Control Systems in Binary Black Hole Evolutions

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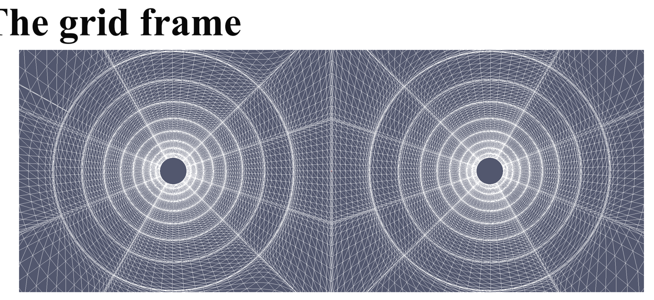
"We cannot direct the wind, but we can adjust the sails." ~Anonymous

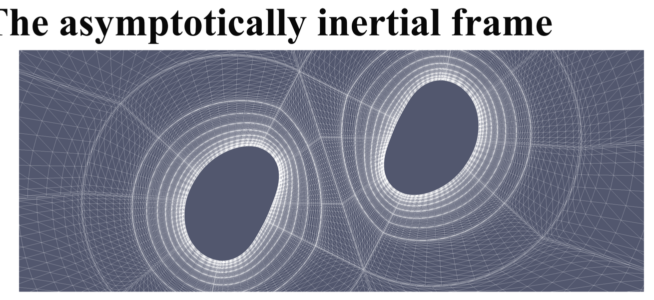
Introduction

Over the past several years, aspects of operations research and control theory have been integrated into the Spectral Einstein Code (SpEC) as part of the dual frames approach to solving the moving excision problem. The two frames are:

The grid frame



The asymptotically inertial frame



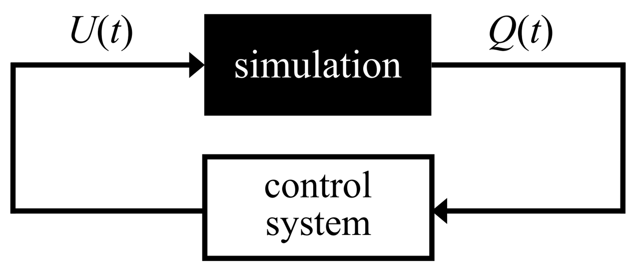
In the grid frame, fixed excision regions necessitate stationary black holes, and the mapping between the two frames is governed by a control system.

Today, we control not just the trajectory of the black holes, but also the size and shape of their apparent horizons. The mapping tracks the horizons as they distort, which is most crucial during the initial relaxation and the merger, to ensure that the excision boundary conditions remain satisfied.

I. Control Basics

The basic function and structure of a control system is easily understood as feedback within the framework of signal processing.

Within the black box of a simulation there is some measure of error, $Q(t)$. This acts as the input for a control system, which then has to find some control signal, $U(t)$, that will minimize the error when fed back into the simulation.



A simple and effective way to compute $U(t)$ is to make it a linear combination of integrals and/or derivatives of the error, $Q(t)$:

- **Proportional**

$$U(t) = K_p\, Q(t)$$

- **Proportional-Derivative**

$$U(t) = K_p\, Q(t) + K_d\, dQ/dt$$

- **Proportional-Integral-Derivative (PID)**

$$U(t) = K_p\, Q(t) + K_i \int Q(t)dt + K_d\, dQ/dt$$

II. Implementation

Control systems are implemented in SpEC through the parameters of the mappings between the asymptotically inertial and non-inertial frames:

- Scaling, a
- Rotation angles, (θ,ϕ)
- Translation, (T_x,T_y,T_z)
- Apparent horizon Y_{lm} coefficients, λ_{lm}
- Characteristic speeds, $d\lambda_{00}/dt$

We represent a generic map parameter, Λ, as an N^{th}-degree Taylor polynomial, where the control signal acts as the N^{th} derivative:

$$\Lambda(t) = \sum_{n=0}^{N-1} \frac{1}{n!}(t-t_i)^n \frac{d^n \Lambda(t_i)}{dt^n} + \frac{1}{N!}(t-t_i)^N U(t_i)$$

Because binary evolutions are a time-dependent problem, the point of expansion must be updated at a frequent set of times t_i.

To update U we use a standard PID controller or a special PD controller:

$$U(t_i) = \sum_{k=0}^{K} a_k \frac{d^k Q(t_i)}{dt^k}$$

The Taylor polynomial is then used to predict the parameters for the next timestep, enabling the map to accurately track the black holes.

III. Analytic Example

Controlling the speed of a simple 1D wave is analogous to controlling characteristic speeds in SpEC. Assume a wave $f = f(x - ct)$ has the following map applied to it:

$$x = x_{grid} + v(t)\, t$$

The wave travels at a velocity $v_{grid} = c - v(t)$ in this frame. We can choose $v(t)$ to attempt to achieve some desired grid velocity, v_d.

Define the error, $Q = v_{grid} - v_d$, and a feedback equation, $d^2v/dt^2 = K_p\, Q + K_d\, dQ/dt$, then the solution has an exponentially damped envelope when $4K_p > K_d^2$, which allows $v_{grid} \to v_d$ as $t \to \infty$.

We can specify $v(t)$ such that v_{grid} is the opposite sign of c and the wave is left-going in the new frame instead of right-going. This is analogous to keeping characteristic speeds positive in SpEC!

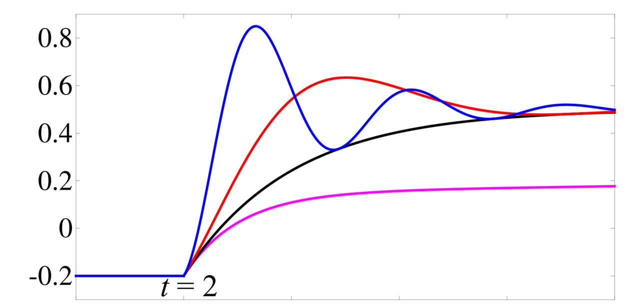


Figure: v_{grid} is plotted for a family of gains K_p. The wave velocity is $c = -0.2$ and the desired velocity is $v_d = 0.5$. The controller turns on at $t = 2$.

IV. Averaging

Whether it is the result of physical processes or numerical effects, rapid variation in the control signal is transfered to the map parameters through their relation in the Taylor polynomial.

In the example of characteristic speed control, this can cause the minimum speed to become negative, in which case the boundary conditions on the excision surface will be violated.

It is sometimes the case that the choice of controller is insufficient to achieve the desired level of averaging, and in these cases we implement direct averaging of the error, Q, in one of two ways:

- Polynomial fit of order N to the previous M measurements of the error, where $M > N$.
- Exponentially-weighted average with timescale τ of all previous control error measurements and their derivatives and integrals, $F(t) = Q$, d^nQ/dt^n, and $\int Q\, dt$, such that they satisfy

$$dF_{avg}/dt = F(t) - F_{avg}/\tau$$

V. Timescales

Control theory is most robust in linear, time-invariant systems. Because binary black hole evolutions are non-linear, dynamic systems, we need to adjust the control law accordingly. We assume the system to be in quasi-static equilibrium on a timescale no less t_d. For the PID controller, if we choose the following gains:

$$K_I = 1/t_d^3$$
$$K_p = 3/t_d^2$$
$$K_d = 3/t_d$$

then the control error will be exponentially damped on this timescale:

$$Q(t) \propto e^{-t/t_d}$$

As the binary black hole system evolves, we automatically increase or decrease t_d in response to the error. Typical behavior of t_d is unsurprising:

- Large → inspiral, late ringdown
- Small → initial relaxation, merger

VI. Results

Waves entering the computational domain from the excision region, which is identified by negative characteristic speeds on the excision boundary, was a major obstacle in merging near-extremal-spin binaries. We can now keep the characteristic speeds at a desired positive value by controlling the size of the apparent horizon. This technique has allowed collaborators in reference [4] to merge binaries with spins above the Bowen-York limit.

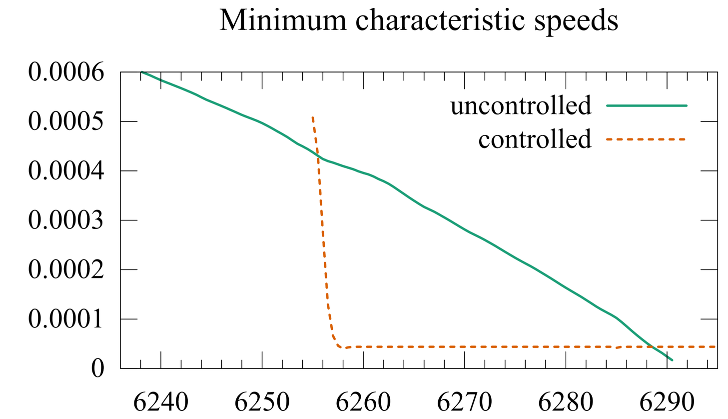


Figure: A nearly merged spin 0.97 evolution is restarted with characteristic speed control to prevent incoming char. fields.

References

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[3] Johnson and Moradi. *PID Control: New Identification and Design Methods*, Springer (2005).

[4] Lovelace, Scheel, and Szilágyi. Simulating merging binary black holes with nearly extremal spins. Phys. Rev. D, **83**, 024010 (2011).

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