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Basis Functions for Neutron Star Interiors

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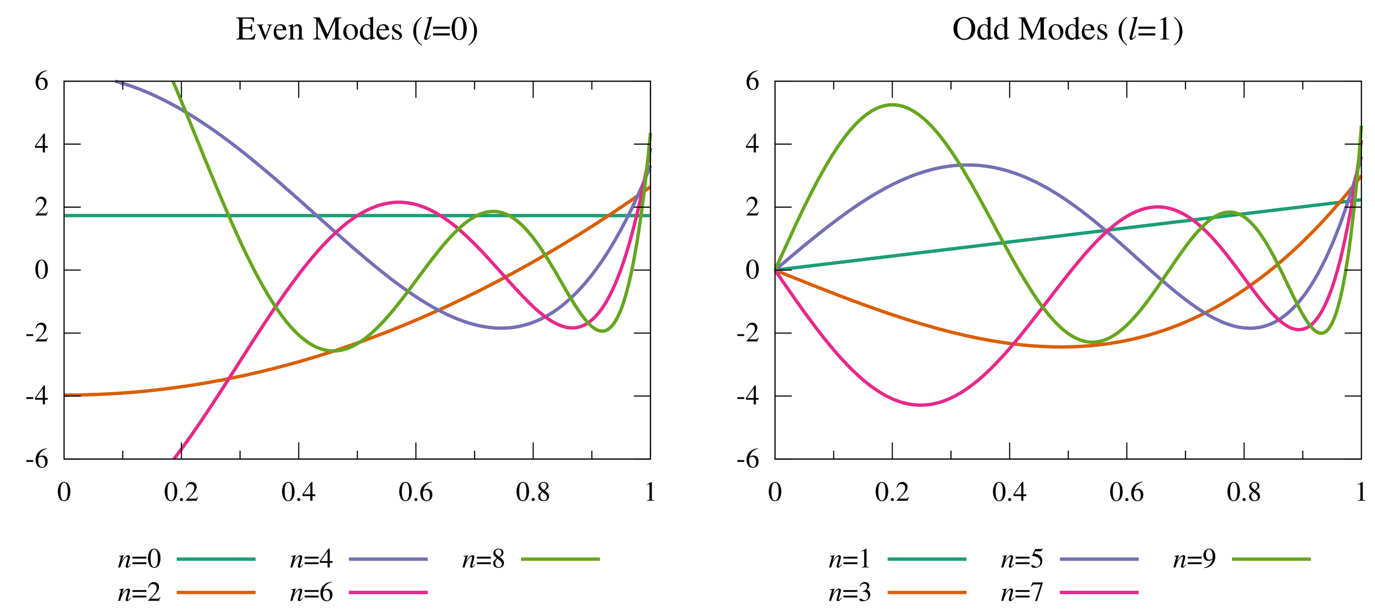
The Pole Condition and Pole Problem

Problem: Find a function basis on B^3 (the filled sphere) in spherical coordinates satisfying the pole condition in the radial direction ($f_{lm}(r) \to r^l$ as $r \to 0$) without severely restricting the CFL stability limit.

Solution: Follow example of spherical harmonics; expand Y_l^m coefficients on $r \in (0, 1]$ in a polynomial basis unique to each l.

One-Sided Jacobi Polynomials

The polynomials $Q_n^l(r) \equiv r^l P^{(0, l+\frac{1}{2})}_{\frac{n}{2}-\frac{l}{2}}(2r^2 - 1)$ are orthogonal w.r.t. the weight $w = r^2$, manifestly satisfy the pole condition, maintain the parity of l, and are solutions to a singular Sturm-Liouville problem – an ideal radial basis for B^3. Furthermore, their low resolution near the origin avoids the "pole problem" of restricted timesteps. We denote their normalized forms as Φ_n^l, some of which are plotted below:



Just as $l \geq |m|$ for spherical harmonics Y_l^m, here $n \geq l$ for Φ_n^l. In both cases, this is just another statement of the pole condition.

The cylindrical variants ($w = r$), proposed by Matsushima & Marcus and Verkley, are discussed thoroughly in the literature. Here we develop the spherical case, filling in the details needed for pseudospectral work.

($P_k^{(\alpha,\beta)}(x)$ is the Jacobi polynomial, whose integration weight is "one sided" for $\alpha = 0$)

Spectral and Physical Resolution

Using Gauss-Radau quadrature to place collocation points on the outer boundary while avoiding the origin, the highest modes we can resolve are:

$$m_{\max} = \lfloor N_\phi/2 \rfloor + 1\,, \quad l_{\max} = N_\theta - 1\,, \quad n_{\max} = 2N_r - 2$$

Combined with the pole condition, this places the following constraints on physical resolution:

$$N_\theta \geq \lfloor N_\phi/2 \rfloor + 2\,, \quad 2N_r \geq N_\theta + 1$$

Other resolutions result in a valid method, but imply higher spectral resolution than will be achieved. **One must be very careful to satisfy constraints both from quadrature and from the pole condition.**

As is the case with spherical harmonics, there are more collocation points than spectral coefficients (roughly 4 times as many). Transforming to spectral space is therefore a *projection* and is not invertible for functions that are not band-limited.

Spectral Decomposition

This basis is not used to represent the radial dependence of collocation values. Rather, it is used to expand the radial dependence of Y_l^m coefficients:

$$f(r,\theta,\phi) \sim \sum_{m=-m_{\max}}^{m_{\max}} \sum_{l=|m|}^{l_{\max}} \left(\sum_{n=l}^{n_{\max}} f_{nlm} \Phi_n^l(r) \right) Y_l^m(\theta,\phi)$$

If $f_{lm}(r_i)$ are the spherical harmonic coefficients for the function evaluated at radius r_i, then the final spectral coefficients are

$$f_{nlm} = \sum_{i=0}^{N_r-1} f_{lm}(r_i) \Phi_n^l(r_i) w_i^r\,,$$

which are easily computed with a matrix multiplication transform.

Interpolation & Differentiation

Interpolation and differentiation are most easily performed in the hybrid space of $f_{lm}(r_i)$. However, as the the order of our polynomials is greater than the number of radial collocation points, we must apply a few tricks to take advantage of the functions' known parity.

Define $x_i \equiv r_i^2$. Then the following functions are of sufficiently low order for interpolation:

$$g_{lm}(x_i) = f_{lm}(r_i) \qquad l \text{ even}$$
$$g_{lm}(x_i) = f_{lm}(r_i)/r_i \qquad l \text{ odd}$$

To get first and second derivatives, evaluate g' and g'' using Fornberg's differentiation matrices. Then:

$$f'_{lm}(r_i) = 2r g'_{lm}(x_i) \qquad f''_{lm}(r_i) = 2[g'_{lm}(x_i) + 2r^2 g''_{lm}(x_i)] \qquad l \text{ even}$$
$$f'_{lm}(r_i) = g_{lm}(x_i) + 2r^2 g'_{lm}(x_i) \qquad f''_{lm}(r_i) = 2r[3g'_{lm}(x_i) + 2r^2 g''_{lm}(x_i)] \qquad l \text{ odd}$$

Note that partial derivatives in r (or θ) are *not* representable in the basis. Instead, one must handle either $r\partial_r$ or Cartesian derivatives.

Integration

Due to our choice of weight function, integration can be efficiently performed as a weighted sum of collocation values. No transform into spectral space is required.

$$\int_0^1 \int_0^\pi \int_0^{2\pi} f(r,\theta,\phi) r^2 \sin(\theta) d\phi d\theta dr = \frac{2\pi}{N_\phi} \sum_{i=0}^{N_r-1} w_i^r \sum_{j=0}^{N_\theta-1} w_j^\theta \sum_{k=0}^{N_\phi-1} f(r_i,\theta_j,\phi_k)$$

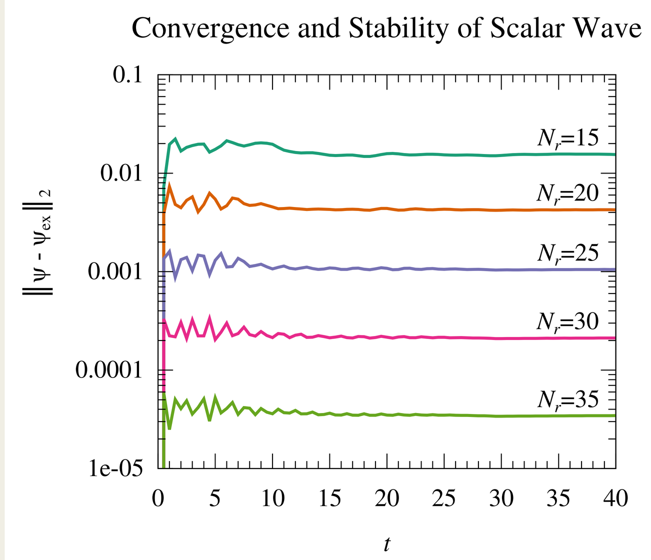
(w_j^θ are the weights associated with the spherical harmonic transform)

Power Monitoring and Filtering

As with spherical harmonics, mode mixing in angular differentiation requires that one filter out the highest l modes after each timestep. This procedure does not interact with the radial basis.

We have not found radial filtering to be necessary for stability, but if required, the notation used here allows consistent filtering on the index n. This index can also label the power in each mode, keeping in mind that any given radial expansion consists of either all even or all odd n. A more holistic approach is to filter and monitor on $\lfloor n/2 \rfloor$.

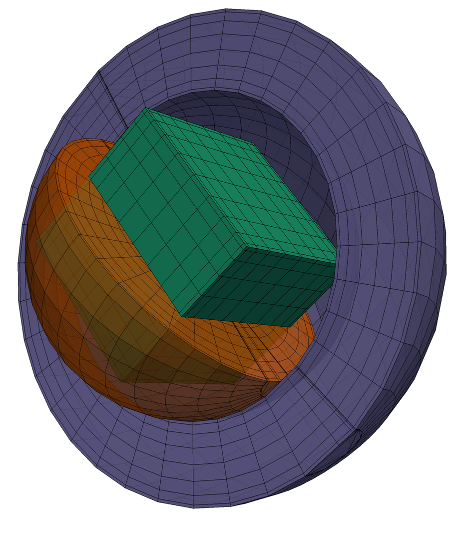
Spherical Scalar Wave

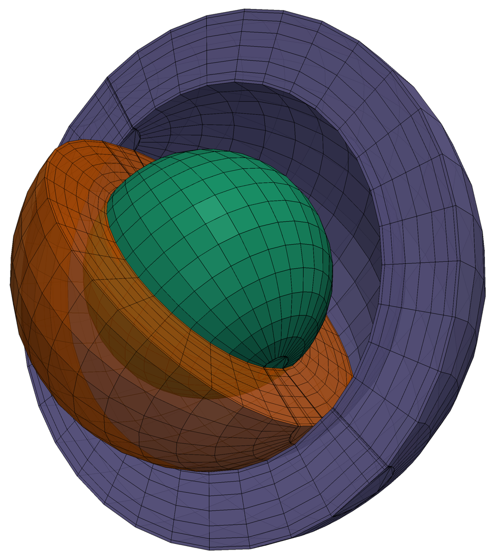


Evolving a spherical scalar wave with a Gaussian profile is both stable and exponentially convergent. Computational cost per timestep is less than that of $I^1 \times S^2$ in our implementation. Filtering is applied to the evolved fields after each full timestep, reducing the required number of spectral transforms.

Neutron Stars

While we still use finite difference methods to evolve neutron star matter, we can now solve for the metric quantities inside the star using these B^3 basis functions. We previously used Chebyshev polynomials on $(I^1)^3$ in the center of the star to avoid the coordinate singularity. Now, neutron stars can now be decomposed into fewer, larger, all-conforming subdomains, eliminating the need to interpolate boundary information and reducing the total number of gridpoints.





Evolving a TOV star shows considerable performance gains. These domains have since been used by Matt Duez and Francois Foucart in simulating black hole – neutron star inspirals.

Conclusions

Advantages
- Elegant and efficient treatment of spherical domains
- Avoids pole problem

Disadvantages
- Interpolation onto hydro grid is expensive
- Implementation requires careful attention to details

References and Acknowledgements

- T. Matsushima and P. S. Marcus, A spectral method for polar coordinates, *J. Comput. Phys.* **120**, 365 (1995)
- P. W. Livermore, C. A. Jones, and S. J. Worland, Spectral radial basis functions for full sphere computations, *J. Comput. Phys.* **227** 1209 (2007)

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