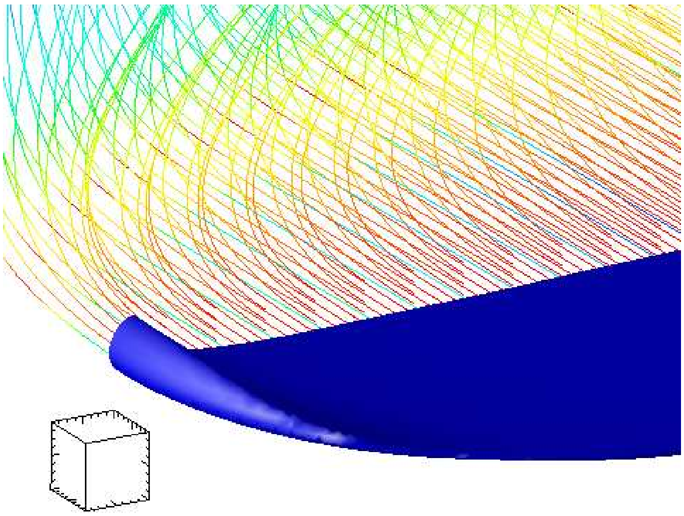


Figure 1.3 The local 3D vortex structures of the upper vortex tube and vortex lines around the maximum vorticity at $t = 17$.

1.3 Numerical evidence of finite time singularity of the 3D model

As we mentioned in the Introduction, the 3D model shares many properties with the full 3D Navier-Stokes equations at the theoretical level. In this section, we will demonstrate that the 3D model without the convection term has a very different behavior from the full Navier-Stokes equation. In particular, we present numerical evidence based on the computations of [23] that seems to suggest that the 3D model develops a potential finite time singularity from smooth initial data with finite energy. However, the mechanism for developing a finite time singularity of the 3D model seems to be destroyed when we add the convection term back to the 3D model. This illustrates the important role played by convection from a different perspective.

By exploiting the axisymmetric geometry of the problem, Hou and Lei obtained a very efficient adaptive solver with effective local resolutions of order 4096^3. More specifically, since the potential singularity must appear along the symmetry axis at $r = 0$, they used the following coordinate transformation along the r-direction to achieve the adaptivity by clustering the grid points near $r = 0$:

$$r = f(\alpha) \equiv \alpha - 0.9 \sin(\pi\alpha)/\pi. \tag{1.11}$$

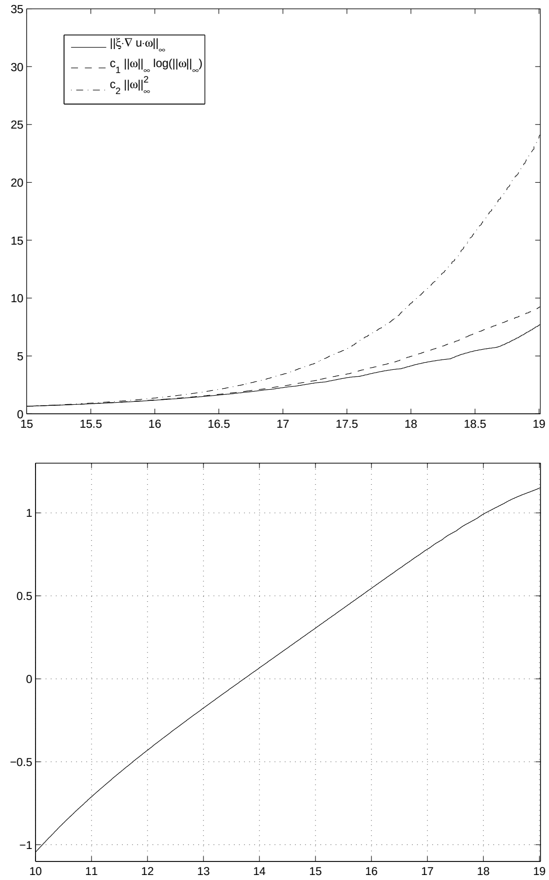


Figure 1.4 Left plot: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. The fact $|\boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}| \le c_1 |\boldsymbol{\omega}| \log |\boldsymbol{\omega}|$ plus $\frac{D}{Dt}|\boldsymbol{\omega}| = \boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}$ implies $|\boldsymbol{\omega}|$ bounded by doubly exponential. Right plot: $\log \log \|\boldsymbol{\omega}\|_\infty$ vs time.

With this level of resolution, they obtained an excellent fit for the asymptotic blowup rate of maximum axial vorticity.

The initial condition we consider in our numerical computations is given by

$$u_1(z, r, 0) = (1 + \sin(4\pi z))(r^2 - 1)^{20}(r^2 - 1.2)^{30}, \tag{1.12}$$

$$\psi_1(z, r, 0) = 0, \tag{1.13}$$

$$\omega_1(z, r, 0) = 0. \tag{1.14}$$

A second order finite difference discretization is used in space, and the classical fourth order Runge-Kutta method is used to discretize in time.

In the following, we present numerical evidence which seems to support the notion that u_1 may develop a potential finite time singularity for the initial condition we consider. In Figure 1.5 we plot the maximum of u_1 in time over the time interval $[0,\ 0.021]$ using the adaptive mesh method with $N_z = 4096$ and $N_r = 400$. The time step is chosen to be $\Delta t = 2.5 \times 10^{-7}$. We observe that $\|u_1\|_\infty$ experiences a very rapid growth in time after $t = 0.02$. In Figure 1.5 (the right plot), we also plot $\log(\log(\|u_1\|_\infty))$ as a function of time. It is clear that $\|u_1\|_\infty$ grows much faster than double exponential in time.

To obtain further evidence for a potential finite time singularity, we study the asymptotic growth rate of $\|u_1\|_\infty$ in time. We look for a finite time singularity of the form:

$$\|u_1\|_\infty \approx \frac{C}{(T-t)^\alpha}. \tag{1.15}$$

We find that the inverse of $\|u_1\|_\infty$ is almost a perfect linear function of time, see Figure 1.6. By using a least square fit of the inverse of $\|u_1\|_\infty$, we find the best fit for α, the potential singularity time T and the constant C. In Figure 1.6 (the left plot), we plot $\|u_1\|_\infty^{-1}$ as a function of time. We can see that the agreement between the computed solution with $N_z \times N_r = 4096 \times 400$ and the fitted solution is almost perfect. In the right box of Figure 1.6, we plot $\|u_1\|_\infty$ computed by our adaptive method against the form fit $C/(T-t)$ with $T = 0.02109$ and $C = 8.20348$. The two curves are almost indistinguishable during the final stage of the computation from $t = 0.018$ to $t = 0.021$. Note that u_1 has the same scaling as the axial vorticity. Thus, the $O(1/(T-t))$ blowup rate of u_1 is consistent with the non-blowup criterion of Beale-Kato-Majda type.

We present the 3D view of u_1 as a function of r and z in Figures 1.7 and 1.8. We note that u_1 is symmetric with respect to $z = 0.375$ and w_1 is anti-symmetric with respect to $z = 0.375$. The support of the solution u_1 in the most singular region is isotropic and appears to be locally self-similar.

To study the dynamic alignment of the vortex stretching term, we plot the solution u_1 on top of $\psi_{1,z}$ along the symmetry axis $r = 0$ at t=0.021 in Figure 1.9. We observe that there is a significant overlap between the support of the maximum of u_1 and that of the maximum of $\psi_{1,z}$. Moreover, the solution u_1 has a strong alignment with $\psi_{1,z}$ near the region of the maximum of u_1. The local alignment between u_1 and $\psi_{1,z}$ induces a strong nonlinearity on the right hand side of the u_1 equation. This strong alignment between u_1 and $\psi_{1,z}$ is the main mechanism for the potential finite time blowup of the 3D model.

It is interesting to see how convection may change the dynamic alignment

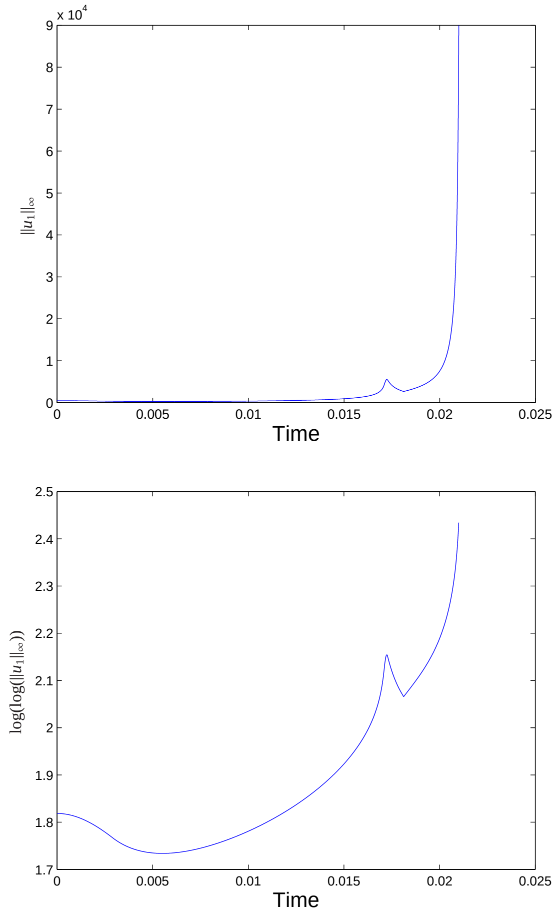
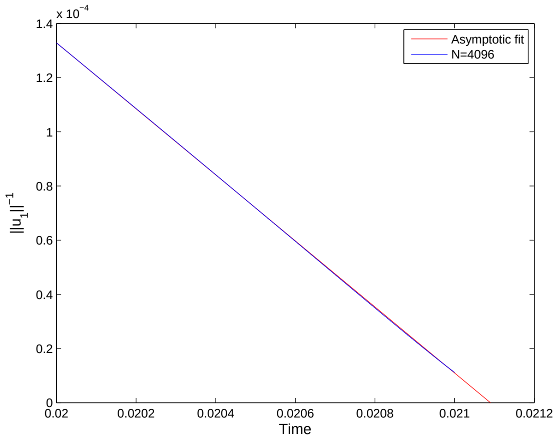


Figure 1.5 Left figure: $\|u_1\|_\infty$ as a function of time over the interval $[0,\ 0.021]$. The right figure: $\log(\log(\|u_1\|_\infty))$ as a function of time over the same interval. The solution is computed by the adaptive mesh with $N_z = 4096,\ N_r = 400, \Delta t = 2.5 \times 10^{-7}, \nu = 0.001$.

of the vortex stretching term in the 3D model. We add the convection term back to the 3D model and use the solution of the 3D model at $t = 0.02$ as the initial condition for the full Navier-Stokes equations. We observe that the local alignment between u_1 and $\psi_{1,z}$ is destroyed for the full Navier-Stokes equations. As a result, the solution becomes defocused and smoother along the symmetry axis, see Figure 1.10. As time evolves, the two focusing centers approach each other. This process creates a strong internal layer orthogonal to the z-axis. The solution forms a jet that moves away from the symmetry axis



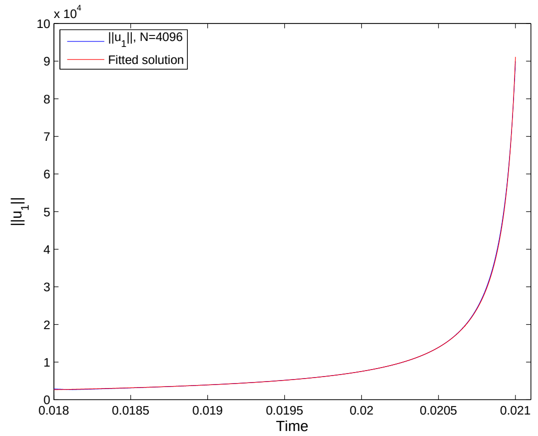


Figure 1.6 The left plot: The inverse of $||u_1||_\infty$ (dark) versus the asymptotic fit (gray) for the viscous model. The right plot: $||u_1||_\infty$ (dark) versus the asymptotic fit (gray). The asymptotic fit is of the form: $||u_1||_\infty^{-1} \approx \frac{(T-t)}{C}$ with $T = 0.02109$ and $C = 8.20348$. The solution is computed by an adaptive mesh with $N_z = 4096$, $N_r = 400$, $\Delta t = 2.5 \times 10^{-7}$. $\nu = 0.001$.

(the z-axis) and generates many interesting vortex structures. By the Caffarelli-Kohn-Nirenberg theory, the singularity of the 3D axisymmetric Navier-Stokes equations must be along the symmetry axis. The fact that the most singular part of the solution moves away from the symmetry axis suggests that the mechanism for generating the finite time singularity of the 3D model has been destroyed by the inclusion of the convection term for this initial condition.

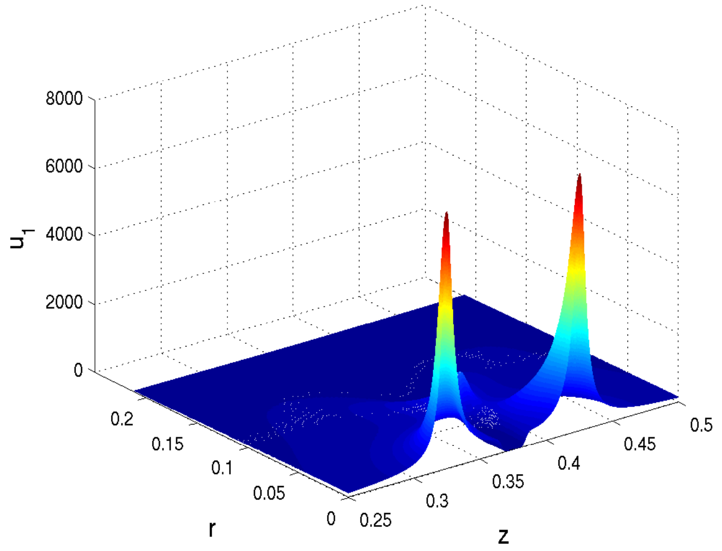


Figure 1.7 The 3D view of u_1 at $t = 0.02$ for the viscous model computed by the adaptive mesh with $N_z = 4096$, $N_r = 400$, $\Delta t = 2.5 \times 10^{-7}$, $\nu = 0.001$.

1.4 Finite time singularities of the 3D model

The numerical evidence of finite time blow-up of the 3D model motivates us to prove finite time singularities of the 3D model rigorously. In a recent paper [26], we developed a new method of analysis and proved rigorously that the 3D model develops finite time singularities for a class of initial boundary value problems with smooth initial data of finite energy. In our analysis, we considered the initial boundary value problem of the generalized 3D model which has the following form (we drop the subscript 1 and substitute (1.6) into (1.5)):

$$u_t = 2u\psi_z, \tag{1.16}$$

$$-\Delta\psi_t = \left(u^2\right)_z, \tag{1.17}$$

where Δ is an n-dimensional Laplace operator with $(\mathbf{x}, z) \equiv (x_1, x_2, ..., x_{n-1}, z)$. Our results apply to any dimension greater than or equal to two ($n \geq 2$). Here we only present our results for $n = 3$. We consider the generalized 3D model in both a bounded domain and in a semi-infinite domain with a mixed Dirichlet Robin boundary condition.

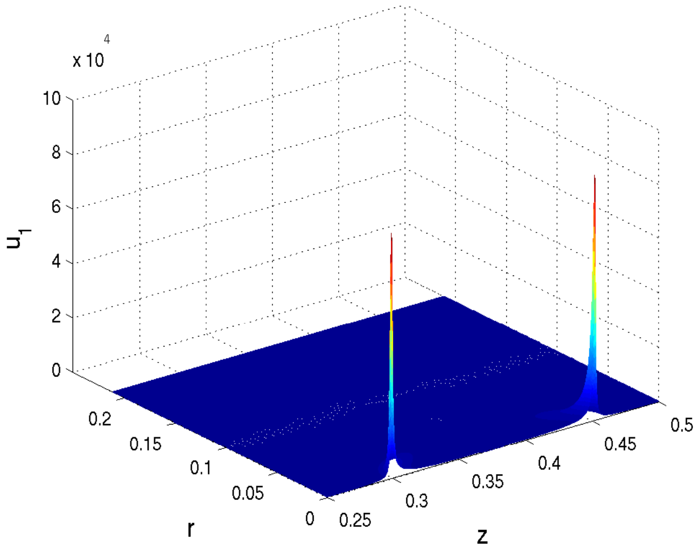


Figure 1.8 The 3D view of u_1 at $t = 0.021$ for the viscous model computed by the adaptive mesh with $N_z = 4096$, $N_r = 400$, $\Delta t = 2.5\times 10^{-7}$, $\nu = 0.001$.

1.4.1 Summary of the main result

In [26], we proved rigorously the following finite time blow-up result for the 3D inviscid model.

Theorem 1.4.1 *Let* $\Omega_{\mathbf{x}} = (0,a)\times(0,a)$, $\Omega = \Omega_{\mathbf{x}}\times(0,b)$ *and* $\Gamma = \{(\mathbf{x},z) \mid \mathbf{x}\in \Omega_{\mathbf{x}},\ z = 0\}$. *Assume that the initial conditions* u_0 *and* ψ_0 *satisfy* $u_0 > 0$ *for* $(\mathbf{x},z)\in\Omega$, $u_0|_{\partial\Omega} = 0$, $u_0\in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$ *and* ψ *satisfies (1.18). Moreover, we assume that* ψ *satisfies the following mixed Dirichlet Robin boundary conditions:*

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \quad (\psi_z + \beta\psi)|_{\Gamma} = 0, \tag{1.18}$$

with $\beta > \frac{\sqrt{2}\pi}{a}\left(\frac{1+e^{-2\pi b/a}}{1-e^{-2\pi b/a}}\right)$. *Define* $\phi(x_1,x_2,z) = \left(\frac{e^{-\alpha(z-b)}+e^{\alpha(z-b)}}{2}\right)\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)$ *where* α *satisfies* $0<\alpha<\sqrt{2}\pi/a$ *and* $2\left(\frac{\pi}{a}\right)^2\frac{e^{\alpha b}-e^{-\alpha b}}{\alpha(e^{\alpha b}+e^{-\alpha b})} = \beta$. *If* u_0 *and* ψ_0 *satisfy the following condition:*

$$\int_\Omega (\log u_0)\phi d\mathbf{x}dz > 0, \quad \int_\Omega \psi_{0z}\phi d\mathbf{x}dz > 0, \tag{1.19}$$

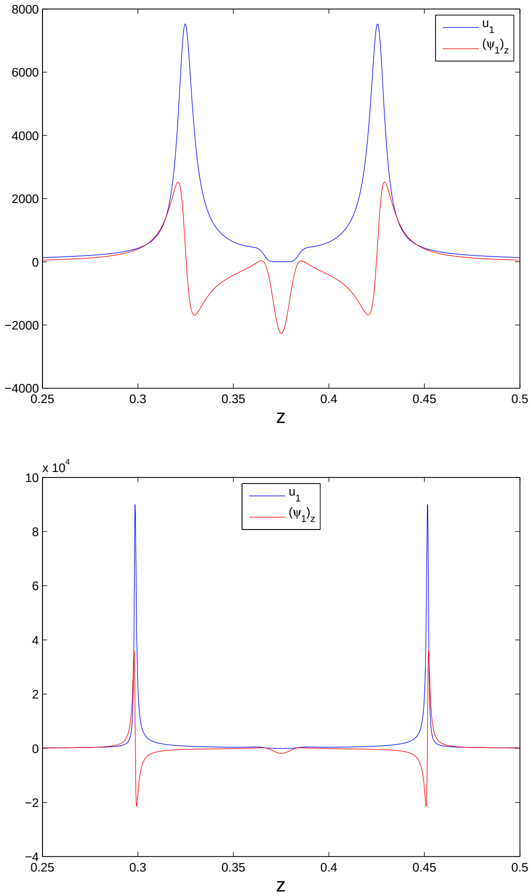
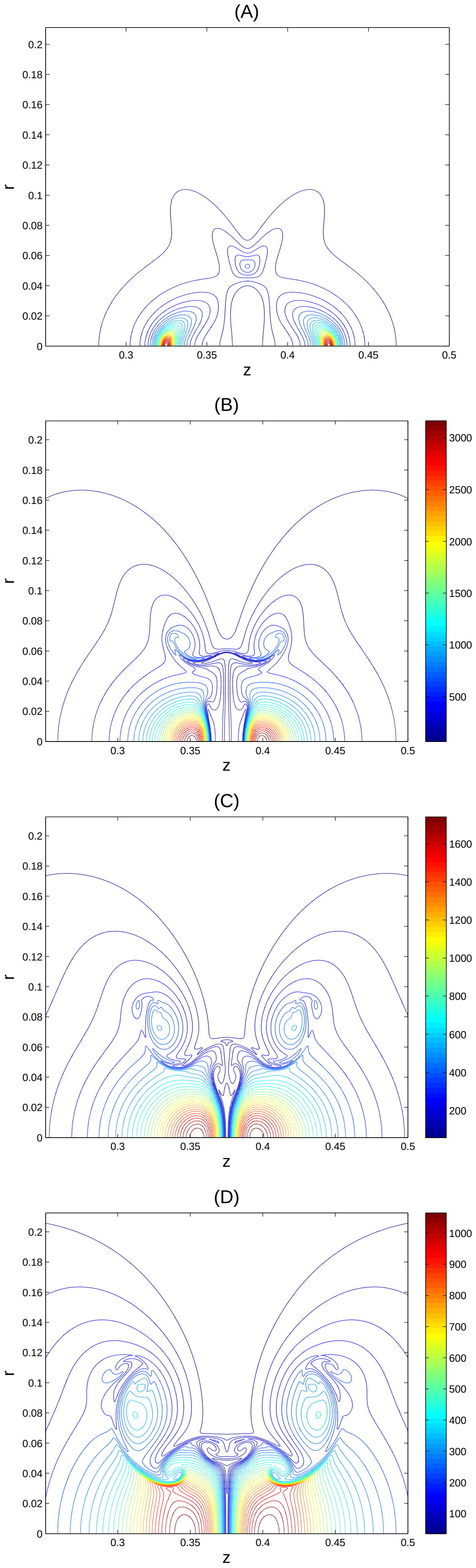


Figure 1.9 u_1 (dark) versus $\psi_{1,z}$ (gray) of the viscous model along the symmetry axis $r = 0$. The left figure corresponds to $t = 0.02$. The right figure corresponds to $t = 0.021$. Adaptive mesh computation with $N_z = 4096$, $N_r = 400$, $\Delta t = 2.5 \times 10^{-7}$, $\nu = 0.001$.

then the solution of the 3D inviscid model (1.16)–(1.17) will develop a finite time singularity in the H^2 norm.

1.4.2 Outline of the singularity analysis

We prove the finite time singularity result of the 3D model by contradiction. The analysis uses the local well-posedness result of the 3D model with the


(A)
r
z
0.3
0.35
0.4
0.45
0.5
(B)
r
z
3000
2500
2000
1500
1000
500
(C)
r
z
1600
1400
1200
1000
800
600
400
200
(D)
r
z
1000
900
800
700
600
500
400
300
200
100

above mixed Dirichlet Robin boundary condition, which will be established in Section 5. By the local well-posedness result, we know that there exists a finite time $T > 0$ such that the initial boundary value problem (1.16)–(1.17) with boundary condition given in the above theorem has a unique smooth solution with $u \in C^1([0,T), H^2(\Omega))$ and $\psi \in C^1([0,T), H^3(\Omega))$. Let T_b be the largest time such that the system (1.16)–(1.17) with initial condition u_0, ψ_0 has a smooth solution with $u \in C^1([0,T_b); H^2(\Omega))$ and $\psi \in C^1([0,T_b); H^3(\Omega))$. We claim that $T_b < \infty$. We prove this by contradiction.

Suppose that $T_b = \infty$. This means that for the given initial data u_0, ψ_0, the system (1.16)–(1.17) has a globally smooth solution $u \in C^1([0,\infty); H^2(\Omega))$ and $\psi \in C^1([0,\infty); H^3(\Omega))$. Note that $u|_{\partial\Omega} = 0$ as long as the solution remains smooth.

There are several important ingredients in our analysis. The first one is that we reformulate the u-equation and use $\log(u)$ as the new variable. With this reformulation, the right hand side of the reformulated u-equation becomes linear. Such reformulation is possible since $u_0 > 0$ in Ω implies that $u > 0$ in Ω as long as the solution remains smooth. We now work with the reformulated system given below:

$$(\log(u))_t = 2\psi_z, \quad (\mathbf{x}, z) \in \Omega, \tag{1.20}$$

$$-\Delta\psi_t = \left(u^2\right)_z. \tag{1.21}$$

The second ingredient is to find an appropriate test function ϕ and work with the weak formulation of (1.20)–(1.21). This test function ϕ is chosen as a positive and smooth eigen-function in Ω that satisfies the following two conditions simultaneously:

$$-\Delta\phi = \lambda_1\phi, \quad \partial_z^2\phi = \lambda_2\phi, \quad \text{for some } \lambda_1, \lambda_2 > 0, \quad (\mathbf{x}, z) \in \Omega. \tag{1.22}$$

Now we multiply ϕ to (1.20) and ϕ_z to (1.21) and integrate over Ω. Upon performing integration by parts, we obtain by using (1.22) that

$$\frac{d}{dt}\int_\Omega (\log u)\phi d\mathbf{x}dz = 2\int_\Omega \psi_z\phi d\mathbf{x}dz, \tag{1.23}$$

$$\lambda_1 \frac{d}{dt}\int_\Omega \psi_z\phi d\mathbf{x}dz = \lambda_2\int_\Omega u^2\phi d\mathbf{x}dz. \tag{1.24}$$

It is interesting to note that all the boundary terms resulting from integration by parts vanish due to the boundary condition of ψ, the property of our eigenfunction ϕ, the specific choice of α defined in Theorem 4.1. We have also used the fact that $u|_{z=0} = u|_{z=b} = 0$. Combining (1.24) with (1.23), we obtain our crucial blow-up estimate:

$$\frac{d^2}{dt^2}\int_\Omega (\log u)\phi \mathrm{d}\mathbf{x}\mathrm{dz} = \frac{2\lambda_2}{\lambda_1}\int_\Omega u^2\phi \mathrm{d}\mathbf{x}\mathrm{dz}. \tag{1.25}$$

Further, we note that

$$\begin{aligned}&\int_\Omega \log(u)\phi \mathrm{d}\mathbf{x}\mathrm{dz} \le \int_\Omega (\log(u))^+\phi \mathrm{d}\mathbf{x}\mathrm{dz} \le \int_\Omega u\phi \mathrm{d}\mathbf{x}\mathrm{dz}\\ &\le \left(\int_\Omega \phi \mathrm{d}\mathbf{x}\mathrm{dz}\right)^{1/2}\left(\int_\Omega \phi u^2 \mathrm{d}\mathbf{x}\mathrm{dz}\right)^{1/2} \equiv \frac{2a}{\pi\sqrt{\alpha}}\left(\int_\Omega \phi u^2 \mathrm{d}\mathbf{x}\mathrm{dz}\right)^{1/2}.\end{aligned} \tag{1.26}$$

From (1.25) and (1.26), we establish a sharp nonlinear dynamic estimate for $(\int_\Omega \phi u^2 \mathrm{d}\mathbf{x}\mathrm{dz})^{1/2}$, which enables us to prove finite time blowup of the 3D model.

This method of analysis is quite robust and captures very well the nonlinear interaction of the multi-dimensional nonlocal system. As a result, it provides a very effective method to analyze the finite time blowup of the 3D model and gives a relatively sharp blowup condition on the initial and boundary values for the 3D model.

1.4.3 Finite time blow-up of the 3D model with conservative BCs

We can also prove finite time blow-up of the 3D model with a conservative boundary condition in a bounded domain. Specifically, we consider the following initial boundary value problem:

$$\begin{cases} u_t &= 2u\psi_z \\ -\Delta\psi_t &= \left(u^2\right)_z \end{cases}, \quad (\mathbf{x}, z) \in \Omega = \Omega_\mathbf{x} \times (0, b), \tag{1.27}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \quad \psi_z|_\Gamma = 0, \tag{1.28}$$

$$\psi|_{t=0} = \psi_0(\mathbf{x}, z), \quad u|_{t=0} = u_0(\mathbf{x}, z) \ge 0,$$

where $\mathbf{x} = (x_1, x_2)$, $\Omega_\mathbf{x} = (0, a)\times(0, a)$, $\Gamma = \{(\mathbf{x}, z) \in \Omega \mid \mathbf{x} \in \Omega_\mathbf{x},\ z = 0 \text{ or } z = b\}$.

The main result is stated in the following theorem.

Theorem 1.4.2 *Assume that the initial conditions u_0 and ψ_0 satisfy $u_0 \in H^2(\Omega)$, $u_0|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $\psi_0 \in H^3(\Omega)$, and ψ satisfies (1.28). Let*

$$\phi(\mathbf{x}, z) = \frac{e^{-\alpha(z-b)} - e^{\alpha(z-b)}}{2} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{1.29}$$

with $\alpha = \frac{\pi}{a}$*, and*

$$A = \int_{\Omega} (\log u_0)\phi \mathrm{d}\mathbf{x}\mathrm{d}z, \quad B = 2\int_{\Omega} \psi_{0z}\phi \mathrm{d}\mathbf{x}\mathrm{d}z,$$

$$r(t) = \frac{2\left(\frac{\pi}{a}\right)^2 (e^{\alpha b} - e^{-\alpha b})}{2\left(\frac{\pi}{a}\right)^2 - \alpha^2} \int_{\Omega_{\mathbf{x}}} (\psi - \psi_0)|_{z=0} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a} \mathrm{d}\mathbf{x} \le \frac{B}{2}.$$

If $A > 0$*,* $B > 0$ *and* $r(t) \le \frac{B}{2}$ *as long as* u, ψ *remain regular, then the solution of (1.27)–(1.28) will develop a finite time singularity in the* H^2 *norm.*

1.4.4 Global regularity of the 3D inviscid model with small data

In this subsection we study the global regularity of the 3D inviscid model for a class of initial data with some appropriate boundary condition. To simplify the presentation of our analysis, we use u^2 and ψ_z as our new variables. We will define $v = \psi_z$ and still use u to stand for u^2. Then the 3D model now has the form:

$$\begin{cases} u_t = 4uv \\ -\Delta v_t = u_{zz} \end{cases}, \quad (\mathbf{x}, z) \in \Omega = (0,\delta)\times(0,\delta)\times(0,\delta). \tag{1.30}$$

We choose the following boundary condition for v:

$$v|_{\partial\Omega} = -4, \tag{1.31}$$

and denote $v|_{t=0} = v_0(\mathbf{x}, z)$ and $u|_{t=0} = u_0(\mathbf{x}, z) \ge 0$.

We prove the following global regularity result for the 3D inviscid model with a family of initial boundary value problems.

Theorem 1.4.3 *Assume that* $u_0, v_0 \in H^s(\Omega)$ *with* $s \ge 4$*,* $u_0|_{\partial\Omega} = 0$*,* $v_0|_{\partial\Omega} = -4$ *and* $v_0 \le -4$ *over* Ω*. Then the solution of (1.30)–(1.31) remains regular in* $H^s(\Omega)$ *for all time as long as the following holds*

$$\delta(4C_s + 1)\left(\|v_0\|_{H^s} + C_s\|u_0\|_{H^s}\right) < 1, \tag{1.32}$$

where C_s *is an interpolation constant. Moreover, we have* $\|u\|_{L^\infty} \le \|u_0\|_{L^\infty} e^{-7t}$*,* $\|u\|_{H^s(\Omega)} \le \|u_0\|_{H^s(\Omega)} e^{-7t}$ *and* $\|v\|_{H^s(\Omega)} \le C$ *for some constant* C *which depends on* u_0, v_0 *and* s *only.*

1.4.5 Blow-up of the 3D model with partial viscosity

In the previous subsections we considered only the inviscid model. In this subsection we show that the 3D model with partial viscosity can also develop finite time singularities. Specifically, we consider the following initial boundary value problem in a semi-infinite domain:

$$\begin{cases} u_t = 2u\psi_z \\ \omega_t = \left(u^2\right)_z + \nu\Delta\omega \quad , \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \\ -\Delta\psi = \omega. \end{cases} \tag{1.33}$$

The initial and boundary conditions are given as follows:

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \quad (\psi_z + \beta\psi)|_\Gamma = 0, \tag{1.34}$$

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)|_\Gamma = 0, \tag{1.35}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x}, z), \quad u|_{t=0} = u_0(\mathbf{x}, z) \geq 0, \tag{1.36}$$

where $\Gamma = \{(\mathbf{x}, z) \in \Omega \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$.

Now we state the main result of this subsection.

Theorem 1.4.4 *Assume that* $u_0|_{\partial\Omega} = 0$, $u_{0z}|_{\partial\Omega} = 0$, $u_0|_\Omega > 0$, $u_0 \in H^2(\Omega)$, $\psi_0 \in H^3(\Omega)$, $\omega_0 \in H^1(\Omega)$, ψ_0 *satisfies (1.34) and* ω_0 *satisfies (1.35). Further, we assume that* $\beta \in S_\infty$ *as defined in Lemma 1.5.1 and* $\beta > \frac{\sqrt{2}\pi}{a}$, $\gamma = \frac{2\pi^2}{\beta a^2}$. *Let*

$$\phi(\mathbf{x}, z) = e^{-\alpha z} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a}, \quad (\mathbf{x}, z) \in \Omega, \tag{1.37}$$

where $\alpha = \frac{2\pi^2}{\beta a^2}$ *satisfies* $0 < \alpha < \sqrt{2}\pi/a$. *Define*

$$A = \int_\Omega (\log u_0)\phi \,\mathrm{d}\mathbf{x}\mathrm{d}z, \quad B = -\int_\Omega \omega_0\phi_z \,\mathrm{d}\mathbf{x}\mathrm{d}z, \quad D = \frac{2}{2\left(\frac{\pi}{a}\right)^2 - \alpha^2}, \tag{1.38}$$

$$I_\infty = \int_0^\infty \frac{\mathrm{d}\mathbf{x}}{\sqrt{x^3 + 1}}, \quad T^* = \left(\frac{\pi\alpha^3 D^2 B}{12a}\right)^{-1/3} I_\infty. \tag{1.39}$$

If $A > 0$, $B > 0$, *and* $T^* < (\log 2)\left(\nu\left(\frac{2\pi^2}{a^2} - \alpha^2\right)\right)^{-1}$, *then the solution of model (1.33) with initial and boundary conditions (1.34)–(1.36) will develop a finite time singularity before* T^*.

1.5 Local well-posedness of the 3D model with mixed Dirichlet Robin Boundary conditions

In this section we prove the local well-posedness of the 3D model with the mixed Dirichlet Robin boundary conditions considered in the previous section. The 3D model with partial viscosity has the following form:

$$\begin{cases} u_t = 2u\psi_z \\ \omega_t = (u^2)_z + \nu\Delta\omega \quad , \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \\ -\Delta\psi = \omega \end{cases} \tag{1.40}$$

where $\Omega_{\mathbf{x}} = (0,a) \times (0,a)$. Let $\Gamma = \{(\mathbf{x},z) \mid \mathbf{x} \in \Omega_{\mathbf{x}},\ z = 0\}$. The initial and boundary conditions for (1.40) are given as following:

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)\,|_{\Gamma} = 0, \tag{1.41}$$

$$\psi|_{\partial\Omega\setminus\Gamma} = 0, \quad (\psi_z + \beta\psi)\,|_{\Gamma} = 0, \tag{1.42}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x},z), \quad u|_{t=0} = u_0(\mathbf{x},z). \tag{1.43}$$

The analysis of finite time singularity formation of the 3D model uses the local well-posedness result of the 3D model. The local well-posedness of the 3D model can be proved by using a standard energy estimate and a mollifier if there is no boundary or if the boundary condition is a standard one, see e.g. [32]. For the mixed Dirichlet Robin boundary condition we consider here, the analysis is a bit more complicated since the mixed Dirichlet Robin condition gives rise to a growing eigenmode.

There are two key ingredients in our local well-posedness analysis. The first one is to design a Picard iteration for the 3D model. The second one is to show that the mapping that generates the Picard iteration is a contraction mapping and the Picard iteration converges to a fixed point of the Picard mapping by using the contraction mapping theorem. To establish the contraction property of the Picard mapping, we need to use the well-posedness property of the heat equation with the same Dirichlet Robin boundary condition as ω. The well-posedness analysis of the heat equation with a mixed Dirichlet Robin boundary has been studied in the literature. The case of $\gamma > 0$ is more subtle because there is a growing eigenmode. Nonetheless, we prove that all the essential regularity properties of the heat equation are still valid for the mixed Dirichlet Robin boundary condition with $\gamma > 0$.

The local existence result of our 3D model with partial viscosity is stated in the following theorem.

Theorem 1.5.1 *Assume that $u_0 \in H^{s+1}(\Omega)$, $\omega_0 \in H^s(\Omega)$ for some $s > 3/2$, $u_0|_{\partial\Omega} = u_{0z}|_{\partial\Omega} = 0$ and ω_0 satisfies (1.41). Moreover, we assume that $\beta \in S_\infty$ (or S_b) as defined in Lemma 1.5.1. Then there exists a finite time $T = T\left(\|u_0\|_{H^{s+1}(\Omega)}, \|\omega_0\|_{H^s(\Omega)}\right) > 0$ such that the system (1.40) with boundary condition (1.41),(1.42) and initial data (1.43) has a unique solution, $u \in C([0,T], H^{s+1}(\Omega))$, $\omega \in C([0,T], H^s(\Omega))$ and $\psi \in C([0,T], H^{s+2}(\Omega))$.*

The local well-posedness analysis relies on the following local well-posedness of the heat equation and the elliptic equation with mixed Dirichlet Robin boundary conditions. First, the local well-posedness of the elliptic equation with the mixed Dirichlet Robin boundary condition is given by the following lemma [26]:

Lemma 1.5.1 *There exists a unique solution $v \in H^s(\Omega)$ to the boundary value problem:*

$$-\Delta v = f, \quad (\mathbf{x}, z) \in \Omega, \tag{1.44}$$
$$v|_{\partial\Omega\setminus\Gamma} = 0, \quad (v_z + \beta v)|_{\Gamma} = 0, \tag{1.45}$$

if $\beta \in S_\infty \equiv \{\beta \mid \beta \neq \frac{\pi|k|}{a}$ for all $k \in \mathbb{Z}^2\}$, $f \in H^{s-2}(\Omega)$ with $s \geq 2$ and $f|_{\partial\Omega\setminus\Gamma} = 0$. Moreover we have

$$\|v\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}, \tag{1.46}$$

where C_s is a constant depending on s, $|k| = \sqrt{k_1^2 + k_2^2}$.

Definition 1.5.1 *Let $\mathcal{K} : H^{s-2}(\Omega) \to H^s(\Omega)$ be a linear operator defined as following: for all $f \in H^{s-2}(\Omega)$,*

$\mathcal{K}(f)$ is the solution of the boundary value problem (1.44)–(1.45).

It follows from Lemma 1.5.1 that for any $f \in H^{s-2}(\Omega)$, we have

$$\|\mathcal{K}(f)\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-2}(\Omega)}. \tag{1.47}$$

For the heat equation with the mixed Dirichlet Robin boundary condition, we have the following result.

Lemma 1.5.2 *There exists a unique solution $\omega \in C([0,T]; H^s(\Omega))$ to the initial boundary value problem:*

$$\omega_t = \nu\Delta\omega, \quad (\mathbf{x}, z) \in \Omega, \tag{1.48}$$
$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)|_{\Gamma} = 0, \tag{1.49}$$
$$\omega|_{t=0} = \omega_0(\mathbf{x}, z), \tag{1.50}$$

for $\omega_0 \in H^s(\Omega)$ with $s > 3/2$. Moreover we have the following estimates in the case $\gamma > 0$

$$\|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s) e^{\nu\gamma^2 t} \|\omega_0\|_{H^s(\Omega)}, \quad t \geq 0, \tag{1.51}$$

and

$$\|\omega(t)\|_{H^s(\Omega)} \leq C(\gamma, s, t) \|\omega_0\|_{L^2(\Omega)}, \quad t > 0. \tag{1.52}$$

Remark 1.5.1 *We remark that the growth factor $e^{\nu\gamma^2 t}$ in (1.51) is absent in the case of $\gamma \leq 0$ since there is no growing eigenmode in this case.*

Proof First, we prove the solution of the system (1.48)–(1.50) is unique. Let $\omega_1, \ \omega_2 \in H^s(\Omega)$ be two smooth solutions of the heat equation for $0 \leq t < T$

satisfying the same initial condition and the Dirichlet Robin boundary condition. Let $\omega = \omega_1 - \omega_2$. We will prove that $\omega = 0$ by using an energy estimate and the Dirichlet Robin boundary condition at Γ:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \omega^2 d\mathbf{x}dz &= \nu \int_\Omega \omega\Delta\omega d\mathbf{x}dz \\
&= -\nu \int_\Omega |\nabla\omega|^2 d\mathbf{x}dz - \nu \int_\Gamma \omega\omega_z d\mathbf{x} \\
&= -\nu \int_\Omega |\nabla\omega|^2 d\mathbf{x}dz + \nu\gamma \int_\Gamma \omega^2 d\mathbf{x} \\
&= -\nu \int_\Omega |\nabla\omega|^2 d\mathbf{x}dz - \nu\gamma \int_\Gamma \int_z^\infty \left(\omega^2\right)_z dzd\mathbf{x} \\
&= -\nu \int_\Omega |\nabla\omega|^2 d\mathbf{x}dz - 2\nu\gamma \int_\Gamma \int_z^\infty \omega\omega_z d\mathbf{x}dz \\
&\le -\nu \int_\Omega |\nabla\omega|^2 d\mathbf{x}dz + \frac{\nu}{2}\int_\Omega |\omega_z|^2 d\mathbf{x}dz + 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz \\
&\le -\frac{\nu}{2}\int_\Omega |\nabla\omega|^2 d\mathbf{x}dz + 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz, \qquad (1.53)
\end{aligned}$$

where we have used the fact that the smooth solution of the heat equation ω decays to zero as $z \to \infty$. Thus, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \omega^2 d\mathbf{x}dz \le 2\nu\gamma^2 \int_\Omega \omega^2 d\mathbf{x}dz. \qquad (1.54)$$

It follows from Gronwall's inequality

$$e^{-4\nu\gamma^2 t}\int_\Omega \omega^2 d\mathbf{x}dz \le \int_\Omega \omega_0^2 d\mathbf{x}dz = 0, \qquad (1.55)$$

since $\omega_0 = 0$. Since $\omega \in H^s(\Omega)$ with $s > 3/2$, this implies that $\omega = 0$ for $0 \le t < T$ which proves the uniqueness of smooth solutions for the heat equation with the mixed Dirichlet Robin boundary condition.

Next, we will prove the existence of the solution by constructing a solution explicitly. Let $\eta(\mathbf{x}, z, t)$ be the solution of the following initial boundary value problem:

$$\eta_t = \nu\Delta\eta, \quad (\mathbf{x}, z) \in \Omega, \qquad (1.56)$$

$$\eta|_{\partial\Omega} = 0, \quad \eta|_{t=0} = \eta_0(\mathbf{x}, z), \qquad (1.57)$$

and let $\xi(\mathbf{x}, t)$ be the solution of the following PDE in $\Omega_\mathbf{x}$:

$$\xi_t = \nu\Delta_\mathbf{x}\xi + \nu\gamma^2\xi, \quad \mathbf{x} \in \Omega_\mathbf{x}, \qquad (1.58)$$

$$\xi|_{\partial\Omega_\mathbf{x}} = 0, \quad \xi|_{t=0} = \overline{\omega}_0(\mathbf{x}), \qquad (1.59)$$

where $\Delta_{\mathbf{x}} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $\overline{\omega}_0(\mathbf{x}) = 2\gamma \int_0^\infty \omega_0(\mathbf{x}, z) e^{-\gamma z} dz$. From the standard theory of the heat equation, we know that η and ξ both exist globally in time.

We are interested in the case when the initial value $\eta_0(\mathbf{x}, z)$ is related to ω_0 by solving the following ODE as a function of z with $\mathbf{x}$ being fixed as a parameter:

$$-\frac{1}{\gamma}\eta_{0z} + \eta_0 = \omega_0(\mathbf{x}, z) - \overline{\omega}_0(\mathbf{x}) e^{-\gamma z}, \quad \eta_0(\mathbf{x}, 0) = 0. \tag{1.60}$$

Define

$$\omega(\mathbf{x}, z, t) \equiv -\frac{1}{\gamma}\eta_z + \eta + \xi(\mathbf{x}, t) e^{-\gamma z}, \quad (\mathbf{x}, z) \in \Omega. \tag{1.61}$$

It is easy to check that ω satisfies the heat equation for $t > 0$ and the initial condition. Obviously, ω also satisfies the boundary condition on $\partial\Omega\backslash\Gamma$. To verify the boundary condition on Γ, we observe by a direct calculation that $(\omega_z + \gamma\omega)|_\Gamma = -\frac{1}{\gamma}(\eta_z)_z|_\Gamma$. Since $\eta(\mathbf{x}, z)|_\Gamma = 0$, we obtain by using $\eta_t = \nu\Delta\eta$ and taking the limit as $z \to 0+$ that $\Delta\eta|_\Gamma = 0$, which implies that $\eta_{zz}|_\Gamma = 0$. Therefore, ω also satisfies the Dirichlet Robin boundary condition at Γ. This shows that ω is a solution of the system (1.48)–(1.50). By the uniqueness result that we proved earlier, the solution of the heat equation must be given by (1.61).

Since η and ξ are solutions of the heat equation with a standard Dirichlet boundary condition, the classical theory of the heat equation [15] gives the following regularity estimates:

$$\|\eta\|_{H^s(\Omega)} \leq C \|\eta_0\|_{H^s(\Omega)}, \quad \|\xi(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})} \leq C e^{\nu\gamma^2 t} \|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})}. \tag{1.62}$$

Recall that $\eta_{zz}|_\Gamma = 0$. Therefore, η_z also solves the heat equation with the same Dirichlet Robin boudary condition:

$$(\eta_z)_t = \nu\Delta\eta_z, \quad (\mathbf{x}, z) \in \Omega, \tag{1.63}$$

$$(\eta_z)_z|_\Gamma = 0, \quad (\eta_z)|_{\partial\Omega\backslash\Gamma} = 0, \quad (\eta_z)|_{t=0} = \eta_{0z}(\mathbf{x}, z), \tag{1.64}$$

which implies that

$$\|\eta_z\|_{H^s(\Omega)} \leq C \|\eta_{0z}\|_{H^s(\Omega)}. \tag{1.65}$$

Putting all the above estimates for η, η_z and ξ together and using (1.61), we obtain the following estimate:

$$\begin{aligned} \|\omega\|_{H^s(\Omega)} &= \left\| -\frac{1}{\gamma}\eta_z + \eta + \xi(\mathbf{x}, t) e^{-\gamma z} \right\|_{H^s(\Omega)} \\ &\leq \frac{1}{\gamma} \|\eta_z\|_{H^s(\Omega)} + \|\eta\|_{H^s(\Omega)} + \left\|\xi(\mathbf{x}, t) e^{-\gamma z}\right\|_{H^s(\Omega)} \\ &\leq C(\gamma, s)\left(\|\eta_{0z}\|_{H^s(\Omega)} + \|\eta_0\|_{H^s(\Omega)} + e^{\nu\gamma^2 t} \|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})}\right). \end{aligned} \tag{1.66}$$

It remains to bound $\|\eta_{0z}\|_{H^s(\Omega)}$, $\|\eta_0\|_{H^s(\Omega)}$ and $\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_{\mathbf{x}})}$ in terms of $\|\omega_0\|_{H^s(\Omega)}$. By solving the ODE (1.60) directly, we can express η in terms of ω_0 explicitly

$$\eta_0(\mathbf{x},z) = -\gamma e^{\gamma z}\int_0^z e^{-\gamma z'} f(\mathbf{x},z')dz' = \gamma \int_z^\infty e^{-\gamma(z'-z)} f(\mathbf{x},z')dz', \quad (1.67)$$

where $f(\mathbf{x},z) = \omega_0(\mathbf{x},z) - \overline{\omega}_0(\mathbf{x})e^{-\gamma z}$ and we have used the property that

$$\int_0^\infty f(\mathbf{x},z)e^{-\gamma z}dz = 0.$$

By using integration by parts, we have

$$\begin{aligned}\eta_{0z}(\mathbf{x},z) &= -\gamma f(\mathbf{x},z) + \gamma^2 \int_z^\infty e^{-\gamma(z'-z)} f(\mathbf{x},z')dz' \\ &= \gamma \int_z^\infty e^{-\gamma(z'-z)} f_{z'}(\mathbf{x},z')dz'. \quad (1.68)\end{aligned}$$

By induction we can show that for any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \geq 0$

$$D^\alpha \eta_0 = \gamma \int_z^\infty e^{-\gamma(z'-z)} D^\alpha f(\mathbf{x},z')dz'. \quad (1.69)$$

Let $K(z) = \gamma e^{-\gamma z}\chi(z)$ and $\chi(z)$ be the characteristic function

$$\chi(z) = \begin{cases} 0, & z \leq 0, \\ 1, & z > 0. \end{cases} \quad (1.70)$$

Then $D^\alpha \eta_0$ can be written in the following convolution form:

$$D^\alpha \eta_0(\mathbf{x},z) = \int_0^\infty K(z'-z) D^\alpha f(\mathbf{x},z')dz'. \quad (1.71)$$

Using Young's inequality (see e.g. page 232 of [17]), we obtain:

$$\begin{aligned}\|D^\alpha \eta_0\|_{L^2(\Omega)} &\leq \|K(z)\|_{L^1(\mathbb{R}^+)} \|D^\alpha f\|_{L^2(\Omega)} \\ &\leq C(\gamma) \left\| D^\alpha \omega_0 - (-\gamma)^{\alpha_3} e^{-\gamma z} D^{(\alpha_1,\alpha_2)} \overline{\omega}_0(\mathbf{x}) \right\|_{L^2(\Omega)} \\ &\leq C(\gamma,\alpha) \left(\|D^\alpha \omega_0\|_{L^2(\Omega)} + \left\| D^{(\alpha_1,\alpha_2)} \overline{\omega}_0(\mathbf{x}) \right\|_{L^2(\Omega_{\mathbf{x}})} \right). \quad (1.72)\end{aligned}$$

Moreover, we obtain by using the Hölder inequality that

$$\begin{aligned}\left\| D^{(\alpha_1,\alpha_2)} \overline{\omega}_0(\mathbf{x}) \right\|_{L^2(\Omega_{\mathbf{x}})} &= \left(\int_{\Omega_{\mathbf{x}}} \left(\int_0^\infty e^{-\gamma z} D^{(\alpha_1,\alpha_2)} \omega_0(\mathbf{x},z)dz \right)^2 d\mathbf{x} \right)^{1/2} \\ &\leq \left(\frac{1}{2\gamma} \int_{\Omega_{\mathbf{x}}} \int_0^\infty \left(D^{(\alpha_1,\alpha_2)} \omega_0(\mathbf{x},z) \right)^2 dzd\mathbf{x} \right)^{1/2} \\ &= \frac{1}{\sqrt{2\gamma}} \left\| D^{(\alpha_1,\alpha_2)} \omega_0(\mathbf{x},z) \right\|_{L^2(\Omega)}. \quad (1.73)\end{aligned}$$

Substituting (1.73) into (1.72) yields

$$\|D^\alpha \eta_0\|_{L^2(\Omega)} \le C(\gamma,\alpha)\left(\|D^\alpha \omega_0\|_{L^2(\Omega)} + \left\|D^{(\alpha_1,\alpha_2)}\omega_0\right\|_{L^2(\Omega)}\right), \tag{1.74}$$

which implies that

$$\|\eta_0\|_{H^s(\Omega)} \le C(\gamma,s)\,\|\omega_0\|_{H^s(\Omega)}\,, \quad \forall\ s \ge 0. \tag{1.75}$$

It follows from (1.73) that

$$\|\overline{\omega}_0(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})} \le C(\gamma)\,\|\omega_0\|_{H^s(\Omega)}\,, \quad \forall\ s \ge 0. \tag{1.76}$$

On the other hand, we obtain from the equation for η_0 (1.60) that

$$\|\eta_{0z}\|_{H^s(\Omega)} = \gamma\,\|f + \eta_0\|_{H^s(\Omega)} \le C(\gamma,s)\,\|\omega_0\|_{H^s(\Omega)}\,, \quad \forall\ s \ge 0. \tag{1.77}$$

Upon substituting (1.75)–(1.77) into (1.66), we obtain

$$\|\omega\|_{H^s(\Omega)} \le C(\gamma,s)e^{\nu\gamma^2 t}\,\|\omega_0\|_{H^s(\Omega)}\,, \tag{1.78}$$

where $C(\gamma,s)$ is a constant depending only on γ and s. This proves (1.51).

To prove (1.52), we use the classical regularity result for the heat equation with the homogeneous Dirichlet boundary condition to obtain the following estimates for $t > 0$:

$$\|\eta\|_{H^s(\Omega)} \le C(t)\|\eta_0\|_{L^2(\Omega)}, \tag{1.79}$$

$$\|\eta_z\|_{H^s(\Omega)} \le C(s,t)\|\eta_{0z}\|_{L^2(\Omega)}, \tag{1.80}$$

$$\|\overline{\omega}(\mathbf{x})\|_{H^s(\Omega_\mathbf{x})} \le C(s,t)e^{\nu\gamma^2 t}\,\|\overline{\omega}_0(\mathbf{x})\|_{L^2(\Omega_\mathbf{x})}\,, \tag{1.81}$$

where $C(s,t)$ is a constant depending on s and t. By combining (1.79)–(1.81) with estimates (1.75)–(1.77), we obtain for any $t > 0$ that

$$\begin{aligned}\|\omega\|_{H^s(\Omega)} &\le C(\gamma,s,t)\left(\|\eta_{0z}\|_{L^2(\Omega)} + \|\eta_0\|_{L^2(\Omega)} + e^{\nu\gamma^2 t}\,\|\overline{\omega}_0(\mathbf{x})\|_{L^2(\Omega_\mathbf{x})}\right)\\ &\le C(\gamma,s,t)\|\omega_0\|_{L^2(\Omega)},\end{aligned} \tag{1.82}$$

where $C(\gamma,s,t) < \infty$ is a constant depending on γ, s and t. This proves (1.52) and completes the proof of the lemma. □

We also need the following well-known Sobolev inequality [18].

Lemma 1.5.3 *Let $u, v \in H^s(\Omega)$ with $s > 3/2$. We have*

$$\|uv\|_{H^s(\Omega)} \le c\|u\|_{H^s(\Omega)}\|v\|_{H^s(\Omega)}. \tag{1.83}$$

Now we are ready to give the proof of Theorem 1.5.1.

Proof **of Theorem 1.5.1** Let $v = u^2$. First, using the definition of the operator $\mathcal{K}$ (see Definition 1.5.1), we can rewrite the 3D model with partial viscosity in the following equivalent form:

$$\begin{cases} v_t &= 4v\mathcal{K}(\omega)_z \\ \omega_t &= v_z + \nu\Delta\omega \end{cases}, \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{1.84}$$

with the initial and boundary conditions given as follows:

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)\,|_\Gamma = 0, \tag{1.85}$$

$$\omega|_{t=0} = \omega_0(\mathbf{x}, z) \in W^s, \quad v|_{t=0} = v_0(\mathbf{x}, z) \in V^{s+1}, \tag{1.86}$$

where $V^{s+1} = \{v \in H^{s+1} : v|_{\partial\Omega} = 0, v_z|_{\partial\Omega} = 0, v_{zz}|_{\partial\Omega} = 0\}$ and $W^s = \{w \in H^s : w|_{\partial\Omega\setminus\Gamma} = 0, (w_z + \gamma w)|_\Gamma = 0\}$.

We note that the condition $u_0|_{\partial\Omega} = u_{0z}|_{\partial\Omega} = 0$ implies that $v_0|_{\partial\Omega} = v_{0z}|_{\partial\Omega} = v_{0zz}|_{\partial\Omega} = 0$ by using the relation $v_0 = u_0^2$. Thus we have $v_0 \in V^{s+1}$. It is easy to show by using the u-equation that the property $u_0|_{\partial\Omega} = u_{0z}|_{\partial\Omega} = 0$ is preserved dynamically. Thus we have $v \in V^{s+1}$.

Define $U = (U_1, U_2) = (v, \omega)$ and $X = C([0, T]; V^{s+1}) \times C([0, T]; W^s)$ with the norm

$$\|U\|_X = \sup_{t\in[0,T]} \|U_1\|_{H^{s+1}(\Omega)} + \sup_{t\in[0,T]} \|U_2\|_{H^s(\Omega)}, \quad \forall U \in X$$

and let $S = \{U \in X : \|U\|_X \le M\}$.

Now, define the map $\Phi : X \to X$ in the following way: let $\Phi(\tilde{v}, \tilde{\omega}) = (v, \omega)$. Then for any $t \in [0, T]$,

$$v(\mathbf{x}, z, t) = v_0(\mathbf{x}, z, t) + 4\int_0^t \tilde{v}(\mathbf{x}, z, t')\mathcal{K}(\tilde{\omega})_z(\mathbf{x}, z, t')dt', \tag{1.87}$$

$$\omega(\mathbf{x}, z, t) = \mathcal{L}(\tilde{v}_z, \omega_0; \mathbf{x}, z, t), \tag{1.88}$$

where $\omega(\mathbf{x}, z, t) = \mathcal{L}(\tilde{v}_z, \omega_0; \mathbf{x}, z, t)$ is the solution of the following equation:

$$\omega_t = \tilde{v}_z + \nu\Delta\omega, \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{1.89}$$

with the initial and boundary conditions:

$$\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)\,|_\Gamma = 0, \quad \omega|_{t=0} = \omega_0(\mathbf{x}, z).$$

We use the map Φ to define a Picard iteration: $U^{k+1} = \Phi(U^k)$ with $U^0 = (v_0, \omega_0)$. In the following, we will prove that there exist $T > 0$ and $M > 0$ such that

1. $U^k \in S$, for all k.
2. $\left\|U^{k+1} - U^k\right\|_X \le \frac{1}{2}\left\|U^k - U^{k-1}\right\|_X$, for all k.

Then by the contraction mapping theorem, there exists $U = (v, \omega) \in S$ such that $\Phi(U) = U$ which implies that U is a local solution of the system (1.84) in X.

First, by Duhamel's principle, we have for any $g \in C([0, T]; V^s)$ that

$$\mathcal{L}(g, \omega_0; \mathbf{x}, z, t) = \mathcal{P}(\omega_0; 0, t) + \int_0^t \mathcal{P}(g; t', t) dt', \tag{1.90}$$

where $\mathcal{P}(g; t', t) = \tilde{g}(\mathbf{x}, z, t)$ is defined as the solution of the following initial boundary value problem at time t:

$$\tilde{g}_t = \nu \Delta \tilde{g}, \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{1.91}$$

with the initial and boundary conditions:

$$\tilde{g}|_{\partial\Omega \setminus \Gamma} = 0, \quad (\tilde{g}_z + \gamma \tilde{g})\,|_\Gamma = 0, \quad \tilde{g}(\mathbf{x}, z, t') = g(\mathbf{x}, z, t'). \tag{1.92}$$

We observe that $g(\mathbf{x}, z, t')$ also satisfies the same boundary condition as ω for any $0 \le t' \le t$ since $g = v_z^k$ and $v^k \in V^{s+1}$.

Now we can apply Lemma 1.5.2 to conclude that for any $t' < T$ and $t \in [t', T]$ we have

$$\|\mathcal{P}(g; t', t)\|_{H^s(\Omega)} \le C(\gamma, s) e^{\nu \gamma^2 (t - t')} \|g(\mathbf{x}, z, t')\|_{H^s(\Omega)}, \tag{1.93}$$

which implies the following estimate for $\mathcal{L}$: for all $t \in [0, T]$,

$$\|\mathcal{L}(g, \omega_0; \mathbf{x}, z, t)\|_{H^s(\Omega)} \le C(\gamma, s) e^{\nu \gamma^2 t} \left(\|\omega_0\|_{H^s(\Omega)} + t \sup_{t' \in [0,t]} \|g(\mathbf{x}, z, t')\|_{H^s(\Omega)} \right). \tag{1.94}$$

Further, by using Lemma 1.5.1 and the above estimate (1.94) for the sequence $U^k = (v^k, \omega^k)$, we get the following estimate: $\forall t \in [0, T]$,

$$\begin{aligned} \left\| v^{k+1} \right\|_{H^{s+1}(\Omega)} &\le \|v_0\|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0,T]} \left\| v^k(\mathbf{x}, z, t) \right\|_{H^{s+1}(\Omega)} \sup_{t \in [0,T]} \left\| \mathcal{K}(\omega^k)_z(\mathbf{x}, z, t) \right\|_{H^{s+1}(\Omega)}, \\ &\le \|v_0\|_{H^{s+1}(\Omega)} + 4T \sup_{t \in [0,T]} \left\| v^k(\mathbf{x}, z, t) \right\|_{H^{s+1}(\Omega)} \sup_{t \in [0,T]} \left\| \omega^k(\mathbf{x}, z, t) \right\|_{H^s(\Omega)}, \end{aligned} \tag{1.95}$$

$$\begin{aligned} \left\| \omega^{k+1} \right\|_{H^s(\Omega)} &\le C(\gamma, s) e^{\nu \gamma^2 t} \left(\|\omega_0\|_{H^s(\Omega)} + t \sup_{t' \in [0,t]} \left\| v_z^k(\mathbf{x}, z, t') \right\|_{H^s(\Omega)} \right) \\ &\le C(\gamma, s) e^{\nu \gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + T \sup_{t \in [0,T]} \left\| v^k \right\|_{H^{s+1}(\Omega)} \right). \end{aligned} \tag{1.96}$$

Next, we will use mathematical induction to prove that if T satisfies the following inequality:

$$8C(\gamma, s) T e^{\nu \gamma^2 T} \left(\|\omega_0\|_{H^s(\Omega)} + 2T \|v_0\|_{H^{s+1}(\Omega)} \right) \le 1 \tag{1.97}$$

then for all $k \geq 0$ and $t \in [0, T]$, we have that

$$\left\|v^k\right\|_{H^{s+1}(\Omega)} \leq 2\left\|v_0\right\|_{H^{s+1}(\Omega)}, \tag{1.98}$$
$$\left\|\omega^k\right\|_{H^s(\Omega)} \leq C(\gamma, s)e^{\nu\gamma^2 T}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + 2T\left\|v_0\right\|_{H^{s+1}(\Omega)}\right). \tag{1.99}$$

First of all, $U^0 = (v_0, \omega_0)$ satisfies (1.98) and (1.99). Assume $U^k = (v^k, \omega^k)$ has this property, then for $U^{k+1} = (v^{k+1}, \omega^{k+1})$, using (1.95) and (1.96), we have

$$\begin{aligned}\left\|v^{k+1}\right\|_{H^{s+1}(\Omega)} &\leq \left\|v_0\right\|_{H^{s+1}(\Omega)} + 4T \sup_{t\in[0,T]}\left\|v^k(\mathbf{x}, z, t)\right\|_{H^{s+1}(\Omega)} \sup_{t\in[0,T]}\left\|\omega^k(\mathbf{x}, z, t)\right\|_{H^s(\Omega)} \\ &\leq \left\|v_0\right\|_{H^{s+1}(\Omega)}\left(1 + 8C(\gamma, s)Te^{\nu\gamma^2 T}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + 2T\left\|v_0\right\|_{H^{s+1}(\Omega)}\right)\right) \\ &\leq 2\left\|v_0\right\|_{H^{s+1}(\Omega)}, \quad \forall t \in [0, T],\end{aligned} \tag{1.100}$$

$$\begin{aligned}\left\|\omega^{k+1}\right\|_{H^s(\Omega)} &\leq C(\gamma, s)e^{\nu\gamma^2 T}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + T\sup_{t\in[0,T]}\left\|v^k\right\|_{H^{s+1}(\Omega)}\right) \\ &\leq C(\gamma, s)e^{\nu\gamma^2 T}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + 2T\left\|v_0\right\|_{H^{s+1}(\Omega)}\right), \quad \forall t \in [0, T].\end{aligned} \tag{1.101}$$

Then, by induction, we prove that for any $k \geq 0$, $U^k = (v^k, \omega^k)$ is bounded by (1.98) and (1.99).

We want to point out that there exists $T > 0$ such that the inequality (1.97) is satisfied. One choice of T is given as following:

$$T_1 = \min\left\{\left[8C(\gamma, s)e^{\nu\gamma^2}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + 2\left\|v_0\right\|_{H^{s+1}(\Omega)}\right)\right]^{-1}, 1\right\}. \tag{1.102}$$

Using the choice of T in (1.102), we can choose

$$M = 2\left\|v_0\right\|_{H^{s+1}(\Omega)} + C(\gamma, s)e^{\nu\gamma^2}\left(\left\|\omega_0\right\|_{H^s(\Omega)} + 2\left\|v_0\right\|_{H^{s+1}(\Omega)}\right).$$

Then we have $U^k \in S$, for all k.

Next, we will prove that Φ is a contraction mapping for some small $0 < T \leq T_1$.

First of all, by using Lemmas 1.5.1 and 1.5.3, we have

$$
\begin{aligned}
\left\|v^{k+1}-v^{k}\right\|_{H^{s+1}(\Omega)} &= \left\|\int_0^t v^k(\mathbf{x},t')\mathcal{K}(\omega^k)_z(\mathbf{x},t')dt' - \int_0^t v^{k-1}(\mathbf{x},t')\mathcal{K}(\omega^{k-1})_z(\mathbf{x},t')dt'\right\|_{H^{s+1}(\Omega)} \\
&\le \left\|\int_0^t \left(v^k - v^{k-1}\right)(\mathbf{x},t')\mathcal{K}(\omega^k)_z(\mathbf{x},t')dt'\right\|_{H^{s+1}(\Omega)} \\
&\quad + \left\|\int_0^t v^{k-1}(\mathbf{x},t')\left(\mathcal{K}(\omega^k)_z - \mathcal{K}(\omega^{k-1})_z\right)(\mathbf{x},t')dt'\right\|_{H^{s+1}(\Omega)} \\
&\le T\sup_{t\in[0,T]}\left\|v^k - v^{k-1}\right\|_{H^{s+1}(\Omega)}\sup_{t\in[0,T]}\left\|\mathcal{K}(\omega^k)_z\right\|_{H^{s+1}(\Omega)} \\
&\quad + T\sup_{t\in[0,T]}\left\|v^{k-1}\right\|_{H^{s+1}(\Omega)}\sup_{t\in[0,T]}\left\|\mathcal{K}(\omega^k-\omega^{k-1})_z\right\|_{H^{s+1}(\Omega)} \\
&\le MT\left(\sup_{t\in[0,T]}\left\|v^k - v^{k-1}\right\|_{H^{s+1}(\Omega)} + \sup_{t\in[0,T]}\left\|\omega^k-\omega^{k-1}\right\|_{H^{s}(\Omega)}\right). \qquad (1.103)
\end{aligned}
$$

On the other hand, Lemma 1.5.2 and (1.90) imply

$$
\begin{aligned}
\left\|\omega^{k+1}-\omega^{k}\right\|_{H^{s}(\Omega)} &= \left\|\mathcal{L}(v_z^k,\omega_0;\mathbf{x},t) - \mathcal{L}(v_z^{k-1},\omega_0;\mathbf{x},t)\right\|_{H^{s}(\Omega)} \\
&\le \left\|\int_0^t \mathcal{P}(v_z^k - v_z^{k-1};t',t)dt'\right\|_{H^{s}(\Omega)} \\
&\le TC(\gamma,s)e^{\nu\gamma^2 T}\sup_{t\in[0,T]}\left\|v_z^k - v_z^{k-1}\right\|_{H^{s}(\Omega)} \\
&\le TC(\gamma,s)e^{\nu\gamma^2 T}\sup_{t\in[0,T]}\left\|v^k - v^{k-1}\right\|_{H^{s+1}(\Omega)}. \qquad (1.104)
\end{aligned}
$$

Let

$$
T = \min\left\{\left[8C(\gamma,s)e^{\nu\gamma^2}\left(\|\omega_0\|_{H^s(\Omega)} + 2\|v_0\|_{H^{s+1}(\Omega)}\right)\right]^{-1}, \left[2C(\gamma,s)e^{\nu\gamma^2}\right]^{-1},\ \frac{1}{2M},\ 1\right\}. \qquad (1.105)
$$

Then, we have

$$
\left\|U^{k+1}-U^{k}\right\|_X \le \frac{1}{2}\left\|U^{k}-U^{k-1}\right\|_X .
$$

This proves that the sequence U^k converges to a fixed point of the map $\Phi : X \to X$, and the limiting fixed point $U = (v,\omega)$ is a solution of the 3D model with partial viscosity. Moreover, by passing to the limit in (1.98)–(1.99), we obtain the following *a priori* estimate for the solution (v,ω):

$$
\|v\|_{H^{s+1}(\Omega)} \le 2\|v_0\|_{H^{s+1}(\Omega)}, \qquad (1.106)
$$

$$
\|\omega\|_{H^{s}(\Omega)} \le C(\gamma,s)e^{\nu\gamma^2 T}\left(\|\omega_0\|_{H^s(\Omega)} + 2T\|v_0\|_{H^{s+1}(\Omega)}\right), \qquad (1.107)
$$

for $0 \le t \le T$ with T defined in (1.105).

It remains to show that the smooth solution of the 3D model with partial viscosity is unique. Let (v_1, ω_1) and (v_2, ω_2) be two smooth solutions of the 3D model with the same initial data and satisfying $\|v_i\|_{H^{s+1}(\Omega)} \leq M$ and $\|\omega_i\|_{H^s(\Omega)} \leq M$ for $i = 1, 2$ and $0 \leq t \leq T$, where M is a positive constant depending on the initial data as well as γ, s, and T. Since $s > 3/2$, the Sobolev embedding theorem [15] implies that

$$\|v_i\|_{L^\infty(\Omega)} \leq \|v_i\|_{H^{s+1}(\Omega)} \leq M, \quad i = 1, 2, \tag{1.108}$$

$$\|\mathcal{K}(\omega_i)_z\|_{L^\infty(\Omega)} \leq \|\mathcal{K}(\omega_i)_z\|_{H^s(\Omega)} \leq C_s \|\omega_i\|_{H^s(\Omega)} \leq C_s M, \quad i = 1, 2. \tag{1.109}$$

Let $v = v_1 - v_2$ and $\omega = \omega_1 - \omega_2$. Then (v, ω) satisfies

$$\begin{cases} v_t &= 4v\mathcal{K}(\omega_1)_z + 4v_2\mathcal{K}(\omega)_z \\ \omega_t &= v_z + \nu\Delta\omega \end{cases}, \quad (\mathbf{x}, z) \in \Omega = \Omega_{\mathbf{x}} \times (0, \infty), \tag{1.110}$$

with $\omega|_{\partial\Omega\setminus\Gamma} = 0, \quad (\omega_z + \gamma\omega)|_\Gamma = 0$, and $\omega|_{t=0} = 0, v|_{t=0} = 0$. By using (1.108)–(1.109), and proceeding as the uniqueness estimate for the heat equation in (1.53), we can derive the following estimate for v and ω:

$$\frac{d}{dt}\|v\|^2_{L^2(\Omega)} \leq C_1(\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}), \tag{1.111}$$

$$\frac{d}{dt}\|\omega\|^2_{L^2(\Omega)} \leq C_3(\|v\|^2_{L^2(\Omega)} + \|\omega\|^2_{L^2(\Omega)}), \tag{1.112}$$

where C_i $(i = 1, 2, 3)$ are positive constants depending on M, ν, γ, C_s. In obtaining the estimate for (1.112), we have performed integration by parts in the estimate of the v_z-term in the ω-equation and absorbing the contribution from ω_z by the diffusion term. There is no contribution from the boundary term since $v|_{z=0} = 0$. We have also used the property $\|\mathcal{K}(\omega)_z\|_{L^2(\Omega)} \leq C_s\|\omega\|_{L^2(\Omega)}$, which can be proved directly by following the argument in the Appendix of [26]. Since $v_0 = 0$ and $\omega_0 = 0$, the Gronwall inequality implies that $\|v\|_{L^2(\Omega)} = \|\omega\|_{L^2(\Omega)} = 0$ for $0 \leq t \leq T$. Furthermore, since $v \in H^{s+1}$ and $\omega \in H^s$ with $s > 3/2$, v and ω are continuous. Thus we must have $v = \omega = 0$ for $0 \leq t \leq T$. This proves the uniqueness of the smooth solution for the 3D model. □

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References

[1] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*. Comm. Math. Phys. **94** (1984), no. 1, 61–66.

[2] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771–831.

[3] R. Caflisch and M. Siegel, *A semi-analytic approach to Euler singularities*, Methods and Appl. of Analysis. **11** (2004), 423–430.

[4] D. Chae, A. Cordoba, D., Cordoba, and M. A. Fontelos, *Finite time singularities in a 1D model of the quasi-geostrophic equation*, Adv. Math. **194** (2005), 203-223.

[5] P. Constantin, *Note on loss of regularity for solutions of the 3D incompressible Euler and related equations*, Commun. Math. Phys. **104** (1986), 311-326.

[6] P. Constantin, C. Fefferman and A. Majda, *Geometric constraints on potentially singular solutions for the 3-D Euler equation*, Commun. in PDEs. **21** (1996), 559-571.

[7] D. Cordoba and C. Fefferman, *Growth of solutions for QG and 2D Euler equations*, J. Amer. Math. Soc., **15:3** (2002), pp. 665 - 670 (electronic).

[8] A. Cordoba, D., Cordoba, and M. A. Fontelos, *Formation of singularities for a transport equation with nonlocal velocity*, Ann. of Math. **162** (2005), 1–13.

[9] P. Constantin, P. D. Lax and A. J. Majda, *A simple one-dimensional model for the three-dimensional vorticity equation*, Comm. Pure Appl. Math. **38** (1985), no. 6, 715–724.

[10] J. Deng and T. Y. Hou and X. Yu, *Geometric properties and non-blowup of 3-D incompressible Euler flow*, Commun. PDEs **30** (2005), 225–243.

[11] J. Deng and T. Y. Hou and X. Yu, *Improved geometric conditions for non-blowup of 3D incompressible Euler equation*, Commun. PDEs **31** (2006), 293–306.

[12] S. De Gregorio, *On a one-dimensional model for the 3-dimensional vorticity equation*, J. Stat. Phys. **59** (1990), 1251–1263.

[13] S. De Gregorio, *A partial differential equation arising in a 1D model for the 3D vorticity equation*, Math. Method Appl. Sci. **19** (1996), no. 15, 1233–1255.

[14] R. J. DiPerna and P. L. Lions *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. Math. **130** (1989), 321-366.

[15] L. C. Evans, *Partial Differential Equations*, American Mathematical Society Publ., 1998.

[16] C. Fefferman, *http://www.claymath.org/millennium/Navier-Stokes equations*.

[17] G. B. Foland, *Real Analysis – Modern Techniques and Their Applications*, Wiley-Interscience Publ., Wiley and Sons, Inc., 1984.

[18] G. B. Foland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, N.J., 1995.

[19] T. Y. Hou and R. Li, *Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations*, J. Nonlinear Science **16** (2006), no. 6, 639–664.

[20] T. Y. Hou and R. Li, *Computing nearly singular solutions using pseudo-spectral methods* J. Comput. Phys. **226** (2007), 379-397.

[21] T. Y. Hou and R. Li, *Blowup or No Blowup? The Interplay between Theorey and Numerics*; Physica D. **237** (2008), 1937-1944.

[22] T. Y. Hou and C. Li, *Dynamic stability of the 3D axi-symmetric Navier-Stokes equations with swirl*, Comm. Pure Appl. Math. **61** (2008), no. 5, 661–697.

[23] T. Y. Hou and Z. Lei, *On the stabilizing effect of convection in 3D incompressible Flows*, Comm. Pure Appl. Math. **62** (2009), no. 4, 501–564.

[24] T. Y. Hou and Z. Lei, *On partial regularity of a 3D model of Navier-Stokes equations*, Commun. Math Phys., **287** (2009), 281-298.

[25] T. Y. Hou, C. Li, Z. Shi, S. Wang, and X. Yu, *On singularity formation of a nonlinear nonlocal system*, Arch. Ration. Mech. Anal **199** (2011), 117-144.

[26] T. Y. Hou, Z. Shi, and S. Wang, *On singularity formation of a 3D model for incompressible Navier-Stokes equations*, arXiv:0912.1316v1 [math.AP], submitted to Adv. Math..

[27] T. Y. Hou and Z. Shi, *Dynamic Growth Estimates of Maximum Vorticity for 3D Incompressible Euler Equations and the SQG Model*, DCDS-A **32 (5)** (2011).

[28] R. Kerr, *Evidence for a singularity of the three dimensional, incompressible Euler equations*, Phys. Fluids **5** (1993), 1725-1746.

[29] R. Kerr, *Velocity and scaling of collapsing Euler vortices*, Phys. Fluids **17** (2005), 075103-114.

[30] D. Li and J. Rodrigo, *Blow up for the generalized surface quasi-geostrophic equation with supercritical dissipation*, Comm. Math. Phys. **286(1)** (2009), 111–124.

[31] D. Li and Y.G. Sinai, *Blow ups of complex solutions of the 3D Navier-Stokes system and renormalization group method*, J. Europ. Math. Soc., **10(2)** (2008), 267-313.

[32] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[33] T. Matsumotoa, J. Becb and U. Frisch, *Complex-space singularities of 2D Euler flow in Lagrangian coordinates*, Physica D. **237** (2007), 1951-1955.

[34] G. Prodi, *Un teorema di unicità per le equazioni di Navier-Stokes.* Ann. Mat. Pura Appl. **48** (1959), 173–182.

[35] J. Serrin, *The initial value problem for the Navier-Stokes equations.* Nonlinear Problems, Univ. of Wisconsin Press, Madison, 1963, 69–98.

[36] R. Temam, *Navier-Stokes Equations*. Second Edition, AMS Chelsea Publishing, Providence, RI, 2001.