A REMARK ON THE PAPER OF M. SCHLESSINGER

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In the conference itself, I spoke on a theorem asserting the existence of "semi-stable" reductions for analytic families of varieties over a disc, smooth outside the origin. This talk turned out to be difficult to transcribe into a paper of moderate size and instead will be incorporated into the notes of a seminar which I am running together with G. Kempf, B. Saint-Donat, and Tai, which we will publish in the Springer Lecture Notes.

Here I would like to add a footnote to Schlessinger's calculations of versal deformations.¹ He studied the situation: $V = \text{complex } n + 1$-dimensional vector space; $\mathbf{P}(V) = n$-dimensional projective space of 1-dimensional subspaces of $V$; $Y \subset \mathbf{P}(V)$ a smooth $r$-dimensional variety, $r \geq 1$; $C \subset V$ the cone over $Y$.

Let $L = \mathcal{O}_Y(1)$. Assume:

$$H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) \to H^0(Y, L^k)$$

is surjective, $k \geq 1$

(We may also assume by replacing $\mathbf{P}(V)$ by a linear space that it is an isomorphism for $k = 1$). Then he proved:

a) There is a natural injection of functors:

$$\tilde{H} = \frac{\text{(Deformations)}}{\text{projective}} \bigg/ \frac{\text{of } Y \text{ in } \mathbf{P}(V)}{\text{automorphisms}} \to \frac{\text{Deformations}}{\text{of } C}$$

b) $T^1_C$ has a natural graded structure

$$T^1_C = \bigoplus_{k = -\infty}^{+\infty} (T^1_C)_k$$

such that $(T^1_C)_0 \cong \text{image of Zariski tangent space to } \tilde{H}$,

c) If $(T^1_C)_k = (0)$ for $k \neq 0$, then $\tilde{H}$ is isomorphic to the functor of deformations of $C$, i.e., all deformations of $C$ remain conical.

d) If $r \geq 2$ and $L$ is sufficiently ample on $Y$, then the condition in (c) is satisfied.

What I would like to show here is:
d') If $r = 1$, $L$ is sufficiently ample on $Y$ and $Y$ has genus $\geq 2$ and is not hyperelliptic, then again the condition in (c) is satisfied.

This gives:

**Corollary.** There exist normal singularities of surfaces with no non-singular deformation!

To prove (d'), we let $U = C - (0)$ and use the exact sequences:

$$\Gamma(V, \theta_V) \xrightarrow{\varphi} \Gamma(C, N_C) \rightarrow T_v^{1} \rightarrow 0$$

Now $C^*$ acts in a natural way on both $\theta_V$ and $N_C$, and if $\pi: V - (0) \rightarrow \mathbb{P}(V)$ is the projection, then both $\pi_* \theta_V$ and $\pi_* N_C$ decompose into direct sums of their eigenspaces for the various characters of $C^*$. Moreover, the $C^*$ invariant sections are:

$$(\pi_* \theta_V)^{C^*} \cong \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{C} V$$

$$(\pi_* N_C)^{C^*} \cong N_Y$$

and $\alpha$ induces the map $\alpha' = \gamma \circ \beta$

$$\alpha': \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{C} V \rightarrow \theta_{\mathbb{P}(V)} \rightarrow N_Y$$

($\beta =$ standard map).

Thus we get:

$$\Gamma(V - (0), \theta_V) \xrightarrow{\varphi} \Gamma(U, N_C)$$

$$\oplus_{v = -\infty}^{\infty} \Gamma(\mathbb{P}(V), \mathcal{O}(v + 1) \otimes_C V) \xrightarrow{\alpha'_v} \Gamma(Y, N_Y(v))$$

So if

$$(T_v^{1})_v = \text{coker} \left[ \Gamma(\mathbb{P}(V), \mathcal{O}(v + 1) \otimes_C V) \xrightarrow{\alpha'_v} \Gamma(Y, N_Y(v)) \right]$$

then $T_v^{1} = \oplus_{v = -\infty}^{\infty} (T_v^{1})_v$. We must compute these groups.

The idea is to determine $N_Y$ explicitly on $Y$ without actually using the embedding of $Y$ defined by $L$. Consider in fact $N_Y^*(1)$ via the dual of $\alpha'$ as a subbundle of $\mathcal{O}_Y \otimes_C V^*$

$$N_Y^*(1) \subset \mathcal{O}_{\mathbb{P}(V)}^*(1) \mid_Y \subset \mathcal{O}_Y \otimes_C V^*$$

hence for every $x \in Y$: 

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\[ [N^*_Y(1) \otimes \mathcal{O}_x/m_x] \subseteq [\theta_{P(V)}^*(1) \otimes \mathcal{O}_x/m_x] \subseteq V^*. \]

It is easy to see that under these inclusions, if \( x' \in C \) lies over \( x \):

\[
\theta_{P(V)}^*(1) \otimes \mathcal{O}_x/m_x = \begin{cases} 
\text{space of linear forms } l \text{ on } V \\
\text{such that } l(x') = 0
\end{cases}
\]

\[
N^*_Y(1) \otimes \mathcal{O}_x/m_x = \begin{cases} 
\text{space of linear forms } l \text{ on } V \\
\text{such that } l(x') = 0 \text{ and } l = 0 \text{ is tangent to } Y \text{ at } x
\end{cases}
\]

But now by assumption:

\[ V^* \cong \Gamma(P(V), \mathcal{O}_{P(V)}(1)) \xrightarrow{\sim} \Gamma(Y, L) \]

and under this isomorphism, the linear forms \( l \) such that \( l = 0 \) and is tangent to \( Y \) at \( x \) go over to the sections of \( L \) vanishing at \( x \) to 2nd order, i.e. \( \Gamma(Y, m_x^2 \cdot L) \). Now consider

\[ \Delta \subset Y \times Y \text{ with } p_1^*L(-2\Delta) \]

\[ \xrightarrow{p_2} Y \text{ with } p_2_*[p_1^*L(-2\Delta)] \]

Then it is easily seen that \( p_2_*[p_1^*L(-2\Delta)] \) is a locally free sheaf on \( Y \) and that

\[ p_2_*[p_1^*L(-2\Delta)] \otimes \mathcal{O}_x/m_x \cong \Gamma(Y \otimes \{y\}, p_1^*L(-2\Delta) \otimes_{\mathcal{O}_y} \mathcal{O}_x/m_x) \]

\[ \cong \Gamma(Y, m_x^2 \cdot L) \]

Thus the two sub-bundles:

a) \( p_2_*[p_1^*L(-2\Delta)] \subset p_2_*[p_1^*L] = \Gamma(Y, L) \otimes_{\mathcal{O}_Y} \mathcal{O}_x \)

b) \( N^*_Y(1) \subset V^* \otimes_{\mathcal{O}_Y} \mathcal{O}_x \cong \Gamma(Y, L) \otimes_{\mathcal{O}_Y} \mathcal{O}_x \)

are equal. Now assume \( r = 1 \), so that \( Y \) is a curve and \( \mathcal{O}(-2\Delta) \) is an invertible sheaf on \( Y \times Y \). Then by Serre duality for the morphism \( p_2 \), we can identify \( N^*_Y(-1) \) as a quotient of \( V \otimes_{\mathcal{O}_Y} \mathcal{O}_x \) or \( \Gamma(Y, L)^* \otimes_{\mathcal{O}_Y} \mathcal{O}_x \):

\[ V \otimes_{\mathcal{O}_Y} \mathcal{O}_x \xrightarrow{\alpha(-1)} N^*_Y(-1) \rightarrow 0 \]

\[ \Gamma(Y, L)^* \otimes_{\mathcal{O}_Y} \mathcal{O}_x \xrightarrow{\alpha(-1)} \text{Hom}(p_2_*[p_1^*L(-2\Delta)], \mathcal{O}_Y) \rightarrow 0 \]

\[ R^1 p_2_*[\text{Hom}(p_1^*L, \Omega_{Y \times Y/Y})] \rightarrow R^1 p_2_*[\text{Hom}(p_1^*L(-2\Delta), \Omega_{Y \times Y/Y})] \rightarrow 0 \]

\[ R^1 p_2_*[p_1^*(\Omega_Y \otimes L^{-1})] \rightarrow R^1 p_2_*[p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)] \rightarrow 0. \]
We want to show that \((T_C^1)_{\nu} = (0)\) if \(\nu \neq 0\), i.e.,
\[
\Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})] \otimes L') \rightarrow \Gamma(Y, R^1 p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)] \otimes L')
\]
is surjective if \(\nu \neq 1\). If \(\text{deg } L > 2g\), then \(p_{2,*}\) of the two sheaves in square brackets is zero, hence by the Leray spectral sequence for \(p_2\), the above map is the same as:
\[
H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L') \rightarrow H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L' \otimes \mathcal{O}(2\Delta)).
\]

We treat the surjectivity in three cases:

**Case I.** \(\nu \geq 2\): Consider the sheaf cokernel
\[
p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L' \rightarrow p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^* L' \otimes \mathcal{O}(2\Delta) \rightarrow K_\nu \rightarrow 0.
\]
It is a sheaf of \(\mathcal{O}_{2\Delta}\)-modules so it lies in an exact sequence between \(\mathcal{O}_\Delta \cong \mathcal{O}_Y\)-modules
\[
0 \rightarrow (\mathcal{O}(\Delta) \otimes \mathcal{O}_\Delta) \otimes L'^{-1} \otimes \Omega_Y \rightarrow K_\nu \rightarrow (\mathcal{O}(2\Delta) \otimes \mathcal{O}_\Delta) \otimes L'^{-1} \otimes \Omega_Y \rightarrow 0
\]

So if \(\text{deg } L > 4g - 4\), \(H^1(K_\nu) = (0)\) when \(\nu \geq 2\).

**Case II.** \(\nu = 0\): Consider the Leray spectral sequence for \(p_1\). Since we have assumed \(Y\) is not hyperelliptic
a) \(p_{1,*} \mathcal{O}_{Y \times Y}(2\Delta) \cong p_{1,*} \mathcal{O}_{Y \times Y}\) and
b) \(R^1 p_{1,*} \mathcal{O}_{Y \times Y}(2\Delta)\) is a locally free sheaf \(\mathcal{E}\) of rank \(g - 2\). Now we have:
\[
0 \rightarrow H^1(Y, \Omega_Y \otimes L^{-1}) \rightarrow H^1(Y \times Y, p_{1,*} \Omega_Y \otimes L^{-1}) \rightarrow H^0(Y, \Omega_Y \otimes L^{-1} \otimes R^1 p_{1,*} \mathcal{O}_{Y \times Y}) \rightarrow
\]

Note that \(\mathcal{E}\) does not depend on \(L\). So by (b) there is an integer \(n_0\) depending only on \(Y\) such that if \(\text{deg } L > n_0\), then \((\Omega_Y \otimes \mathcal{E}) \otimes L^{-1}\) has no sections.

**Case III:** \(\nu \leq -1\): Surjectivity in this case always follows from surjectivity when \(\nu = 0\). In fact, if we know that
\[
L \rightarrow \Gamma(Y, N_Y \otimes L^{-1}) \rightarrow 0
\]
is surjective, I claim \(\Gamma(Y, N_Y \otimes L^{-1}) = (0), \nu \geq 2\). If not, \(N_Y \otimes L^{-2}\) has
a non-zero section \( s \). Then for all \( t \in \Gamma(Y, L) \cong V^* \), \( t \otimes s \) is a non-zero section of \( N_Y \otimes L^{-1} \). Thus we must get all sections of \( N_Y \otimes L^{-1} \) in this way. But this means that all these sections are proportional, hence do not generate \( N_Y \otimes L^{-1} \). But since

\[
V \otimes \mathcal{O}_Y \rightarrow N_Y \otimes L^{-1}
\]

is surjective and \( V \otimes \mathcal{O}_Y \) is generated by its sections, so is \( N_Y \otimes L^{-1} \). This is a contradiction, so \( s = 0 \).

This completes the proof of (d'). Finally two remarks:

(A) If you look at the case \( Y = \mathbb{P}^1 \), \( L = \mathcal{O}_{\mathbb{P}^1}(k) \), then \( C = \) cone over the rational curve of degree \( n \) in \( \mathbb{P}^n \) and the sequences we have used enable us to compute \( T_C^\dim \) easily. In fact it turns out that if \( k \geq 3 \),

\[
(T_C^\dim)_l = (0), \quad \text{if } l \neq -1
\]

\[
\dim (T_C^\dim)_{-1} = 2k - 4.
\]

It seems most reasonable to conjecture that the versal deformation space of this \( C \) is a non-singular \((k - 1)\)-dimensional space but with a 0-dimensional embedded component at the origin if \( k \geq 4 \).

(B) What happens in the hyperelliptic case? If, for instance, \( \pi: Y \rightarrow \mathbb{P}^1 \) is the double covering and \( L = \pi^* \mathcal{O}_{\mathbb{P}^1}(k) \), then \( C \) is itself a double covering of the rational cone considered in (A) which is known to have non-singular deformations. Do these lift to deformations of this \( C \)?

NOTES


2. H. Pinkham has recently proved that this is true if \( k \geq 5 \), but if \( k = 4 \), the versal deformation space has two components, a smooth 3-dimensional one and a smooth 1-dimensional one crossing normally at the origin! (Cf. “Deformations of cones with negative grading,” J. of Algebra, to appear.)