Constraint preserving boundary conditions for the Z4 formulation of general relativity

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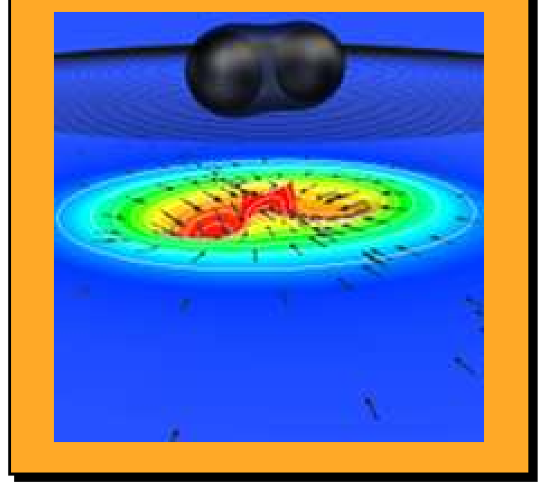
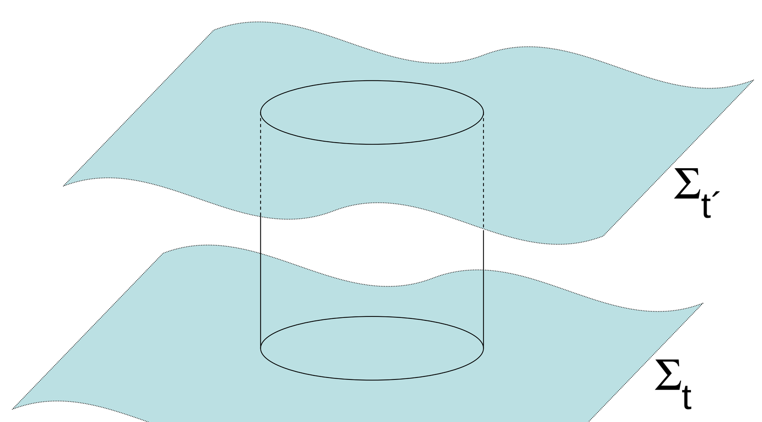


We discuss high order absorbing constraint preserving boundary conditions for the conformal decomposition of the Z4 formulation of general relativity coupled to the moving puncture family of gauges. Using a Kreiss' theory we prove well-posedness of the initial boundary value problem with a particular choice of the puncture gauge in the frozen coefficient approximation. Numerical evidence for the efficacy of the first and second order boundary conditions in constraint preservation and absorption is compared with the standard sommerfeld conditions in the evolution of flat, spherical black-hole and neutron star spacetimes.

1. Introduction

Numerical simulations of general relativity typically introduce an artificial time-like outer boundary $\partial\Sigma$.



This boundary requires conditions which ought to render the initial boundary value problem (IBVP) well-posed. Well-posedness is the requirement that the solution of the IBVP should be unique and depend continuously upon given initial and boundary data.

What is the challenge?

The boundary should be specified so that

- is compatible with the constraints,
- controls the incoming modes,
- yields a well-posed IBVP.

What is the problem?

- The system is constrained.
- Gauge freedom
- No local expressions for in/outgoing modes,
- The geometry is not known.

Formulations

The two most popular choices of GR in use in numerical relativity today are the generalized harmonic gauge (GHG) and the BSSN formulations [1, 2, 3, 4].

- GHG: since the system has a very simple wave-equation structure in the principal part, significant progress has been made in the construction of both continuum and discrete boundary conditions [5, 6, 7, 8, 9].
- BSSN: Sommerfeld boundary conditions are the most common in use in applications, despite the fact that it is not known whether or not they result in a well-posed IBVP. Recently, Núñez and Sarbach have proposed in [11] CPBCs for this system.
- Z4: this formulation is formally equivalent to GHG when it is coupled to the generalized harmonic gauge. Additionally, it is possible to recover the BSSN from Z4 by freezing one of the constraint variables. In this sense Z4 may be thought of as a generalization of both BSSN and GHG. Boundary conditions for a first order reduction of Z4 have been specified and tested numerically in [10].

2. The Z4 formulation

The Z4 formulation [12, 13, 14] takes the 4-dimensional Einstein equations and replaces them by

$$R_{ab} + \nabla_a Z_b + \nabla_b Z_a = 8\pi\left(T_{ab} - \frac{1}{2} g_{ab} T\right), \qquad (1)$$

where Z_a is a 4-vector of constraints. Solutions of Eq. (1) are also valid solutions of the Einstein equations when the constraints Z_a vanish. From the PDEs point of view, the most important part of the constraint addition is that of the partial derivatives.

We $3+1$ decompose the system. This has the undesirable effect of breaking the 4-covariance of the Z4 formulation. However, the evolution equations can be written very similarly to BSSN. The time-evolution equations are

$$\partial_t \gamma_{ij} = \mathcal{L}_\beta \gamma_{ij} - 2\alpha K_{ij}, \qquad (2)$$

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha\left[R_{ij} - 2K_{ik}K^k_j + K_{ij}K + 2\,\partial_{(i}Z_{j)}\right] + \mathcal{L}_\beta K_{ij} + 4\pi\alpha[\gamma_{ij}(S - \rho_{\text{ADM}}) - 2S_{ij}], \qquad (3)$$

$$\partial_t \Theta = \alpha\left[\frac{1}{2}H + \partial_k Z^k\right] + \beta^i \Theta_{,i}, \qquad (4)$$

$$\partial_t Z_i = \alpha\, M_i + \alpha\,\Theta_{,i} + \beta^j Z_{i,j}\,. \qquad (5)$$

where Z_i is just the spatial projection of Z_a and $\Theta = -n_a Z^a$.

- The conformal decomposition of this formulation was recently presented [15].

Puncture gauge conditions

The most popular gauge choice in the numerical evolution of dynamical spacetimes is the puncture gauge. By introducing scalar functions $(\mu_L, \mu_S, \epsilon_\alpha, \epsilon_\chi)$ the general form of the gauge (without introducing the additional field B^i) is

$$\partial_t \alpha = \beta^i \alpha_{,i} - \mu_L \alpha^2 \hat{K}\,, \qquad (6)$$

$$\partial_t \beta^i = \beta^j \beta^i{}_{,j} + \mu_S \tilde{\Gamma}^i - \eta\,\beta^i - \epsilon_\alpha\,\alpha\,\alpha^{,i} + \epsilon_\chi\,\tilde{\gamma}^{ij}\partial_j \chi\,. \qquad (7)$$

Note that in this condition we have included a new term proportional to the spatial derivative of χ [16].

- The standard choices are $\mu_L = 2/\alpha$, $\mu_S = 3/4$ and $\epsilon_\alpha = \epsilon_\chi = 0$.

High order constraint preserving boundary conditions

To specify boundary conditions we define the background outgoing characteristic vectors and a null vector

$$\mathring{l}^a = \frac{1}{\sqrt{2}}\left(\mathring{n}^a + \mathring{s}^a\right), \qquad \mathring{k}^a = \frac{1}{\sqrt{2}}\left(\mathring{n}^a + \sqrt{\mu_S}\,\mathring{s}^a\right), \qquad (8)$$

$$\mathring{m}^a = \frac{1}{\sqrt{2}}\left(\mathring{n}^a + \sqrt{\mu_L}\,\mathring{s}^a\right), \qquad \hat{m}^a = \frac{1}{\sqrt{2}}\left(\mathring{\omega}^a + i\,\mathring{v}^a\right). \qquad (9)$$

Since the constraints Θ, Z_i satisfy wave equations, the constraint preserving boundary conditions in the linear regime around the background are given by

$$\left(r^2\,\mathring{l}^a \partial_a\right)^L \Theta \mathrel{\hat{=}} 0\,, \qquad \left(r^2\,\mathring{l}^a \partial_a\right)^L Z_i \mathrel{\hat{=}} 0\,. \qquad (10)$$

where $L \geq 0$ is an integer and $\hat{=}$ denotes equality in the boundary $\mathcal{T}$. Note that this conditions are also absorbing boundary conditions.

We assume that, both the physical and background metrics, are sufficiently close to flat so that the full system has ten incoming characteristic variables at the boundary. This determines the number of boundary conditions we may specify. The boundary conditions (10) give four of the total. We take for the remaining

$$\left(r^2\,\mathring{m}^a \partial_a\right)^{L+1} \alpha \;\hat{=}\; h_\alpha\,, \qquad \left(r^2\,\mathring{k}^a \partial_a\right)^{L+1} \beta_s \mathrel{\hat{=}} h_s\,, \qquad (11)$$

$$\left(r^2\,\mathring{l}^a \partial_a\right)^{L+1} \beta_A \;\hat{=}\; h_A\,, \qquad \left(r^2\,\mathring{l}^a \partial_a\right)^{L+1} \gamma^{\text{TF}}_{AB} \mathrel{\hat{=}} h^{\text{TF}}_{AB}\,, \qquad (12)$$

where $h_\alpha, h_i, h^{\text{TF}}_{AB}$ are given boundary data. One can extend the above results to consider the standard puncture gauge. We consider the so-called freezing-Ψ_0 boundary condition in term of the electric and magnetic part of Z4 [17]

$$\Psi_0 = \left(E^{\text{TT}}_{ij} - i\,B^{\text{TT}}_{ij}\right)\hat{m}^i\,\hat{m}^j \mathrel{\hat{=}} q_{\hat{m}\,\hat{m}}\,. \qquad (13)$$

Numerical Test

The necessity of CPBCs for the Z4 system is not only motivated by the requirement of having a mathematical well-posed system, but also by the numerical evidence of artifacts and/or instabilities related to a bad or naive implementation of the boundary conditions. An example of this numerical artifact is show in Fig. 1 where the time evolution of the central rest-mass density of a equilibrium model of spherical compact star obtained with the Z4c formulation.

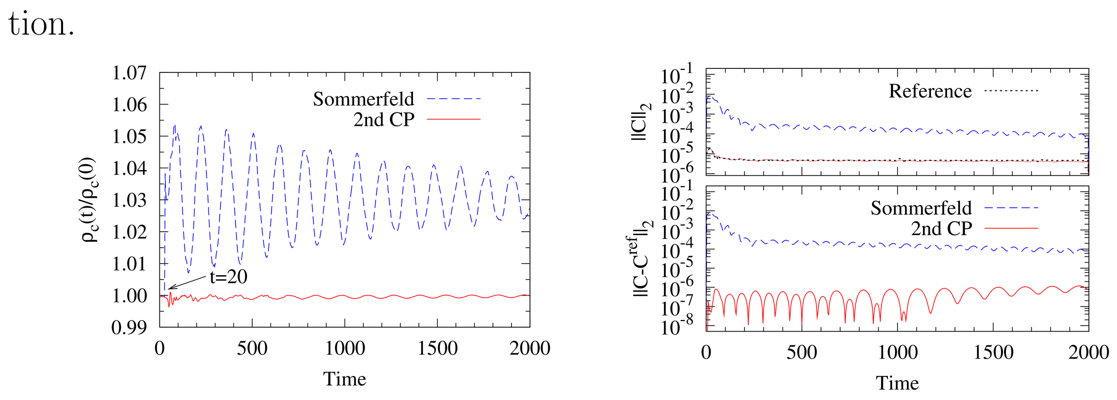


FIGURE 1: **Left:** *radial oscillations of a compact stars' central rest mass density in time. The outer boundary is at $r_{out} = 20$. Incoming constraint violation from the outer boundary perturbs the star at $t = 20$.* **Right:** *norm of the constraint violation in the same numerical test.*

- We perform several tests, in each case with Sommerfeld, first and second order constraint preserving boundary conditions.
- To examine the stability and the effect of the BCs we consider simulations with very a close outer boundary ($r_{\text{out}} \simeq 20\,M$) and compare the results with a *reference simulation* [8], in which the outer boundary is placed far away ($r'_{\text{out}} \simeq 1000\,M$) from the origin.
- To globally monitor the constraint violation we define the quantity:

$$C \equiv \sqrt{H^2 + M^i M_i + \Theta^2 + Z^i Z_i}\,, \qquad (14)$$

and we will refer as the *constraint monitor.* We will make often use of 2-norms of quantities:

$$||C(\cdot,t)||_2 \equiv \sqrt{\int dr\, r^2 C(r,t)^2}\,. \qquad (15)$$

where in practical computations the integral is performed on the grid by the trapezium rule. For a fair comparison with the "investigated" solution, the norm of the reference solution is taken only on the domain, $r \in (0, r_{\text{out}})$.

Since most of the presented analytical results have been obtained by using the new shift condition ($\epsilon_\chi = 1/2$), we numerically investigated this gauge as well as standard puncture gauge. We found in all the cases comparable results (see *e.g.* Fig. 4), so most of the results are presented for the more popular puncture gauge in order to give numerical evidence of what we cannot prove.

Perturbed flat spacetime. Evolution of constraint violating initial data on flat space. Here we focus on convergence and constraint absorption. We find near-perfect constraint transmission of the constraints when using the second order CPBCs.

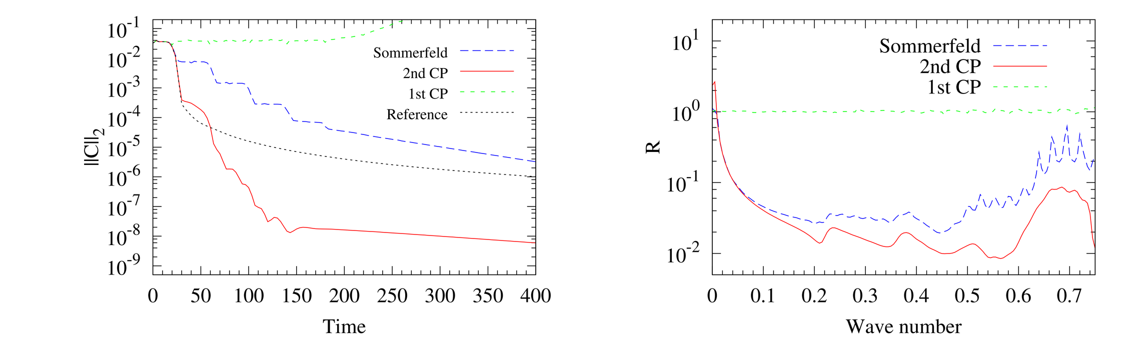


FIGURE 2: **Left:** *Constraint violation in flat spacetime test. The 2-norm of the constraint monitor is showed in time for different BCs implemented. The same quantity for the reference simulation is showed.* **Right:** *Experimental reflection coefficient in flat spacetime test. The experimental reflection coefficient, defined in Eq. (??), is plotted versus the wave number for different BCs implemented.*

Star spacetime. Evolution of a stable compact star. In the Sommerfeld case, non convergent reflections from the boundary effect the dynamics of the star. The absorbing CPBCs completely solve this problem.

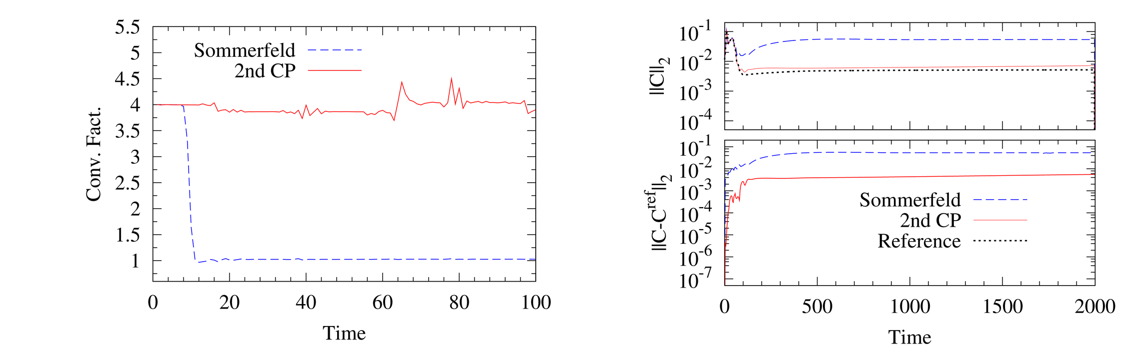


FIGURE 3: **Left:** *Convergence factor in flat spacetime test. The self-convergence factor is compute from the 2-norms of three simulations at different resolutions and showed in time for 2nd order CPBCs and Sommerfeld BCs.* **Right:** *Constraint violation in black hole spacetime test. Upper panel: The 2-norm of the constraint monitor is showed in time for different BCs implemented. The same quantity for the reference simulation is showed (black dotted line). Bottom panel: The 2-distance of the constraint monitor with the reference simulation is showed in time for different BCs implemented.*

Black hole spacetime. Evolution of black hole initial data. The robustness and performances of CPBCs have been tested against black holes spacetime with different initial data and gauges.

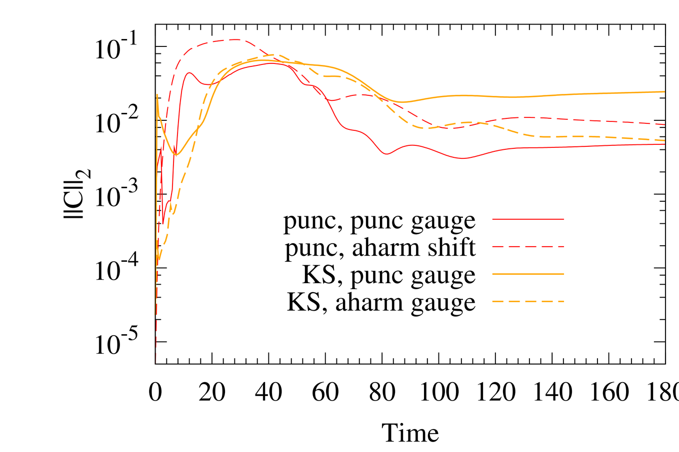


FIGURE 4: *Constraint violation in black hole spacetime test, comparison of different initial data and gauges. The 2-norm of the constraint monitor is showed in time for 2nd order CPBCs. The evolutions refer to puncture initial data showed with puncture shift (red solid line) and asymptotically harmonic shift (red dashed line) and to Kerr-Schild initial data (evolved with excision) with puncture shift (thick orange solid line) and asymptotically harmonic shift (thick orange dashed line).*

3. Well-posedness

A method to demonstrate well-posedness is based on the frozen coefficient principle. In this approach one freezes the coefficients of the equations of motion and the boundary operators. Therefore, the IBVP is simplified to a linear, constant coefficient problem which is solved using a Laplace-Fourier transformation. Sufficient conditions for the well-posedness of the frozen coefficient problem were developed by Kreiss in [18] if the system is strictly hyperbolic. Using that theory, it can be constructed a smooth symmetrizer with which well-posedness can be shown using an energy estimate in the frequency domain. We use those results to prove the well-posedness of the IBVP for Z4 with high order constraint preserving boundary conditions.

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