Mesh Generation With Large Deformations and Topological Changes

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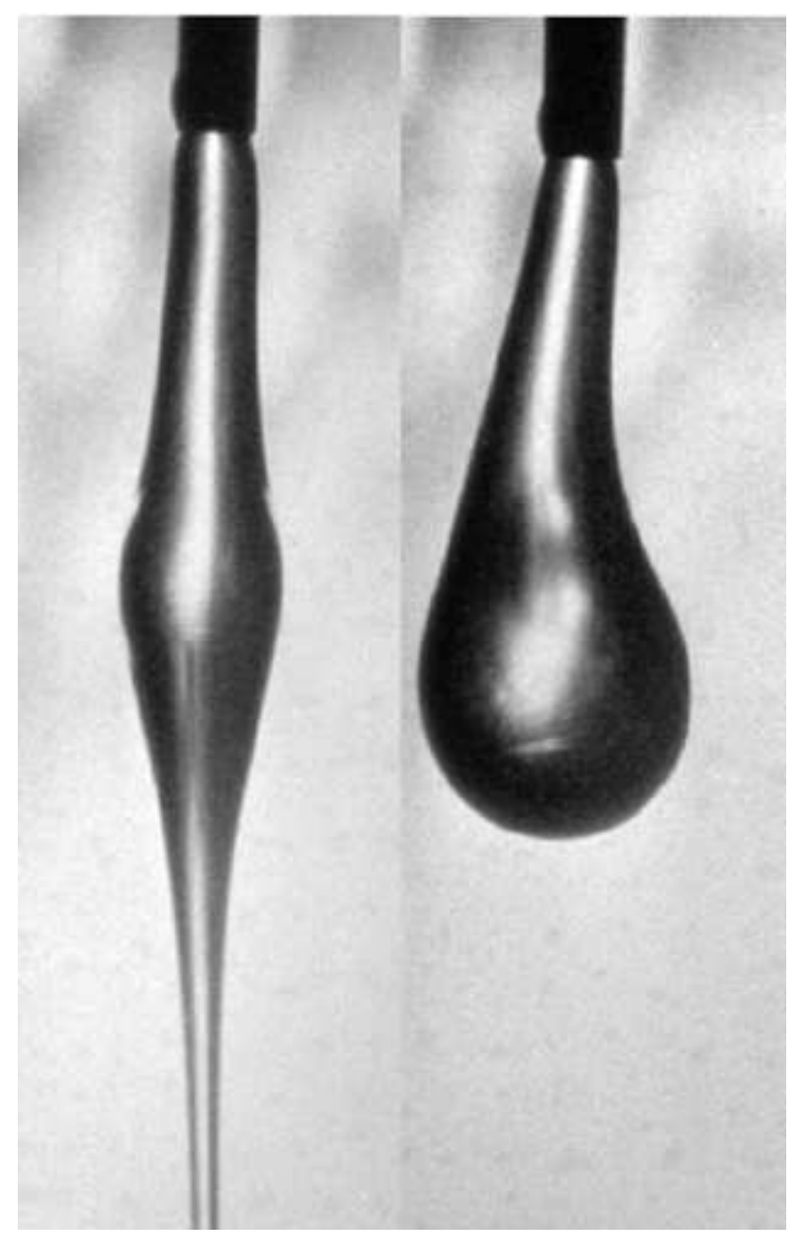
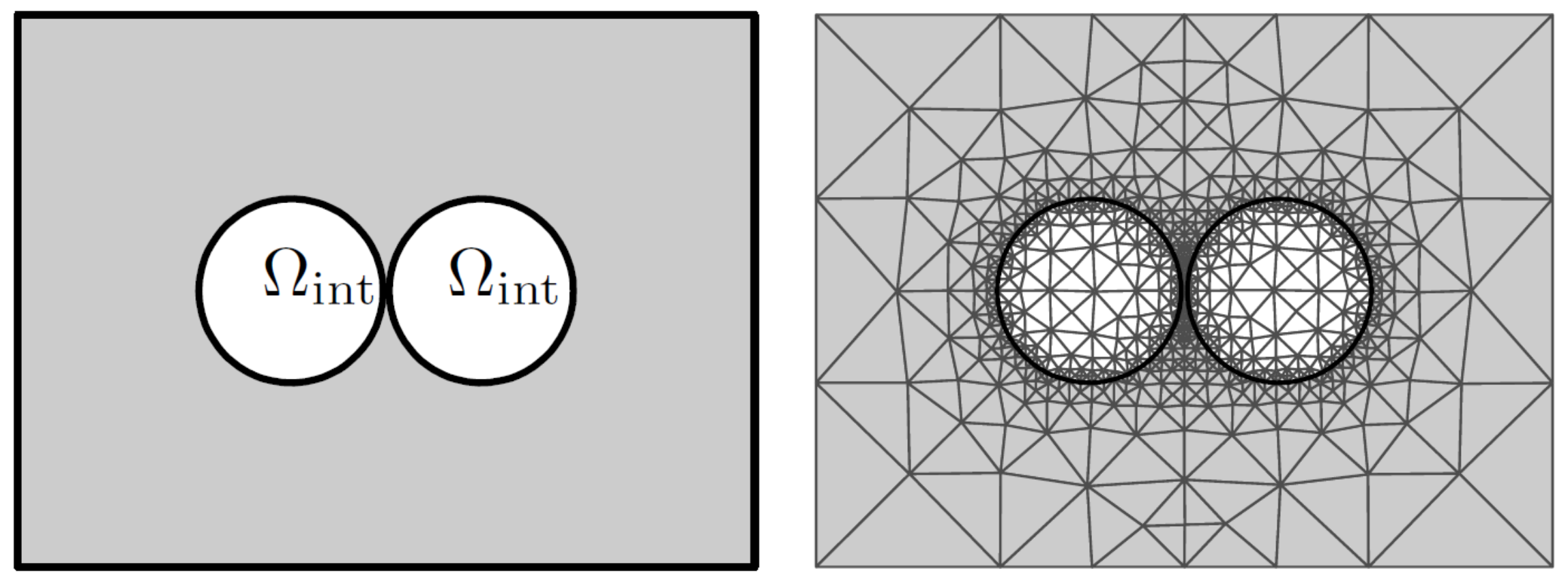
Tracking Time-Dependent Domains



Free boundary problems arise in many areas of mathematics, physics, and engineering. Understanding free surface dynamics is important for applications such as coating flows [4], simulating water wave dynamics for computer graphics [9], and surface tension/curvature driven flows in micro-fluidic devices such as Hele-Shaw flow [7, 10, 16, 14]. Other examples involve fluid-structure interactions [13, 17, 1].
However, in any application with a moving boundary, the deformation of the domain is the main obstacle in obtaining a tractable physical model. In addition, some of these applications exhibit topological changes (i.e. pinching or joining of disjoint parts of the interface) and prove even more difficult to model [3, 6, 15, 8].

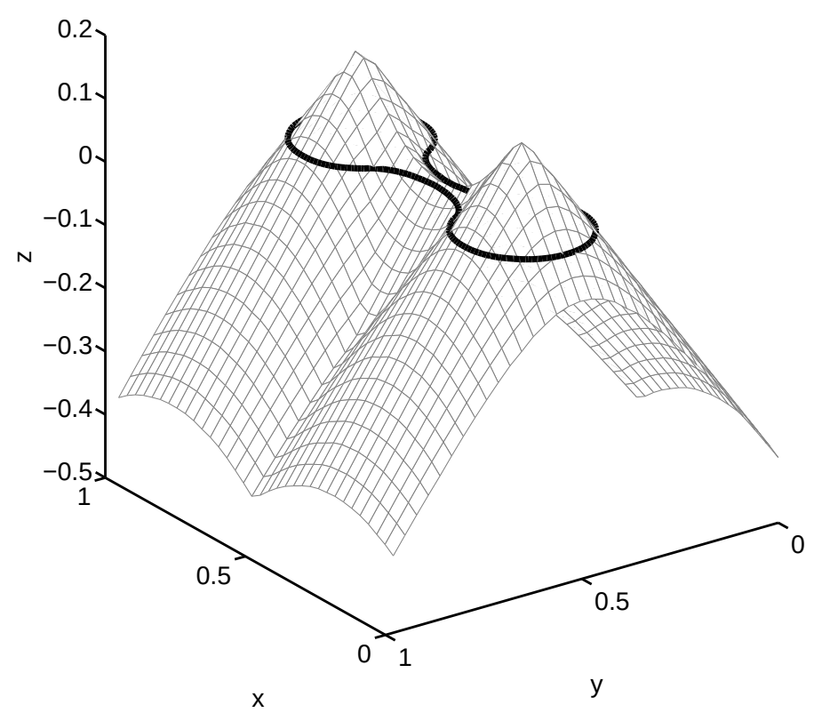
Result: We develop a method for generating explicit unstructured triangular meshes in 2-D that conform to a smooth closed curve and can be used with Arbitrary-Lagrangian-Eulerian (ALE) methods [11]. The method can also be generalized to 3-D tetrahedral meshes [5, 12, 2].

Mesh Generation



- FEM is a flexible tool to solve many kinds of multi-physics problems.
- General domain shapes can be considered, but generating a mesh for an arbitrary domain can be expensive. This issue is further exacerbated in free boundary problems.

Large Deformations and Topological Changes

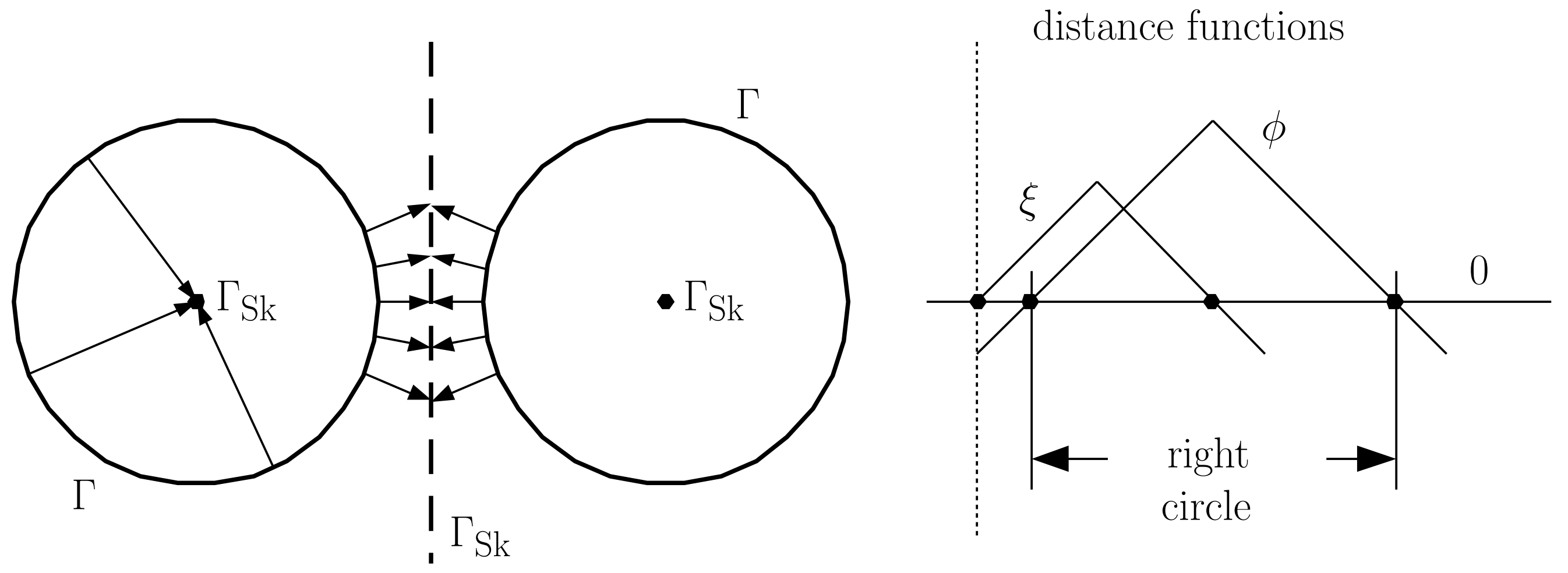


- **Level Set Methods:**
- Advantages: Eulerian Grid. Automatic handling of topological changes.
- Issues: Accuracy of boundary quantities. Mass conservation.
- **Variational/Front Tracking:**
- Advantages: Geometry is accurately represented. Mass is conserved.
- Issues: Mesh distortion and remeshing.
- **Hybrid Mesh Generation:** W. and Nochetto, JCP, 2010.

Main Ideas

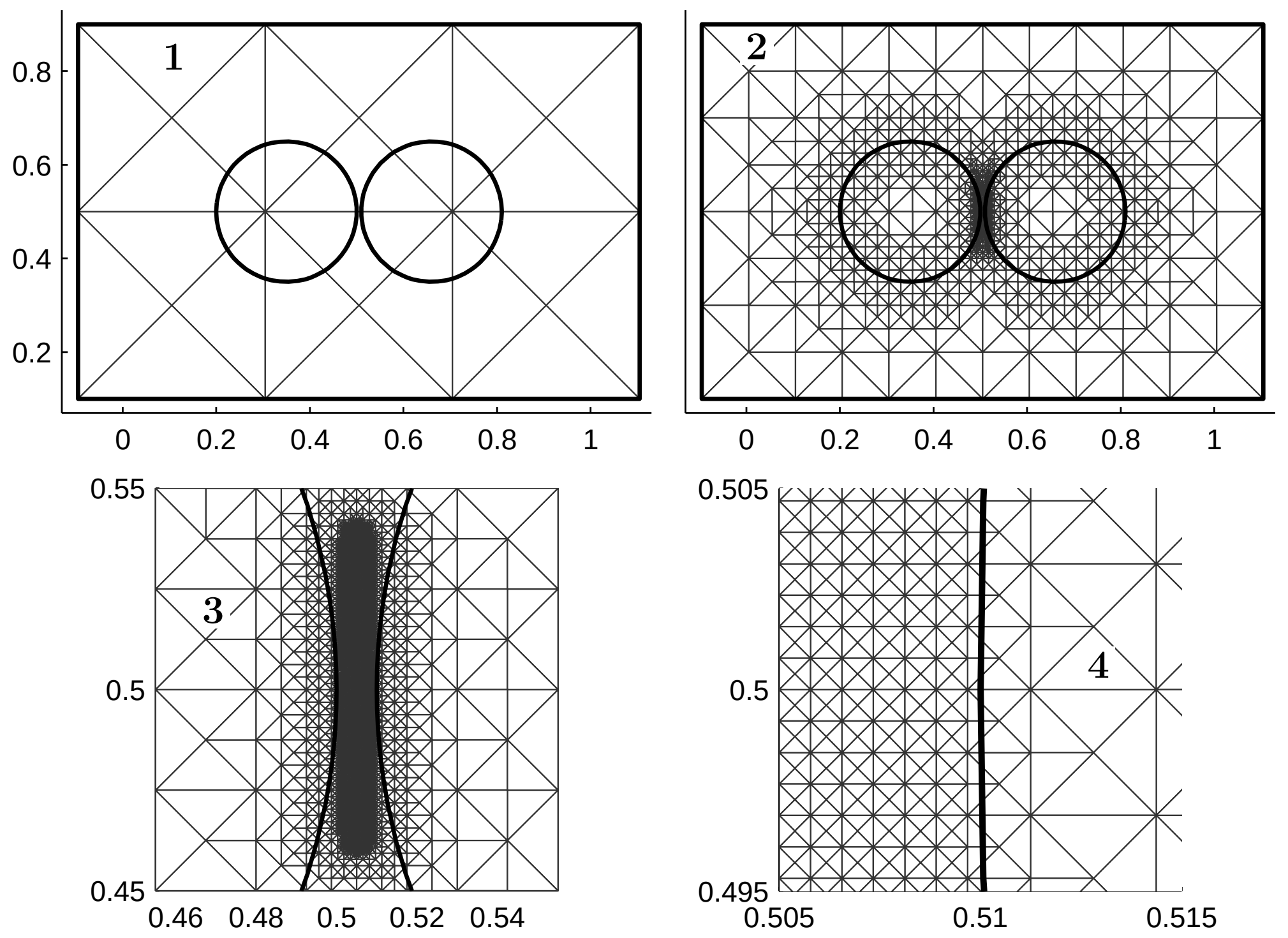
- Simulate physics on a deforming domain by a variational/ALE front-tracking method.
- Use a level set formulation **only** when a new mesh must be generated.
- The *geometry* of the domain is encoded implicitly by the distance function to the manifold Γ.
- The *topology* of the domain is captured by the shape skeleton, which derives directly from the signed distance function.
- Generate mesh by adaptive refinement guided by the distance function and shape skeleton.

Distance Functions And Shape Skeletons



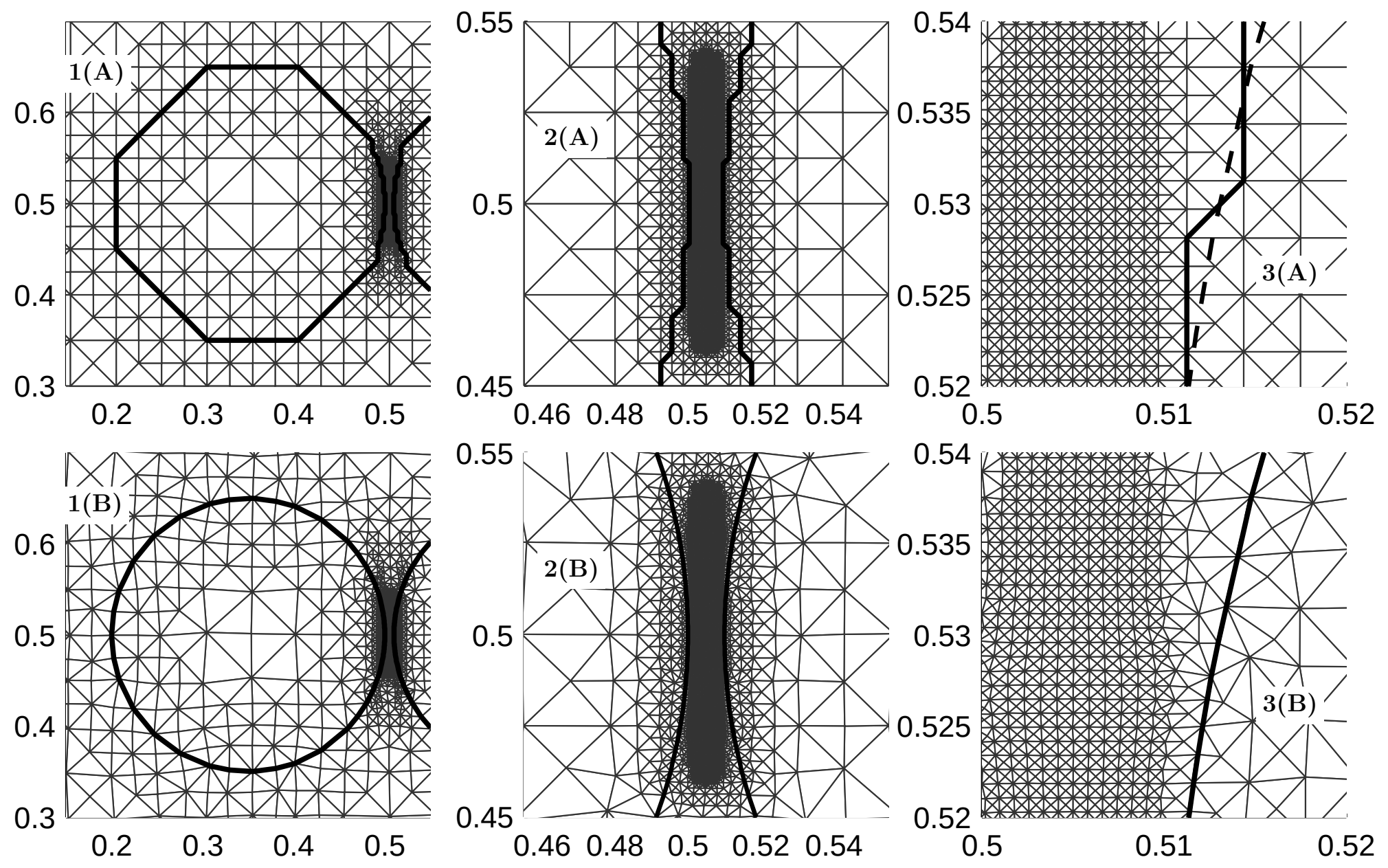
- Compute the signed distance function ϕ (to Γ) on the previous unstructured triangular mesh; $O(N_V^{1+1/d})$ operations. Possibility of parallelizing.
- Straightforward processing of ϕ gives an approximation of the **shape skeleton**, i.e. look for jumps in $\nabla\phi$; $O(N_V + N_E + N_T)$ operations.
- $\Rightarrow$ Let Γ_{Sk} be the approximate shape skeleton.
- **Crucial step:** compute the distance function ξ (to Γ_{Sk}) on the previous mesh.

Adaptive Refinement



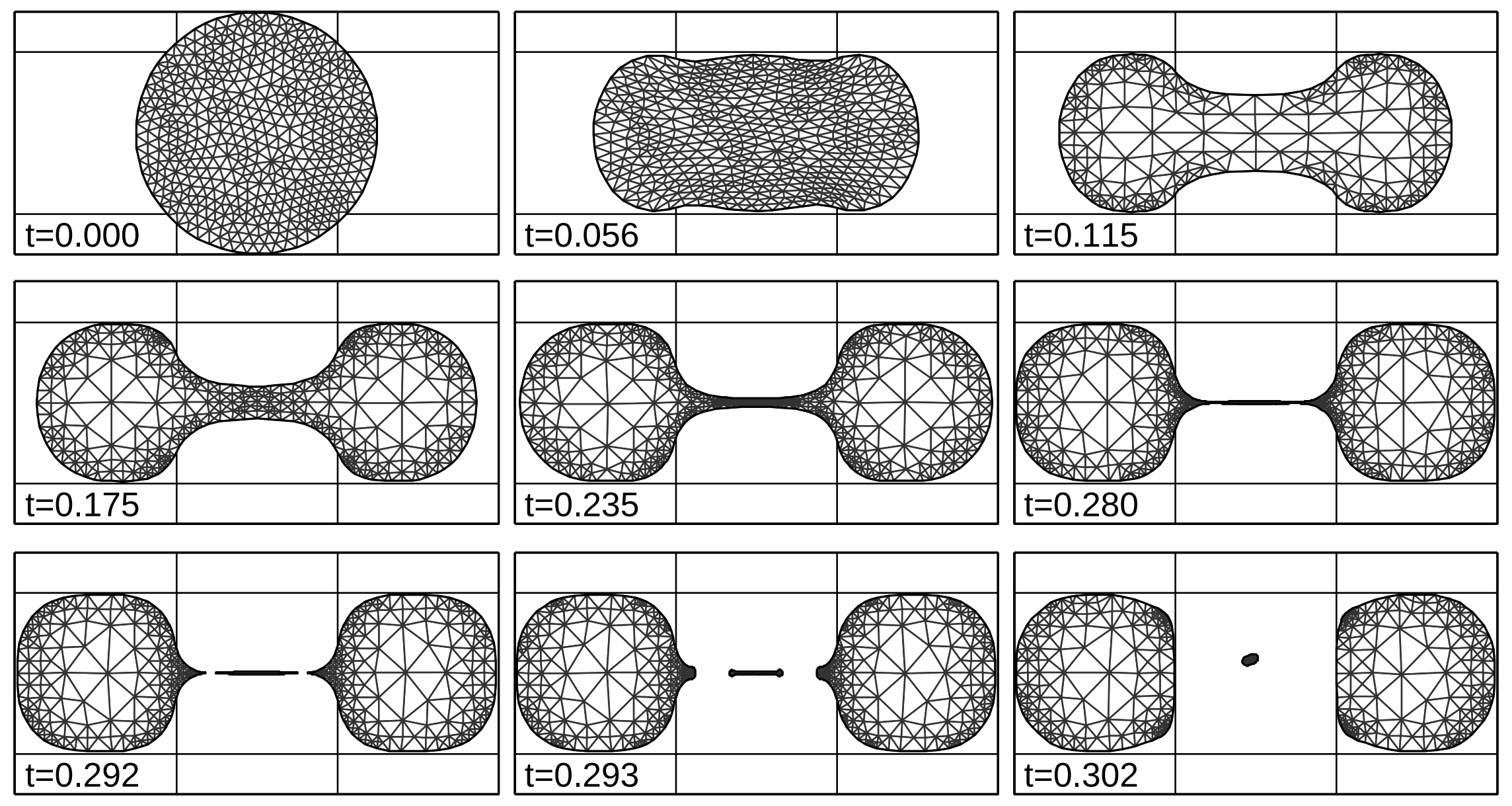
- Interpolate distance functions ϕ and ξ onto the new mesh vertices. Excessive searching can be avoided by taking advantage of the sequentially refined grids.
- Adaptively refine (by Rivara bisection) all triangles until no triangle overlaps both Γ and Γ_{Sk}
- This is an $O(N_V^{\text{new}})$ algorithm and is *guaranteed to terminate* for smooth manifolds. Resulting grid resolves shape topology, e.g. fingering.
- Further refinement achieved via *curvature* estimation; $O(N_V^{\text{new}}(\Gamma))$.

Candidate Manifold Selection

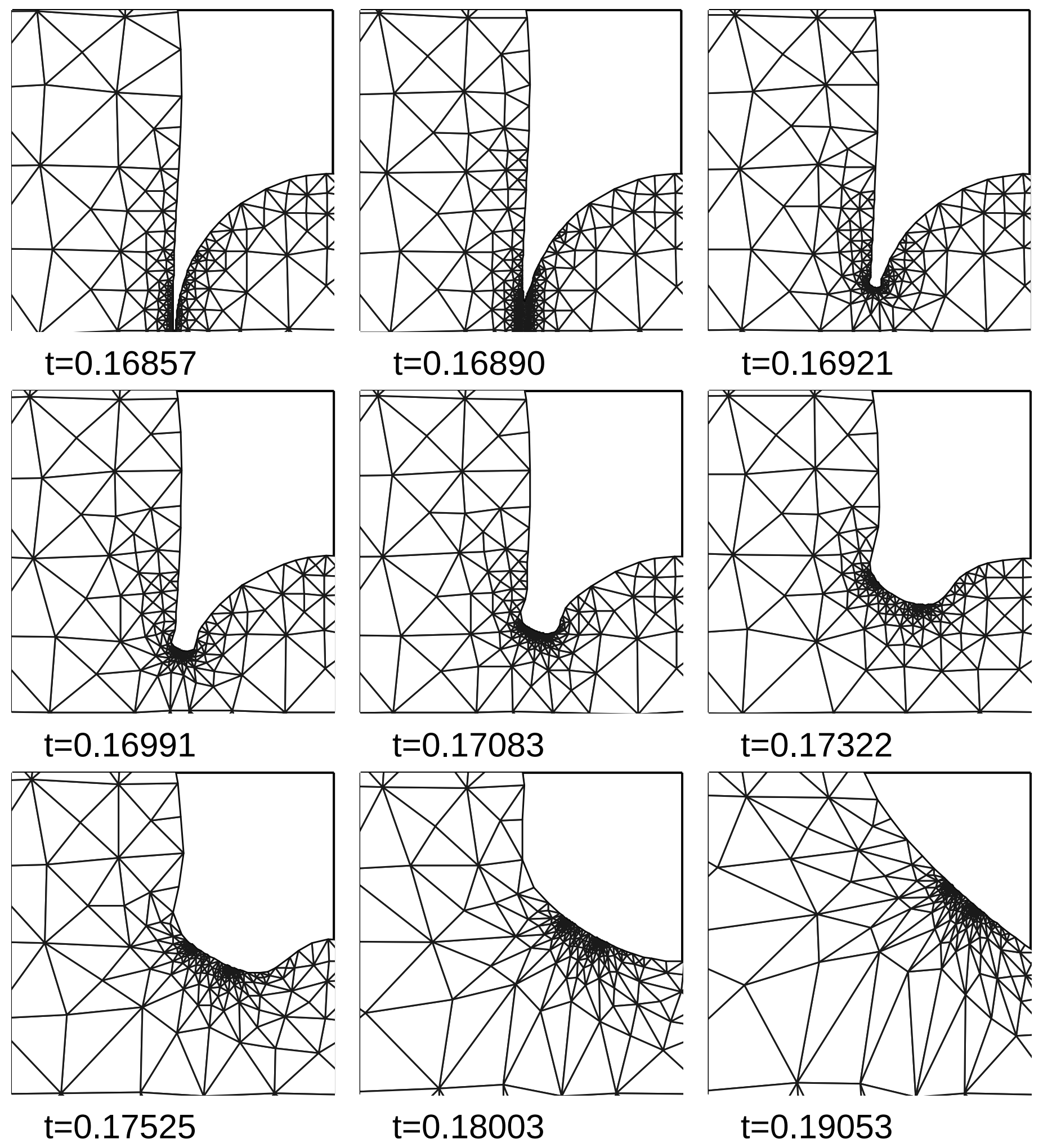


- Interior and exterior triangles are selected via the distance function. This produces a **candidate manifold** of edge segments; $O(N_T^{\text{new}})$. This is the most critical part.
- Even with minor pruning, this will introduce an **aliasing effect**.
- Mesh conformity is ensured via a *shape optimization* technique; $O(N_V^{\text{new}}(\Gamma))$.

Results



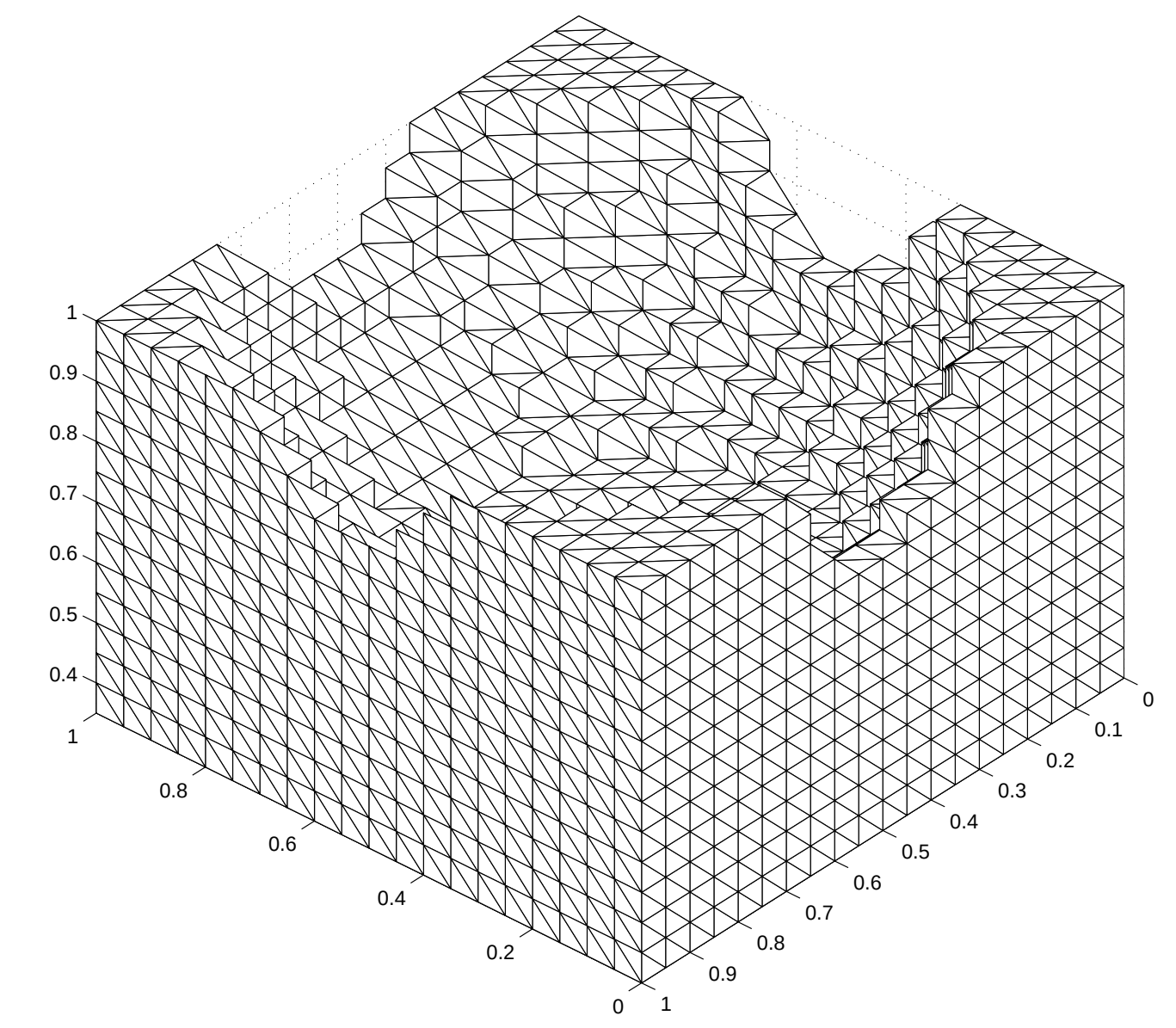
- EWOD driven droplet splitting; only the interior mesh is shown.
- The droplet pinches in two places (symmetrically) resulting in an elongated satellite droplet.



- Two droplets joining by surface tension; zoom-in of sharp reconnection region. Relaxation is very fast due to high curvature.

Extension To 3D

Selecting The Candidate Manifold



- Everything generalizes to 3-D, *except* the candidate manifold selection process. Must ensure that no tetrahedra will get crushed during the mesh conforming phase.
- This will require a more complicated manifold selection process, which may include edge/face swapping to give a well-formed polyhedral surface.

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