Geometric Structure Estimation of Axially Symmetric Pots from Small Fragments

Yan Cao
Division of Applied Mathematics
Brown University
Providence, RI 02912, USA
email: cao@cfm.brown.edu

David Mumford
Division of Applied Mathematics
Brown University
Providence, RI 02912, USA
email: David_Mumford@brown.edu

ABSTRACT

The recovery of geometric structure from noisy data poses difficult non-linear statistical estimation problems. This paper describes a novel, robust, low computational cost approach for finding the geometric structure of an axially symmetric pot from a small fragment of it (an unorganized set of 3D points). This problem is of great archaeological importance to the study of the hundreds and thousands of shards found at excavation sites. Our method is based on the following fact: for each point on the surface, the center of the sphere of principal curvature corresponding to the circles of revolution is on the symmetric axis. By finding the line which minimizes the weighted least-squares distance to the estimated centers, we can find first the symmetric axis and then the profile curve. Because of the special properties of a surface of revolution, we can do this using only *first* derivatives, hence this method is robust to noisy data. We then use bootstrap methods to find confidence bounds for the axis and the profile curve. These confidence bounds are essential if the estimations are used for the assembly of the full pot from multiple sherds.

KEY WORDS

Shape analysis, 3D data analysis, Surface of revolution, Symmetric axis

1 Introduction

Estimating geometric structures from noisy 3D data is a highly non-linear statistical estimation problem. This paper concerns a particular problem of estimating axes which arises in archaeology. This is part of the effort of the Brown Shape Lab to understand and reconstruct 3D shapes from archaeological finds [1] [2] [3] [4]. In particular, many of these finds are potsherds. Potsherds are fragments of a pot, which is ideally a surface of revolution, i.e. a surface formed by rotating a generator curve about an axis. Reconstructing complete or nearly complete pots from a large collections of sherds is one of the most time consuming and difficult problems that archaeologists face. At present, this is done manually and often takes a few days to reconstruct one pot. Instead, most pottery is approximately classified using two dimensional drawing and measuring techniques. We can assist them by finding accurate estimates (i) of the axis hence (ii) of the distance of each point on the sherd to the axis (which define the profile curve). With this information, the problem of finding adjacent sherds belonging to the same pot is greatly simplified, since these distance values must agree on the break curve between the sherds and the axes must match. Of course, it is also necessary to compute confidence bounds for both the axis and distance function or correct matches will be rejected when the estimates are only slightly different.

Given a perfect surface of revolution, theoretically, there are many ways to find its axis: one of the 2 families of lines of curvature are circles of revolution with centers on the axis and the other are the generator curves, lying in planes containing the axis. But still, finding an reliable algorithm working on unorganized 3D data is not an easy task. Unfortunately, potsherds are far from perfect, due to imperfections in the production process and subsequent erosion. In other words, they are noisy surfaces of revolution. Even worse, each sherd is only a small fragment of the whole surface, which makes this problem more challenging. What then is the best way to find their approximate axis and confidence bounds for this estimate?

As far as we know, not much work has been done for this problem. Lukacs *et. al.* [5] proposed geometric least-squares fitting methods for spheres, cylinders, cones and tori, however, each geometric type is treated separately. Potmann *et. al.* [6] proposed an approach to reconstruct helical surfaces or surfaces of revolution using concepts from line geometry. Their algorithm is based on the fact that the normals of these surfaces lie in "linear complexes", which are given by linear equations in the Plűcker coordinates. Using these coordinates, the least squares distance between a line and the set of normals can be represented by a positive semidefinite quadratic form. Minimizing that form is then reduced to a generalized eigenvalue problem. This method can be applied to our problem and it is fast, but lack of accuracy sometimes since they didn't use all the possible information. We use this solution as our initial starting point.

Our approach is inspired by the study of the contact

between surfaces and spheres [7]. Spheres of curvature are defined to be spheres tangent to the surface and having radius equal to the inverse of one of the principal curvatures there. For any point on a surface of revolution, the symmetry axis contains the center of the sphere of curvature corresponding to the parallel circles. By finding the line which minimizes the least squares distance from the estimated centers to it, we can find the symmetry axis. After we get the axis of revolution, we can calculate the distance from each data point to the symmetric axis, and do a cubic spline fit [8] [9] to get the profile curve. Because of the special properties of a surface of revolution, we will show that this can be done using only first derivatives of the data, hence this method is quite robust with noisy data. We also use weighted least squares instead of least squares to achieve robustness to outlier data points. This algorithm is computationally efficient. The goal function is easy to calculate, and the minimization can be carried out by general iterative methods. We apply this method to real data from potsherds from an archaeological site in Petra, Jordan [10]. After the estimate is made, we use bootstrap methods to obtain confidence bounds on both the axis and the profile curve. Note that a line in 3-space is described by 4 parameters, hence the confidence interval for the axis is best described as a hyperellipsoid centered at the estimate. Fortunately, it typically takes a special form and can be readily described. The results are very promising and we expect our approach can be integrated into a system for the full reconstruction of pots being built here at Brown.

2 Facts about Surfaces of Revolution

A surface of revolution is formed by rotating a plane curve about a line in $\mathbf{R}^3$. The plane curve is called the profile curve, the line is called the axis of revolution [11]. For convenience we let the axis of revolution be the z-axis and consider the profile curve in the xz-plane. Then the surface of revolution can be parameterized as:

$$X(u,v) = (\phi(v)\cos u, \phi(v)\sin u, \psi(v)) \qquad (1)$$

Then the coordinate curves $u =$ cnst., $v =$ cnst. are the principal curves. The unit surface normal is given by

$$n(u,v) = \text{sign}(\phi)\frac{(\psi'\cos u, \psi'\sin u, \phi')}{\sqrt{\phi'^2+\psi'^2}} \qquad (2)$$

and the two principal curvatures are given by

$$\begin{cases} \kappa_\mu = \frac{\text{sign}(\phi)(\phi''\psi'-\phi'\psi'')}{(\phi'^2+\psi'^2)^{\frac{3}{2}}}, \\ \kappa_\pi = \frac{-\psi'}{|\phi|\sqrt{\phi'^2+\psi'^2}}. \end{cases} \qquad (3)$$

Here κ_π is the curvature of the parallel $u \longmapsto X(u,v)$ and κ_μ is the curvature of the meridian $v \longmapsto X(u,v)$.

It is easy to show that the centers of the spheres of curvature corresponding to κ_π are in the axis of revolution.

3 Formation of the Problem

Given a set of m 3D data points from a surface of revolution, let $\mathbf{p_i}$ and $\mathbf{n_i}$ be the m 3D data points and their corresponding estimated normals. Suppose the axis of revolution is the line L. L is determined by a point $\mathbf{p_0}$ on L and a unit vector $\mathbf{v}$ corresponding to the direction of L. We make the point $\mathbf{p_0}$ unique by requiring $\mathbf{p_0}\cdot\mathbf{v} = 0$. Note that now the axis has four degrees of freedom. For any point $\mathbf{p}$ on the surface, suppose the normal at that point is $\mathbf{n}$. (See Figure 1) We shall show that κ_π can be calculated from

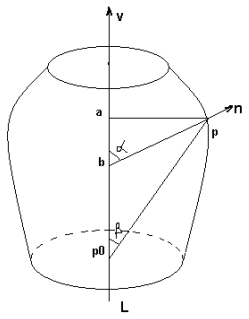

Figure 1.

the axis L and the vectors $\mathbf{p}, \mathbf{n}$, without using any higher derivatives. κ_π is very important in our algorithm. We can calculate it in general coordinates as follows. From Figure 1, notice that the radius of curvature corresponding to κ_π is just $\|\overline{\mathbf{bp}}\|$,

$$\begin{aligned} \|\overline{\mathbf{ap}}\| &= \|\overline{\mathbf{p_0p}}\|\sin\beta = \|(\mathbf{p}-\mathbf{p_0})\times\mathbf{v}\| \\ \text{and} \quad \sin\alpha &= \|\mathbf{n}\times\mathbf{v}\| \end{aligned}$$

Hence,

$$\|\overline{\mathbf{bp}}\| = \frac{\|\overline{\mathbf{ap}}\|}{\sin\alpha} = \frac{\|(\mathbf{p}-\mathbf{p_0})\times\mathbf{v}\|}{\|\mathbf{n}\times\mathbf{v}\|}$$

and thus

$$\kappa_\pi = \frac{\|\mathbf{n}\times\mathbf{v}\|}{\|(\mathbf{p}-\mathbf{p_0})\times\mathbf{v}\|} \qquad (4)$$

Thus the center of the i^{th} sphere of curvature is

$$\mathbf{c_i} = \mathbf{p_i} - \frac{1}{\kappa_{\pi,i}}\mathbf{n_i}$$

All the centers $\mathbf{c_i}$ should be in the axis of revolution. Hence, we seek to minimize the following function:

$$\begin{aligned} f(\mathbf{p_0},\mathbf{v}) &= \sum_{i=1}^{m}\|(\mathbf{c_i}-\mathbf{p_0})\times\mathbf{v}\|^2 \\ &= \sum_{i=1}^{m}\|(\mathbf{p_i}-\mathbf{p_0})\times\mathbf{v} - \frac{\|(\mathbf{p_i}-\mathbf{p_0})\times\mathbf{v}\|}{\|\mathbf{n_i}\times\mathbf{v}\|}(\mathbf{n_i}\times\mathbf{v})\|^2 \end{aligned}$$

The arguments $\mathbf{p_0}, \mathbf{v}$ of f depend on 6 real parameters but they are not independent as they satisfy the two constraints:

$$\mathbf{p_0}\cdot\mathbf{v} = 0, \qquad \|\mathbf{v}\| = 1.$$

It is convenient to introduce 4 new independent parameters. Define the matrix R as follows:

$$R = \begin{bmatrix} \cos\alpha\cos\beta & \sin\alpha\cos\beta & \sin\beta \\ -\sin\alpha & \cos\alpha & 0 \end{bmatrix}$$

and then the vector $\mathbf{v}$ can be represented as

$$\mathbf{v} = \begin{bmatrix} -\cos\alpha\sin\beta & -\sin\alpha\sin\beta & \cos\beta \end{bmatrix}^T$$

Define $\mathbf{w} = (x_0, y_0)^T = R \cdot \mathbf{p_0}$. Note that $\|R \cdot \mathbf{x}\| = \|\mathbf{x} \times \mathbf{v}\|$ because $\begin{pmatrix} R \\ \mathbf{v}^T \end{pmatrix}$ is the rotation matrix which makes the z-axis parallel to the axis of revolution. Hence, we have

$$\begin{aligned} \|(\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v}\| &= \|R \cdot (\mathbf{p_i} - \mathbf{p_0})\| = \|R \cdot \mathbf{p_i} - \mathbf{w}\| \\ \text{and} \qquad \|\mathbf{n_i} \times \mathbf{v}\| &= \|R \cdot \mathbf{n_i}\| \end{aligned}$$

Also notice that $(\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v}$ should be parallel to $\mathbf{n_i} \times \mathbf{v}$, so the function f can be represented as a function of 4 parameters: α, β, x_0, y_0.

$$f(\alpha, \beta, x_0, y_0) = \sum_{i=1}^{m} r_i^2 \tag{5}$$

where

$$r_i = \|(R \cdot \mathbf{p_i} - \mathbf{w}) - \frac{\|(R \cdot \mathbf{p} - \mathbf{w})\|}{\|R \cdot \mathbf{n_i}\|}(R \cdot \mathbf{n_i})\| \tag{6}$$

The above is the least squares formation, which is known as sensitive to outliers. To achieve robustness, we want an estimator which is not affected by the outliers. One way to do it is using weighted least squares [12], i.e. replace $\min_{\hat{\theta}} \sum_i r_i^2$ by

$$\min_{\hat{\theta}} \sum_i w_i r_i^2 \tag{7}$$

where

$$w_i = \begin{cases} 1 & \text{if } |r_i/\hat{\sigma}| \le c; \\ 0 & \text{if } |r_i/\hat{\sigma}| > c. \end{cases} \tag{8}$$

$$\text{and} \qquad \hat{\sigma} = 1.4826 \text{median}_i |r_i| \tag{9}$$

The constant c is chosen depending on the data. In most cases, we use $c = 2.5$. We implement this as an iterative reweighted procedure: for each iteration, we use the weight function based on the solution from last iteration.

The algorithm is as following:

1. get an initial estimation of the axis of revolution $L^{(0)}$ using Plűcker coordinates by the method of [6]. (The Plűcker coordinates of a line goes through point $\mathbf{p}$ with direction unit vector $\mathbf{v}$ may be taken as $(\mathbf{v}; \mathbf{v} \times \mathbf{p})$.
2. compute the weight function $w_i^{(k)}$ using Eq. (8), and Eq. (6) from the current estimation of the axis of revolution.
3. get a new estimation of the axis of revolution $L^{(k+1)}$ by minimizing Eq. (7).
4. if the difference between $L^{(k+1)}$ and $L^{(k)}$ is small enough, stop, otherwise, go back to step 2.

For simplicity, in step 3, we used a MatLab minimization routine based on an iterative line search algorithm [13].

4 Comparison with the Plűcker method (Potmann's method)

The Plűcker method is based on the fact that all the surface normals intersect with the axis of revolution. It minimizes the least squares distance between a line (the axis) and the set of all surface normals, which is

$$\begin{aligned} f(\mathbf{p_0}, \mathbf{v}) &= \sum_{i=1}^{m} \|(\mathbf{p_i} - \mathbf{p_0}) \cdot (\mathbf{n_i} \times \mathbf{v})\|^2 \\ &= \sum_{i=1}^{m} \|((\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v}) \times (\mathbf{n_i} \times \mathbf{v})\|^2 \end{aligned}$$

(using $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.) Compare this with the goal function of our algorithm, which is

$$f(\mathbf{p_0}, \mathbf{v}) = \sum_{i=1}^{m} \|(\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v} - \frac{\|(\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v}\|}{\|\mathbf{n_i} \times \mathbf{v}\|}(\mathbf{n_i} \times \mathbf{v})\|^2$$

We can see that the Plűcker method requires that the two vectors $(\mathbf{p_i} - \mathbf{p_0}) \times \mathbf{v}$ and $\mathbf{n_i} \times \mathbf{v}$ be parallel, while our method requires the two vectors not only be parallel, but also satisfy a length relation. The intuition is the Plűcker method doesn't use the extra constraint that the normals of the surface points from the same revolution circle intersect at the same point on the axis, but our method uses this also.

5 Experimental Results

Experiments are performed on various sherds from the Great Temple site of Petra, Jordan [10] scanned by a Shapegrabber laser scanner [14]. The data for each sherd is a dense set of unorganized 3D points and their corresponding normals. The pixel size is 1mm. We have used of sherds of very different shapes. For example, the sherd p1313 is highly curved and not convex. The sherd p654 is very flat. The sherd p642 is also flat but has a ridge. The sherd p967 is very small. The sherd 1135 has multi-valued profile curve.

In order to get the error estimation, we do the bootstrap [15] for each sherd. We generate 500 independent bootstrap samples by "resampling with replacement", i.e. each bootstrap sample has the same number of data points as the original with points being drawn with replacement from the original data. The axis has 4 degree of freedom and in order to make the 4 parameters comparable to each other, we use the following parameterization:

$$x = a/z_0 \cdot z + b \tag{10}$$

$$y = c/z_0 \cdot z + d \tag{11}$$

where z_0 is the size of the sherd in z direction. To normalize the data, we do a transformation such that the z-axis is the mean estimate of the axis of the 500 bootstrap samples, and the x-axis goes through the centroid of the sherd. We apply our algorithm to each bootstrap sample, get an estimate of the 4 parameters a, b, c, d of the axis of revolution and the profile curve. Then we estimate the standard error by computing the covariance matrix $\mathbf{S}$ of the 4 parameters based on the 500 bootstrap samples.

Suppose $l^{(i)} = (a_i, b_i, c_i, d_i)$, $i = 1, 2, ..., n$ is a random sample of size n from a multivariate normal distribution $N_4(\mu, \mathbf{\Sigma})$, then the mean estimate $\overline{l} \sim N_4(\mu, \mathbf{\Sigma}/n)$. The statistic $n(\overline{l} - \mu)'\mathbf{\Sigma}^{-1}(\overline{l} - \mu)$ therefore has a χ^2 distribution with 4 degrees of freedom and can be used to make inferences about μ. Given a sample mean vector $\overline{l}$, the equation $n(\overline{l} - \mu)'\mathbf{\Sigma}^{-1}(\overline{l} - \mu) = \chi^2_{\alpha;4}$ describes an ellipsoid with center at $\overline{l}$. This equation provides a $100(1-\alpha)\%$ confidence ellipsoid for μ.

Since $\mathbf{\Sigma}$ is unknown, we can replace $\mathbf{\Sigma}$ by $\mathbf{S}$. Hotelling's T^2

$$T^2 = n(\overline{l} - \mu)'\mathbf{S}^{-1}(\overline{l} - \mu)$$

has distribution $4(n-1)F_{4,n-4}/(n-4)$, where $F_{p,n-p}$ denotes an F distribution with p and $(n-p)$ degrees of freedom. Therefore, the ellipsoid $n(\overline{l} - \mu)'\mathbf{S}^{-1}(\overline{l} - \mu) = 4(n-1)F_{\alpha;4,n-4}/(n-4)$ provides a $100(1-\alpha)\%$ confidence ellipsoid for μ. Moreover, using properties of the Hotelling T^2, the individual confidence intervals for the element of μ can be given as following:

$$\overline{l_j} \pm \sqrt{\frac{4(n-1)}{(n-4)} F_{\alpha;4,(n-4)}} \sqrt{s_j^2/n}$$

where s_j^2 is the sample variance of the j^{th} parameter [16].

Figure 2 to Figure 6 are some examples of the experimental results. All these figures are organized the same way: the top left shows the picture of the sherd examined. The top right shows the function of estimated distance to the axis shown on the potsherd itself. The points with the same distance are marked with the same color. We use several colors to separate different distances. We can see the parallel circles clearly from the plot. The second row left shows the estimated profile curve (solid line) and its 95% confidence interval (dash lines) using 500 independent bootstrap samples. The second row right shows the standard deviation of the estimated profile curve as a function of height along the axis of revolution, based on axis estimates using 500 independent bootstrap samples. The bottom left shows the covariance matrix of the 4 parameters of axis using 500 independent bootstrap samples. The bottom right shows the 4 eigenvectors of the covariance matrix with eigenvalues in decreasing order. The numbers on top of the plot are the corresponding eigenvalues for the eigenvectors. For most sherds, the results are good. The confidence interval for each parameter of the axis is small, hence the error of the estimated axis is small. The error of the estimated profile curve is also small comparing to the radius of the parallel circles of the pot.

From the bootstrap results, we can also see the uncertainty when deciding each parameter of the axis of revolution. For sherd p1135, all the eigenvalues of the covariance matrix are very small, and therefore, the algorithm is quite certain about where the axis should be. This may because this sherd looks like a bottom, and has a clear preferred orientation. For all other sherds, there always one big eigenvalue, whose corresponding eigenvector is very nearly a linear combination of the parameters a and b. But the coefficients of the linear combination vary for different sherds due to different geometry properties. Therefore, the algorithm is most uncertain in one direction and the confidence ellipsoid is "cigar-shaped". Note that a is determined by the angle between that sherd and the axis at the centroid, b is x-distance between an estimation and the mean estimation. Therefore, the one direction which the algorithm is most uncertain is related to the angle and the distance between the sherd and the axis. But the algorithm is quite certain about c and d. This is reasonable because the sherd is somehow symmetric about yz-plane.

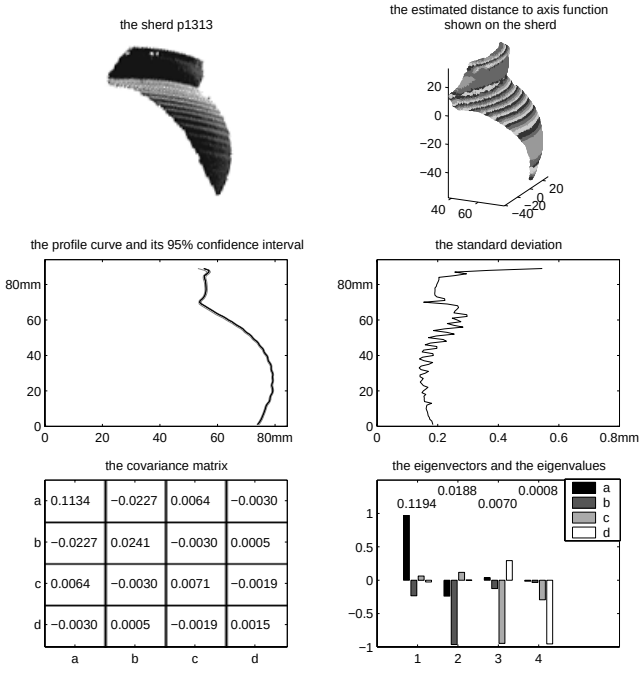

Figure 2. experimental results for sherd p1313

The sherd p654 and p642 are similar in shape, texture, thickness and clay type, so we think they are from the same kind of pot. Figure 7 is a comparison of the two profile curves. The two profile curves are very similar.Though they don't match exactly, there is an small turning angle difference, the confidence interval for sherd p642 falls into the confidence interval for sherd p654, so the error is under control. From Figure 3 and Figure 4, we can see that the pattern of error of the two sherd is also similar, yet the esti-

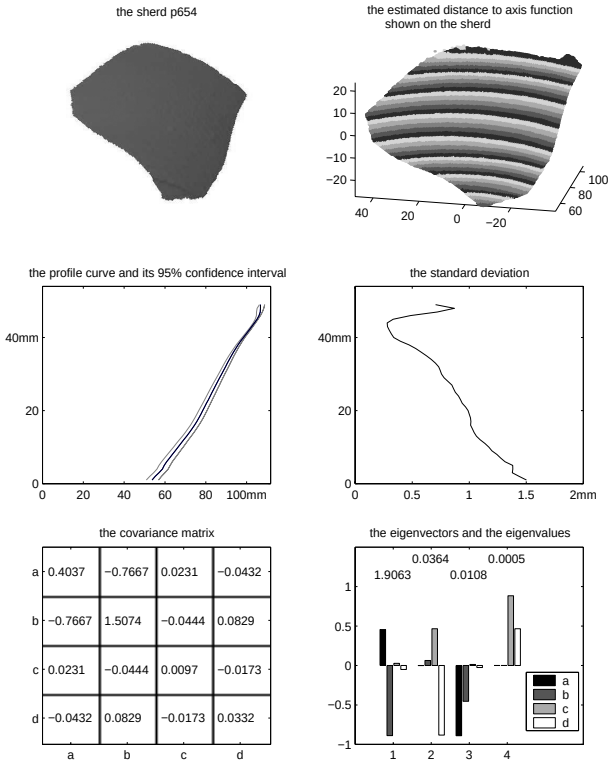

Figure 3. experimental results for sherd p654

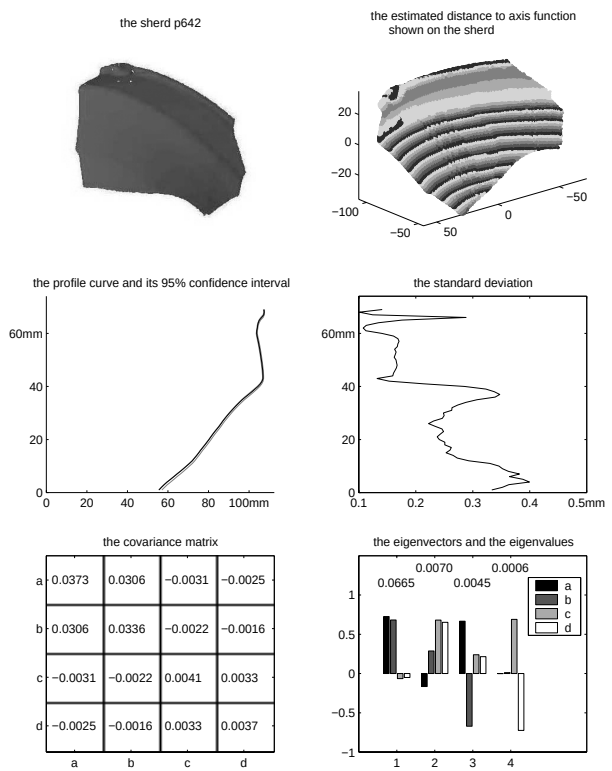

Figure 4. experimental results for sherd p642

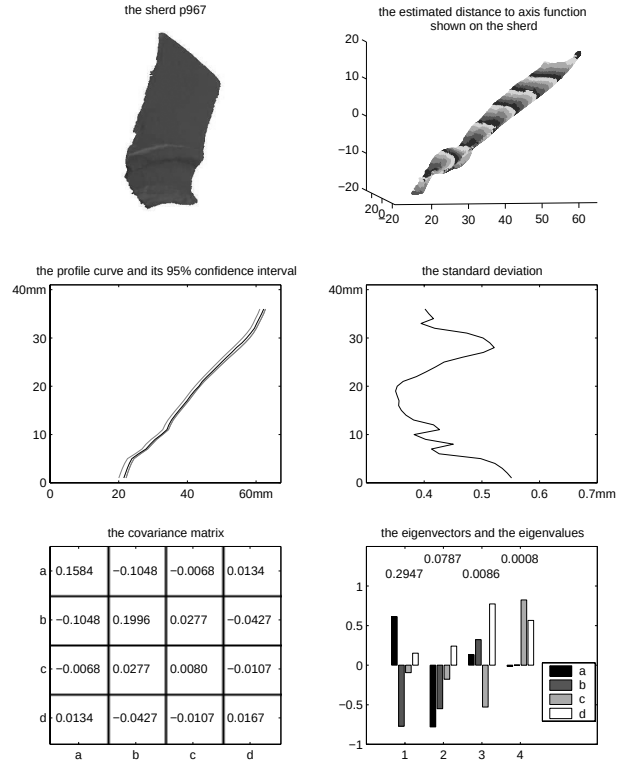

Figure 5. experimental results for sherd p967

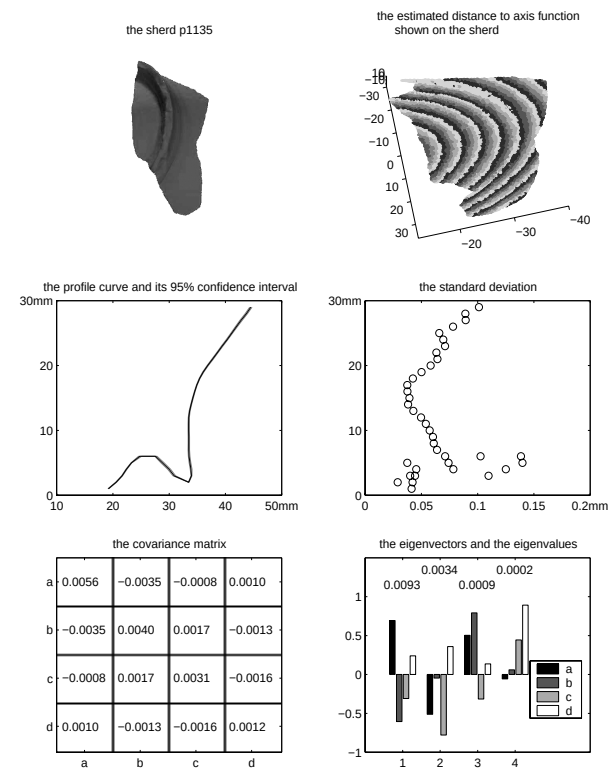

Figure 6. experimental results for sherd p1135

mate for sherd p642 is much better than that for sherd p654. This is consistent to that the sherd p642 is a little bigger and it has a ridge, which gives a clue where the axis should be , hence it contains more information. The similarity shows that our methods are reliable in this sense.

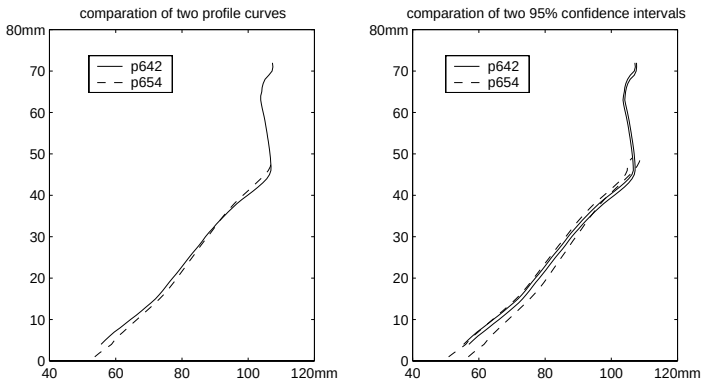

Figure 7. The left shows the profile curves of p642 and p654. The right shows the 95% confidence intervals for the profile curves of p642 and p654. We can see that the two profile curves are very similar and the confidence interval for p642 falls into the confidence interval for p654.

6 Conclusion

In this paper, we presented a novel method to extract symmetric axis and associated profile curve of a surface patch from a surface of revolution. This method is robust to noisy data and computationally efficient. The experimental results are very promising. We also give the confidence bounds for the axis and the profile curve using bootstrap method, which are essential for using the estimations for the full assembly of a pot. This method may face difficulties if the surface is very close to a sphere. This is because for any point on a sphere, the corresponding sphere of curvature is the sphere itself, hence we will get only one point (the center of the sphere). If that happens, any diameter could be claimed as the symmetric axis.

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