Homework #10 Solutions

10.3.3

Analyze the long-term behavior of the map $x_{n+1} = \frac{rx_n}{1+x_n^2}$, where $r > 0$.

Solutions

The fixed points are given by solutions of the equation

$$x = \frac{rx}{1+x^2}$$

$$\Rightarrow x_0^* = 0 \text{ or } x_{\pm}^* = \pm \sqrt{r-1}.$$

Therefore, if $r > 1$ we have three fixed points otherwise there are two.

Let $f(x) = \frac{rx}{1+x^2}$, we have that

$$f'(0) = r \text{ and } f'(\pm \sqrt{r-1}) = \frac{2}{r} - 1.$$

Therefore the fixed point 0 is unstable if and only if $r > 1$ while the points $x_{\pm}^* = \pm \sqrt{r-1}$ are stable if and only if $r > 1$.

We will analyze the existence of non-trivial periodic orbits in two cases.

Case 1:

If $r \leq 1$ then $|f(x)| = \frac{|rx|}{1+x^2} \leq r|x| \leq |x|$, with equality if and only if $x = 0$. Therefore,

$$|f^n(x)| = |f(f^{n-1}(x))| \leq |f^{n-1}(x)|$$

Continuing by induction we have $\forall x \neq 0$ that

$$|f^n(x)| = |x|.$$

Therefore, if $r \leq 1$ there are no periodic orbits. However, since $x_n = f^n(x_0)$ satisfies $|x_n| \leq |x_{n-1}|$ it follows that all sequences are monotone decreasing. Consequently

$$\lim_{n \to \infty} x_n = 0.$$ Therefore, there is no Chaos in this regime.

Case 2:

If $r > 1$, the nonexistence of periodic orbits is less clear. Graphically, the function $f^n(x)$ looks something like:

![Graphical representation]

Therefore no periodic orbits exist.
Consider the quadratic map \( x_{n+1} = x_n^2 + c \).

\( \text{a.} \) Find and classify all the fixed points as a function of \( c \).

**Solution:**

Solving the equation

\[
x^2 - x + c = 0
\]

we have that fixed points correspond to:

\[
x^*_3 = \frac{1 \pm \sqrt{1 - 4c}}{2}
\]

These points exist if and only if \( 1 - 4c > 0 \Rightarrow c < \frac{1}{4} \).

Letting \( f(x) = x^2 + c \) we have that:

\[
f'(x^*_3) = 1 \pm \sqrt{1 - 4c}
\]

Therefore, when these points exist one is stable and the other is unstable.

\( \text{b.} \) Find the values of \( c \) at which the fixed points bifurcate.

**Solution:**

At \( c = \frac{1}{4} \) there is a saddle-node bifurcation.

\( \text{c.} \) For what values of \( c \) is there a stable 2-cycle?

**Solution:**

\[
f(f(x)) = (x^2 + c)^2 + c = x^4 + 2x^2c + c^2 + c.
\]

Dividing, we have that:

\[
x^2 - x + c + \frac{1}{c+1}
\]

\[
x^2 - x + c
\]

\[
x^4 - x^3 + x^2 + c
\]

\[
x^2 + x^2
\]

\[
x^2 - x^2 + c(1 + x)
\]

\[
(c + 1)x^2 - (c + 1)x + c + 1
\]

Therefore, period 2-orbits satisfy

\[
x^2 + x + c + 1 = 0
\]

\[
\Rightarrow x^*_3 = -1 \pm \sqrt{1 + 4(c + 1)}
\]

Therefore, period 2-cycles exist if \( c < -\frac{1}{4} \). To analyze stability we know that

\[
\frac{d}{dx} (f(f(x^*_3))) = f'(x^*_3) + f'(x^*_3) = (-1 + \sqrt{1 + 4(c + 1)})(-1 - \sqrt{1 + 4(c + 1)}) = 4(c + 1).
\]
11.4.1
Find the box dimension of the von Koch snowflake.

**Solution:**

\[ S_0: \quad N_1 = 3 \quad \varepsilon_1 = 1 \]

\[ S_1: \quad N_2 = 12 = 4 \cdot 3 \quad \varepsilon_2 = \frac{1}{3} \]

\[ \Rightarrow N_n = 3 \cdot (4^n) \]

\[ \varepsilon_n = \left( \frac{1}{3} \right)^n \]

Therefore, \( d = \lim_{n \to \infty} \frac{\ln(4^n)}{\ln(3^n)} = \lim_{n \to \infty} \frac{\ln(4) + \ln(3)}{\ln(3)} = \frac{\ln(4)}{\ln(3)} \)

11.4.2
Find the box dimension of the Sierpinski carpet.

**Solution:**

\[ \varepsilon_n = \left( \frac{1}{3} \right)^n \]

\[ N_n = (8)^n \]

\[ \Rightarrow d = \lim_{n \to \infty} \frac{\ln(8^n)}{\ln(3^n)} = \frac{\ln(8)}{\ln(3)} \]
Therefore, the two-cycle is stable if 
\[-\frac{3}{4} < c < -\frac{1}{4}\).

\#10.3.7

Consider the decimal shift map on \(I = [0, 1]\) defined by 
\[x_{n+1} = 10x_n \pmod{1}\)

b.) Find all fixed points

**Solution:**

Any \(x \in I\) with decimal expansion \(x = \cdots a a a a \cdots\), where \(a \in \{0, 1, \ldots, 9\}\).

c.) Show that the map has periodic orbits of all periods, but all of them are unstable.

**Solution:**

Any \(x \in I\) with a repeating decimal expansion is a periodic orbit. Let \(f(x) = 10x \pmod{1}\). Since \(f'(x) = 10\) it follows that all periodic orbits are unstable.

d.) Show that the map has infinitely many aperiodic orbits.

**Solution:**

The periodic orbits correspond to \(Q \cap I\). Consequently, the aperiodic orbits correspond to \((\mathbb{R} \setminus Q) \cap I\), i.e. the set of irrational numbers in \([0, 1]\) which is infinite.

e.) Show that the map has sensitive dependence on initial conditions.

**Solution:**

The Lyapunov exponent is given by
\[
\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln |10| = \ln(10).
\]