Chapter 11: Fractals

Dimension - How do we measure the dimension of a set?
One idea is to count the number of coordinates needed to describe set.

A smooth manifold of dimension $n$ is a set $M^n$ that locally looks like $\mathbb{R}^n$. I.e., for each $p \in M^n$, there exists an open set $O_p$ containing $p$ and a smooth map $f: O_p \to \mathbb{R}^n$ with smooth inverse $f^{-1}$.

$(f_p, O_p) \to \text{coordinate chart}$ (This is like a map of the set)

*The collection of all coordinate charts is called an atlas.*

What about non-smooth sets??

Example:

![Diagram of a sphere and a coordinate chart]

The sphere is two-dimensional.
Example:
\[ f(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{\pi k^2}, \text{ on interval } [0, 1]. \]

\[ |f(x)| \leq \sum_{k=1}^{\infty} \frac{1}{\pi k^2} = \pi \int_0^1 \frac{dx}{6k^2} \Rightarrow f \text{ is continuous.} \]

However:
\[ f'(x) = \sum_{k=1}^{\infty} \frac{\cos(\pi k^2 x)}{k} \]

For large \( k \) \( \cos(\pi k^2 x) \approx 1 \) i.o.
\[ \Rightarrow |f'(x)| = \infty. \]

\( f \) is not differentiable almost everywhere.

\[ \Rightarrow \text{Consequence: } f \text{ has infinite arc length:} \]
\[ L = \int_0^1 \sqrt{1 + [f'(x)]^2} \, dx = \infty. \]

\[ \Rightarrow \text{Consequence: We cannot define dimension in the classical sense.} \]

Size of Sets.

- **Countable:** finite or can be put in one to one correspondence with \( \mathbb{N} \). (Set can be indexed).
- **Uncountable:** not countable.

Examples:
- \( \mathbb{Z} \) - countable
- \( [0, 1] \) - uncountable
- \( \mathbb{Q} \) - countable
- \( \mathbb{Z} \times \mathbb{Z} \) - countable
- \( \mathbb{Q} \times [0, 1] \) - countable
- \( \mathbb{R} \) - countable
- \( [0, 1] \times [0, 1] \) - uncountable

Binary representation of \( [0, 1] \).
Sets of measure 0 - A set \( S \) has measure 0 if \( \forall \varepsilon > 0 \), \( S \) is a subset of a union of \( \varepsilon \)-open cubes the sum of whose volume is less than \( \varepsilon \).

**Example:**

\( Q \) is a set of measure 0.

**Proof:**

We can index \( Q \) by points \( (r_1, r_2, \ldots) \). Let \( b_i = (r_i - \frac{\varepsilon}{2^i}, r_i + \frac{\varepsilon}{2^i}) \). Then,

\[
V \left( \bigcup_{i=1}^{\infty} b_i \right) \leq \sum_{i=1}^{\infty} V(b_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \leq \pi \varepsilon.
\]

**Consequence:** There are two ways to measure the size of a set.

**Example: Cantor Set**

The Cantor set is formed by removing middle third of sets.

\[
\begin{align*}
S_0 &= [0,1] \setminus \left( \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right) \right) \\
S_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
S_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\
\vdots
\end{align*}
\]

Do this to \( \infty \), \( \Rightarrow S_\infty = \text{Cantor Set} \).

\[X \in S_\infty \iff X \in \bigcap_{n=1}^{\infty} S_n\]

1. \( S_\infty \) is uncountable \( \Rightarrow \) can be put into correspondence with \([0,1] \)
   by binary representation.
2. \( S_\infty \) has measure 0 \( \Rightarrow \) take balls of volume \((\frac{1}{3})^n \cdot 2^n = (\frac{2}{3})^n \)
   take limit \( n \to \infty \)
Let \( A \subset \mathbb{R}^n \), take a mesh of boxes of length \( \varepsilon \). Let \( N(\varepsilon) \) be the number of boxes that intersect with \( A \).

- \( 1 \)-dim.: \( N(\varepsilon) \sim \frac{1}{\varepsilon} \)
- \( 2 \)-dim.: \( N(\varepsilon) \sim \frac{1}{\varepsilon^2} \)

Assume there is some scaling law:

\[ N(\varepsilon) \sim \frac{1}{\varepsilon^d} \]

then

\[ d = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \varepsilon} \]

**Example:**

What is the box dimension of the Cantor set?

We can construct a sequence of coverings. Let

\[ \varepsilon_n = (\frac{1}{3})^n \rightarrow \text{width of boxes.} \]

\[ 0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1 \quad n = 1 \]

Then

\[ N(\varepsilon) = 2^n \]

\[ d = \lim_{n \to \infty} \frac{\log (2^n)}{\log (3^n)} = \frac{\log (2)}{\log (3)} \]

**Example:**

\( F(\mathbb{R}^2) = \mathbb{R}^2 \)

\[ F(x,y) = \begin{cases} 
(\frac{2x}{3}, \frac{y}{3}) & 0 \leq x \leq \frac{1}{3} \\
(\frac{2x-1}{3}, \frac{y}{3} + \frac{2}{3}) & \frac{1}{3} < x \leq 1
\end{cases} \]

1. **Squish:**
   
   \[ 0 \quad \frac{1}{3} \rightarrow \frac{1}{3} \]

2. **Stretch:**
   
   \[ \frac{1}{3} \rightarrow \frac{1}{3} \]

3. **Stack:**
   
   \[ \frac{1}{3} \rightarrow \frac{1}{3} \]

\[ y \rightarrow \frac{1}{3} \]
The planar baker’s map is chaotic. What is its attracting set? 
\[ A = \bigcup_{n=0}^{\infty} F^n([0,1] \times [0,1]) \]

Cross sections:

The attracting set 
\[ A = [0,1] \times C \rightarrow C \] is the Cantor set.

Box dimension is 
\[ 1 + \frac{\ln(2)}{\ln(3)} \]

Example:

What is the box dimension of \( Q \cap [0,1] \)?
No matter how you cover this set \( N(\varepsilon) = \frac{1}{\varepsilon} \Rightarrow d = 1 \).

Hausdorff Dimension

Intuition:

Cover a set with disks

\[ A \sim \prod_{n=1}^{\infty} \frac{2n}{N} \rightarrow 2 \text{-dimensional measure} \]

\[ A \sim \prod_{n=1}^{\infty} \frac{2n}{N} \rightarrow 0 \text{, However what if we changed the power?} \]

\[ H \sim \prod_{n=1}^{\infty} 1 \sim \Pi \rightarrow 1 \text{ dimensional object} \]

Let \( \mathcal{F}(\varepsilon) \) be covering of \( A \subset \mathbb{R}^n \) at closed balls \( B_i \) of radii \( r_i = \varepsilon \). For \( \varepsilon \in \mathcal{F}(\varepsilon) \) set

\[ H_0(A,\varepsilon) = \inf_{\mathcal{F}(\varepsilon)} \sum r_i^0 \] - Hausdorff measure

\[ H_0(A) = \lim_{\varepsilon \to 0} H_0(A,\varepsilon). \]
There exists unique $d \geq 0$ such that
$$H_\alpha(A) = \begin{cases} 0, & \alpha > d \\ \infty, & \alpha < d \end{cases}$$

$d$ is the Hausdorff dimension.

Each countable set has Hausdorff dimension $0$.

**Proof:**

For $d > 0$, let $\delta(\varepsilon) = \{B_i : B_i = B_{\varepsilon_i}(x_i)\}$, where $x_i$ is a sequence satisfying $x_i \in A$ and $\bigcup x_i = A$. Then
$$H_\alpha(A, \varepsilon) \leq \sum_{i=1}^{\infty} \frac{\varepsilon_i^\alpha}{(2\varepsilon_i)^\alpha} = \sum_{i=1}^{\infty} \frac{\varepsilon_i^\alpha}{(2\varepsilon)^\alpha} \leq C \varepsilon^\alpha.$$

$$\Rightarrow \lim_{\varepsilon \to 0} H_\alpha(A, \varepsilon) = H_\alpha(A) = 0.$$

**Example (Fat Fractal)**

Does every fractal have measure $0$?

No.

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$C_{\text{Fat}} = \bigcap_{i=1}^{n} S_i$

**Box Dimension:**

Let $\varepsilon_n = \frac{3}{2^{n+2}}$, this corresponds to $N = 2^n$.

The box dimension is then
$$d = \lim_{n \to \infty} \frac{\log N}{\log \left(\frac{2^n}{3}\right)} = \frac{1}{2}.$$

The width removed is
$$\sum_{i=1}^{n} \frac{1}{4^i} = 2^4 = 2^4 \times \frac{1}{2^4} = \frac{1}{2}. $$
The fat fractal has length $\frac{1}{2}$.

Strange repellor

$$x_{n+1} = \begin{cases} 
3x, & x < \frac{1}{2} \\
3x - 2, & x > \frac{1}{2}
\end{cases}$$