Degeneration of algebraic theta functions

§1. 2-adic theta functions, values in a complete valued field

Problem:

Given $K$: complete algebraically closed valued field; integers $\mathcal{O}$, residue field $k = \mathcal{O}/\mathfrak{m}$, absolute value $| \cdot | : k^\times \to \mathbb{R}_{>0}$, char.(k) $\neq 2$.

Given $V$: $2g$-dimensional vector space over $\mathbb{Q}_2$, plus $e$, $e^*$, $\Lambda$.

1This is a lightly edited version of a set of hand written notes by Mumford. It contains an essentially complete proof of the results in the undated letter [???] to Grothendieck, despite the following disclaimer in that letter: 'I say "think" because I haven't written down the details systematically. In fact one should get a rather complete "structure theorem" for these abelian varieties (I hope).' Mumford lectured on these results in the 1967 Summer School at Bowdoin, see paper [???] in this volume. Appendix II in the 1984 Ph. D. dissertation of C.-L. Chai, London Math. Soc. Lecture Notes Series 107, 1985, pp. 237–286 is an adulterated version of the same set of notes.

2The notes come in two groups, reproduced as two sections. The first section contains the key results on the structure of 2-adic theta functions associated to abelian varieties over a local field. This structure theory is applied in §2 to 2-adic monodromy of abelian varieties over local fields. Two pages of the original notes are essentially the same as the last section of the Bowdoin lecture notes [???]; they are not reproduced here.

3The notations and results in Equations defining abelian varieties I, II, III, abbreviated as [Eq I, II, III] in the footnotes, are used extensively in this set of notes. In particular

\[ e: V \times V \longrightarrow \mu_{2^\infty}(K) \]

is a skew-symmetric bi-multiplicative non-degenerate pairing from $V \times V$ to the group of all roots of unity whose order is a power of 2, $\Lambda$ is a maximal isotropic $\mathcal{O}$-lattice in $V$, and

\[ e^*: \frac{1}{2}\Lambda/\Lambda \longrightarrow \{\pm\} \]

is a quadratic character such that

\[ e^*(\alpha + \beta) e^*(\alpha) e^*(\beta) = e(\alpha, \beta)^2 \quad \forall \alpha, \beta \in \frac{1}{2}\Lambda. \]
Given $\Theta : V \to K$, a \textit{theta function} w.r.t. $e, e_*$,\footnote{That $\Theta : V \to K$ is a theta function for $(V, \Lambda, e, e_*)$ means that it satisfies
- (theta transformation law) $\Theta(\alpha + \beta) = e_*(\beta/2) e(\beta/2, \alpha) \Theta(\alpha)$ $\forall \alpha \in V, \forall \beta \in \Lambda$. 
- (symmetry) $\Theta(-\alpha) = \Theta(\alpha)$ $\forall \alpha \in V$. 
- (Riemann theta relation) For all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in V$ we have 
  \[ \prod_{i=1}^{4} \Theta(\alpha_i) = 2^{-g} \cdot \sum_{\eta \in \frac{1}{2} \Lambda/\Lambda} e(\gamma, \eta) \prod_{i=1}^{4} \Theta(\alpha_i + \gamma + \eta) \]
  where $\gamma = -\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$.}
coarse support($\Theta$) = $V$.\footnote{The \textit{coarse support} of an algebraic theta function $\Theta : V \to K$ is the set of all for every $\alpha \in V$, there exists $\eta \in \frac{1}{2} \Lambda$ such that $\Theta(\alpha + \eta) \neq 0$.} 

\textbf{Analyze structure of $\theta$}

\textbf{(I)} All values $\Theta(\alpha)$ are integrally dependent on $\{\Theta(\beta) \mid \beta \in \frac{1}{2} \Lambda\}$. Hence $\max |\Theta(\alpha)|$ exists and is taken on for some $\alpha \in \frac{1}{2} \Lambda$.\footnote{See Prop. 1 of Abstract theta functions, paper [???] in this volume.} So multiply $\Theta$ by a constant s.t.

(a) $\Theta(\alpha) \in \mathcal{O}$ for all $\alpha \in \frac{1}{2} \Lambda$
(b) $\exists \alpha \in \frac{1}{2} \Lambda$ with $\Theta(\alpha) \notin \mathfrak{m}$, or equivalently $|\Theta(\alpha)| = 1$.

:. Get a non-zero theta function $\overline{\Theta}(\alpha) := [\Theta(\alpha) \mod \mathfrak{m}] \in k$.

\textbf{(II)} Say coarse support($\Theta$) = $W + \frac{1}{2} \Lambda$, $W \subseteq V$ a \textit{cusp}.\footnote{See Theorem on p. 230 of [Eq III] for this assertion. A \textit{cusp} is a vector subspace $W \subseteq V$ such that $W^\perp \subseteq W$; see page 229 of loc. cit.}

(* Choose a symplectic translation $T$ of $V$ s.t. $T(\Lambda) = \Lambda$, $e_* \equiv 1$ on $T(W^\perp) \cap \frac{1}{2} \Lambda$. Change $\Lambda$ by this: Then 0 is an \textit{origin} for $W$. Later, will have to apply $T$ in reverse to the structure $\Theta$.}

$\sim$ OK:

$$\overline{\Theta}(\alpha) = e_*(\eta/2) e(\eta/2, \alpha) \overline{\Theta}(\alpha_0^*)$$

if $\alpha = \eta + \alpha_0, \eta \in \Lambda, \alpha_0 \in W$

$\alpha_0^* = \text{image of } \alpha_0 \text{ in } W/W^\perp$

$\overline{\Theta} = k$-valued non-degen. thetafcn. on $W/W^\perp$
Choose:
\[
V = W_1 \oplus W_2 \\
\Lambda = \Lambda_1 \oplus \Lambda, \quad \Lambda_i = \Lambda \cap W_i \\
e_* = 1 \text{ on } \frac{1}{2}\Lambda_i
\]
standard decomp. of \(V\)
\[
(0) \subset W_2 \subset W_1 \subset W \subset V, \text{ so } W = W_1 \oplus W_2, \, \tilde{W}_2 \subset W_2.
\]

Given \(V = W_1 \oplus W_2\) and \(W = W_1 \oplus \tilde{W}_2\) as above:
\[\exists \text{1-1 correspondence between}\]

(a) \(\mathcal{O}\)-valued theta fcns. \(\Theta\) on \(V\) s.t. coarse \(\text{supp}(\Theta) = W + \frac{1}{2}\Lambda\).

(b) \(\mathcal{O}\)-valued Gaussian measures \(\mu\) on \(W_2\) s.t. \(\text{supp}(\mu) = \tilde{W}_2\)

In fact\(^9\)
\[
\mu(a_2 + 2^n\Lambda_2) = 2^{-ng} \sum_{a_1 \in 2^{-n}\Lambda_2/\Lambda_2} e(a_2, a_1/2) \cdot \Theta(a_1 + a_2) \quad \forall a_2 \in W_2, \forall n \in \mathbb{N}
\]
\[
\Theta(a_1 + a_2) = e(a_1, a_1/2) \int_{a_2 + \Lambda_2} e(a_1, \beta) \cdot d\mu(\beta) \quad \forall a_1 \in W_1, \forall a_2 \in W_2
\]

Esp.
\[
\sup \left\{ |\mu(U)| : U \subset a_2 + \tilde{W}_2 + \frac{1}{2}\Lambda_2, \, U \text{ compact open} \right\}
\]
\[
= \sup \left\{ |\mu(a_2' + 2^n\Lambda_2)| : a_2' \in a_2 + \tilde{W}_2 + \frac{1}{2}\Lambda_2, \, n \geq 0 \right\}
\]
\[
= \sup \left\{ |\Theta(a_1 + a_2')| : a_1 \in W_1, \, a_2 \in \tilde{W}_2 + \frac{1}{2}\Lambda_2 \right\}
\]
\[
= \sup \left\{ \Theta(b) : b \in a_2 + W + \frac{1}{2}\Lambda \right\}
\]

(III) Show that \(\forall \Theta\) or \(\mu\), and \(\forall a_2\), this \(\text{sup}\) is a \(\text{max}\).

Proof. Associate \(\Phi\) to \(\Theta\) s.t.
\[
\Phi(\alpha) \Phi(\beta) = \sum_{\zeta \in \frac{1}{2}\Lambda_1/\Lambda_1} e(\alpha, \zeta) \Theta(\alpha + \beta + \zeta) \Theta(\alpha - \beta + \zeta)
\]
\[
2^g \Theta(2\alpha) \Theta(2\beta) = \sum_{\zeta \in \frac{1}{2}\Lambda_2/\Lambda_2} e(\alpha, \zeta)^2 \Phi(\alpha + \beta + \zeta) \Phi(\alpha - \beta + \zeta)
\]

\(^9\)An \(k\)-valued \text{even} measure on \(W_2\) is a \text{Gaussian measure} if there exists a \(k\)-valued measure \(\nu\) on \(W_2\) such that \((\mu \times \mu)(U) = (\nu \times \nu)(\xi(U))\) for all compact open subsets \(U\) in \(W_2 \times W_2\), where \(\xi: W_2 \times W_2 \rightarrow W_2 \times W_2\) is defined by \(\xi(x, y) = (x + y, x - y)\); see page 118 of [Eq II].

\(^{10}\)See pp. 116–117 of [Eq II].
\[
\therefore |\Phi(\alpha)| \cdot |\Phi(\beta)| \leq \max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Theta(\alpha + \beta + \zeta)| \cdot |\Theta(\alpha - \beta + \zeta)|
\]
\[
|\Theta(\alpha + \beta)| \cdot |\Theta(\alpha - \beta)| \leq \max_{\zeta \in \frac{1}{2} \Lambda_{2}} |\Phi(\alpha + \zeta)| \cdot |\Phi(\beta + \zeta)|
\]
\[
\therefore \max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Theta(\alpha + \beta + \zeta)| \cdot |\Theta(\alpha - \beta + \zeta)| = \max_{\zeta \in \frac{1}{2} \Lambda_{2}} |\Phi(\alpha + \zeta)| \cdot |\Phi(\beta + \zeta)|
\]

So
\[
\max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Theta(\alpha + \beta + \zeta)| \cdot \max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Theta(\alpha + \beta + \zeta)| = \max_{\zeta \in \frac{1}{2} \Lambda_{2} + \frac{1}{4} \Lambda_{1}} |\Phi(\alpha + \zeta)| \cdot |\Phi(\beta + \zeta)|
\]

Now assume \( \beta \in W \). Use
\[
\{ \forall x \in W + \Lambda_{2} \exists \eta \in \frac{1}{2} \Lambda_{1} \text{ s.t. } |\Phi(x + \eta)| = 1 \}
\]

Let \( \tau(\gamma) = \max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Theta(\gamma + \zeta)| \)

\[
\therefore \tau(\alpha + \beta) \cdot \tau(\alpha - \beta) = \max_{\zeta_{1}, \zeta_{2} \in \frac{1}{2} \Lambda_{2} + \frac{1}{4} \Lambda_{1}, \zeta_{1} + \zeta_{2} \in \frac{1}{4} \Lambda_{1}} |\Phi(\alpha + \zeta_{1})| \cdot |\Phi(\beta + \zeta_{2})|
\]

**Def.** \( \alpha \in V \) is normal if
\[
\max_{\zeta \in \frac{1}{2} \Lambda_{1}} |\Phi(\alpha + \zeta)| = \max_{\zeta \in \frac{1}{2} \Lambda_{2} + \frac{1}{4} \Lambda_{1}} |\Phi(\alpha + \zeta)|.
\]

\[
[ \forall \alpha \exists \eta \in \frac{1}{2} \Lambda_{2} \text{ s.t. } \alpha + \eta \text{ is normal }]
\]

So if \( \alpha \) normal, \( \beta \in W \), then
\[
\tau(\alpha + \beta) \cdot \tau(\alpha - \beta) = x \max_{\zeta \in \frac{1}{2} \Lambda_{1} + \frac{1}{2} \Lambda_{2}} |\Phi(\alpha + \zeta)| =: \rho(\alpha)
\]

Esp.
\[
\tau(\alpha + \beta) \cdot \tau(\alpha - \beta) = \tau(\alpha)^{2}.
\]

**Note:** If \( \eta \in \frac{1}{2} \Lambda_{2}, \alpha + \eta \) normal, then \( \tau(\alpha + \eta) \geq \tau(\alpha) \).

**Proof of Note.**
\[
\tau(\alpha)^{2} = \max_{\zeta_{1}, \zeta_{2} \in \frac{1}{2} \Lambda_{2} + \frac{1}{4} \Lambda_{1}, \zeta_{1} + \zeta_{2} \in \frac{1}{4} \Lambda_{1}} |\Phi(\alpha + \zeta_{1})| \cdot |\Phi(\zeta_{2})|
\]
\[
\leq \max_{\zeta \in \frac{1}{2} \Lambda_{2} + \frac{1}{4} \Lambda_{1}} |\Phi(\alpha + \zeta)| = \rho(\alpha)
\]
\[
\alpha + \eta \text{ normal } \implies \tau(\alpha + \eta)^{2} = \rho(\alpha + \eta) \geq \tau(\alpha)^{2}. \text{ Q.E.D.}
\]
Now suppose \( \alpha_n \in a + W + \frac{1}{2} \Lambda \) s.t.

\[
|\Theta(\alpha_n)| \longrightarrow \sup \left\{ |\Theta(\beta)| : \beta \in a + W + \frac{1}{2} \Lambda \right\} =: s.
\]

W.l.o.g. can assume \( |\Theta(\alpha_n)| = \tau(\alpha_n) \) & \( \alpha \) normal (in view of Note above).

OK : Pass to subsequence s.t.

\[
\alpha_n - \alpha_m \in W + \Lambda \quad (\text{all } n, m).
\]

W.l.o.g. may assume \( \alpha_n - \alpha_m \in W \) for all \( n, m \). Now if

\[
\tau(\alpha_n) = |\Theta(\alpha_n)| > \sqrt{s \cdot |\Theta(\alpha_1)|} = \text{geom. mean of } s, |\Theta(\alpha_1)|,
\]

then

\[
s \cdot \tau(\alpha_1) < \tau(\alpha_n)^2 = \tau(\alpha_n + (\alpha_1 - \alpha_n)) \cdot \tau(\alpha_n - (\alpha_1 - \alpha_n)) < \tau(\alpha_1) \cdot s,
\]

contradiction. \( \therefore s = \tau(\alpha_n) \) for all \( n \). Step (III.) is proved.

We conclude **Proposition 1.** \( \forall \mathcal{O} \)-valued Gaussian measure \( \mu \) on \( W_2 \), let \( \tilde{W}_2 = \text{supp}(\mu) \). Then \( \forall \) compact open subgroup \( \Lambda'_2 \subset V_2 \) and \( \forall a \in W_2 \),

\[
\sup \left\{ |\mu(U)| : U \subseteq \tilde{W}_2 + \Lambda'_2 + a \right\}
\]

is attained.\(^{11}\)

**IV) Theorem 2.** Let \( \mathcal{V} \) be a finite-dimensional vector space over \( \mathbb{Q}_2 \), \( W \) be a vector subspace of \( \mathcal{V} \), and let \( \Lambda \subset \mathcal{V} \) be a compact open subgroup.\(^{12}\) Let \( \mu \) be a Gaussian measure on \( \mathcal{V} \) with values in \( \mathcal{O} \). Let \( \nu \) be the dual Gaussian measure of \( \mu \), i.e. \( \xi_*(\mu \times \mu) = \nu \).\(^{13}\)

Assume that

1. \( \overline{\mu}, \overline{\nu} \) have support \( W \subset \mathcal{V} \).
2. \( \forall w \in W \)

\[
\max \left\{ |\mu(V)| : V \subset w + \Lambda + W \right\} =: \sigma(w)
\]

exist.

If \( w \in \mathcal{V}, c \in \mathcal{O} \), and \( |c| = \sigma(w) = \max_{\eta \in \frac{1}{2} \Lambda} (\sigma(w + \eta)) \), then

\[
\text{supp} \left\{ \frac{\mu}{c} \Big|_{w+\Lambda+W} \right\} = w + \eta_0 + W
\]

for some \( \eta_0 \in \Lambda \).

\(^{11}\)Prop. 1 has been proved for \( \Lambda' = \frac{1}{2} \Lambda_2 \). Apply an automorphism \( A \) of \( V_2 \) such that \( A(\frac{1}{2} \Lambda_2) \subset \Lambda'_2 \)

\(^{12}\)The general notation for \( \S 1 \) is suspended in Step (IV). In application the triple \( (\mathcal{V}, \Lambda, W) \) in this theorem will be \( (W_2, \Lambda'_2, \tilde{W}) \). Also the meaning of \( \tau \) is different from that in the proof of Step (III).

\(^{13}\)As before, \( \xi : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V} \times \mathcal{V}, \ \xi : (x, y) \mapsto (x + y, x - y) \).
Proof. Claim 1: $\tau(w)^2 = \sigma(w)$.

1. $\exists U \subset w + \Lambda + W$ s.t. $|\mu(U)| = \sigma(W)$.

$$\therefore |\mu \times \mu(U \times \Lambda)| = \sigma(w), \quad \therefore |\nu \times \nu_0(U \times \Lambda)| = \sigma(w).$$

But $\xi(U \times \Lambda) \subset (w + \Lambda + W) \times (w + \Lambda + W)$,

$$\therefore \exists U_1, U_2 \subset w + \Lambda + W \text{ s.t. } |\nu(U_1)| \cdot |\nu(U_2)| = |\nu \times \nu(U_1 \times U_2)| = \sigma(w)$$

$$\therefore \tau(w) \geq \max\{|\nu(U_1)|, |\nu(U_2)|\} \geq \sqrt{\sigma(w)}$$

2. $\exists U \subset w + \Lambda + W$ s.t. $|\nu(U)| = \tau(W)$.

$$\therefore |\nu \times \nu(U \times U)| = \tau(w)^2$$

But

$$(w + \Lambda + W) \times (w + \Lambda + W) = \bigcup_{\text{disjoint}} \xi((w + \Lambda + W + \eta) \times (\Lambda + W + \eta)) \quad \eta \in \frac{1}{2} \Lambda$$

$$\therefore \exists \eta, U_1 \subset w + \Lambda + W + \eta, \exists U_2 \in \Lambda + W + \eta \text{ s.t.}$$

$$\tau(w)^2 = |\nu \times \nu_0(\xi(U_1 \times U_2))| = |\mu(U_1)| \cdot |\mu(U_2)|$$

$$\leq |\mu(U_1)| \leq \sigma(w + \eta) \leq \sigma(w).$$

We have proved Claim 1.

Look at measures

$$\mu \varepsilon|_{w + \Lambda + W} =: \mu_w, \quad \nu \sqrt{c}|_{w + \Lambda + W} =: \nu_w.$$

Claim 2. (a) $\xi_0(\mu_w \times \nu) = (\nu_w \times \nu_w)|_{\xi((w + \Lambda + W) \times (\Lambda + W))}$.

(b) The restriction of the measure $\nu \times \nu$ to

$$(w + \Lambda + W) \times (w + \Lambda + W) - \xi((w + \Lambda + W) \times (\Lambda + W))$$

has absolute values strictly less than $\sigma(w)$.

(c) $\xi_0(\mu_w \times \nu) = \nu_w \times \nu_w$ as measures on $(w + \Lambda + W) \times (w + \Lambda + W)$.

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14Here $U$ is compact open; the same for the $U_1$, $U_2$ and $U$ below.
Clearly (a) holds, and (b) implies (c). To see (b), suppose that $U_1 \subset w + \Lambda + W + \eta$, $U_2 \subset \Lambda + W + \eta$, $U_1, U_2$ compact open, $\eta \in \frac{1}{2}\Lambda$, $\eta \not\in \Lambda + W$. Then

$$|\nu \times \nu(\xi(U_1 \times U_2))| = |\mu(U_1)| \cdot |\mu(U_2)| \leq \sigma(w + \eta) \cdot \sigma(\eta) \leq \sigma(w + \eta) \quad (\because \eta \not\in \Lambda + W) \leq \sigma(w) \quad (\because \text{assumption on } w)$$

Claim (b) is proved.\(^{15}\)

Theorem 2 is a formal consequence of (c): Let $S := \supp(\mu_w)$, $T := \supp(\nu_w)$. So

$$T \times T = \xi(S \times W) = T \times T = \{(a + u, a - u) \mid a \in S, \ u \in W\}$$

because $\xi_* (\mu_w \times \mu_w) = \nu_w \times \nu_w$. Start with $a \in S, \ u \in W$. Then $a + u \in T$, so $(a + u, a + u) \in T \times T$. and $a + u \in S$ too because $\xi^{-1}(a + u, a + u) = (a + u, 0)$. We have shown that $a + W \subset S$ for all $a \in S$. If $b \in S$ also, then $b + u \in T$ as before, and $(a + u, b + u) \in T \times T$.

$$\therefore \frac{1}{2}(a - b) \in W \quad \therefore \xi^{-1}(a + u, b + u) \in S \times W.$$ 

So $a - b \in W$ for all $a, b \in S$. Step (IV) is proved.

(V) We reformulate what have been proved so far, and what is expected. Let $\mathcal{V}$ be a vector space over $\mathbb{Q}_2$, $W \subset \mathcal{V}$ a vector subspace, $\pi: \mathcal{V} \rightarrow \mathcal{V}/W$, $\dim\mathcal{V} = g$, $\dim W = g - r$. Let $\mu$ be an $\mathcal{O}$-valued Gaussian measure on $\mathcal{V}$ s.t. $\supp(\mu) = W$. We have proved:

1. For all compact open subset $U \subset \mathcal{V}/W$,

$$\sup \left\{ |\mu(U')| : U' \subset \pi^{-1}(U), \ U' \text{ compact open} \right\}$$

is reached by some compact open subset $U'$.

2. For all compact open $U \subset \mathcal{V}/W$, let

$$\sigma_U = \max \left\{ |\mu(U')| : U' \subset \pi^{-1}(U), \ U' \text{ compact open} \right\},$$

let $c_U \in K$ be s.t. $|c_U| = \sigma_U$, and let

$$\mu_U = \left[ \frac{\mu}{c_U} \right]_{\pi^{-1}(U)}.$$

Then $\supp(\mu_U)$ is a finite union of cosets of $W$.

\(^{15}\)Because each compact open subset of $(w + \Lambda + W) \times (w + \Lambda + W) - \xi((x + \Lambda + W) \times (\Lambda + W))$ is a finite disjoint union of subsets of the form $\xi(U_1 \times U_2)$ satisfying the above conditions.
Expectation 3: \( \exists S \subset \mathcal{V}/W, \quad S \xrightarrow{=} V/W \)

\( \cong \quad \cong \)

\( \mathbb{Z}[1/2]^h \xrightarrow{=} \mathbb{Q}_2^h \)

and \( \exists \) a function\(^{16} \) \( \sigma: S \rightarrow \mathbb{R} \) of the form \( \sigma(x) = e^{-Q(x+y)} \), \( Q \) a pos. def. quad. form on \( S \), s.t. \( \forall U \subset \mathcal{V}/W \) compact open, let \( \sigma_U = \max_{x \in U \cap S} \sigma(x) \), and let \( c_U \in K \) be s.t. \( |c_U| = \sigma_U \). Then\(^{17} \)

(a) \( |\mu(U')| \leq \sigma_U \) for all compact open \( U' \subset \pi^{-1}(U) \).

(b) \( \mu_U = \left[ \frac{\nu}{c_U} \right]_{\pi^{-1}(U)} \) is a \( k \)-valued measure whose support is exactly

\[ \bigcup_{y \in S \cap U, \sigma(y) = \sigma_U} \pi^{-1}(y). \]

**Def.** The *singular set* \( S = S(\mu) \) of \( \mu \) is defined by

\[ S = S(\mu) := \{ x \in \mathcal{V}/W : \exists \text{ open neigh. } U \ni x \in \mathcal{V}/W \text{ s.t. supp}(\mu_U) = \pi^{-1}(x) \} \]

**Def.** Define \( \sigma: S \rightarrow \mathbb{R} \) by

\[ \sigma(x) = \max \{ |\mu(U')| : U' \subset \pi^{-1}(U) \} \]

for any \( x \in S \), where \( U \) is an open neighborhood of \( x \) in \( \mathcal{V}/W \) s.t. \( \text{supp}(\mu_U) = \pi^{-1}(x) \). This definition is independent of the choice of \( U \).

It remains to show that

\[ S \xrightarrow{=} V/W \]

\( \cong \quad \cong \)

\( \mathbb{Z}[1/2]^r \xrightarrow{=} \mathbb{Q}_2^r \)

and

\[ \sigma(x) = e^{-Q(x)}, \quad Q \text{ pos. def.} \]

**Proof of Expectation 3.** Let \( \nu \) be the \( O \)-valued Gaussian measure dual to \( \mu \), i.e. \( \xi_*(\mu \times \mu) = \nu \times \nu \). Let \( T = S(\nu) \subset \mathcal{V}/W \) be the singular set of \( \nu \), and let \( \tau = \sigma(\nu): T \rightarrow \mathbb{R} \) be the sup. map for \( \nu \). Then

\[ \xi(S \times S) = T \times T \]

and

\[ \sigma(x) \cdot \sigma(y) = \gamma(x+y) \cdot \gamma(x-y) \quad \text{for all } x, y \in S. \]

From these we deduce

\(^{16}\)The function \( \sigma \) on \( S \subset \mathcal{V}/W \) here is different from the function \( \sigma \) on \( W \) in Theorem 2.

\(^{17}\)In terms of the function \( \sigma: S \rightarrow \mathbb{R} \) here, for \( w \in \mathcal{V} \), the positive number \( \sigma(w) \) in Theorem 1 is equal to \( \max_{x \in S \cap \pi(w + \Lambda + W)} \sigma(x) \).
(a) $S$ is a subgroup of $V/W$ and $2S = S$.

(b) $Q := -\log \sigma$ is a quadratic form from $S$ to non-negative real numbers.\textsuperscript{18}

Let $\Lambda \subset V/W$ be a neighborhood of 0, and let $S_0 := \Lambda \cap S$, a subgroup of $S$ s.t.

$$\bigcup_{n \in \mathbb{N}} 2^{-n} S_0 = S.$$ 

(c) Let $x_1, \ldots, x_n \in S_0$ be a $\mathbb{Z}$-linearly independent subset in $S_0$. Look at the maximal $H$ s.t. \exists

$$\begin{align*}
\mathbb{Z}^n &\xrightarrow{\zeta} H \xrightarrow{\phi} \mathbb{Q}^n \\
S_0 &\xrightarrow{\phi} \mathbb{Q}^n
\end{align*}$$

Let $Q'$ be the quadratic form on $\mathbb{Q}^n$ s.t.

$$Q'(a, a) := -\log \sigma(\phi(a_1, \ldots, a_n)) \quad \text{for } a = (a_1, \ldots, a_n) \in H$$

Note that

$$Q \text{ is a pos. semi-definite quad. form on } \mathbb{Q}^n$$

$$\begin{align*}
Q(a) = 0, &\quad a \in \mathbb{Q}^n \quad \Rightarrow \quad a = 0
\end{align*}$$

(c\textsubscript{1}) $[H : \mathbb{Z}^n] < \infty$.

If not, \exists $\mathbb{Q}$-vector subspace $L \subset \mathbb{Q}^n$ s.t. $H \cap L$ is dense in $L$ in classical topology. But \forall $a \in \mathbb{Z}^n$, $\phi(a) \in S_0$,

$$\therefore \text{ in } \phi(a) + 2^m \Lambda, \quad \sigma(\phi(a)) \geq \sigma(b) \quad \text{for all } b \in \phi(b) + 2^m \Lambda$$

if $m$ is large enough. Thus

$$Q'(a, a) \leq Q'(b, b) \quad \text{for all } b \in a + 2^m H.$$ 

Take $a \in L \cap \mathbb{Z}^n$ and $b \in (a + 2^m H) \cap L$ in particular: then the possible $b$’s are dense in $L$. So there are some $b$’s for which $Q'(b, b) <$ any given $\epsilon$, and get a contradiction.

**Corollary.** $H$ is a finitely generated abelian group: w.l.o.g. $H = \mathbb{Z}^n$.

(c\textsubscript{2}) $Q'$ is positive definite.

If not, get

$$\begin{align*}
\mathbb{R}^n &\xrightarrow{\text{proj.}} \mathbb{R}^m \\
H &\xrightarrow{\phi} \mathbb{Q}^n
\end{align*}$$

\textsuperscript{18}For $s \in S$, we have $\sigma(s) = 1 \iff s \in W = \text{supp}(\overline{\mu})$. 

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and a quadratic form $Q''$ on $\mathbb{R}^m$ s.t. $Q'(a) = Q''(\pi(a))$ for all $a \in \mathbb{R}^n$, and $\pi(H) \subset \mathbb{R}^m$ is not discrete. i.e. $\exists$ $\mathbb{R}$-vector subspace $L \subset \mathbb{R}^m$ s.t. $\pi(H) \cap L$ is dense in $L$. Get the same contradiction as above.

(d) $S_0$ is a free abelian group of rank $r = \dim(V/W)$.

**Proof.** Define $r := \dim(V/W)$, $d := \dim_Q(S_0 \otimes \mathbb{Q})$. Then $d$ finite $\iff S_0$ fin. gen. by $(c_1)$ and $(c_2)$.

If $d < r$, then $S_0$ is too small to be dense$^{19}$ in $\Lambda$, OUT. If $d > r$, well $S_0/2S_0 \subset \Lambda/2\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^r$.

Q.E.D.

\[
\therefore S \xrightarrow{\cong} \mathbb{Z}[1/2]^r
\]

and $\sigma = e^{-Q(a,a)}$, $Q$ pos. def. quad. form on $\mathbb{R}^r$. Expectation 3 is proved.

**(VI) Theorem.** 4 (1) Every Gaussian measure $\mu$ on $\mathcal{V}$ (as above) can be written as

$$
\mu = \sum_{x \in S} \mu_x
$$

where each $\mu_x$ is an $\mathcal{O}$-valued measure on $\mathcal{V}$ with

$$
supp(\mu_x) = \pi^{-1}(x) \quad \forall x \in S
$$

$$
\sup\left\{ |\mu_x(U)| : U \subset \mathcal{V} \text{ compact open} \right\} \quad \forall x \in S
$$

Similarly, the dual measure $\nu$ can be written as

$$
\nu = \sum_{x \in S} \nu_x
$$

with similar properties as above. Moreover

$$
\xi_x(\mu_x \times \mu_y) = \nu_{x+y} \times \nu_{x-y} \quad \forall x, y \in S.
$$

---

$^{19}$ $S$ is dense in $\mathcal{V}/W$ because $\text{supp}(\mu) = \mathcal{V}$.

$^{20}$ For $x \in S$, the measure $\mu_x$ is the push-forward to $\mathcal{V}$ of a measure $\mu'_x$ on $\pi^{-1}(x)$, defined as follows. For any compact open subset $U'$ of $\pi^{-1}(x)$, let $\{U_i\}_{i \in \mathbb{N}}$ be a decreasing family of compact open subsets of $\mathcal{V}$ such that $\bigcap_{i \in \mathbb{N}} U_i = U'$. Then $\mu'_x(U') = \lim_{i \to \infty} \mu(U_i)$. 

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(2) Correspondingly: if \((0) \subset W^\perp \subset W \subset V\) is the set-up for \(\pi : V \to V/W \subset S\), and \(\Theta\) is an \(O\)-valued theta function w.r.t. \(e, e^*, \Lambda\) as in the beginning of §1. Then

\[
\Theta(\alpha) = \sum_{x \in \mathcal{S}} \Theta_x(\alpha),
\]

where each \(\Theta_x\) is a function on \(V\) such that

(a) \(\Theta_x(\alpha + \beta) = e^*(\beta/2) e(\beta/2, \alpha) \Theta_x(\alpha) \quad \forall \beta \in \Lambda,\)

(b) \(\text{supp}(\Theta_x) \subset \pi^{-1}(x) + \Lambda,\)

(c) \(\Theta_x(\alpha + \beta) = e(\beta, \gamma_x - \alpha/2) \Theta_x(\alpha) \quad \forall \beta \in W^\perp\) if \(\gamma_x \in V\) satisfies \(\pi(\gamma_x) = x.\)

---

**Defining an associated tower of toroidal groups**

\(\Theta\) on \(V\) gives

\[
\begin{aligned}
(0) &\subset W^\perp \subset W \subset V, \quad \pi : V \to (V/W) \quad \text{(we assume} \ e^*(\alpha/2) = 1, \forall \alpha \in W^\perp \cap (1/2)\Lambda) \\
S &\subset V/W \\
\Theta_x &\text{ on } \pi^{-1}(x) + \Lambda
\end{aligned}
\]

(1) \(\Theta_0\) on \(W/W^\perp\) defines a tower of abelian varieties

\(B_\alpha\) indexed by compact open subsets \(U_\alpha \subset W/W^\perp\)

(2) If \(U \subset V\) is a compact open subgroup, get

(a) \(U_\alpha = (U \cap W)/(U \cap W^\perp)\), hence \(B_\alpha\).

(b) \(\pi(U) \cap S = S_0\), a lattice in \(S\).

\[\forall x \in S_0, \text{choose } \gamma_x \in U \cap \pi^{-1}(x). \text{ Set} \]

\[\Phi_x(\alpha) = e(\gamma_x/2, \alpha) \cdot \Theta_x(\alpha + \gamma_x), \quad \alpha \in W,\]

a function on \(W/W^\perp\) "related to" \(\Theta_0\).

\[\therefore \Phi_x \text{ defines a point } P_\alpha(x) \in B_\alpha.\]

\[\text{[If } \gamma_x' = \gamma_x + \eta, \eta \in U + W, \text{ then } \phi_{\alpha}'(\alpha) = \text{const. } e(\eta/2, \alpha) \phi_x(\alpha + \eta), \text{ so } P_\alpha(x) \text{ doesn’t change.]}\]

---

\(^{21}\)The quotient \(V/W\) here corresponds to the quotient \(V/W\) in (1).

\(^{22}\)The functions \(\Theta_x\) is related to the measures \(\mu_x\) as in (II):

\[
\mu_x(a_2 + 2^n \Lambda_2) = \sum_{a_1 \in 2^n \Lambda_1/\Lambda_1} e(a_2, a_1/2) \Theta_x(a_1 + a_2).
\]

\[
\Theta_x(a_1 + a_2) = e(a_2, a_1/2) \int_{a_2 + \Lambda_2} e(a_1, \beta) \cdot d\mu_x(\beta) \quad a_1 \in V_1, a_2 \in V_2.
\]
Get a homomorphism

\[ S_0 \xrightarrow{P_{\alpha}} B_{\alpha} \]

\[ x \xlongleftarrow{\rightarrow} P_{\alpha}(x) \]

(c) \( G_{\alpha} = \text{Spec}_{B_{\alpha}} \left( \bigoplus_{x \in S_0} \left\{ T_{P_{\alpha}(x)}^{-1} L_{\alpha} \right\} \right) \)

A class of rigid analytic maps

Given: \( K = \) complete valued field, \( C = \widehat{\mathbb{K}} \),

Given: \( G \), a comm. alg. grp. over \( K \) of type

\[ \xymatrix{ G \ar[d]_{\mathbb{G}_m} \ar[r]^\pi & A \ar[d] \text{ abel. var.,} \\
L \text{, ample inv. sheaf on } A \text{, all rational over } K.} \]

Now

\( G \cong \text{Spec}_A \left\{ \bigoplus_{n \in \mathbb{Z}^r} \left( K_1^{n_1} \otimes \cdots \otimes K_r^{n_r} \right) \right\} \)

where \( K_1, \ldots, K_r \) are invertible sheaves on \( A \), alg. equiv. to \( \mathcal{O}_A \).

To define a rigid analytic map \( \phi: G_C \longrightarrow \mathbb{P}^m_C \), need \( m + 1 \) analytic sections of \( \pi^*(L) \) over \( G_C \).

\[ \sim \quad m+1 \text{ Laurent-type expressions } L_i = \sum_{n \in \mathbb{Z}^r} s(n, i), \quad 0 \leq i \leq m \]

\[ s(n, i) \in \Gamma(A, L \otimes K_1^{n_1} \otimes \cdots \otimes K_r^{n_r}) \]

CONVERGENCE: \( \forall x \in G_C \), get \( \pi(x) = y \in A_C \), plus \( K_i(y) \xrightarrow{\sim} C \) for \( i = 0, 1, \ldots m \).

Then evaluate:

\[ s(n, i) \xlongleftarrow{\rightarrow} \text{Val}_x[s(n, i)] \in L(y) \]

Ask that

\[ \sum_n \text{Val}_x[s(n, i)] \begin{cases} \text{exists in } L(y) \text{ for all } i \\ & \text{& not be 0 for all } i \end{cases} \]

Hence \( \phi \) comes out.
§2. Application to monodromy: method of theta functions

Given

(a) an abelian variety $X$ over $K$ → get $T_2(X)$, a module over $\mathbb{Z}_2[\text{Gal}(\overline{K}/K)]$,

(b) a principal polarization on $X$ plus an even symmetric theta-divisor $D_\theta$ representing it → get a theta function $\Theta: V_2(X) \to \overline{K}$ s.t. $\Theta(\sigma x) = \Theta(x)^\sigma \ \forall \sigma \in \text{Gal}(\overline{K}/K)$.

[State converse: all such $(V, \Lambda, e, e_*, \Theta)$ come from $(X, D_\theta)$.]

Problem is:

if $K = \text{local field, alg. cl. res. field } k$, $\text{char}(k) \neq 2$,

& if $\Gamma := \text{Gal}((\overline{K}/K)$ acts on $T_2(X)$ via its tamely ramified quotient $\Gamma_{\text{tame}}$,

show that

$\exists$ an open subgroup $U \subset \Gamma_{\text{tame}}$ s.t. $\gamma$ operates unipotently on $T_2(X) \ \forall \gamma \in U$

Method: a complete description to the solutions of the theta functional equations over a local field. viz. $\exists$

(i) $(0) \subset W^\perp \subset W \subset V$ subspaces, $\pi: V \to V/W$

(ii) $S \hookrightarrow V/W \xrightarrow{\cong} \mathbb{Z}[1/2]^r \xrightarrow{\cong} \mathbb{Q}_r^r$

(iii) $Q: S \to \mathbb{R}$ pos. def. quad. form

s.t.

$\Theta = \sum_{s \in S} \Theta_x$

(a) $\text{supp}(\Theta_x) \subset \pi^{-1}(x) + \Lambda$.

(b) $\max_y |\Theta_x(y)| = e^{-Q(x,x)}$.

(c) $\Theta_x(\alpha + \beta) = e_{*}(\beta/2) \cdot e(\beta/2, \alpha) \cdot \Theta_x(\alpha)$ for all $\beta \in \Lambda$.

(d) $\Theta_x(\alpha + \beta) = e(\beta, \gamma_x - \frac{a}{2}) \cdot \Theta_x(\alpha)$ for all $\beta \in W^\perp$ if $\gamma_x \in \pi^{-1}(x)$.

---

$\Gamma_{\text{tame}} \cong \prod_\ell \mathbb{Z}_\ell(1)$, where $\ell$ runs through all prime numbers which are invertible in $k$. 
Claim: It follows that
\[ \gamma = \text{id. on } W \text{ and on } V/W^\perp \quad \forall \gamma \in U, \]
i.e. the matrix representation of \( \gamma \) has the form
\[
\begin{pmatrix}
I & 0 & * \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\]
This Claim will be proved in two steps below.

**Step 1.** Assume that \( \Theta : V \to k \) is an algebraic theta function for \((V, \Lambda, e, e^*), \sigma \in \text{Sp}(V, \Lambda)\) such that
- \( \forall x \in V, \exists \eta \in \frac{1}{2}\Lambda \text{ s.t. } \Theta(x + \eta) \neq 0, \)
- \( \Theta(\sigma x) = \Theta(x) \text{ for all } x \in V \)

Then \( \sigma \) is of finite order.

**Proof of Step 1.** Replace \( \sigma \) by a suitable tower so that \((\sigma - 1)\Lambda \subseteq 4\Lambda\). We will show that for any \( n \geq 2, \)
\[(\sigma - 1)\Lambda \subseteq 2^n \Lambda \implies (\sigma - 1)\Lambda \subseteq 2^{n-1} \Lambda \]
For any \( x \in 2^{-n}\Lambda \), we have
\[
\Theta(x) = \Theta(\sigma x) = \Theta((\sigma - 1)x) = e_*\left(\frac{\sigma x - x}{2}\right) \cdot e(\frac{\sigma x - x}{2}, x) \cdot \Theta(x)
\]
\[ \therefore e(\sigma x - x, x) = 1 \text{ if } \Theta(x) \neq 0. \]
Pick an \( \eta \in \frac{1}{2}\Lambda \) such that \( \Theta(x + \eta) \neq 0. \) Then
\[ 1 = e((\sigma - 1)(x + \eta), (x + \eta)) = e((\sigma - 1)x, x) \cdot e((\sigma - 1)\eta, x) \cdot e((\sigma - 1)x, \eta) \cdot e((\sigma - 1)\eta, \eta) . \]
The last term is 1 because \((\sigma - 1)\eta \in 2^{n-1}\Lambda \subseteq 2\Lambda\). The product of the two middle terms is
\[
eq e((\sigma - 1)^2\eta + 2(\sigma - 1)\eta, \sigma x) = 1
\]
because \((\sigma - 1)^2\eta \in 2^{n-1}\Lambda \subseteq 2\Lambda\) and \(2(\sigma - 1)\eta \in 2^n\Lambda\). So
\[
Q(x) := e((\sigma - 1)x, x) = 1 \quad \forall x \in 2^{-n}\Lambda.
\]
Now we have
\[
1 = \frac{Q(x + y)}{Q(x) \cdot Q(y)} = e((\sigma - 1)x, y) \cdot e((\sigma - 1)y, x) = e(x, \sigma^{-1}y) \cdot e(\sigma y, x) = e(x, \sigma^{-1}y - \sigma y)
\]
for all \( x, y \in 2^{-n}\Lambda \), therefore \( \sigma^{-1}y - \sigma y \in 2^n\Lambda \) for all \( y \in 2^{-n}\Lambda \). Write \( \sigma = 1 + \tau \), we have
\[
2^n\Lambda \ni \sigma^2 y - y = 2\tau y + \tau^2 y \quad \forall y \in 2^{-n}\Lambda.
\]
But \( \tau^2 y \in 2^n\Lambda \), therefore \( \tau y \in 2^{n-1}\Lambda \) for all \( y \in 2^{-n}\Lambda \), i.e. \((\sigma - 1)\Lambda \subseteq 2^{2n-1}\Lambda\). Q.E.D.

\[ \text{This statement was formulated for } n = 2 \text{ in the original notes.} \]
We go back to the algebraic theta function $\Theta$ for $(V = V_2(X), \Lambda = T_2(X), e, e_*)$ attached to $(X, D_\theta)$. Let $W \subset V$ be the associated cusp, $W^\perp \subset W$. An element $\gamma \in \Gamma_{tame}$ operates on $V$ via an element of $\text{Sp}(V, \Lambda)$ s.t. $\gamma(W) \subset W$, $\gamma(W^\perp) \subset W^\perp$. By Step 1, there exists an open subgroup $U \subset \Gamma_{tame}$ such that every $\gamma \in U$ has a matrix representation of the form

$$\begin{pmatrix}
  tA^{-1} & B & * \\
  0 & I & C \\
  0 & 0 & A
\end{pmatrix}$$

i.e. $\gamma$ operates on $W/W^\perp$ as the identity.

**Step 2.** $A = I$, $B = 0$, i.e. $\gamma|_W = \text{id}_W$.

We need the following facts for the proof of Step 2; they are consequences of the results in §1, summarized at the beginning of this section.

**Fact (a).** $\sigma(x) := \sup_{\eta \in \frac{1}{2}\Lambda} |\Theta(x + \eta)|$ depends only on the image of $x$ in $V/(W + \frac{1}{2}\Lambda)$.

**Fact (b).** $\forall x \in V, \exists \xi_x \in V$, depending only on the image of $x \in V/W$, s.t.

$$|\Theta(x + u) - e(\xi_x, u)\Theta(x)| < \sigma(x) \quad \forall u \in W^\perp.$$ 

We know that $\Theta(\gamma x) = \Theta(x)^\gamma$ $\forall x \in V, \gamma(\Lambda) \subset \Lambda$ and $(\gamma - 1)W \subset W^\perp$. Replacing $U$ by an open subgroup, we may assume$^{25}$ that $U \cong \mathbb{Z}_2$ and

$$(\sigma - 1)(\Lambda) \subset 8\Lambda + W^\perp, \quad (\sigma - 1)(\Lambda \cap W) \subset 8\Lambda$$

i.e. $P(3)$ holds, where $P(n)$ stands for the statement

$$(\sigma - 1)(\Lambda) \subset 2^n\Lambda + W^\perp \quad \text{and} \quad (\sigma - 1)(\Lambda \cap W) \subset 2^n\Lambda \quad \forall \gamma \in U.$$ 

It is clear that Step 2 follows from Claim 2 below.

**Claim 2.** Suppose $P(n)$ holds, $n \geq 3$, then $P(2n - 1)$ holds.$^{26}$

The first part of $P(2n - 1)$ implies that if $x \in \Lambda \cap W$, $\xi = 2^{-2n+1}\lambda + w$, $\lambda \in \Lambda$, $w \in W$, then $\gamma^{-1}\xi \in \Lambda + W^\perp$, so

$$e(\gamma x, \xi) = e(x, \gamma^{-1}\xi) = 1.$$ 

i.e. the second part of $P(2n - 1)$ follows from the first part.

$^{25}$Because $\text{Sp}(V, \Lambda)$ is an extension of a finite group by a pro-2 group.

$^{26}$Claim 2 was formulated for $n = 3$ in the original notes.
Let $x \in 2^{-n} \Lambda$, $n \geq 3$. Changing $x$ by an element of $\frac{1}{2} \Lambda$, we may assume that $|\Theta(x)| = \sigma(x)$. Write $(\gamma - 1)x = \eta + u$, $\eta \in \Lambda$, $u \in W^\perp$. Then

$$\Theta(x)^\gamma = \Theta(\gamma x) = \Theta(x + \eta + u) = e_*(\eta/2) e(\eta/2, x + u) e(\xi_x, u) \Theta(x).$$

Change $x$ to $x'$ with $w := x' - x \in 2^{-n} \Lambda \cap W$. Then $|\Theta(x')| = \sigma(x') = \sigma(x)$ too by Facts (a), (b) above. We know\textsuperscript{27} that

$$\frac{\Theta(x')^\gamma}{\Theta(x')} = \frac{\Theta(x)^\gamma}{\Theta(x)}.$$

Then $e_*(\eta/2) e(\eta/2, x + u) e(\xi_x, u) = e_*(\eta'/2, x' + u) e(\xi_x, u)$.

We have $(\gamma - 1)x = \eta + u$, $(\gamma - 1)x' = \eta' + u$, $\eta' - \eta = (\gamma - 1)w \in W^\perp \cap \Lambda$ by $P(n)$.

$$e(\eta, x + u) = e(\eta', x' + u).$$

We have shown that $\forall y \in 2^{-n} \Lambda$, $\exists x \in y + \frac{1}{2} \Lambda$ s.t. $(\gamma - 1)x \in 2^{n-1} \Lambda + W^\perp$ for all $\gamma \in U$. So $(\gamma - 1)\Lambda \subset 2^{2n-1} \Lambda + W^\perp$.

Q.E.D.

\textsuperscript{27}from the structure of tamely ramified extensions of local fields.