A Discontinuous Galerkin Method for BSSN-Type Systems

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Outline

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Introduction

- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) system is a popular formulation of the Einstein equations used for numerical evolutions
- Typical applications include binary black hole simulations
- We work with the Generalized BSSN (GBSSN) system[1]

What are the differences from traditional BSSN?

[1] David Brown, arXiv: 0501092

Metric in ADM form

We may write the full spacetime metric metric as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(\alpha^2 - \gamma_{ij}\beta^i\beta^j)dt^2 + 2\gamma_{ij}\beta^j dt dx^i + \gamma_{ij} dx^i dx^j,$$

Lapse α, shift β^i, and spatial metric γ_{ij}

- **Conformal spatial metric** (χ's weight to be specified)

$$\gamma_{ij} \equiv \chi^{-1}\bar{\gamma}_{ij}$$

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- Thus the conformal metric is a usual tensor
- Not necessarily unit determinant
- Must specify how the conformal metric's determinant evolves

The GBSSN choice leads to... (a very small sampling)

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$$\mathcal{L}_n \bar{A}_{ij} = \frac{1}{3}\bar{A}_{ij}\mathcal{L}_n \ln\bar{\gamma} + K\bar{A}_{ij} - 2\bar{A}_{ik}\bar{A}^k{}_j + \chi\left(R_{ij} - \frac{1}{\alpha}D_i D_j \alpha\right)^{\mathrm{TF}}$$

Preview of results

- To date BSSN-type codes are based on finite difference methods
- We present a high-order accurate discontinuous Galerkin scheme for GBSSN
- We directly discretize the second order spatial operators. Fewer variables and no extra constraints to worry about

We will specialize to spherically symmetric solutions with comments towards a 3D solver

Outline

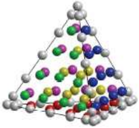
The discontinuous Galerkin method: A hybrid of methods

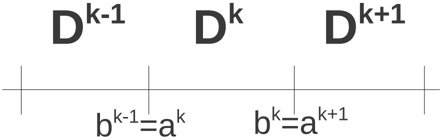
- **Spectral methods:** approximate solutions by expanding them in a basis
- **Finite element methods:** integrate the residual against a set of test functions
- **Finite volume methods:** elements coupled via FV *numerical* fluxes, when the basis functions are constants dG formally is a FV method

Will develop the dG method in 4 steps, with 1 step per slide

DG method: space (step 1 of 4)

- Approximate physical domain Ω by subdomains D^k such that $\Omega \sim \Omega_h = \cup_{k=1}^{K} D^k$
- In general the grid is unstructured. We choose lines, triangles, and tetrahedrons for 1D, 2D, and 3D respectively.





DG method: solution (step 2 of 4)

- Local solution expanded in set of basis functions

$$x \in D^k : \Psi_h^k(x,t) = \sum_{i=0}^{N} \Psi_h^k(x_i,t) l_i^k(x)$$

- Polynomials span the space of polynomials of degree N on D^k.
- Global solution is a direct sum of local solutions

$$\Psi_h(x,t) = \bigoplus_{k=1}^{K} \Psi_h^k(x,t)$$

- Solutions double valued along point, line, surface.

DG method: residual (step 3 of 4)

- Consider a model PDE

$$L\Psi = \partial_t \Psi + \partial_x f = 0,$$

  where Ψ and $f = f(\Psi)$ are scalars.

- Integrate the residual $L\Psi_h$ against all basis functions on D^k

$$\int_{D^k} (L\Psi_h)\, l_i^k(x) dx = 0 \qquad \forall i \in [0, N]$$

- We still must couple the subdomains D^k to one another...

DG method: numerical flux (step 4 of 4)

- To couple elements first perform IBPs

$$\int_{D^k} \left(l_i^k \partial_t \Psi_h - f\left(\Psi_h\right) \partial_x l_i^k \right) dx = - \oint_{\partial D^k} l_i^k \hat{n} \cdot f^*\left(\Psi_h\right)$$

  where the *numerical* flux is $f^*\left(\Psi_h\right) = f^*\left(\Psi^+, \Psi^-\right)$
- Ψ^+ and Ψ^- are the solutions exterior and interior to subdomain D^k, restricted to the boundary
- **Example**: Central flux $f^* = \frac{f(\Psi^+)+f(\Psi^-)}{2}$
- Passes information between elements, implements boundary conditions, and ensures stability of scheme
- Choice of f^* is, in general, problem dependent

We have finished

Remark: The term 'nodal discontinuous Galerkin' should now be clear. We seek a global discontinuous solution interpolated at nodal points and demand this solution satisfy a set of integral (Galerkin) conditions.

Final comments

- Timestep with a classical 4^{th} order Runge-Kutta
- Robust for hyperbolic equations as we *directly* control the scheme's stability through a numerical flux choice
- For a smooth enough solution, numerical error decays exponentially with polynomial order N

Stable treatment of second order operators

Semi-discrete stability

Generic framework in place, specification of numerical flux remains

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Generic framework in place, specification of numerical flux remains

- We should hope the result is semi-discrete stable
  - Stablity after spatial discretization
- Extensive literature on fully first order hyperbolic systems (e.g. Lax-Friedrichs flux)
- Strange terms like χ'', $(\chi')^2$, and $\alpha'\chi'$. What to do?

Second order operators: Key new feature

Consider a model problem ($a \geq 1$ for real speeds)

$$\begin{aligned} \partial_t u &= u' + av - u^3 \\ \partial_t v &= u'' + v' - (u+v)(u')^2 + v^2 u^2, \end{aligned}$$

- Techniques used to treat this system used for GBSSN

Second order operators: Key new feature

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- First rewrite as

$$\partial_t u = Q + av - u^3$$
$$\partial_t v = Q' + v' - (u+v)Q^2 + v^2 u^2$$
$$Q = u' \qquad Q \textit{ not } \text{evolved}$$

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- First rewrite as, and we presently specialize to

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$$\partial_t v = Q' + v' - (u+v)Q^2 + v^2 u^2$$
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Follow the previous discontinuous Galerkin construction

$$\int_{\mathsf{D}^k} l_i^k \partial_t u_h = \int_{\mathsf{D}^k} l_i^k (Q_h + a v_h)$$

$$\int_{\mathsf{D}^k} l_i^k \partial_t v_h = -\int_{\mathsf{D}^k} l_i^{k\prime} (Q_h + v_h) + \int_{\partial \mathsf{D}^k} l_i^k (Q^* + v^*)$$

$$\int_{\mathsf{D}^k} l_i^k Q_h = -\int_{\mathsf{D}^k} l_i^{k\prime} u_h + \int_{\partial \mathsf{D}^k} l_i^k u^*,$$

- Q_h is constructed and substituted, thus we see Q is not evolved
- The key is specifying a form for Q^*, u^*, and v^*, such that the resulting scheme is stable.

Stability of the continuum system

Notice that the continuum system (with $Q = u'$) with periodic boundary conditions satisfies

$$\frac{1}{2}\partial_t \int_\Omega \left[av^2 + Q^2\right] = 0.$$

- Mimic this estimate for our dG scheme. We seek...

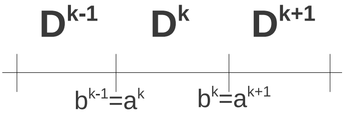
Stability of the continuum system

Notice that the continuum system (with $Q = u'$) with periodic boundary conditions satisfies

$$\frac{1}{2}\partial_t \int_\Omega \left[a v^2 + Q^2 \right] = 0.$$

- Mimic this estimate for our dG scheme. We seek...

$$\frac{1}{2}\partial_t \sum_{k=1}^{k_{\max}} \int_{\mathsf{D}^k} (Q_h^2 + a v_h^2) \leq 0$$

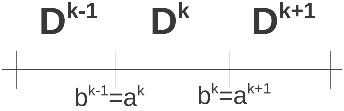


At each subdomain interface

$$\{\{v_h\}\} = \frac{1}{2}\left(v^L_{k+1/2} + v^R_{k+1/2}\right)$$
$$[[v_h]] = v^L_{k+1/2} - v^R_{k+1/2}.$$

Consider numerical fluxes of the form (No need to diagonalize!)

$$Q^* = \{\{Q_h\}\} - \frac{\tau_Q}{2}[[Q_h]]$$
$$v^* = \{\{v_h\}\} - \frac{\tau_v}{2}[[v_h]]$$
$$u^* = \{\{u_h\}\} - \frac{\tau_u}{2}[[u_h]]$$



Integrate to internal boundaries

$$\frac{1}{2}\partial_t \sum_{k=1}^{k_{\max}} \int_{\mathrm{D}^k} (Q_h^2 + a v_h^2) = \sum_{k=1}^{k_{\max}-1} \left(\text{interface terms}\right)\Big|_{\mathrm{I}^{k+1/2}}$$

With our choice of numerical flux each subdomain interface term is

$$-\frac{a\tau_v}{2}\left[\!\left[v_h\right]\!\right]^2 - \frac{a(\tau_u+\tau_Q)}{2}\left[\!\left[Q_h\right]\!\right]\left[\!\left[v_h\right]\!\right] - \frac{\tau_u}{2}\left[\!\left[Q_h\right]\!\right]^2.$$

Role of penalties

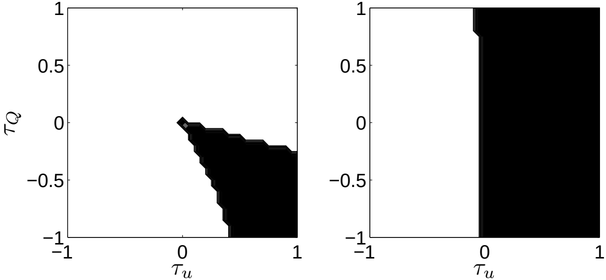


Figure: The left ($\tau_v = 10^{-6}$) and right ($\tau_v = 1 + \sqrt{2}$) plots depict stable choices (determined empirically) of τ_u and τ_Q for the linear model system. The stable regions are colored black, but the jagged edges result from the discretization of the (τ_u, τ_Q)-plane.

Back to GBSSN

We work with the spherically symmetric version to demonstrate the general method. Discontinuous Galerkin method directly applies

- Introduce *locally* constructed auxiliary variables
  - For example $Q_\chi = \chi'$
- Solution is a sum over interpolating polynomials
- Integrate against test functions
- We use the same penalty choice as discussed for model problem

Outline

Choices

- Evolution for conformal metric's determinant $\partial_t \bar{\gamma} = 0$
  - Used to replace $\mathcal{L}_n \ln \bar{\gamma}$ throughout system
- 1+log and Gamma-driver evolution for the lapse and shift (standard choice)

Conformal Kerr-Schild Initial data

Physical metric is

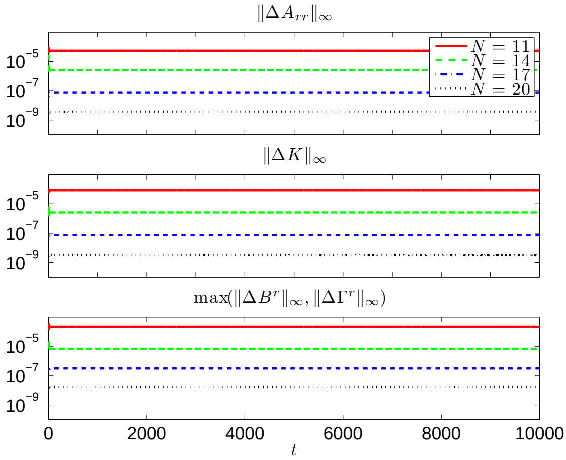
$$ds^2 = -\alpha^2 dt^2 + (1 + 2M/R)(dR + \beta^R dt)^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2$$

the lapse $\alpha = (1 + 2M/R)^{-1/2}$ and shift $\beta^R = 2M/(R + 2M)$

- Conformal metric determinant $\bar{\gamma}$ is not unity
- Spherically symmetric, analytic, coordinates pass through the horizon
- Inner boundary is outflow, singularity treated by excision

Stability and convergence (with polynomial order N)

$\Omega = [.3, 4]$ and $M = 1$, left boundary inside the event horizon



Stability and convergence

- Other fields show similar convergence
- Hamiltonian, momentum, and conformal connection constraints converge
- A variety of M were tested, similarly a variety of domain sizes and locations
- Perturbing all fields leads to a stable scheme

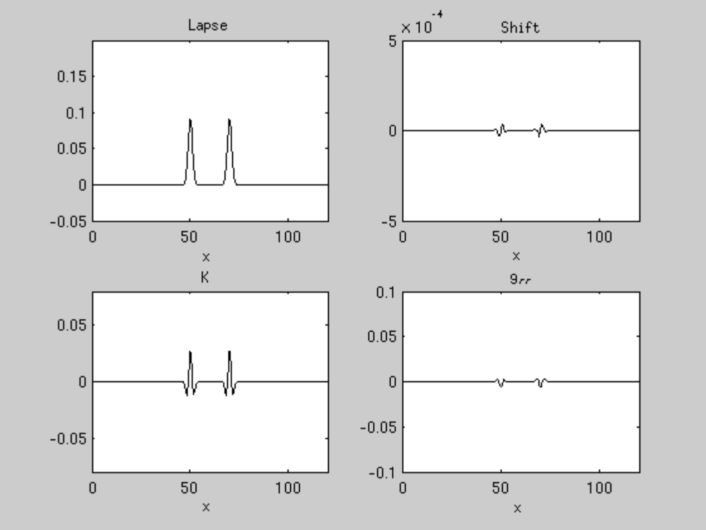
Main result: We conclude that the scheme is stable in 1D

Time dependent solutions

We can perturb the initial data

$$\alpha = \alpha_{\mathrm{KS}} + \frac{1}{10}\exp\left(-\frac{1}{2}(R-50)^2\right) + \frac{1}{10}\exp\left(-\frac{1}{2}(R-70)^2\right)$$

Results


Lapse
0.15
0.1
0.05
0
-0.05
0
50
100
x
Shift
× 10^-4
5
0
-5
K
0.05
0
-0.05
g_rr
0.1
0.05
0
-0.05
-0.1

Outline

The GBSSN System

A Nodal Discontinuous Galerkin Method

Results

Conclusions and Outlook

Future work: Puncture evolutions

If one does not use excision...

- Quantities diverge like powers of $1/r$ near a singularity
- Very successful in finite difference codes

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As we use subdomains, $r = 0$ should be included. Some ideas to try[2]

- Gauss-Radau points remove the $r = 0$ node, likely a 1D trick
- When our basis functions are constants, dG is a first order finite volume method
- Turducken (smooth stuffing) around the singularity, perhaps repeatedly

These may require singularity tracking (vanishing lapse, distribution of solution's modes, etc)

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Future work: 3D solver

- Both theory and applications are well-developed for 3D hyperbolic problems
  - Open source projects like HEDGE are available (Andreas' Sunday talk)
- To-do list: punctures and generalization of our numerical flux choice
- Questions...
  - What elements to use? Cubes? Tetrahedrons? Spheres?
  - Polynomial or tensor product basis?

Potential benefits of a 3D solver

Potentially useful when...

- High-order accuracy needed
- Matter fields are present (including shocks)[3]
- Different length scales are present, can use local timestepping techniques
  - Δt might be different in each subdomain

[3] David Radice and Luciano Rezzolla, arXiv: 1103.2426

What has been done

- Brief remarks on (G)BSSN system
- Introduced a discontinuous Galerkin method
- Developed a stable and exponentially convergent scheme
  - Key part is treatment of second order spatial operators
- Highlighted potential future work and challenges

QUESTIONS?

Metric in ADM form

We may write the full spacetime metric metric as

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -(\alpha^2 - \gamma_{ij}\beta^i\beta^j)dt^2 + 2\gamma_{ij}\beta^j dt dx^i + \gamma_{ij}dx^i dx^j,$$

- γ_{ij} is the **spatial metric** for 3D spatial slice
- α is the **lapse**
- β^i is the **shift**

Extrinsic Curvature:

$$K_{ij} \equiv -\frac{1}{2}\mathcal{L}_n\gamma_{ij} = -\frac{1}{2}\frac{1}{\alpha}\left(\partial_t - \mathcal{L}_\beta\right)\gamma_{ij}$$

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$$K_{ij} = \chi^{-1}\left(\bar{A}_{ij} + \frac{1}{3}\bar{\gamma}_{ij}K\right)$$

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3. **Conformal connection functions**

$$\bar{\Gamma}^i \equiv \bar{\gamma}^{jk}\bar{\Gamma}^i_{jk}$$

The variables are χ, $\bar{A}_{ij}$, K, $\bar{\gamma}_{ij}$, α, β^i, $\bar{\Gamma}^i$

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GBSSN evolution system (a small sampling)

$$\mathcal{L}_n\chi = \frac{\chi}{3}\left(\mathcal{L}_n \ln\bar{\gamma} + 2K\right),$$

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