1. Suppose we are sending a digital signal which is a string of 0s and 1s of length five. (Example strings are 00101, 11000, 10101.) When we send the message, each bit is sent independently and there is some chance that the bit is corrupted. Namely, each time we send a 0 there is a 5% chance that a 1 is received and each time we send a 1 there is a 5% chance that a 0 is received.

(a) Suppose we send a message of length 5, what is the probability that an incorrect message is received?

Upon receiving a signal, we are not able to tell if some part is incorrect. To aid in error correction we decide to send each bit three times. So if the original signal is 00101 we now send 000001111000111.

When the message is received, the string is broken into pieces of length three and whichever bit appears more often is recorded. For example, if 001000101000111 is received, then we consider

001 000 101 000 111

and 00101 is recorded. If 110000101001001 is received, then we consider

110 000 101 010 001

and 10100 is recorded.

(b) Under this scheme, what is the probability that an incorrect message is recorded?

2. Suppose we are choosing a number from \{1, 2, 3, 4, 5, 6\} such that the probability of choosing a particular number is proportional to the number itself. For example, drawing a six is three times as likely as drawing a two. What is the probability we pick an even number?

3. Suppose a class has 300 students.

(a) The professor will pick 100 students equally likely at random and give these 100 students a “free” A. In lecture we computed the probability that a given group of 100 students would be chosen. Suppose you are in the class, what is the probability that you will be one of the 100 students chosen and hence receive a “free” A?
(b) The exams are handed back one by one in a random order (i.e. an ordering of the 300 students is chosen equally likely at random from all orderings and the exams are handed back in this order). If Alice and Bob are two students in class, what is the probability that Alice will get her test back before Bob?

*Note:* You may have intuition about the solutions to both parts of this problem. Be sure to set up an appropriate probability space, define the relevant events, and justify their probability.

4. In a city, streets are laid out as a grid. Your home is at location (0, 0) and work is at location (10, 10). You walk to work taking only steps up and to the right.
   (a) How many distinct ways can you walk to work?
   Suppose you choose your path uniformly at random.
   (b) If someone is standing at location (10, 0), what is the probability you will pass them? At location (9, 1)?

5. Suppose your materials science roommate managed to make a two-headed coin. Your roommate has the two-headed coin and a regular two-sided fair coin in their pocket. They draw a coin from their pocket equally likely at random and flip the coin. Given that the coin comes up heads, what is the probability that the fair coin was flipped?

6. Suppose we have two identical biased coins: coin1 and coin2. For both coins, the probability of Heads is $p$ and the probability of Tails is $q = 1 - p$.

   Suppose we flip both coins. If the coins match, namely if we get two Heads or two Tails, we record nothing and flip both coins again. If the coins do not match, we stop and record the outcome of coin2. What is the probability of recording Heads?

7. Occasionally we wonder if a class is effective. Suppose one out of every 20 Brown students who has not taken 1650 can pass a statistics exam. Further suppose that 10 Brown students who have not previously taken 1650 are chosen at random and forced to take the class. Afterwards, nine out of the ten people who were forced to take the class pass the
statistics exam. If the class is completely ineffective (teaches you nothing) what is the probability of nine people passing the exam? (Does that seem like a good hypothesis?)

8. You and your roommate are playing a coin tossing game (good times!). You each flip a fair coin. If the coins are both $H$ you win 1 dollar. If the coins are both $T$ you win 2 dollars. If they don’t match you lose 1 dollar. Let $X$ be the random variable recording your winnings.

(a) What is the expectation of $X$ if you play 1 time? 10 times?

Suppose you have to pay $d$ dollars up front to play the game. Your net winnings is then defined as $W = X - d$. In general, a fair game is one in which the expectation of the net winnings is 0.

(b) How much should you pay to play 1 time, 10 times, to make the game fair?

(Moral of this problem: what happens when you add or subtract a constant from a random variable? What is the expectation of a constant function?)

9. As on the last homework, suppose we have two identical biased coins: coin1 and coin2. For both coins, the probability of Heads is $p$ and the probability of Tails is $q = 1 - p$. Suppose we flip both coins. If the coins match, namely if we get two Heads or two Tails, we record nothing and flip both coins again. If the coins do not match, we stop and record the outcome of coin2.

What is the expected number of iterations before we stop?