Consider the system \( \dot{x} = y^3 - 4x, \dot{y} = y^3 - y - 3x \).

a.) Find all the fixed points and classify them.

**Solution:**

The fixed points satisfy the equation
\[
0 = y^3 - 4x \\
0 = y^3 - y - 3x
\]

The null-clines are then given by
\[
y = (4x)^{\frac{1}{3}} \\
x = \frac{y^3 - y}{3}
\]

The null-clines intersect at the point:
\[
y = \frac{4x^3 - 4y}{3}
\]
\[
\Rightarrow y^3 - 4y = 0 \\
\Rightarrow y = 0, \pm 2
\]

The fixed points are then 
\((0, 0), (2, 2), (-2, -2)\)

The Jacobian matrix is then
\[
J = \begin{pmatrix}
-4 & 3y^2 \\
-3 & 3y^2 - 1
\end{pmatrix}
\]

\[
J|_{(0,0)} = \begin{pmatrix}
-4 & 0 \\
-3 & -1
\end{pmatrix}
\]

Therefore, \((0, 0)\) is a stable fixed point.

\[
J|_{(2,2)} = \begin{pmatrix}
-4 & 12 \\
-3 & 11
\end{pmatrix}
\]

\[\Rightarrow \text{det}(J|_{(2,2)}) = -8 < 0\]

Therefore, \((2, 2)\) is a saddle node.
From odd symmetry we have that $J(2,2) = -J(-2,-2)$

$\Rightarrow \det(J(2,2)) = \det(J(-2,-2))$.

Therefore $(-2,2)$ is a saddle point as well.

b.) Show that the line $y = x$ is invariant.

Solution:

$\dot{y} - \dot{x} = x - y = -(y-x)$

Therefore, if we let $z = y - x$ it follows that

$\dot{z} = -z$

Which has a fixed point at $z = 0 \Rightarrow y - x = 0$ for all $t$ if $x(0) = y(0)$.

c.) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all trajectories.

Solution:

For the one-dimensional system $\dot{z} = -z$, $z = 0$ is a stable fixed point.

Consequently, $\lim_{t \to \infty} |x(t) - y(t)| \to 0$.

d.) Sketch the phase portrait.

Solution:
6.3.14
Classify the fixed point at the origin for the system \( \dot{x} = -y + ax^3 \) and \( \dot{y} = x + ay^3 \).

**Solution:**
Let \( r = x^2 + y^2 \). Then,
\[
\dot{r} = \frac{x \dot{x} + y \dot{y}}{r} = \frac{a(x^4 + y^4)}{r}
\]
Therefore, \((0,0)\) is a stable fixed point if \( a < 0 \), unstable if \( a > 0 \) and a linear center if \( a = 0 \).

6.5.2
Consider the system \( \dot{x} = x - x^2 \).

(a) Find and classify the equilibrium points.

**Solution:**
This is a conservative system with potential \( V(x) = -\frac{x^2}{2} + \frac{x^3}{3} \) which is plotted below.

Therefore, \( x = 1 \) is a nonlinear center and \( x = 0 \) is a saddle.

(b) Sketch the phase portrait.

**Solution:**
From the potential we have that...
C.) Find an equation for the homoclinic orbit.

Solution:
The energy is given by \( E = \frac{1}{2} v^2 - \frac{x^2}{2} + \frac{x^3}{3} \). At \( x = 0, v = 0 \) — the start of the homoclinic orbit — we have that \( E = 0 \). Consequently, the homoclinic orbit is given by the curve:

\[ v = \pm \sqrt{x^2 - \frac{2}{3} x^3}, \quad x > 0. \]

# 6.8.6

A closed orbit in the phase plane encircles \( S \) saddles, \( N \) nodes, \( F \) spirals, and \( C \) centers. Show that \( N + F + C = 1 + S \).

Solution:
The index of the curve is \( 1 \) so we have that

\[ N + F + C - S = 1 \]

\[ \Rightarrow N + F + C = 1 + S. \]

# 6.8.13

Consider a smooth vector field \( \dot{x} = f(x, y), \dot{y} = g(x, y) \) and let \( C \) be a simple closed curve in the plane that does not pass through any fixed points. Let \( \theta = \tan^{-1}\left( \frac{y}{x} \right) \).

a.) Show that

\[ d\theta = \frac{f \, dx - g \, dy}{x^2 + y^2} \]

Solution:

\[ d\theta = \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{f \, dx - g \, dy}{x^2 + y^2} = \frac{f \, dx - g \, dy}{x^2 + y^2} \]

b.) Derive the formula

\[ I_C = \frac{1}{2\pi} \oint_C \frac{f \, dx - g \, dy}{x^2 + y^2} \]

Solution:

\[ I_C = \frac{1}{2\pi} \oint_C d\theta = \frac{1}{2\pi} \oint_C \frac{f \, dx - g \, dy}{x^2 + y^2} \]