Chapter 8: Bifurcations Part Deux'

\[ \dot{x} = f(x, y, \mu) \]
\[ \dot{y} = g(x, y, \mu) \]

A bifurcation point \( \mu^* \) is a point where the topology of the phase portrait changes.

1. Let \( (x^*, y^*) \) denote an equilibrium point.
2. Let \( \lambda_1, \lambda_2 \) denote the eigenvalues associated with \( \lambda_1, \lambda_2 \).

Bifurcations occur if one or both of the eigenvalues \( \lambda_1, \lambda_2 \) lie on the imaginary axis.

1. \( \lambda_{1,2} = \pm i \mu \rightarrow \text{Hopf bifurcation (new stuff)} \)
2. \( \lambda_1 = 0, \lambda_2 \neq 0 \rightarrow 1-D \) bifurcation
3. \( \lambda_{1,2} = 0, J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), never really happens.

1-D - Bifurcations

1. Saddle Node:
   \[ \dot{x} = \mu - x^2 \]
   \[ \dot{y} = -y \]

<table>
<thead>
<tr>
<th>Left</th>
<th>Right</th>
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<tbody>
<tr>
<td>( \mu &lt; 0 )</td>
<td>( \mu &gt; 0 )</td>
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<tr>
<td>( \lambda_{1,2} = \pm i \mu )</td>
<td>( \lambda_1 = 0, \lambda_2 \neq 0 )</td>
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More Generally: One null-cline slips through another

\( \mu < \mu^* \) (blue null-cline intersects pink one)
2. Transcritical:
\[
\begin{cases}
x = u x - x^2 \\
y = -y
\end{cases}
\]
*Fixed point at origin switches stability.

3. Pitchfork:
\[
\begin{align*}
\dot{x} &= u x \pm x^3 \\
\dot{y} &= -y
\end{align*}
\]
Take $-\text{sign} \rightarrow$

Example:
\[
\begin{align*}
S &= r_s S \left(1 - \frac{S}{k_s E}\right) \\
E &= r_E E \left(1 - \frac{E}{k_E}\right) - P \frac{B}{S}
\end{align*}
\]

$S =$ size of the forest
$B =$ worm population
$k_s =$ spruce tree carrying capacity when $E = k_E$
$k_E =$ energy reserve carrying capacity.
$B =$ budworm population
$P =$ Rate energy reserve is eaten by worms.
$r_s =$ growth rate of forest
$r_E =$ growth rate of energy reserves.
Rescale
\[ x = \frac{S}{k_s} \]
\[ y = \frac{E}{k_E} \]
\[ \tau = r_s t \]

\[ \Rightarrow \frac{dx}{d\tau} = x(1-\frac{x}{y}) \]
\[ \frac{dx}{d\tau} = xy(1-y) - \frac{\beta}{x} \]
\[ x = \frac{r_E}{r_s} \]
\[ \beta = \frac{PB}{k_s r_s} \]

Nullclines
\[ \frac{dx}{d\tau} = 0, \quad x = 0, \quad y = x \]
\[ \frac{dx}{d\tau} = 0, \quad x = \frac{\beta}{\alpha y (1-y)} \]

Let's sketch the nullclines for this system. First lets figure out what the non-trivial nullcline looks like.
Therefore, we have two types of phase portraits:

Small $B$
$\rightarrow$ Forest can live with tree beds

Large $B$
$\rightarrow$ Forest dies

How can we determine bifurcation point? Too hard...
However, the bifurcation is a saddle-node
Hopf-Bifurcation

Let \( w > 0 \)

\[
\begin{align*}
\dot{x} &= ux - w y \pm (x^3 + x y^2) + b (-x^2 y - y^3) \\
\dot{y} &= w x + b y \pm (x^2 y + y^3) + b (x^3 + x y^2)
\end{align*}
\]

linear rotations, \( R \), cubic nonlinearity

\( J(0,0) = \begin{pmatrix} w & -w \\ w & w \end{pmatrix} \), eigenvalues \( \lambda_{1,2} = \mu \pm i w \)

Convert to polar coordinates:

\[
\begin{align*}
\dot{r} &= w r \pm r^3 \\
\dot{\theta} &= w + b r^2
\end{align*}
\]

1. **Supercritical Hopf-bifurcation for \( \mu > 0 \) sign**

   Fixed points:

   \( r = 0, \ r = \sqrt{\mu} \)

   \( R < 0 \)

   \( R > 0 \)

2. **Subcritical Hopf-bifurcation for \( \mu > 0 \) sign**

   Fixed points:

   \( r = 0, \ r = \sqrt{-\mu} \)

   \( R < 0 \)

   \( R > 0 \)
Example

Chemical reaction

\[ \begin{align*}
    \dot{x} &= a - x + x^2 \\
    y &= b - x^2
\end{align*} \]

Nullclines

\[ \begin{align*}
    \frac{dx}{dt} = 0: & \quad y = \frac{x-a}{x^2} \\
    \quad \frac{dy}{dt} = 0: & \quad y = \frac{b}{x^2}
\end{align*} \]

Fixed Point

\( (a+b, \frac{b}{a+b^2}) \)

\[ \begin{align*}
    dx & \sim -1 \text{ for large } x \\
    \frac{dy}{dx} & < -1 \Rightarrow \frac{b-x^2}{a-x+x^2y} < -1 \Rightarrow x > b+a
\end{align*} \]

\[ \begin{align*}
    \frac{dy}{dx} & \sim \frac{b}{a} \text{ near } x = a \\
    \text{Solve: } & \quad \frac{b-x^2}{a-x+x^2y} < \frac{b}{a} \Rightarrow y > \frac{1}{x(a+1)}
\end{align*} \]
Since we have a trapping region we now check stability of the fixed point.

\[
J = \begin{pmatrix}
-1 + 2xy & x^2 \\
-2xy & -x^2
\end{pmatrix}
\]

\[
J(\alpha + \beta, \frac{1}{\beta(\alpha + \beta)^2}) = \begin{pmatrix}
-1 + \frac{2b}{\alpha \beta} (\alpha + \beta)^2 \\
-2\frac{b}{\alpha \beta} - (\alpha + \beta)^2
\end{pmatrix}
\]

**Hopf bifurcation occurs when**

\[
\text{Tr}[J((\alpha + \beta, \frac{1}{\beta(\alpha + \beta)^2})] = 0
\]

\[
\Rightarrow -1 + 2\frac{2b}{\alpha \beta} - (\alpha + \beta)^2 = 0
\]

\[
\Rightarrow b - \alpha = (\alpha + \beta)^3
\]

This is a super-critical Hopf bifurcation. 

*Really need a numerical scheme to understand complete dynamics.*

The period can be estimated by assuming a circular orbit at the bifurcation point.

\[
\text{det}(J(\alpha + \beta, \frac{1}{\beta(\alpha + \beta)^2}) = (\alpha + \beta)^2 - 2b(\alpha + \beta) + 2b(\alpha + \beta) = -\lambda_1
\]

at the bifurcation point,

\[
\Rightarrow \lambda_1 = \lambda(\alpha + \beta).
\]

The period is then

\[
T = \frac{2\pi}{\alpha + \beta}
\]

Since the linear solution is:

\[
x = A \cos(\lambda_1 t)
\]

\[
y = A \sin(\lambda_1 t)
\]
Other Periodic Bifurcations

1. Saddle Node:
\[ \begin{align*}
\dot{r} &= \nu r - r^3 + r^5 \\
\dot{\theta} &= \nu 
\end{align*} \rightarrow \text{Quintic normal form} \]

\[ \begin{array}{ll}
N > \frac{1}{4} & \quad \text{limit cycle} \\
N < \frac{1}{4} & \quad \text{no limit cycle}
\end{array} \]

2. Infinite Period:
\[ \begin{align*}
\dot{r} &= r (1-r^2) \\
\dot{\theta} &= \nu - \sin \theta \rightarrow \text{Saddle node bifurcation}
\end{align*} \]

\[ \begin{array}{ll}
N > \frac{1}{4} & \quad \text{no limit cycle} \\
N < \frac{1}{4} & \quad \text{limit cycle}
\end{array} \]
3. Homoclinic Bifurcation.

\[ \dot{\phi} + \alpha \dot{\phi} + \sin(\phi) = I, \quad I, \alpha > 0. \]
\[ \phi \in S^1 \]

Let \( \nu = \dot{\phi} \) then
\[ \dot{\nu} = I - \alpha \nu - \sin(\phi) \]
\[ \phi = \nu \]

Fixed point only exist if \( I < 1 \), \( \nu = 0 \), \( \sin(\phi) = I \).

Jacobian:
\[
\begin{bmatrix}
-\alpha & 1 \\
-\cos(\phi) & -\alpha
\end{bmatrix} \Rightarrow \text{The eigenvalues are}
\]
\[ 2 \lambda, \lambda_1 = -\alpha \pm \sqrt{\alpha^2 - 4 \cos^2(\phi)} \]
\[ = -\alpha \pm \sqrt{\alpha^2 + 4 \sqrt{1 - I^2}} \]

\( \rightarrow \) 1 fixed point is a saddle, the other a stable fixed point of some type.
Analogous with Josephson junction

\[ \Phi_1 e^{i \phi_1} - \Phi_2 e^{i \phi_2} \to \text{phase difference} \]

If I < 1, junction acts as if it had zero resistance. There is a phase difference between the states.

If I > 1, junction acts as a resistor. The voltage is given by:

\[ V = \frac{\Phi}{2e} \]

Case 1': I > 1

\[ \Phi = \Phi_1 - \Phi_2 \]

\[ V = I - \alpha V - \sin(\phi) \]

Nullcline

\[ V = 0 \]

\[ V = \frac{1}{\alpha}(I - \sin \phi) \]

The Poincaré map \( P: \mathbb{R}^+ \to \mathbb{R}^+ \) maps initial \( V \) at \( \phi = 0 \) to value of \( V \) at \( \phi = 2\pi \) for a solution trajectory.

1. \( P(V_1) < V_1 \)
2. \( P(V_2) > V_2 \)
Let's sketch $P(v)$.

$P(v)$ must be monotone otherwise trajectories could cross.

$\Rightarrow \exists v^* \text{ such that } P(v^*) = v^$. This implies the existence of a limit cycle.

**Case 2'.**

$I_c < I < 1, \alpha \ll 1$

Limit cycle still exists, however we also have a stable fixed point.

Stable fixed point.
Case 3':

$I = I_c < 1, \alpha << 1$.

Limit cycle merges with stable manifold.

Infinite period bifurcation
Homeoclinic bifurcation

Case 4':

$\alpha >> 1, I < 1$.

No limit cycle. Pendulum dies down.