5.53. Independent, since \( f(y_1, y_2) \) can be factored.

5.59 a. Because of the independence of \( Y_1 \) and \( Y_2 \),
\[
f(y_1, y_2) = f(y_1)f(y_2) = \frac{1}{9} e^{-(y_1+y_2)/3}
\]
for \( y_1 > 0, y_2 > 0 \).

b. The probability of interest is the shaded area in Figure 5.11. Hence
\[
P(Y_1 + Y_2 \leq 1) = \int_0^1 \int_0^{1-y_1} f(y_1, y_2) \, dy_1 \, dy_2
\]
\[
= \int_0^1 \left[ 1 - e^{-(1-y_2)/3} \right] \frac{1}{3} e^{-y_2/3} \, dy_2
\]
\[
= \int_0^1 \left( \frac{1}{3} e^{-y_2/3} - \frac{1}{3} e^{-1/3} \right) \, dy_2 - e^{-1/3} \left[ \frac{1}{3} - \frac{1}{3} e^{-1/3} \right] = 1 - \frac{4}{3} e^{-1/3}
\]

Figure 5.11

5.61 Let \( Y_1 \) = calling time to the switchboard of the first call, then
\( f(y_1) = 1; \quad 0 \leq y_1 \leq 1 \)

\( Y_2 \) = calling time to the switchboard of the second call, then
\( f(y_2) = 1; \quad 0 \leq y_2 \leq 1 \)

Then we have \( f(y_1, y_2) = 1 \).

a. \( P(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{1}{2}) = \left( \int_0^{1/2} 1 \, dy_1 \right) \left( \int_0^{1/2} 1 \, dy_2 \right) = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{4} \)

(since \( Y_1 \) and \( Y_2 \) are independent).

b. Note that 5 minutes = \( \frac{1}{12} \) of 1 hour.
\( P(|Y_1 - Y_2| < \frac{1}{12}) = \int_0^{\frac{1}{12}} \int_0^{\frac{1}{12}} dy_2 \, dy_1 + \int_{\frac{1}{12}}^{\frac{11}{12}} \int_{\frac{1}{12}}^{\frac{11}{12}} dy_2 \, dy_1 + \int_{\frac{1}{12}}^{\frac{1}{12}} \int_{\frac{1}{12}}^{\frac{11}{12}} dy_1 \, dy_1
\]
\[
= \left( \frac{11}{2} \right) \left( \frac{11}{2} \right) + \frac{2}{12} \int_0^{11/12} dy_1 + (\frac{13}{12}) \left( \frac{11}{12} \right) + \frac{3}{2} \left( \frac{11}{12} \right) = \frac{46}{288} = \frac{23}{144}.
\]
5.64 Refer to Exercises 5.22. Recall \( f_1(y_1) = 2y_1 \) for \( 0 \leq y_1 \leq 1 \).

a. \( E(Y_1) = \int_0^1 2y_1y_1 \, dy_1 = \int_0^1 2y_1^2 \, dy_1 = \frac{2}{3} \)

b. \( E(Y_1^2) = \int_0^1 2y_1^3 \, dy_1 = \frac{1}{2} \) so that \( V(Y_1) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \).

c. Since \( E(Y_2) = \int_0^1 2y_2^2 \, dy_2 = \frac{3}{5} \), \( E(Y_1 - Y_2) = 0 \).

In all the above, we use the following computation (which is also required for 5.22):
\[
f_1(y_1) = \int_{-\infty}^\infty f(y_1, y_2) \, dy_2 = \int_0^1 4y_1y_2 \, dy_2
= (4y_1) \left[ \frac{y_2^2}{2} \right]_0^1 = 2y_1 \quad \text{for} \quad 0 \leq y_1 \leq 1.
\]

which provides the marginal density function for \( Y_1 \).

5.69 Since \( Y_1 \) and \( Y_2 \) are independent, with \( f_1(y_1) = \frac{1}{4} y_1 e^{-y_1/2} \) and \( f_2(y_2) = \frac{1}{2} e^{-y_2/2} \),
\[
E\left(\frac{Y_1}{Y_2}\right) = E\left(\frac{1}{Y_2}\right) E(Y_2) = \frac{1}{8} \int_0^\infty e^{-y_2/2} dy_1 \int_0^\infty y_2 e^{-y_2/2} \, dy_2
= \frac{1}{8} \left[ 2e^{-y_2/2} \right]_0^\infty = \frac{1}{4} = 1
\]
since the second integral is the variable factor of a gamma distribution with \( \alpha = 2, \beta = 2 \) and integrates to \( \Gamma(2)2^2 = 4 \).

5.70 The marginal distribution of \( Y_1 \) is \( f_1(y_1) = 1 \) for \( 0 \leq y_1 \leq 1 \), so that \( E(Y_1) = \int_0^1 y_1 \, dy_1 = \frac{1}{2} \)

Using the joint distribution of \( Y_1 \) and \( Y_2 \), we obtain
\[
E(Y_2) = \int_0^1 \int_0^y y_1 \, dy_1 \, dy_2 = \int_0^1 \frac{y_1^2}{2} \, dy_1 = \frac{1}{3} \quad \text{for} \quad 0 \leq y_1 \leq 1.
\]

Thus, \( E(Y_1 - Y_2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \).

5.77 From Exercise 5.64, \( E(Y_1) = E(Y_2) = \frac{3}{4} \). Then
\[
E(Y_1 Y_2) = \int_0^1 \int_0^1 4y_1^2y_2^2 \, dy_1 \, dy_2 = \int_0^1 \frac{4}{3} y_2^2 \, dy_2 = \frac{4}{9}
\]

\[
\text{Cov}(Y_1, Y_2) = \frac{4}{9} - \frac{4}{9} = 0.
\]

No, this is not surprising since \( Y_1 \) and \( Y_2 \) are independent.
\[ 5.80 \quad \text{Cov}(U_1, U_2) = E\{(Y_1 + Y_2)(Y_1 - Y_2) - [E(Y_1) + E(Y_2)][E(Y_1) - E(Y_2)]\} \\
= -E(Y_1 Y_2) + E(Y_1^2) - E(Y_1)E(Y_2) - E(Y_2^2) - [E(Y_1)]^2 - E(Y_1)E(Y_2) \\
+ E(Y_1)E(Y_2) + [E(Y_2)]^2 \\
= \sigma_1^2 - \sigma_2^2 \\
\]

Now
\[ V(U_1) = E\{U_1^2\} - [E(U_1)]^2 \\
= E\{Y_1^2 + 2Y_1 Y_2 + Y_2^2\} - [(EY_1)^2 + 2(EY_1)(EY_2) + (EY_2)^2] \\
= V(Y_1) + V(Y_2) + 2\{E(Y_1 Y_2) - (EY_1)(EY_2)\} \\
= \sigma_1^2 + \sigma_2^2 + 2\text{Cov}(Y_1, Y_2) \\
= \sigma_1^2 + \sigma_2^2 \\
\]

since \( Y_1 \) and \( Y_2 \) are uncorrelated. A similar calculation yields \( V(U_2) = \sigma_1^2 + \sigma_2^2 \). Hence
\[ \rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{\sigma_1^2 + \sigma_2^2}(\sigma_1^2 + \sigma_2^2)} = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \]

5.81 The marginal distributions for \( Y_1 \) and \( Y_2 \) are shown in the accompanying tables.

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( p_1(y_1) )</th>
<th>( y_2 )</th>
<th>( p_2(y_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{2}{3} )</td>
</tr>
<tr>
<td>0</td>
<td>( \frac{1}{3} )</td>
<td>1</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
<td>( \frac{1}{3} )</td>
</tr>
</tbody>
</table>

Since, for example, \( p(-1, 0) \neq p(-1)p(0) \), \( Y_1 \) and \( Y_2 \) are not independent. However, \( E(Y_1) = -1 \left( \frac{1}{4} \right) + 0 \left( \frac{1}{4} \right) + 1 \left( \frac{1}{4} \right) = 0 \)
\[ E(Y_1 Y_2) = (-1)(0) \left( \frac{1}{3} \right) + (0)(1) \left( \frac{1}{3} \right) + (1)(0) \left( \frac{1}{3} \right) = 0 \]
so that \( \text{Cov}(Y_1, Y_2) = 0 \).

5.87 Refer to Theorem 5.12.
\[ E(3Y_1 + 4Y_2 - 6Y_3) = 3(2) + 4(-1) - 6(4) = -22 \]
\[ V(3Y_1 + 4Y_2 - 6Y_3) = 9(4) + 16(6) + 36(8) + (2)(3)(4)(1) + (2)(3)(-6)(1) \\
+ 2(4)(-6)(0) = 480 \]
5.92 \( f_1(y_1) = y_1 e^{-y_1} \), which is a gamma distribution with \( \alpha = 2, \beta = 1 \). Hence \( E(Y_1) = \gamma(1) = 2 \) and \( V(Y_1) = \alpha \beta^2 = 2 \).

\[
\int_{y_1} \int_{y_2} e^{-y_1} dy_1 = \int_{y_2} e^{-y_1} \left|_{y_2}^{\infty} \right| = e^{-y_2}
\]

which has a gamma distribution with \( \alpha = \beta = 1 \). Hence \( E(Y_2) = V(Y_2) = 1 \). Finally,

\[
E(Y_1 Y_2) = \int_0^\infty \int_0^\infty y_1 y_2 e^{-y_1} dy_2 dy_1 - \int_0^\infty \frac{y_2^2}{2} e^{-y_1} dy_1 = \frac{\Gamma(2)^2}{2} = 3
\]

\[
\text{Cov}(Y_1, Y_2) = 3 - (1)(2) = 1 \quad E(Y_1 - Y_2) = 2 - 1 = 1
\]

\[
V(Y_1 - Y_2) = 2 + 1 - 2(1) = 1
\]

Note: \( f_1(y) = \int_0^y e^{-y_1} dy_2 = y_1 e^{-y_1} \)

It is unlikely that a customer would spend more than 4 minutes at the service window because this is 3 standard deviations above the mean.

5.98 Let \( Y = X_1 + X_2 \), the total sustained load on the footing.

\textbf{a.} Since \( X_1 \) and \( X_2 \) have gamma distributions, \( E(X_1) = \alpha_1 \beta_1 = 100 \) and \( E(X_2) = \alpha_2 \beta_2 = 40 \). Also, \( V(X_1) = \alpha_1 \beta_1^2 = 200 \) and \( V(X_2) = \alpha_2 \beta_2^2 = 80 \). Thus \( E(Y) = E(X_1 + X_2) = 100 + 40 = 140 \).

Since \( X_1 \) and \( X_2 \) are independent,

\( V(Y) = V(X_1 + X_2) = V(X_1) + V(X_2) = 200 + 80 = 280 \).

\textbf{b.} Consider Tchebysheff's theorem with \( k = 4 \), \( P(|Y - \mu| \geq 4\sigma) \leq \frac{1}{16} \). The corresponding interval is \( (140 - 4\sqrt{280}, 140 + 4\sqrt{280}) \) or \( (73.07, 206.93) \). Thus the sustained load will exceed 206.93 with a probability less than \( \frac{1}{16} \).