Section 9.3
For the problems in this section, you may use “stability/instability table 9.3.1”, page 513, when needed, to draw your final conclusions.

- Work out problems 1, 3, 5, 7, 13. Note that in the question (d) of problems 5, 7, and 13, try as much as possible to sketch a phase portrait at least locally near the critical point(s). You may use computers to draw these if you know how to do so, but not required.
- Work out problem 21 parts (a), (b), and (c), and problems 27 & 28.

Stability/instability questions

Lemma 1. (Stability) Let \( x_0 \) be a critical point of
\[
x'(t) = f(x), \quad t \geq 0.
\]
Assume that there are two positive constants \( C_0, \delta_0 \) and a nonnegative constant \( \alpha \) so that whenever \( x(0) \) satisfies \( \| x(0) - x_0 \| \leq \delta_0 \), the solution \( x(t) \) with the initial data \( x(0) \) exists for all time \( t \geq 0 \) and satisfies
\[
\| x(t) - x_0 \| \leq C_0 e^{-\alpha t} \| x(0) - x_0 \|, \quad \forall \ t \geq 0.
\]
Then prove by the stability definition given in the book or in class that the critical point \( x_0 \) is stable if \( \alpha \geq 0 \) and asymptotically stable if \( \alpha > 0 \).

- Once you have proved the above lemma, it might serve as a definition of stability of the critical point throughout the class.

- Show that the critical point \( x = 0 \) of each of the following systems of differential equations is (asymptotically) stable or unstable. In this problem, it’s asked you to show your proof without using the “stability/instability table 9.3.1”. You may use the above lemma, that is to verify the assumptions of the lemma for each of the systems:

(a) \[
\begin{align*}
x'(t) &= -x + \sin y + y^3 \\
y'(t) &= -\sin x - y
\end{align*}
\]
(b) \[
\begin{align*}
x'(t) &= 2x - x^2 - xy \\
y'(t) &= -y - 2y^2 + x^2 y
\end{align*}
\]