Statistical Inference 1650/1655 Homework 5
Due Thursday October 29th Homework is due by 1pm in the dropoff boxes located in the first floor of 182 George St. Show all of your work in deriving your solutions.

APMA 1650: Complete all unstarred problems.
APMA 1655: Complete all unstarred and single starred problems. Double starred problems are particularly challenging and I do not necessarily expect you to answer them to completion. You should work on them and write up your (partial) solutions/attempts/computations, etc.

1. Let $Y$ be a random variable, define the random variable

$$Z = \frac{Y - E[Y]}{\sqrt{V(Y)}}$$

Find the expectation and variance of $Z$. Give purely numerical answers.

2. The Normal distribution is often used to approximate the binomial distribution for large $n$. It is also a good approximation for relatively small values of $n$ as long as $p$ is not too small or too large.

Let $n = 25$ and $p = .4$
(a) Suppose $X$ is a binomial random variable with parameters $n$ and $p$ as above. Compute the probability $p(X = 8)$.
(b) Suppose $Y$ is a Normal random variable with $\mu = np$ and $\sigma^2 = np(1 - p)$. Compute the probability that $p(7.5 \leq Y \leq 8.5)$.

The “Rule of Thumb” is that the Normal approximation is good enough if $p \pm 3\sqrt{pq/n}$ is between 0 and 1.
(c) Show that $p \pm 3\sqrt{pq/n}$ is between 0 and 1 if and only if

$$n > 9(\frac{p}{q}) \text{ and } n > 9(\frac{q}{p}).$$

Moral: The Normal approximation is considered good enough if

$$n > 9(\frac{\max\{p, q\}}{\min\{p, q\}})$$

(d) How large should $n$ be to approximate the binomial distribution with $p = .5$? .8? .99? .999?

3. Suppose we are playing a game of darts, throwing a dart at a circular board of radius 1. Model the dart board as a circle of radius 1; Board = \{(x, y) : x^2 + y^2 \leq 1\}. Suppose the probability of landing the dart in any subset $E$ of the board is given by the fraction of board area lying in $E$.

$$P(E) = \frac{\text{area of } E}{\pi} = \int_E \frac{1}{\pi} dx$$

(a) What is the probability that the dart hits exactly the center?
(b) What is the probability that the dart lies within distance $\frac{1}{2}$ of the center of the board?
(c) What is the probability that the dart lies between a distance $a$ and $b$ ($a < b < 1$) from the center?

*(d) Now suppose you throw a dart at a square dart board. What is the probability that the dart lands closer to the center than any edge?

4. Now consider a random dart board.

(a) Suppose that the radius of the dart board is uniformly distributed between 0 and 1, what are the mean and variance of the area of the dart board?

b) Unfazed by a dartboard which somehow has a random radius, you throw a dart uniformly at random in the circle of radius 1. What is the probability you miss the board entirely?

c) Suppose you repeat part (b) until you hit the board, with a new random dartboard generated independently for each throw. What is the expected number of darts thrown?

*d) Suppose you repeat part (b) until you hit the board, this time using the same random dartboard for every throw (i.e., the radius of the dartboard is chosen, then you make throws until hitting the board). Argue that the expected number of darts thrown diverges. Why is this different from (c)?

5. (a) Suppose a stick of length 1 is broken in two places, each chosen uniformly at random along the length of the stick. Find the probability that the three resulting pieces can be arranged to form a triangle (i.e. all triangle inequalities are satisfied; i.e no piece is longer than the sum of the other two).

**(b) Suppose a stick of length 1 is broken in five places, each chosen uniformly at random along the length of the stick. Find the probability that the six resulting pieces can be arranged as the edges of a tetrahedron (triangular based pyramid). (This problem does not have a closed solution and might more accurately be labeled a 3 or 4 star problem. As always, give us your thoughts on how you would approach it.)