Exercises

4.1 Let $Y$ be a random variable with $p(y)$ given in the table below.

<table>
<thead>
<tr>
<th>$y$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(y)$</td>
<td>0.4</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
</tr>
</tbody>
</table>

a. Give the distribution function, $F(y)$. Be sure to specify the value of $F(y)$ for all $y, -\infty < y < \infty$.
b. Sketch the distribution function given in part (a).

c. What is $P(1 < Y < 3)$? $P(Y < 3)$?

d. If $Y$ is a continuous random variable, we argued that, for all $-\infty < a < \infty, P(Y = a) = 0$.

4.2 A box contains five keys, only one of which will open a lock. Keys are randomly selected and tried, one at a time, until the lock is opened (keys that do not work are discarded before another is tried). Let $Y$ be the number of the trial on which the lock is opened.

a. Find the probability function for $Y$.
b. Give the corresponding distribution function.
c. Find $P(Y = 1)$.
d. If $P(Y = 1)$ is small, does part (c) give support to the claim that $Y$ takes on only the values 1, 2, 3, 4, 5? Explain your reasoning.

4.3 A Bernoulli random variable is one that assumes only two values, 0 and 1, with $p(1) = p$ and $p(0) = 1 - p = q$.

a. Sketch the corresponding distribution function.
b. Show that this distribution function has the properties given in Theorem 4.1.

c. Let $Y$ be a binomial random variable with $n = 10$ and success probability $p$.

4.4 a. Find the probability and distribution function for $Y$.
b. Compute the distribution function from part (a) with that in Exercise 4.3(a). What do you conclude?

4.5 Suppose that $Y$ is a random variable that takes on only integer values 1, 2, ..., and has distribution function $F(y)$. Show that the probability function $p(y) = P(Y = y)$ is given by

\[ p(y) = \begin{cases} F(y) & y = 1, \\ F(y) - F(y-1) & y = 2, 3, \ldots \end{cases} \]

4.6 Consider a random variable with a geometric distribution (Section 3.5), that is

\[ p(y) = q^{y-1}p, \quad y = 1, 2, 3, \ldots, 0 < p < 1, \]

a. Show that $Y$ has distribution function $F(y)$ such that $F(1) = 1 - q$, $F(2) = 0$, $1 - q^2$, $i \leq y < i+1$, for $i = 0, 1, 2, \ldots$.

b. Show that the preceding cumulative distribution function has the properties given in Theorem 4.1.

4.7 Let $Y$ be a binomial random variable with $n = 10$ and $p = 0.5$.

a. Use Table 1, Appendix 3, to obtain $P(2 < Y < 5)$ and $P(2 \leq Y < 5)$. Are the probabilities that $Y$ falls in the intervals $(2, 5)$ and $[2, 5)$ equal? Why or why not?

Exercises 167

4.8 Suppose that $Y$ has density function

\[ f(y) = \begin{cases} ky(1-y) & 0 \leq y \leq 1, \\ 0 & \text{elsewhere}. \end{cases} \]

a. Find the value of $k$ that makes $f(y)$ a probability density function.
b. Find $P(Y \leq 1)$.
c. Find $P(1 < Y < 2)$.
d. Find $P(Y \leq 2, Y > 1)$.
e. Find $P(Y < 1, Y < 2)$.

4.9 A random variable $Y$ has the following distribution function:

\[ F(y) = \begin{cases} 0, & \text{for } y < 2, \\ 1/8, & \text{for } 2 \leq y < 2.5, \\ 3/16, & \text{for } 2.5 \leq y < 4, \\ 1/2, & \text{for } 4 \leq y < 5.5, \\ 5/8, & \text{for } 5.5 \leq y < 6, \\ 11/16, & \text{for } 6 \leq y < 7, \\ 1, & \text{for } y \geq 7. \end{cases} \]

a. Is $Y$ a continuous or discrete random variable? Why?
b. What values of $Y$ are assigned positive probabilities?
c. Find the probability function for $Y$.
d. What is the median, $\phi_0$, of $Y$?

4.10 Refer to the density function given in Exercise 4.8.

a. Find the 0.95 quantile, $\phi_{0.95}$, such that $P(Y < \phi_{0.95}) = 0.95$.
b. Find a value $\phi_0$ so that $P(Y > \phi_0) = 0.95$.
c. Compare the values of $\phi_0$ and $\phi_{0.95}$ that you obtained in parts (a) and (b). Explain the relationship between these two values.

4.11 Suppose that $Y$ possesses the density function

\[ f(y) = \begin{cases} cy, & 0 \leq y \leq 2, \\ 0, & \text{elsewhere}. \end{cases} \]

a. Find the value of $c$ that makes $f(y)$ a probability density function.
b. Find $P(Y)$.
c. Graph $f(y)$ and $P(Y)$.
d. Use $F(y)$ to find $P(1 \leq Y \leq 2)$.
e. Use $f(y)$ and geometry to find $P(1 \leq Y \leq 2)$.

4.12 The length of time to failure (in hundreds of hours) for a transistor is a random variable $Y$ with distribution function given by

\[ F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2}, & y \geq 0. \end{cases} \]
4.13 A supplier of kerosene has a 150-gallon tank that is filled at the beginning of each week. His weekly demand shows a relative frequency behavior that increases steadily up to 100 gallons and then levels off between 100 and 150 gallons. If $Y$ denotes weekly demand in hundreds of gallons, the relative frequency of demand can be modeled by

$$f(y) = \begin{cases} 
0 & 0 \leq y \leq 1, \\
1 & 1 < y \leq 1.5, \\
0 & \text{elsewhere}.
\end{cases}$$

a. Find $F(y)$.
b. Find $P(0 \leq Y \leq 1)$.
c. Find $P(1 \leq Y \leq 1.2)$.

d. Find $P(Y > 100)$.

e. Use geometry and the graph for $f(y)$ to calculate $P(1 \leq Y \leq 2)$.

4.14 A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a random variable $Y$ (measured in 10,000 gallons) with a probability density function given by

$$f(y) = \begin{cases} 
y & 0 < y < 1, \
2 - y & 1 \leq y < 2, \\
0 & \text{elsewhere}.
\end{cases}$$

a. Graph $f(y)$.
b. Find $F(y)$ and graph it.
c. Find the probability that the station will pump between 8000 and 12,000 gallons in a particular month.
d. Given that the station pumped more than 10,000 gallons in a particular month, find the probability that the station pumped more than 15,000 gallons during the month.

4.15 As a measure of intelligence, mice are timed when going through a maze to reach a reward of food. The time (in seconds) required for any mouse is a random variable $X$ with a density function given by

$$f(x) = \begin{cases} 
b \sqrt{x} & 0 < x < b, \\
0 & \text{elsewhere}.
\end{cases}$$

where $b$ is the minimum possible time needed to traverse the maze.

a. Show that $f(x)$ has the properties of a density function.
b. Find $F(x)$.
c. Find $F(Y > b + c)$ for a positive constant $c$.
d. If $c$ and $d$ are both positive constants such that $d > c$, find $P(Y > b + d)$.

4.16 Let $Y$ possess a density function

$$f(y) = \begin{cases} 
(c - 2y) & 0 \leq y \leq 2, \\
0 & \text{elsewhere}.
\end{cases}$$

a. Find $c$.
b. Find $F(y)$.
c. Graph $f(y)$ and $F(y)$.
d. Use $F(y)$ in part (b) to find $P(1 \leq Y \leq 2)$.
e. Use geometry and the graph for $f(y)$ to calculate $P(1 \leq Y \leq 2)$.

4.17 The length of time required by students to complete a one-hour exam is a random variable with a density function given by

$$f(y) = \begin{cases} 
cy^2 + y & 0 \leq y \leq 1, \\
0 & \text{elsewhere}.
\end{cases}$$

a. Find $c$.
b. Find $F(y)$.
c. Graph $f(y)$ and $F(y)$.
d. Use $F(y)$ in part (b) to find $F(-1)$, $F(0)$, and $F(1)$.
e. Find the probability that a randomly selected student will finish in less than half an hour.
f. Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

4.18 Let $Y$ have the density function given by

$$f(y) = \begin{cases} 
2 & -1 \leq y \leq 0, \\
2 + cy & 0 < y \leq 1, \\
0 & \text{elsewhere}.
\end{cases}$$

a. Find $c$.
b. Find $F(y)$.
c. Graph $f(y)$ and $F(y)$.
d. Use $F(y)$ in part (b) to find $P(-1)$, $F(0)$, and $F(1)$.
e. Find $P(0 \leq Y \leq 1)$.
f. Find $P(Y > 1)$.

4.19 Let the distribution function of a random variable $Y$ be

$$F(y) = \begin{cases} 
0 & y \leq 0, \\
\frac{y^2}{2} & 0 < y < 2, \\
\frac{y^2}{2} + \frac{y^4}{16} & 2 \leq y < 4, \\
1 & y \geq 4.
\end{cases}$$

a. Find the density function of $Y$.
b. Find $P(1 \leq Y \leq 3)$.
c. Find $P(Y \geq 1.5)$.
d. Find $P(Y > 1)$.
Exercises

4.20 If, as in Exercise 4.16, $Y$ has density function

$$f(y) = \begin{cases} \frac{1}{2}(2-y), & 0 \leq y \leq 2, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of $Y$.

4.21 If, as in Exercise 4.17, $Y$ has density function

$$f(y) = \begin{cases} (3/2)y + y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of $Y$.

4.22 If, as in Exercise 4.18, $Y$ has density function

$$f(y) = \begin{cases} 2, & -1 < y \leq 0, \\ 2 + (1.2)y, & 0 < y \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

find the mean and variance of $Y$.

4.23 Prove Theorem 4.5.

4.24 If $Y$ is a continuous random variable with density function $f(y)$, use Theorem 4.5 to prove that $\theta^2 = E(Y^2) - [E(Y)]^2$.

4.25 If, as in Exercise 4.19, $Y$ has distribution function

$$F(y) = \begin{cases} 0, & y \leq 0, \\ \frac{2}{3}, & 0 < y < 2, \\ \frac{2}{3} + y^2, & 2 \leq y < 4, \\ 1, & y \geq 4, \end{cases}$$

find the mean and variance of $Y$.

4.26 If $Y$ is a continuous random variable with mean $\mu$ and variance $\sigma^2$ and $a$ and $b$ are constants, use Theorem 4.5 to prove the following:

a. $E(aY + b) = aE(Y) + b = a\mu + b$.

b. $V(aY + b) = a^2V(Y) = a^2\sigma^2$.

4.27 For certain ore samples, the proportion $Y$ of impurities per sample is a random variable with density function given in Exercise 4.21. The dollar value of each sample is $W = 5 - 5Y$. Find the mean and variance of $W$.

4.28 The proportion of time per day that all checkout counters in a supermarket are busy is a random variable $Y$ with density function

$$f(y) = \begin{cases} y^2(1 - y)^2, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the value of $c$ that makes $f(y)$ a probability density function.

b. Find $E(Y)$.

4.29 The temperature $Y$ at which a thermostatically controlled switch turns on has probability density function given by

$$f(y) = \begin{cases} \frac{1}{2}, & 59 \leq y \leq 61, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $E(Y)$ and $V(Y)$.

4.30 The proportion of time $Y$ that an industrial robot is in operation during a 40-hour week is a random variable with probability density function

$$f(y) = \begin{cases} 2y, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find $E(Y)$ and $V(Y)$.

b. For the robot under study, the profit $X$ for a week is given by $X = 2000Y - 60$. Find $E(X)$ and $V(X)$.

c. Find an interval in which the profit should lie for at least 75% of the weeks that the robot is in use.

4.31 Daily total solar radiation for a specified location in Florida in October has probability density function given by

$$f(y) = \begin{cases} \frac{5}{32}(y - 2)(6 - y), & 2 \leq y \leq 6, \\ 0, & \text{elsewhere,} \end{cases}$$

with measurements in hundreds of calories. Find the expected daily solar radiation for October.

4.32 Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} \frac{3}{64}(y - 4)^2, & 0 \leq y \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find the expected value and variance of weekly CPU time.

b. The CPU time costs the firm $200 per hour. Find the expected value and variance of the weekly cost for CPU time.

c. Would you expect the weekly cost to exceed $600 very often? Why?

4.33 The pH of water samples from a specific lake is a random variable $Y$ with probability density function given by

$$f(y) = \begin{cases} \frac{3}{64}(7 - y)^2, & 5 \leq y \leq 7, \\ 0, & \text{elsewhere.} \end{cases}$$

a. Find $E(Y)$ and $V(Y)$.

b. Find an interval shorter than (5, 7) in which at least three-fourths of the pH measurements must lie.

c. Would you expect to see a pH measurement below 5.5 very often? Why?

4.34 Suppose that $Y$ is a continuous random variable with density $f(y)$ that is positive only if $y \geq 0$. If $F(y)$ is the distribution function, show that

$$E(Y) = \int_0^\infty y f(y) \, dy = \int_0^\infty [1 - F(y)] \, dy.$$

HINT: If $Y > 0$, $y = \int_0^y f(y) \, dy$, and $E(Y) = \int_0^\infty y f(y) \, dy = \int_0^\infty \left[ \int_0^y f(y) \, dy \right] f(y) \, dy$. Exchange the order of integration to obtain the desired result.

4.35 Exercises preceded by an asterisk are optional.
4.59 If \( Z \) is a standard normal random variable, find the value \( z_0 \) such that:

a. \( P(Z > z_0) = .5 \)

b. \( P(Z < z_0) = .3843 \)

c. \( P(-z_0 < Z < z_0) = .90 \)

d. \( P(-z_0 < Z < z_0) = .99 \)

4.60 A normally distributed random variable has density function \( f(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \), \( -\infty < y < \infty \).

Using the fundamental properties associated with any density function, argue that the parameter \( \sigma \) must be such that \( \sigma > 0 \).

4.61 What is the median of a normally distributed random variable with mean \( \mu \) and standard deviation \( \sigma \)?

4.62 If \( Z \) is a standard normal random variable, what is:

a. \( P(Z^2 < 1) \)

b. \( P(Z^2 < 3.84146) \)

4.63 A company that manufactures and bottles apple juice uses a machine that automatically fills 16-ounce bottles. There is some variation, however, in the amounts of liquid dispensed into the bottles that are filled. The amount dispensed has been observed to be approximately normally distributed with mean 16 ounces and standard deviation 1 ounce.

Exercises

4.64 The weekly amount of money spent on maintenance and repairs by a company was observed, over a long period of time, to be approximately normally distributed with mean \( \$400 \) and standard deviation \( \$20 \). If \( \$450 \) is budgeted for next week, what is the probability that the actual costs will exceed the budgeted amount?

a. Answer the question, using Table 4, Appendix 3.

b. Applet Exercise Use the applet Normal Probabilities to obtain the answer.

c. Why are the labeled values different on the two horizontal axes?

4.65 In Exercise 4.64, how much should be budgeted for weekly repairs and maintenance to provide that the probability the budgeted amount will be exceeded in a given week is only .17?

4.66 A machining operation produces bearings with diameters that are normally distributed with mean 3.0005 inches and standard deviation .0010 inch. Specifications require the bearing diameters to lie in the interval 3.000 to 3.0020 inches. Those outside the interval are considered scrap and must be remanufactured. With the existing machine setting, what fraction of total production will be scrap?

a. Answer the question, using Table 4, Appendix 3.

b. Applet Exercise Obtain the answer, using the applet Normal Probabilities.

4.67 In Exercise 4.66, what should the mean diameter be in order that the fraction of bearings scrapped be minimized?

4.68 The grade point averages (GPAs) of a large population of college students are approximately normally distributed with mean 2.4 and standard deviation .8. What fraction of the students will possess a GPA in excess of 3.07?

a. Answer the question, using Table 4, Appendix 3.

b. Applet Exercise Obtain the answer, using the applet Normal Tail Areas and Quantiles.

4.69 Refer to Exercise 4.68. If students possessing a GPA less than 1.9 are dropped from college, what percentage of the students will be dropped?

4.70 Refer to Exercise 4.68. Suppose that three students are randomly selected from the student body. What is the probability that all three will possess a GPA in excess of 3.07?

4.71 Wires manufactured for use in a computer system are specified to have resistances between .12 and .14 ohms. The actual measured resistances of the wires produced by company A have a normal probability distribution with mean .13 ohm and standard deviation .005 ohm.

a. What is the probability that a randomly selected wire from company A’s production will meet the specifications?

b. If four of these wires are used in each computer system and all are selected from company A, what is the probability that all four in a randomly selected system will meet the specifications?

4.72 One method of arriving at economic forecasts is to use a consensus approach. A forecast is obtained from each of a large number of analysts; the average of these individual forecasts is the consensus forecast. Suppose that the individual 1966 January prime interest-rate forecasts of all economic analysts are approximately normally distributed with mean 7% and standard
deviation 2.6%. If a single analyst is randomly selected from among this group, what is the probability that the analyst's forecast of the prime interest rate will
a. exceed 11%?
 b. be less than 9%?

4.73 The width of bolts of fabric is normally distributed with mean 950 mm (millimeters) and standard deviation 10 mm.
   a. What is the probability that a randomly chosen bolt has a width of between 947 and 958 mm?
   b. What is the appropriate value for C such that a randomly chosen bolt has a width less than C with probability .8531?

4.74 Scores on an examination are assumed to be normally distributed with mean 78 and variance 36.
   a. What is the probability that a person taking the examination scores higher than 72?
   b. Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?
   c. What must be the cutoff point for passing the examination if the examiner wants only the top 28.1% of all scores to be passing?
   d. Approximately what proportion of students have scores 5 or more points above the score that cuts off the lowest 25%?
   e. Applet Exercise. Answer parts (a)-(d), using the applet Normal Tail Areas and Quantiles.
   f. If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

4.75 A soft-drink machine can be regulated so that it discharges an average of $\mu$ ounces per cup. If the ounces of fill are normally distributed with standard deviation 0.3 ounce, give the setting for $\mu$ so that 8-ounce cups will overflow only 1% of the time.

4.76 The machine described in Exercise 4.75 has standard deviation $\sigma$ that can be fixed at certain levels by carefully adjusting the machine. What is the largest value of $\sigma$ that will allow the actual amount dispensed to fall within 1 ounce of the mean with probability at least .95?

4.77 The SAT and ACT college entrance exams are taken by thousands of students each year. The mathematics portions of each of these exams produce scores that are approximately normally distributed. In recent years, SAT mathematics exam scores have averaged 540 with standard deviation 100. The average and standard deviation for ACT mathematics scores are 18 and 6, respectively.
   a. An engineering school sets 550 as the minimum SAT math score for new students. What percentage of students will score below 550 in a typical year?
   b. What score should the engineering school set as a comparable standard on the ACT math test?

4.78 Show that the maximum value of the normal density with parameters $\mu$ and $\sigma$ is $1/(\sigma \sqrt{2\pi})$ and occurs when $x = \mu$.

4.79 Show that the normal density with parameters $\mu$ and $\sigma$ has inflection points at the values $\mu - \sigma$ and $\mu + \sigma$. (Recall that an inflection point is a point where the curve changes direction from concave up to concave down, or vice versa, and occurs when the second derivative changes sign. Such a change in sign may occur when the second derivative equals zero.)

4.80 Assume that $Y$ is normally distributed with mean $\mu$ and standard deviation $\sigma$. After observing a value of $Y$, a mathematician constructs a rectangle with length $L = |Y|$ and width $W = 3|Y|$. Let $A$ denote the area of the resulting rectangle. What is $E(A)$?
4.85 **Applet Exercise** Use the applet Comparison of Gamma Density Functions to compare gamma density functions with \((\alpha = 1, \beta = 4), (\alpha = 1, \beta = 2),\) and \((\alpha = 1, \beta = 2).\)

a. What is another name for the density functions that you observed?

b. Do these densities have the same general shape?

c. The parameter \(\beta\) is a "scale" parameter. What do you observe about the "spread" of these three density functions?

4.86 **Applet Exercise** When we discussed the \(\chi^2\) distribution in this section, we presented (with justification to follow in Chapter 6) the fact that if \(Y\) is gamma distributed with \(\alpha = m/2\) for some integer \(m\), then \(2Y/\beta\) has a \(\chi^2\) distribution. In particular, it was stated that when \(m = 1.5\) and \(\beta = 4\), if \(W = Y/2\) has a \(\chi^2\) distribution with 3 degrees of freedom.

a. Use the applet Gamma Probabilities and Quantities to find \(P(Y < 3.5).\)

b. Use the applet Gamma Probabilities and Quantities to find \(P(W < 1.75).\) (Hint: Recall that the \(\chi^2\) distribution with \(v\) degrees of freedom is just a gamma distribution with \(\alpha = v/2\) and \(\beta = 2.\))

c. Compare your answers to parts (a) and (b).

4.87 **Applet Exercise** Let \(Y\) and \(W\) have the distributions given in Exercise 4.86.

a. Use the applet Gamma Probabilities and Quantities to find the 0.05-quantile of the distribution of \(Y.\)

b. Use the applet Gamma Probabilities and Quantities to find the 0.05-quantile of the \(\chi^2\) distribution with 3 degrees of freedom.

c. What is the relationship between the 0.05-quantile of the gamma \((\alpha = 1.5, \beta = 4)\) distribution and the 0.05-quantile of the \(\chi^2\) distribution with 3 degrees of freedom? Explain this relationship.

4.88 The magnitude of earthquakes recorded in a region of North America can be modeled as having an exponential distribution with mean 2.4, as measured on the Richter scale. Find the probability that an earthquake striking this region will

a. exceed 3.0 on the Richter scale.

b. fall between 2.0 and 3.0 on the Richter scale.

4.89 If \(Y\) has an exponential distribution and \(P(Y > 2) = 0.821\), what is

a. \(\beta = E(Y)\)

b. \(P(Y \leq 1.7)\)?

4.90 Refer to Exercise 4.88. Of the next ten earthquakes to strike this region, what is the probability that at least one will exceed 5.0 on the Richter scale?

4.91 The operator of a pumping station has observed that demand for water during early afternoon hours has an approximately exponential distribution with mean 100 cfs (cubic feet per second).

a. Find the probability that the demand will exceed 200 cfs during the early afternoon on a randomly selected day.

b. What water-pumping capacity should the station maintain during early afternoons so that the probability that demand will exceed capacity on a randomly selected day is only 0.01?

4.92 The length of time \(Y\) necessary to complete a key operation in the construction of houses has an exponential distribution with mean 10 hours. The formula \(C = 100 + 40Y + 3Y^2\) relates

4.93 Historical evidence indicates that times between fatal accidents on scheduled American domestic passenger flights have an approximately exponential distribution. Assume that the mean time between accidents is 44 days.

a. If one of the accidents occurred on July 1 of a randomly selected year in the study period, what is the probability that another accident occurred that same month?

b. What is the variance of the times between accidents?

4.94 One-hour carbon monoxide concentrations in air samples from a large city have an approximately exponential distribution with mean 3.6 ppm (parts per million).

a. Find the probability that the carbon monoxide concentration exceeds 9 ppm during a randomly selected one-hour period.

b. A traffic-control strategy reduced the mean to 2.5 ppm. Now find the probability that the concentration exceeds 9 ppm.

4.95 Let \(Y\) be an exponentially distributed random variable with mean \(\beta\). Define a random variable \(X\) in the following way: \(X = k\) if \(k - 1 < Y < k\) for \(k = 1, 2, \ldots\)

a. Find \(P(X = k)\) for each \(k = 1, 2, \ldots\).

b. Show that your answer to part (a) can be written as

\[ P(X = k) = (e^{-\beta})^{k-1} (1 - e^{-\beta}) \quad k = 1, 2, \ldots \]

and that \(X\) has a geometric distribution with \(p = 1 - e^{-\beta}.\)

4.96 Suppose that a random variable \(Y\) has a probability density function given by

\[ f(y) = \begin{cases} ky^2 e^{-y/\beta}, & y > 0, \\ 0, & \text{elsewhere}. \end{cases} \]

a. Find the value of \(k\) that makes \(f(y)\) a density function.

b. Does \(Y\) have a \(\chi^2\) distribution? If so, how many degrees of freedom?

c. What are the mean and standard deviation of \(Y\)?

d. **Applet Exercise** What is the probability that \(Y\) lies within 2 standard deviations of its mean?

4.97 A manufacturing plant uses a specific bulk product. The amount of product used in one day can be modeled by an exponential distribution with \(\beta = 4\) (measurements in tons). Find the probability that the plant will use more than 4 tons on a given day.

4.98 Consider the plant of Exercise 4.97. How much of the bulk product should be stock sized so that the plant's chance of running out of the product is only 0.05?

4.99 If \(\lambda > 0\) and \(\alpha\) is a positive integer, the relationship between incomplete gamma integrals and sums of Poisson probabilities is given by

\[ \frac{1}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y} dy = \sum_{i=0}^{\alpha-1} \frac{\lambda^i e^{-\lambda}}{i!}. \]