Chapter 7: Limit Cycles

Definition: A closed trajectory is a limit cycle if it is separated from all other closed trajectories.

a) A limit cycle is stable if there is a tubular neighborhood such that trajectories that enter the neighborhood approach the limit cycle as $t \to \infty$.

b) A limit cycle is unstable if it is not stable.

Example:

$$\begin{align*}
\dot{x} &= -y + x(1 - \sqrt{x^2 + y^2}) \\
\dot{y} &= x + y(1 - \sqrt{x^2 + y^2}) \\
x &= r \cos \theta \\
y &= r \sin \theta \\
\dot{x} &= r \cos \theta - r \sin \theta \\
\dot{y} &= r \sin \theta + r \cos \theta \\
\left[\begin{array}{c}
\dot{r} \\
\dot{\theta}
\end{array}\right] &= \frac{1}{r} \left[\begin{array}{cc}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{array}\right] \left[\begin{array}{c}
\dot{x} \\
\dot{y}
\end{array}\right]
\end{align*}$$

$$\begin{align*}
\dot{r} &= \cos \theta (-r \sin \theta + r \cos \theta (1 - r)) + \sin \theta (r \cos \theta + r \sin \theta (1 - r)) \\
\dot{\theta} &= -\sin \theta (-r \sin \theta + r \cos \theta (1 - r)) + \cos \theta (r \cos \theta + r \sin \theta (1 - r)) \\
\Rightarrow r &= r(1 - r) \\
\Rightarrow \dot{\theta} &= -1
\end{align*}$$

The curve $x^2 + y^2 = 1$ is a stable limit cycle.
Poincare-Bendixson Theorem: Consider $\dot{x} = F(x)$, with $F$ continuously differentiable. Assume $R \subset \mathbb{R}^2$ is closed and bounded.

(i) $R$ does not contain any fixed points.
(ii) There exists $x(0) \in R$ so that $x \in R$ for all $x \geq 0$. Then $R$ contains a limit cycle.

**Typical Application:**

![Diagram](image)

$R$ = trapping region.

**Example:**

\[
\begin{align*}
\dot{x} &= -x + ay + x^2y \\
\dot{y} &= b - ay - x^2y
\end{align*}
\]

**Global Analysis:**

Null clines:

\[
\begin{align*}
y &= \frac{x}{a+x^2}, \quad \dot{x} = 0 \\
y &= \frac{b}{a+x^2}, \quad \dot{y} = 0
\end{align*}
\]
When is \( \frac{dx}{dt} < -1 \)
\[
\Rightarrow \quad b - ay - x^2 y < -x - ay - x^2 \\
\Rightarrow \quad b < x
\]

Therefore, if \( x > b \) we know \( \frac{dx}{dt} < -1 \).

The fixed point is \( \alpha = x = b, \quad y = \frac{b}{a+b^2} \).

\[J = \begin{pmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{pmatrix} \]

\[J \left| \begin{array}{cc} x, y \end{array} \right| = \begin{pmatrix} -1 + \frac{2b}{a+b^2} & a + b^2 \\ -\frac{2b}{a+b^2} & -a - b^2 \end{pmatrix} = \mathcal{A} \]

\[\text{det}(\mathcal{A}) = a + b^2 > 0 \]

\[\text{Tr}(\mathcal{A}) = -1 - a - b^2 + \frac{2b^2}{a+b^2} \]

\[\text{Tr}(\mathcal{A}) = 0 \iff a^2 + a(2b^2 + 1) + b^2(b^2 - 1) = 0 \]

![Graph showing a repeller and attractor with a limit cycle.]

Van der Pol Oscillator
\[
x'' + \frac{1}{\varepsilon}(x^2 - 1)x' + x = 0
\]

\( \frac{1}{\varepsilon} x^2 - 1 \) is like a damping or pumping term.

- If \( x^2 < 1 \) energy is pumped in (forcing)
- If \( x^2 > 1 \) energy is pumped out (friction)

In fact:
\[
\frac{d}{dt} E = \frac{d}{dt} \left( \frac{1}{2} x^2 + \frac{x^2}{2} \right) = -\frac{1}{\varepsilon} (x^2 - 1) x'^2
\]
We expect the damping and pumping to lead to oscillations.

\[ \dot{x} + \frac{1}{2} (x^2 - 1) \dot{x} = \frac{d}{dt} \left( x + \frac{1}{3} \left( x^3 - x \right) \right) \]

Let \( F(x) = \frac{x^3}{3} - x \), \( y = \varepsilon x + \frac{1}{3} x^3 - x \)

\[ \Rightarrow \begin{cases} \dot{x} = \frac{1}{\varepsilon} (y - F(x)), & 0 < x < 1 \\ \dot{y} = -3x \end{cases} \]

Let's try to analyze this system.

\[ |y| \sim \varepsilon \]

\[ |\dot{y}| \sim 3 \]

Has to hug the nullcline

Let's plot the dynamics of \( x' \).

\[ \text{Firing}, \text{Firing}, \text{Firing}, \text{Firing} \]

\[ \text{Relaxation}, \text{Relaxation}, \text{Relaxation} \]
Can we estimate the time scales. Essentially:
\[ \dot{x} \sim \frac{1}{\epsilon} \] away from the nullclines.
At the nullclines,
\[ \dot{x} \sim \epsilon, \quad \dot{y} \sim \epsilon \]
as the two terms balance each other.

**Example:**
\[ \dot{x} = (1 - 3x^2 - 2\dot{x}^2) \dot{x} - x \]

Let \( v = \dot{x} \),
\[ \dot{x} = v, \quad \dot{v} = (1 - 3x^2 - 2v^2)v - x. \]

The phase portrait analysis is done in Mathematica.
How can we prove there is a limit cycle?

Convert to polar coordinates:
\[
\begin{align*}
\dot{r} &= \cos \theta \dot{x} + \sin \theta \dot{v} \\
\dot{\theta} &= \cos \theta \left[ (1 - 3\cos^2 \theta - 2r^2 \sin^2 \theta) \sin \theta - \cos \theta \right]
\end{align*}
\]

\[ \Rightarrow \dot{r} = \sin \theta \left[ \cos \theta + (1 - r^2 - \cos^2 \theta) \sin \theta - \cos \theta \right] \]
\[ \dot{r} = r \sin^3 \theta \left[ 1 - 2r^2 + r^2 \cos^2 \theta \right] \]

Let \( r = \frac{1}{2} \),
Then \( \dot{r} = \frac{1}{2} \sin^2 \theta \left[ 1 - \frac{1}{2} - \frac{1}{2} \cos^2 \theta \right] \neq 0 \)

Let \( r = \frac{1}{\sqrt{2}} \),
Then \( \dot{r} = \frac{1}{\sqrt{2}} \sin^3 \theta \left[ -\frac{1}{2} \cos^2 \theta \right] \leq 0 \)

Therefore, we have constructing a trapping region.

\[ \Rightarrow \text{A limit cycle exists!} \]
Lienard Systems

\[ \ddot{x} + f(x)\dot{x} + g(x) = 0. \]

\( f(x) \) is damping force from external

\( V(x) = \int_{x_0}^{x} g(y) \, dy \)

We saw from the Van-der pol oscillator that we need several conditions for a limit cycle.

1. We need \( V \) to be an even function.

\[ \rightarrow \text{Oscillations for conservative systems} \]

2. We need \( V \) to satisfy \( \lim_{|x| \to \infty} V(x) = \infty \).

\( \rightarrow \text{Trajectories are trapped for conservative system} \)

Rewrite

\[ \frac{d}{dt}(x + F(x)) = -g(x), \quad F(x) = \int_{x_0}^{x} f(y) \, dy \]

Let

\[ y = x + F(x) \]

\[ \Rightarrow \begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases} \]

3. We need \( F(x) \) to be odd.

4. We need \( F(x) \) to satisfy:

\( F(x) \) is increasing on \( \mathbb{R}^+ \) and, \( \exists a \in \mathbb{R} \) such that \( F(x) > 0 \)

if \( x > a \)

\*These four conditions guarantee the existence of a limit cycle. This is a sufficient condition.