6.3-5 (a) primal and dual

<table>
<thead>
<tr>
<th>primal LP problem</th>
<th>dual LP problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>max $Z = 2x_1 + 4x_2$</td>
<td>min $W = y$</td>
</tr>
<tr>
<td>$x_1 - x_2 \leq 1$</td>
<td>$y \geq 0$</td>
</tr>
<tr>
<td>$x_1 \geq 0$</td>
<td>$y \geq 2$</td>
</tr>
<tr>
<td>$x_2 \geq 0$</td>
<td>$-y \geq -4$</td>
</tr>
</tbody>
</table>

We can see the optimal of dual LP problem is 2 when $y=2$.

(b) complete slackness

By Complete slackness theorem, we know optimal $x$ and dual optimal $y$ satisfy those equations:

$$(b - Ax)^T y = 0 \text{ and } (A^T y - c)^T x = 0.$$ 

Replace these vectors by what they are in this problem, we can get:

$$x_1 - x_2 = 1$$

$$(0, 2)(x_1, x_2)^T = 0$$

it’s easy to find the solution of this group equations, that is $x_1 = 1, x_2 = 0$.

(c) reconstaint

Now the dual LP problem is:

Minimize $w = y$ subject to $y \geq 0, y \geq c_1, -y \geq -4$

we can see when $c_1 > 4$, there is no feasible solution for dual problem. Notice that changes in objective function don’t affect the feasible region. Thus the primal problem is unbounded.
The final simplex tableau is

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Row</th>
<th>Z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>x₆</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3/2</td>
<td>0</td>
<td>3/2</td>
<td>1/2</td>
</tr>
<tr>
<td>x₄</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td>x₁</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>15</td>
</tr>
<tr>
<td>x₂</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-3/2</td>
<td>0</td>
<td>-1/2</td>
<td>1/2</td>
<td>5</td>
</tr>
</tbody>
</table>

(a) change the right hand side

The only change that happens is in the last column. The problem is whether or not the optimal solution now is still feasible. Let’s check out $B^{-1} \cdot \vec{b}$

$$
B^{-1} \cdot \vec{b} = \begin{pmatrix}
1 & -1 & -2 \\
0 & 1/2 & 1/2 \\
0 & -1/2 & 1/2
\end{pmatrix} \cdot \begin{pmatrix}
70 \\
20 \\
10
\end{pmatrix} = \begin{pmatrix}
30 \\
15 \\
-5
\end{pmatrix}
$$

Which tells us that the optimal solution now is not feasible.

(c) change the coefficients of $x₃$

Noticing $x₃$ is not a basic variable, thus this change only affects the $x₃$ column. The coefficient of $x₃$ in row(0) will change into $C_B^T B^{-1} \vec{A}_j - \vec{c}_j$, the column except row(0) will be $B^{-1} \vec{A}_j$. Check out these values by replacing vectors into it.

$$
C_B^T B^{-1} \vec{A}_j - \vec{c}_j = \begin{pmatrix}
0 & 3/2 & 1/2
\end{pmatrix} \cdot \begin{pmatrix}
3 \\
-2
\end{pmatrix} = -3/2
$$

So now the solution is not optimal any more. since

$$
B^{-1} \vec{A}_j = \begin{pmatrix}
1 & -1 & -2 \\
0 & 1/2 & 1/2 \\
0 & -1/2 & 1/2
\end{pmatrix} \cdot \begin{pmatrix}
3 \\
-2
\end{pmatrix} = \begin{pmatrix}
6 \\
-3/2
\end{pmatrix}
$$

The new simplex table will be

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Row</th>
<th>Z</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>x₆</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-3/2</td>
<td>0</td>
<td>3/2</td>
<td>1/2</td>
<td>25</td>
</tr>
<tr>
<td>x₄</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>1</td>
<td>-1</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td>x₁</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1/2</td>
<td>0</td>
<td>1/2</td>
<td>1/2</td>
<td>15</td>
</tr>
<tr>
<td>x₂</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-3/2</td>
<td>0</td>
<td>-1/2</td>
<td>1/2</td>
<td>5</td>
</tr>
</tbody>
</table>
iteration(0)

<table>
<thead>
<tr>
<th>Basic Variable</th>
<th>Row</th>
<th>Z</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>5/4</td>
<td>0</td>
<td>55/2</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/6</td>
<td>-1/6</td>
<td>-1/3</td>
<td>5/3</td>
</tr>
<tr>
<td>$x_1$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>5/12</td>
<td>1/3</td>
<td>95/6</td>
</tr>
<tr>
<td>$x_2$</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/4</td>
<td>-3/4</td>
<td>0</td>
<td>15/2</td>
</tr>
</tbody>
</table>

see there is no negative number in first row, thus this is the optimal solution.

(d) change the objective
Check out the new simplex form. Just reconstruct the first row

$$C^T_B B^{-1} A - c^T = (0 \ 3 \ -2) \cdot \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1/2 \\ 0 & 1 & -3/2 \end{pmatrix} - (3 \ -2 \ 3) = (0 \ 0 \ 3/2)$$

$$C^T_B B^{-1} = (0 \ 3 \ -2) \cdot \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 1/2 \\ 0 & -1/2 & 1/2 \end{pmatrix} = (0 \ 5/2 \ 1/2)$$

$$C^T_B B^{-1} b = (0 \ 5/2 \ 1/2) \cdot (60 \ 10 \ 20)^T = 35$$

Now we see the first row is not negative, so it’s still optimal after changing some coefficients of basic variable.

(f) introduce a new variable
just reconstruct the simplex form, taking the new variable as a non-basic variable.

$$C^T_B B^{-1} A_8 - c_8 = (0 \ 3/2 \ 1/2) \cdot \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} + 1 = 7/2$$

which positive, shows that the solution is still optimal.