Index Theory

How can we be sure no periodic orbits exist? Consider

\[ \dot{x} = F(x) \]

with \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) continuously differentiable.

Take a closed curve \( \Gamma \) with no self-intersections, that does not pass through a fixed point.

1. Start at \( x_0 \), traverse \( \Gamma \) counter-clockwise and take the angle \( \theta \) of \( F(x) \). The angle changes continuously as \( \Gamma \) is traversed.

2. After one pass we again end up at \( x_0 \) with an angle \( \theta' = \theta + 2\pi n \); \( n \in \mathbb{Z} \)

\[ I_{\Gamma} = \frac{1}{2\pi} (\theta' - \theta_0) \]
Examples

a.)

$\Gamma_{p} = 1$

b.)

$\Gamma_{p} = 1$

c.)

$\Gamma_{p} = -1$

d.)

Periodic Orbit $\Rightarrow \Gamma_{p} = 1$

e.)

$\begin{cases} \dot{x} = x^2y \\
\dot{y} = x - y^2 \end{cases}$; $\Gamma = \text{unit circle}$

$\Gamma_{p} = 0$
Properties of the Index

1. If \( I \) can be deformed continuously into \( \bar{I} \) without passing through any equilibrium points then
\[
I_p = \bar{I}_p
\]

Proof:
\( I_p \) varies continuously as \( I \) is deformed, but \( I_p \) is integer valued.

2. If \( I \) does not contain any fixed points then \( I_p = 0 \)

Proof:
Property 1 implies we can shrink \( I \) to a point without changing the index.

3. If we replace \( F(x) \) by \( F(-x) \) the index is not changed.

Proof:
Each angle is replaced by \( \pi + \pi \), hence \( \psi_1 - \psi_0 \) is the same.

4. The index of a periodic orbit is one.

5. If \( F(x) \) is deformed continuously without creating any fixed points on \( I \), \( I_p \) stays the same.

Theorem - Assume \( F \) is continuously differentiable. Inside each periodic orbit, there is at least one equilibrium.

Proof:
Follows from items 2 and 4.

Index of isolated fixed point - Let \( \hat{x}^* \) be an isolated fixed point of \( \hat{x} = F(x) \). Define
\[
I(\hat{x}^*) = \text{index of simple closed curve that encloses } \hat{x}^* \text{ and no other fixed points}
\]

\( I(\hat{x}^*) \) is well defined by property 4.
Consequences:
1. If \( x^* \) is an attractor or repeller then \( I(x^*) = 1 \).
2. If \( x^* \) is a saddle point then \( I(x^*) = -1 \).

Proof:
Follows from examples b and c and properties 1, 3, 5.

Theorem - If \( f \) is a closed simple curve that contains in isolated fixed points \( \bar{x}_1, \ldots, \bar{x}_n \) then \( \int f = I(\bar{x}_1) + \ldots + I(\bar{x}_n) \).

Proof:

Contribution cancel in the limit.

Corollary: A periodic orbit must enclose fixed points whose indices sum to +1.

Omnivore example:
The index of all the fixed points is \( -2 \).

Sheep and Rabbits:
The index of all the fixed points is 0.
Conservative Systems:

Inertial Systems of the Form:

\[ \dot{x} = F(x) \]

A first integral can be found as follows:

\[ \dot{x} \dot{x} = \dot{x} F(x) \]
\[ \Rightarrow \frac{1}{2} \frac{d}{dt}(\dot{x}^2) = \frac{dx}{dt} \left( -\frac{dV}{dx} \right), \quad \text{(For any solution curve)} \]

where \[ V(x) = -\int_x^x F(x')dx' \quad \text{(x can be chosen) \} \]
\[ \Rightarrow \frac{d}{dt} \left( \frac{\dot{x}(t)^2}{2} + V(x(t)) \right) = 0, \quad \text{arbitrarily} \]

For any solution curve, there is a constant \( E \) such that

\[ \frac{\dot{x}(t)^2}{2} + V(x(t)) = E \]

We can also write as a system:

\[ \dot{x} = \nu \]
\[ \dot{\nu} = -\frac{F(x)}{2} \]

Phase portrait \( \leftrightarrow \) contour plot

Theorem - A conservative system cannot have any attractors or repellors.

Proof:
Suppose there exists \( (x^*, \nu^*) \) that is an attracting point with a basin of attraction \( \mathcal{A} \). Then, for all \( (x_1, \nu_1), (x_2, \nu_2) \in \mathcal{A} \), it follows that \( E(x_1, \nu_1) = E(x_2, \nu_2) \). Since

\[ E(x_1, \nu_1) = \lim_{t \to \infty} E(x_1, \nu_1, t) \]

\[ = E(x^*, \nu^*(t)) \]

\[ = \lim_{t \to \infty} (x_2(t), \nu_2(t)) \]

\[ = E(x_2, \nu_2). \]
Therefore, \( E \) must be constant in entire basin of attraction which we preclude by definition.

Example.

\[ \dot{x} + \sin(x) = 0 \]
\[ V(x) = -\cos(x) \]

\[ E = \frac{1}{2} v^2 - \cos(x) \]

\[ \Rightarrow v = \pm \sqrt{2E + 2\cos(x)} \]

\[ v = \pm \frac{2(\cos(x) - \cos(x_0))}{2(\cos(x) - \cos(x_0)) + \frac{1}{2} v_0^2} \quad (\text{if } v(0) = 0) \]

\[ v = \pm \frac{2v_0}{2(\cos(x) - \cos(x_0)) + \frac{1}{2} v_0^2} \quad (\text{if } v(0) \neq 0) \]

In this region, velocity can be nonzero.
Example
\[ \dot{x} = x - x^3 \]
\[ V(x) = -\frac{x^2}{2} + \frac{x^4}{4} \]
\[ E = \frac{1}{2} V^2 - x^2 + \frac{x^4}{2} \]
\[ \Rightarrow V = \pm \sqrt{\frac{x^2 - \frac{x^4}{2} - x^2}{x^4}} \quad (\text{If } V(0) = 0). \]
\[ \dot{x} = -\nabla V, \quad \text{where } V : \mathbb{R}^{2} \to \mathbb{R} \]

**Lemma** - Gradient systems cannot have closed orbits.

Let \( x(t) \) be a closed orbit with period \( T \). Then,
\[
\frac{d}{dt} V(x(t)) = \nabla V \cdot \frac{dx}{dt} = -|\nabla V|^2 \leq 0.
\]
and \( V(t) \) decreases strictly unless \( x(t) = x^* \) is an equilibrium point. Therefore, \( V(x(0)) \geq V(x(T)) \) which is a contradiction.

**Example**

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = 
\begin{pmatrix}
2x + \sin(y) \\
x \cos(y)
\end{pmatrix}
\]
does not have any periodic orbits.

\[-V_x = 2x + \sin(y) \]
\[-V_y = x \cos(y) \]
\Rightarrow \(-V = x^2 + \sin(y) x + g(y) \)
\Rightarrow \text{Setting } g(y) = 0 \text{ yields the result.}
\[
V = x^2 - \sin(y) x
\]

**How can we know beforehand if a system is potentially a gradient system?**

\[ V_{xy} = V_{yx} \]

\Rightarrow \cos y = \cos(y) \quad (\text{True in our case})
Solitons

Shallow water waves in a narrow canal

\[ u_t + uu_x + u_{xx} + u_{xxx} = 0 \]

Let \( z = x - ct \),

\[
\begin{align*}
\frac{\partial}{\partial x} &= \frac{\partial z}{\partial x} \frac{d}{dz} = \frac{d}{dz} \\
\frac{\partial}{\partial t} &= \frac{\partial z}{\partial t} \frac{d}{dz} = -c \frac{d}{dz}
\end{align*}
\]

\[
\Rightarrow (1 - c) \frac{d}{dz} u + \frac{1}{2} \frac{d}{dz} \left( u^2 \right) + u_{zzz} = 0
\]

\[
\Rightarrow (1 - c) u + \frac{1}{2} u^2 + u_{zz} = \mu
\]

\[
\Rightarrow u_{zz} = -(1 - c) u - \frac{1}{2} u^2 + \mu
\]

This is a conservative system with potential:

\[
\mathcal{V}(u) = (1 - c) u^2 + \frac{1}{6} u^3 - \mu u
\]

The fixed points are:

\[
u = \frac{(1 - c) \pm \sqrt{(1 - c)^2 + 2\mu}}{2}
\]

\[
u_z = 0
\]
Generically the potential looks like.

Separatrix/homoclinic orbit.
Lyapunov Functions
\[ \dot{x} = F(x) \]

A continuously differentiable function \( L : \mathbb{R}^2 \to \mathbb{R} \) is called a Lyapunov function if \( L(x(t)) \) strictly decreases along each solution of \( \dot{x} = F(x) \) that is not an equilibrium.

**Lemma** - If \( \dot{x} = F(x) \) admits a Lyapunov function, then it cannot have any periodic orbits.

**Example**
\[ \dot{x} + \alpha x = g(x)x, \quad \alpha > 0. \]
Let \( V(x) = \frac{1}{2}\int_{x_0}^x g(x) \, dx \). Then,
\[ \frac{1}{2} \frac{d}{dt}(\dot{x}^2) + \alpha \dot{x}^2 = -\frac{d}{dt}(V(x(t))) \]
\[ \Rightarrow \frac{d}{dt}(\frac{1}{2} \dot{x}^2 + V(x(t))) = -\alpha \dot{x}^2 < 0. \]
The function \( L(x, x) = \frac{1}{2} x^2 + V \) is a Lyapunov function.

**Summary:**
1. \( \dot{x} = -\frac{dV}{dx} \rightarrow \) conservative \( E(x, \dot{x}) \) is conserved \( \Rightarrow \) many periodic orbits.
2. \( (\frac{\dot{x}}{\dot{y}}) = -\nabla V \rightarrow \) gradient system, \( V \) decreases along solutions \( \Rightarrow \) no periodic solutions.
3. \( \dot{x} + \alpha \dot{x} = -\frac{dV}{dx} \rightarrow E(x, \dot{x}) \) decreases
The travelling wave corresponds to the separatrix. Sketch of the solution.