Chapter 9: Lorenz Equations.

3-D dynamics.

Old tools:
1. Local linearization
2. Global analysis

* Trajectories can be attracted to lower dimensional subsets.

Surfaces:
   a.) Spheres
   b.) Torus.

Example:

\[ x = -y + \frac{x(1-x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} \]

\[ y = x + y(1-x^2+y^2+z^2) \sqrt{x^2+y^2+z^2} \]

\[ z = z(1-x^2-y^2-z^2) \sqrt{x^2+y^2+z^2} \]

Convert to spherical polar:

\[ \rho = 1 - \rho \]

\[ \theta = 1 \]

\[ \phi = 0 \]

* All trajectories go to sphere of radius 1 and form limit cycles of constant angle θ.

* Approach sphere along a cone.
Example:

\[
\begin{align*}
\dot{x} &= \frac{xz}{\sqrt{x^2 + y^2}} + \frac{x(1 - (x^2 + y^2 + z^2))}{\sqrt{x^2 + y^2 + z^2}} \\
\dot{y} &= \frac{yz}{\sqrt{x^2 + y^2}} + \frac{y(1 - (x^2 + y^2 + z^2))}{\sqrt{x^2 + y^2 + z^2}} \\
\dot{z} &= \frac{z(1 - (x^2 + y^2 + z^2))}{\sqrt{x^2 + y^2 + z^2}}
\end{align*}
\]

Convert to spherical polar:

\[
\begin{align*}
\dot{\tilde{x}} &= 1 - \tilde{r}^2 \\
\dot{\theta} &= 0 \\
\dot{\phi} &= 1
\end{align*}
\]

Converge to latitudes \(\Rightarrow\) This violates existence and uniqueness. This occurs because \(x^2 + y^2\) can equal 0.

Example:

\[
\begin{align*}
\tilde{x} &= (1 - r^2) \\
\dot{\theta} &= 1 \\
\dot{\phi} &= 1
\end{align*}
\]

This forms Lissajous cycles.

*In these examples the long term behaviour is attracted to surfaces which we can analyze dynamics on.*
Lorenz Equations:

\[
\begin{align*}
\dot{x} &= \sigma(y-x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{align*}
\]

- \(\sigma\) - Prandtl number \(\rightarrow\) ratio of viscosity \(\rightarrow\) thermal diffusivity
- \(r\) - Rayleigh number: \(r < 1\) conduction, \(r \gg 1\) convection
- \(b\) - dimensionless aspect ratio.

Next stuff happens as we play with \(r\):

Fixed Points:
1. \((0, 0, 0) \rightarrow\) Pure conduction
2. \((\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1) \rightarrow\) left and right moving rolls

\[
J(0,0,0) = \begin{pmatrix}
-\sigma & \sigma & 0 \\
0 & r & -1 \\
0 & 0 & -b
\end{pmatrix}
\]

2 \(\lambda_{1,2} = -\sigma - 1 \pm \sqrt{(\sigma + 1)^2 - 4(r-\sigma r)}\)
\[
= -\sigma - 1 \pm \sqrt{(\sigma + 1)^2 - 4(r-1)(1-r)}
= -\sigma - 1 \pm \sqrt{(r-1)^2 + 4(r-1)}
\]

Case 1:

\(r < 1\), only one fixed point and it is a stable node.

How can we eliminate closed orbits? Construct a Lyapunov function.
Let \( V = \sqrt{\frac{1}{a} x^2 + y^2 + z^2} \)

\[ V = \frac{1}{a} x \dot{x} + y \dot{y} + z \dot{z} \]

\[ = x(y - x) + y(rx - y - xz) + z(zy - b z) \]

\[ = (r+1) xy - x^2 - y^2 - b z^2 \]

\[ = -(x^2 - (r+1) xy + \left( \frac{r+1}{2} \right)^2 y^2 \frac{r+1}{2}) + [(\frac{r+1}{2}) - 1] y^2 - b z^2 \]

\[ = -(x - \left( \frac{r+1}{2} \right) y)^2 - (1 - \left( \frac{r+1}{2} \right)) y^2 - b z^2 \]

If \( r < 1 \), \( V < 0 \), and \( V \geq 0 \) with \( V = 0 \) if and only if \( x = y = z = 0 \).

\[ \lim_{t \to \infty} V(t) = 0. \]

If \( r < 1 \), all trajectories go to the origin.

**Case 2:**

\( r > 1 \)

\((0, 0, 0)\) has eigenvalues \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0. \)

\( \rightarrow \) This is like a saddle point.

\( (\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1) \) now exists. \( \Rightarrow \) pitchfork bifurcation

We should do local analysis.

From now on fix \( b = \frac{g}{3}, \quad c = -10 \). We denote new fixed points by \( C^+ \) and \( C^- \).

The characteristic polynomial is given by:

\[
3 P(\lambda) = -\lambda^3 - 41 \lambda^2 - 88 \lambda - 80 (r-1).
\]

Plotting this can tell us interesting behavior.

\[ \Rightarrow \text{Changing } r \text{ shifts this graph down} \]

\[ \Rightarrow \text{There exists } r_g \text{ such that if } 1 < r < r_g \text{ then } C_r, C^- \text{ are stable} \]
Case 3:
\[ v_s < r < v_o \]

Stability analysis tells us that they are stable spirals around the fixed points.

Case 4:
\[ r = v_o \]

Unstable limit cycle born in a homoclinic bifurcation.

Case 5:
\[ \mu > v_o \]

This creates a strange invariant set. Spirals may loop around a lot. Impossible to predict which fixed point we go to.
Case 6:
A Hopf bifurcation occurs at some critical $r^*$. The unstable limit cycle vanishes leaving behind two unstable spirals.

\[ \nabla = F(\nabla), \text{let } V, \text{ denote an initial volume of initial conditions} \]
\[ \Rightarrow \dot{V} = \int_S \nabla \cdot F \, dA \]
\[ = \int_V \nabla \cdot F \, dV \]
\[ = \int_V (-a - l - b) \, dV \]
\[ = (-a - l - b) V \]
\[ \Rightarrow V(t) = V_0 \exp \left( (-a - l - b) t \right) . \]

All volumes shrink to nothing.

\Rightarrow The trajectories must be attracted to something...

\rightarrow 1. cannot be a surface
\rightarrow 2. cannot be a fixed point (all unstable if $r > r^*$)
\rightarrow 3. possible to eliminate limit cycles
\rightarrow 4. cannot be quasiperiodic

\Rightarrow Must be a new object!

Goals:
1. Find out what new object is.
2. Try to study chaos.
What is Chaos?

* Chaos - Aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.

1. Aperiodic - There exist trajectories which do not go to fixed points, periodic orbits, or quasi-periodic orbits.
2. Sensitive Dependence - We say $\dot{x} = f(x)$ has sensitive dependence on initial data on a set $A$ if $\forall x_0 \in A, \exists \varepsilon > 0$ such that $\forall \delta > 0, \exists x_0'$ and $T > 0$ with $|x_0 - x_0'| < \delta$ and $|x(T) - x'(T)| > \varepsilon$.

Trajectories remain bounded

Consider a spherical surface $S_R = x^2 + y^2 + (z - r - R)^2 = R^2$.

For a trajectory starting on this surface we have that

$$\frac{d}{dt} \left[ x^2 + y^2 + (z - r - R)^2 \right] = 2x \ddot{x} + 2y \ddot{y} + 2(z - r - R) \ddot{z}$$

$$= -2 \left[ a \dot{x}^2 + \dot{y}^2 + b \left( z - \frac{r + R}{2} \right)^2 - \frac{b(r + R)^2}{4} \right]$$

Pick $R$ large enough that the ellipse

$$a \dot{x}^2 + \dot{y}^2 + b \left( z - \frac{r + R}{2} \right)^2 = \frac{b(r + R)^2}{4}$$

is enclosed in $S_R$. This guarantees $S_R$ is a trapping region.

$\Rightarrow$ Trajectories remain bounded and must go to something.
The Lorenz map looks at the relationship between $Z_n$ and $Z_{n+1}$.

This is awesome! We can extract some order from chaos $Z_{n+1} = S(Z_n)$.

Note $|f'(Z)| > 1$. Now suppose we have a stable limit cycle. There is one closed orbit at $Z = Z^*$.
Consider
\[ z_0 = z^* + \delta_0 \]
\[ \delta_0 = \text{initial perturbation.} \]
\[ \Rightarrow z_1 = f(z^* + \delta_0) \]
\[ \Rightarrow f(z) \approx f(z^*) + f'(z^*) (z - z^*) \]
\[ \Rightarrow z_1 = f(z_0) \approx z^* + f'(z^*) \delta_0 \]
\[ \Rightarrow \delta_1 = z_1 - z^* \approx f'(z^*) \delta_0 \]
\[ \Rightarrow |\delta_1| > |\delta_0| \]

The trajectory grows away from the set. This fixed point is 
unstable.
\[ \Rightarrow \text{No stable limit cycles.} \]

**Attractor**

An attractor is a closed set \( A \) with the following properties.

1. \( A \) is invariant. Any \( x(t) \) that starts in \( A \) remains in \( A \).
2. \( A \) attracts an open set of initial conditions. \( \exists \) exists an open set \( U \) containing \( A \) such that if \( x(0) \in U \), then \( d(x(t), A) \to 0 \) as \( t \to \infty \). The largest \( U \) is called the basin of attraction.
3. \( A \) is minimal. There is no proper subset of \( A \).

**What about Double Periodic Iterations?**

\[ z_n = f(f(Z_n)) = f^2(Z_n) \]

\[ f^2(\frac{1}{2}) \approx 0, \ f^2(\frac{1}{4}) \approx \frac{1}{4} \]

3 double period orbits all unstable.
General Case

We can continue forever

→ There exists an infinite number of unstable limit cycles.