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31
Elastica and Computer Vision

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31.1 Introduction

I want to discuss the problem from differential geometry of describing those plane curves C which minimize the integral

$$\int_C (\alpha\kappa^2 + \beta)ds. \tag{1}$$

Here α and β are constants, κ is the curvature of C, ds the arc length and, to make the fewest boundary conditions, we mean minimizing for infinitesimal variations of C on a compact set not containing the endpoints of C. Alternately, one may minimize

$$\int_C \kappa^2 ds$$

over variations of C which preserve the total length.

This problem has a very long history: it was first proposed and solved by Euler [4] in 1744 in the appendix "*De curvis elastica*" to his monumental work "*Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*". These curves have been called "elastica" ever since then, at least by those who knew of Euler's work. Curiously for such an elementary problem, Euler's solution is not even mentioned in any textbooks on variational calculus and therefore it has been rediscovered innumerable times by people like me who needed a description of these curves. I want to thank Garrett Birkhoff for steering me first to Love's treatist [11] on elasticity, thence to Born's prize-winning Ph.D. thesis on these curves and finally to Euler. I guess that the reason these beautiful curves have remained so obscure is a) because they can only be described by non-elementary functions and b) because they were developed chiefly by applied mathematicians and pure mathematicians never looked at this literature.

Among pure mathematicians, a recent but surely not the only rediscovery is due Bryant and Griffiths [3]. Among applied mathematicians, they were proposed as potential interpolating curves by Birkhoff and collaborators and called non-linear splines (cf. [2], p. 171, and [1]). This application has been pursued by Golumb and Jerome (cf. for instance [5] and the references cited there). My own interest in them was motivated by computer vision problems and I discovered later not only that I was redoing Euler but also an earlier computer vision scientist, [8] who had similar ideas. Maybe this paper will serve to forestall yet more rediscoveries of this 200 year old topic!

In this paper, in §§32.1-32.3, I want to present the problem from the point of view of Bayerian computer vision, which leads to elastica as a maximum likelihood reconstruction of occluded edges. Then, in §§32.4-32.5, I want to present what I think are new formulae for elastica using theta functions. The manipulation of elliptic functions has remained something of a black art ever since Euler and I believe that in spite of sporadic efforts for 200 years, my formulae for them are in some ways the simplest, yet don't seem to appear in the literature.

31.2 Edges in Computer Vision

From one point of view, the central problem of computer vision is this: One is given a function $I(x,y)$ representing the light intensity produced by a 3D world and striking a lens from direction x, y. One seeks to compute a second function $d(x,y)$, the distance from the lens to the nearest opaque surface in direction x, y, hence reconstructing the 3D geometry. A simplified but still useful idealization of the problem assumes that the 3D world consists of a small number of opaque objects with smooth boundaries with smoothly varying albedo illuminated by a smoothly varying light source. Then both the intensity image $I(x,y)$ and the range image $d(x,y)$ will be piecewise smooth functions, with discontinuities along "edges" Γ in their domain R. These edges are the directions where a more distant object is just visible along a ray from the lens grazing a closer object (see Figure 1)[1] Thus discontinuities in I are the first major clue to the 3D-geometry.

A second clue are the singularities of Γ. Suppose for instance that there are 3 objects: a nearest one A, a further one B and a farthest object, or background C. Suppose the edge of B, viewed against C, disappears behind A. Then the locus of visible edges Γ looks like the letter "T": the edges of A and B are smooth curves but the edge of B ends at a point on the edge of A. See Figure 2 which depicts schematically 3 blades of grass against a distant background. In fact, from a generic viewpoint, our simplified world produces an edge Γ with only 2 types of singularities: these so-called *T-junctions* and *cuspidal crack tips* which arise when a smoth object develops a crease. The latter is the "elementary catastrophe" given in suitable coorindates by

[1] One may think of Γ as the visible part of the branch locuses of the maps of surfaces $p : \partial A_i \to R$, if A_i are 3D objects and $p : U \to R$, $U \subset \mathbf{R}^3$, maps the points of the 3D-world to rays through the lens.

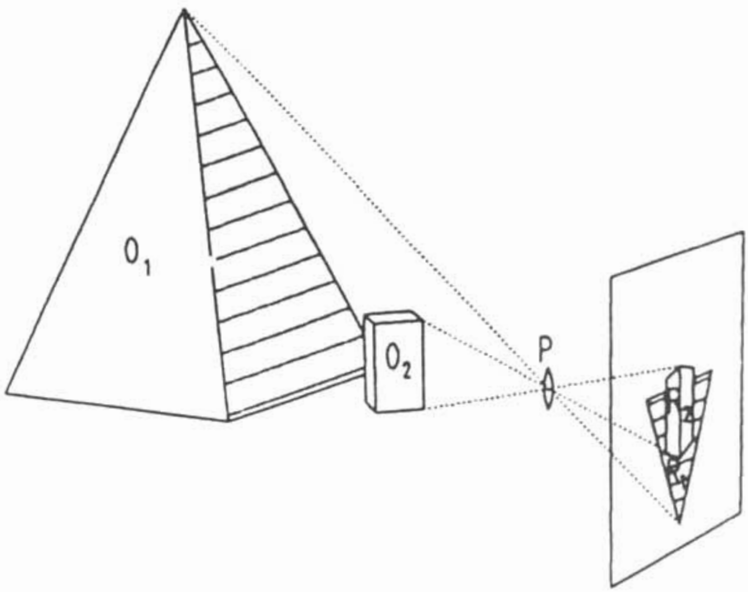

FIGURE 1. An Image of a 3D Scene

$$d(x, y) = (\text{smallest solution } z \text{ of } z^3 - xz - y = 0) + d(0, 0).$$

This $d(x, y)$ has discontinuities along the curve $27y^2 = 4x^3$, $y \geq 0$ ending in a cusp at $x = y = 0$ (see Figure 3). Note that although Γ comes to an endpoint, a crack tip, at which it has a cusp, it is the visible part of a smooth 3-dimensional curve $\Gamma^* : x = 3z^2, y = z^3$ and only gets a singularity because Γ^* turns directly away from the lens at $z = 0$ and afterwards becomes invisible in the crease.

A third clue to 3D geometry comes from the fact that an edge of an object A which disappears at a point P behind object B may often reappear further on at another point Q. This is seen in Figure 2 where the edges of the blades of grass reappear and its signature is a pair of T-junctions in Γ which nearly "match up", i.e., 2 T's as in Figure 4 where the dotted line is the inferred invisible part of the edge of object A behind B. Unlike the previous clues, this one cannot be "read off" from Γ because the 2 pieces of Γ that end at the T's are unlikely to line up exactly. What we need is to measure the relative likelihood of two disappearing edges to be matched up by an invisible edge in terms of how long and how curved an invisible edge is needed to link them. To measure this probability requires a stochastic process to model the space of all possible edges. This is an approach to computer vision that has been pioneered by Grenander and

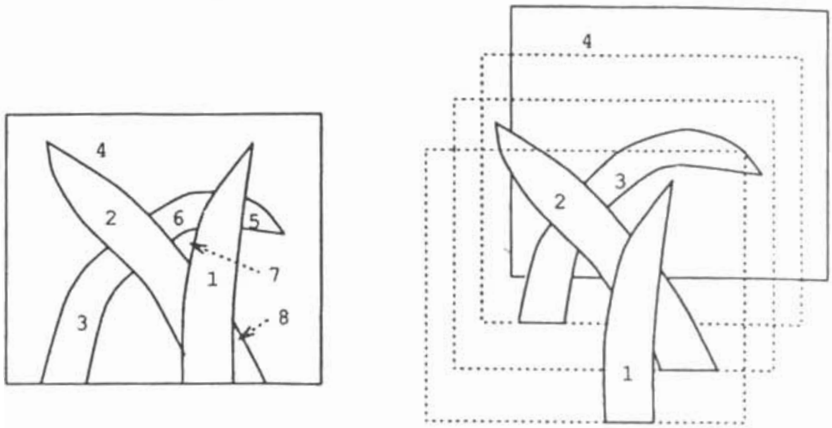

FIGURE 2. Blades of Grass

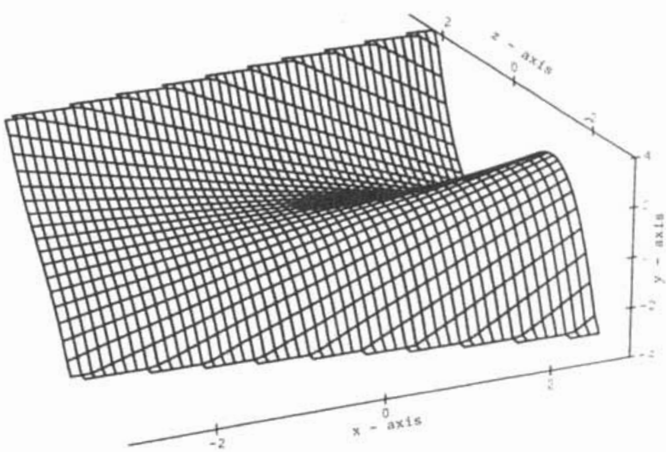

FIGURE 3. $z^3 - xz - y = 0$

FIGURE 4. An Illusion of Kanizsa

other statisticians and probabilists (see [7] and [6]).

It may be worthwhile to try to convince the skeptic of the psychological reality of these probabilistic reconstructions of 3D scenes. Some very striking "optical illusions" are due to the fact that reconstructions of the above type are made automatically at an early stage in visual processing so strongly later stages cannot reverse them if they are absurd on the basis of one's knowledge of plausible worlds. An example is the man and woman entwined in the fence in Figure 4. This and many other beautiful demonstrations are due to Kanizsa [10].

31.3 A Brownian Prior for Edges

What sort of stochastic process is a plausible candidate for modelling the relative likelihood of different edges appearing in a scene of the world? Our edges are to be continuous and almost everywhere differentiable so that, when occluded in part, they will tend to reappear with approximately the same tangent line. The simplest way to do this is to allow curvature $\kappa(s)$, as a function of arc length, to be white noise $n(s)$, so that once integrated, the tangent direction $\theta(s)$ is a Brownian motion $W(s)$. Then the positiion is given by:

$$\begin{aligned} x(s) &= \int_0^s \cos\theta(t)dt + x(0) \\ y(s) &= \int_0^s \sin\theta(t)dt + y(0). \end{aligned} \tag{2}$$

Let's also assume that the total length of the curves is exponentially distributed.

Choosing a very large N, we may approximate this by a discrete time system in which the curve Γ is a polygon of length ℓ:

$$\Gamma = \bigcup_{i=0}^{N} \overline{P_i P_{i+1}}$$

whose sides have length ℓ/N:

$$P_{i+1} = P_i + \frac{\ell}{N}(\cos\theta_i, \sin\theta_i), \quad 0 \leq i < N$$

and the θ_i are discrete Brownian motion scaled down by $\frac{\ell}{N}$:

$$\theta_{i+1} = \theta_i + \sqrt{\frac{\ell}{N}} n_{i+1}, \quad 0 \leq i < N-1,$$

n_i independent normal random variables with mean 0, standard deviation σ and the random variable ℓ has exponential distribution $\lambda \cdot e^{-\lambda\ell} d\ell$. Then

$$\begin{aligned} Pr(\Gamma) &= \frac{\lambda}{(\sqrt{2\pi \cdot \sigma})^{N-1}} e^{-\Sigma n_i^2/2\sigma^2 - \lambda\ell} \cdot dn_1 \cdots dn_{N-1} d\ell \\ &= \text{cnst.} e^{-\Sigma \frac{\ell}{N} \cdot \left(\frac{\theta_{i+1}-\theta_i}{\ell/N}\right)^2 / 2\sigma^2 - \lambda\ell} \cdot dn_1 \cdots dn_{N-1} d\ell \end{aligned} \tag{3}$$

which is a discrete approximation to

$$e^{-\int(\alpha\kappa^2+\beta)ds}$$

if $\alpha = \frac{1}{2\sigma^2}, \beta = \lambda$. Thus we see that elastica have the interpetation of being the *mode* of the probability distribution underlying this stochastic process restricted to curves with prescribed boundary behavior, e.g., the maximum likelihood curve with which to reconstruct hidden contours. Some typical elastica are shown in Figure 5.

For applications to computer vision, another invariant of this stochastic process is even more important. When you find some set of T-junctions in an image, you must decide which pairs are most likely to represent the disappearance and reappearance of a single contour behind a nearer object. For this we want the generating function of the stochastic process (2):

$$p(x,y,\theta,t)\Delta x \Delta y \Delta\theta = Pr\left[\begin{array}{c|c} x \leq x(t) \leq x + \Delta x & x(0) = 0 \\ y \leq y(t) \leq y + \Delta y & y(0) = 0 \\ \theta \leq \theta(t) \leq \theta + \Delta\theta & \theta(0) = 0 \end{array}\right].$$

The forwards and backwards diffusion equations for p are:

$$\begin{aligned} \frac{\partial p}{\partial t} &= \sigma \cdot \frac{\partial^2 p}{\partial \theta^2} - \cos\theta \cdot \frac{\partial p}{\partial x} - \sin\theta \cdot \frac{\partial p}{\partial y} \quad \text{and} \\ \frac{\partial p}{\partial t} &= \sigma\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \frac{\partial}{\partial \theta}\right)^2 p - \frac{\partial p}{\partial x} \end{aligned} \tag{4}$$

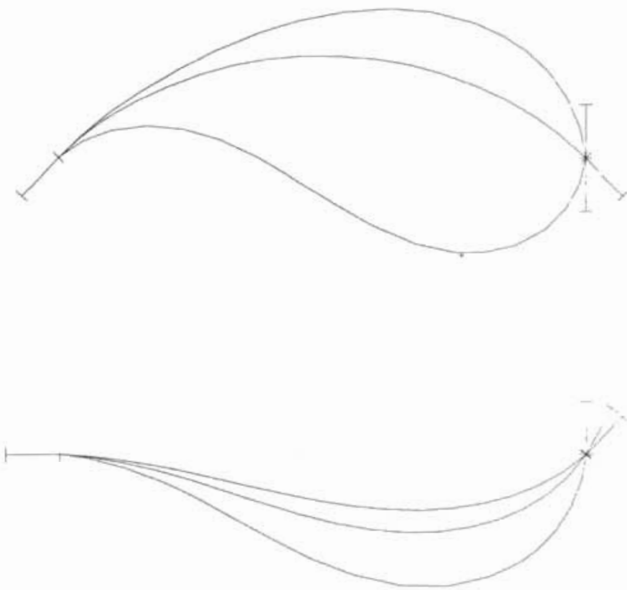

FIGURE 5. Some Elastica

where σ measures the amount of noise in the curvature. For each t, $p(x, y, \theta, t)$ is a probability distribution on $\mathbf{R}^2 \times S^1$ in the variables x, y, θ. For $t = 0$, it starts at the delta function

$$\delta(x)\delta(y)\delta(\theta)$$

and then spreads out. It spreads in each of 3 directions in totally different ways – it is transported in the $(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y})$-direction, it diffuses in the θ-direction and it fills x, y, θ-space only as a 2nd order effect, via the non-integrability of the $(\sin\theta \cdot dx - \cos\theta \cdot dy)$-foliation. We can also integrate the backwards equation with respect to θ and get a diffusion equation for the marginal probability $\bar{p}(x, y)$:

$$\frac{\partial \bar{p}}{\partial t} = \sigma \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - \frac{\partial}{\partial \theta} \right)^2 \bar{p} - \frac{\partial \bar{p}}{\partial x}.$$

I have looked for an explicit formula for p and $\bar{p}$ but in vain. Still, on the basis of the results of §32.2, I would conjecture that a formula exists, in terms of elliptic functions of some kind.

The function we want for computer vision is the probability that an occluded curve reappears at a particular point (x, y, θ) before dying its exponential death, i.e.,

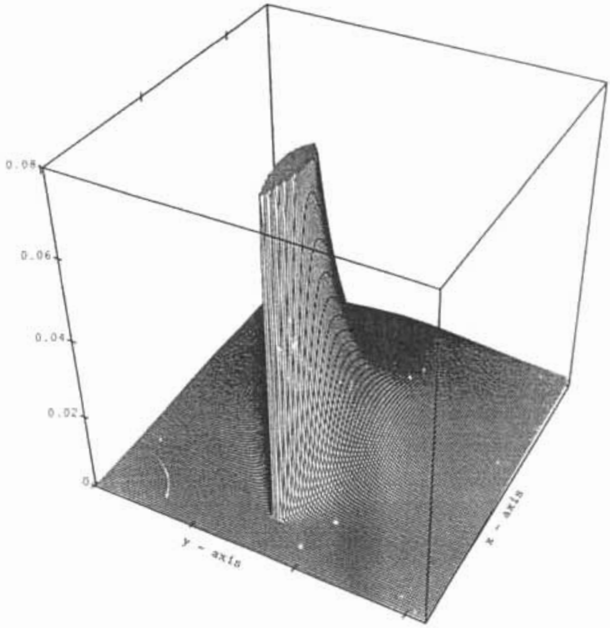

FIGURE 6. Graph of q(x,y)

$$q(x, y, \theta) = \int_0^\infty e^{-\lambda t} \cdot p(x, y, \theta, t) dt$$

which solves

$$\lambda q = \sigma \cdot \frac{\partial^2 q}{\partial \theta^2} - \cos\theta \cdot \frac{\partial q}{\partial x} - \sin\theta \cdot \frac{\partial q}{\partial y}.$$

We have integrated this equation numerically. The marginal $\bar{q}(x, y)$ is shown in Figure 6 and the conditional $q(4, y, \theta)$ is shown in Figure 7, both with $\lambda = 0.17$ (half-life of 4 units of distance) and $\sigma = \pi^2/128$ (in time 1, standard deviation of θ is $\pi/8$). Note the interesting singularity of $\bar{q}$ at $x = y = 0$ where its true value is infinite. The peak of $q(4, y, \theta)$ corresponds to a horizontal curve following the x-axis, i.e., $y = \theta = 0$, and as y increases or decreases, the corresponding most probable θ also increases or decreases.

31.4 Alternate Priors

The stochastic process (2) has a major failing as a model of edges in images: images are 2D projections of 3D scenes, so edges are 2D projections of 3D curves. In particular, this means that edges have near singularities when

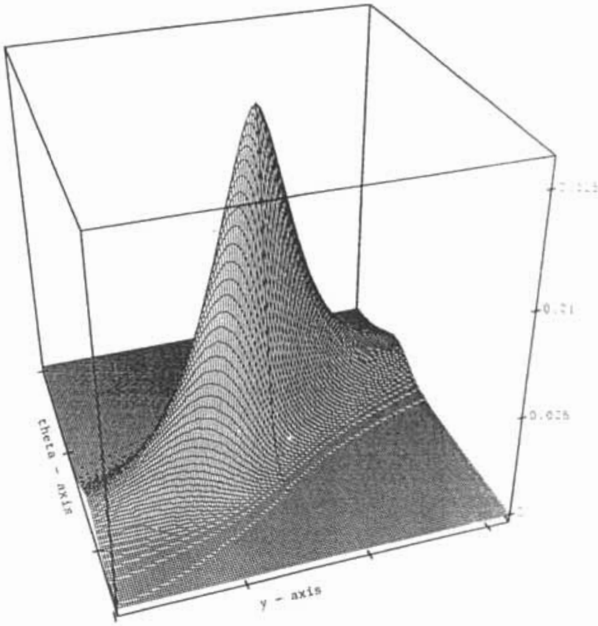

FIGURE 7. Graph of q(4,y, θ)

the 3D curve moves approximately towards or away from the imaging system. An alternate model is to generate sample paths in $\mathbf{R}^3$ by integrating Brownian motion on the sphere S^2 of 3D unit tangent vectors. This leads to:

$$\begin{aligned}
x(s) &= \int_0^s \cos\theta(t)\cdot\sin\phi(t)\cdot dt \\
y(s) &= \int_0^s \sin\theta(s)\cdot\sin\phi(t)\cdot dt \\
\phi(t) &= B_1(t) + \int_0^t \frac{ds}{\tan\theta(s)}, \qquad 0 < \phi < \pi \qquad \text{almost surely} \\
\theta(t) &= B_2\left(\int_0^t \frac{ds}{\sin^2\phi(s)}\right), \qquad 0 \le \theta \le 2\pi
\end{aligned} \tag{5}$$

where ϕ and θ are colatitude and longitude on S^2 and B_1 and B_2 are independent Brownian motions. The point is that to get Brownian motion on S^2, θ must be rescaled to go faster near the north and south poles and ϕ must have an extra drift term pushing it towards the equator (cf. [12], §4.3). This is analytically a bit of a mess, but a simple approximation to it is given by the Uhlenbeck process: this models a particle in the plane subject to white noise random forces plus friction pulling its velocity back to 0. The equations are:

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \alpha\cdot n_1(t) - \beta\frac{dx}{dt} \\
\frac{d^2y}{dt^2} &= \alpha\cdot n_2(t) - \beta\frac{dy}{dt}
\end{aligned} \tag{6}$$

where n_1 and n_2 are independent white noise. This process has the advantage of being a Gaussian process, hence its generating function $p(x, y, \dot{x}, \dot{y}, t)$ can be readily calculated and applied to image analysis. If an exponential death is included, the maximum likelihood curves which reconstruct occluded edges for this process minimize integrals of the form:

$$\int_C \left\{ \alpha \left[\left(\frac{d^2x}{dt^2} \right)^2 + \left(\frac{d^2y}{dt^2} \right)^2 \right] + \beta \left[\left(\frac{dx}{dt} \right)^2 + z \left(\frac{dy}{dt} \right)^2 \right] + \gamma \right\} dt. \quad (7)$$

If $\beta = \gamma = 0$, there are cubic splines and otherwise they are transcendental curves generalizing exponential spirals of the form

$$\begin{aligned} x(s) &= a_1 e^{\lambda_1 s} + a_2 e^{\lambda_2 s} + a_3 e^{-\lambda_2 s} + a_4 e^{-\lambda_1 s} \\ y(s) &= b_1 e^{\lambda s} + b_2 e^{\lambda_2 s} + b_3 e^{-\lambda_2 s} + b_4 e^{-\lambda_1 s}, \qquad \lambda_1 > \lambda_2 > 0 \end{aligned}$$

or

$$\begin{aligned} x(s) &= a_1 e^{\lambda s} \sin(\mu s + a_2) + a_3 e^{-\lambda s} \sin(\mu s + a_4) \\ y(s) &= b_1 e^{\lambda s} \sin(\mu s + b_2) + b_3 e^{-\lambda s} \sin(\mu s + b_4). \end{aligned}$$

Curves which minimize the above integral (7) *plus* an "external force" term $\int_C I(x, y) dt$ have been used in computer vision under the name of "snakes" [9] to reconstruct edges which are mostly visible but obscured by noise, gaps, blur, etc.

In Figure 7, we show some sample paths from all 3 priors for comparison.

31.5 The Differential Equation of Elastica

We would like to give an explicit formula for elastica in terms of theta functions. The first step is to characterize them by a differential equation for their curvature $\kappa(s)$ as a function of arc length. Using vector notation, start with the curve itself $\vec{x}(s)$ in arc-length parametrization, its tangent and normal vectors are

$$\begin{aligned} \vec{t}(s) &= \dot{\vec{x}}(s) \\ \vec{n}(s) &= \vec{t}(s)^{\perp}. \end{aligned}$$

Then

$$\dot{\vec{t}} = \kappa \cdot \vec{n}, \quad \dot{\vec{n}} = -\kappa \cdot \vec{t}.$$

Now consider an infinitesimal deformation:

$$(\text{new}\vec{x}) = \vec{x}(s) + \delta(s) \cdot \vec{n}(s).$$

Then

$$\begin{aligned} (\text{new}\vec{x})^{\cdot} &= \vec{t} + \delta' \cdot \vec{n} - \delta\kappa \cdot \vec{t} \\ &= (1 - \delta\kappa) \cdot (\vec{t} + \delta' \cdot \vec{n}) \end{aligned}$$

(since $\delta \cdot \delta'$ is effectively zero). Then

$$\begin{aligned}(\text{new}ds) &= (1-\delta\kappa)ds\\ (\text{new}\vec{t}) &= \vec{t}+\delta'\cdot\vec{n}\\ (\text{new}\vec{n}) &= \vec{n}-\delta'\cdot\vec{t}.\end{aligned}$$

We want the deformation to preserve arc length, i.e.,

$$\int(\text{new}ds) = \int ds$$

or

$$\int_C \delta\kappa \cdot ds = 0.$$

Now

$$(\text{new}\kappa)\cdot(\text{new}\vec{n}) = \frac{d(\text{new}\vec{t})}{\text{new}ds}$$

or

$$\begin{aligned}(\text{new}\kappa)(\vec{n}-\delta'\cdot\vec{t}) &= \tfrac{1}{1-\delta\kappa}\tfrac{d}{ds}(\vec{t}+\delta'\cdot\vec{n})\\ &= (1+\delta\kappa)(\kappa\vec{n}+\delta''\cdot\vec{n}-\delta'\kappa\cdot\vec{t})\\ &= (\kappa+\delta''+\delta\kappa^2)(\vec{n}-\delta'\vec{t}).\end{aligned}$$

Therefore

$$\text{new}\kappa = \kappa+\delta''+\delta\kappa^2.$$

Thus

$$\begin{aligned}\text{new}(\textstyle\int\kappa^2 ds) &= \textstyle\int(\kappa+\delta''+\kappa^2\delta)^2\cdot(1-\delta\kappa)ds\\ &= \textstyle\int\kappa^2 ds+\int(2\kappa\delta''+\kappa^3\delta)ds\\ &= \textstyle\int\kappa^2 ds+\int(2\kappa''+\kappa^3)\delta ds\\ &\quad \text{(integration by parts).}\end{aligned}$$

If C is an elastica, $\int\kappa^2 ds$ is a minimum for all deformations which preserve $\int ds$, hence for some constant R:

$$2\kappa''+\kappa^3 \equiv \frac{R}{2}\kappa. \tag{8}$$

Conversely, this differential equation implies that C is a critical point for $\int(\kappa^2+\frac{R}{2})ds$. We won't consider the question of which critical points are minima and which are not, but go on to write down the solutions of (8).

31.6 Solving for Elastica

Multiplying (8) by κ' and integrating, we get

$$4(\frac{d\kappa}{ds})^2+\kappa^4-R\kappa^2-S=0$$

for some constant S, so that s is an elliptic integral in κ

$$s = \int \frac{2d\kappa}{\sqrt{-\kappa^4 + R\kappa^2 + S}}.$$

Now let $\mathcal{E}$ be the real elliptic curve

$$v^2 = -u^4 + Ru^2 + S.$$

What this means is that the map

$$s \longmapsto (\kappa(s), 2\kappa'(s))$$

identifies the elastica with the real points $\mathcal{E}_{\mathbf{R}}$ of $\mathcal{E}$. Under this map, ds becomes the differential $2du/v$, the differential on $\mathcal{E}$, unique up to scalars, without zeroes or poles. If Λ is the lattice of periods of ds, then, as usual, $\mathcal{E}$ is uniformized

$$\begin{aligned} \mathcal{E} &\cong \mathbf{C}/\Lambda \\ \mathcal{E}_{\mathbf{R}} &\cong \mathbf{R}/\mathbf{R}\cap\Lambda \end{aligned}$$

and arc length s on the elastica is just the additive coordinate on the elliptic curve, the variable of the universal cover $\mathbf{C}$. The curve $\mathcal{E}$ has 2 conjugate imaginary points P and $\bar{P}$ at ∞ and the function $\kappa(s)$ on the elastica is now the restriction to $\mathcal{E}_{\mathbf{R}}$ of the meromorphic function u on $\mathcal{E}$ with 2 simple poles at P and $\bar{P}$. What we want to do is to integrate κ *twice* to get the x and y coordinates of the elastica as functions on $\mathcal{E}_{\mathbf{R}}$. To do this, combine them into the complex-valued function

$$z(s) = x(s) + \sqrt{-1}\, y(s)$$

on $\mathcal{E}_{\mathbf{R}}$. We must solve:

$$\begin{aligned} \frac{dz}{ds} &= e^{i\theta(s)} \\ \frac{d\theta}{ds} &= u(s), \end{aligned}$$

which determine z up to rotation and translation.

Note on $\mathcal{E}$, the identities

$$\begin{aligned} [v + i(u^2 - \tfrac{R}{2})] \cdot [v - i(u^2 - \tfrac{R}{2})] &= v^2 + (u^4 - Ru^2 + \tfrac{R^2}{4}) \\ &= S + \tfrac{R^2}{4}. \end{aligned}$$

This shows that $v \pm i(u^2 - \frac{R}{2})$ are functions on $\mathcal{E}$ with no zeroes or poles at finite values of (u, v). Examining their behavior at the infinite points $P, \bar{P}$, we compute their divisors (possibly interchanging $P, \bar{P}$):

$$\begin{aligned} (v + i(u^2 - \tfrac{R}{2})) &= 2P - 2\bar{P} \\ (v - i(u^2 - \tfrac{R}{2})) &= 2\bar{P} - 2P. \end{aligned}$$

FIGURE 8. Sample Paths forom the 3 priors

i.e., $P - \bar{P}$ is a divisor class of order 2. Now let $s_0 \in \mathbf{C}$ map to P, so $\bar{s}_0$ maps to $\bar{P}$. Then $s_0 - \bar{s}_0 \in \frac{1}{2}\Lambda$ expresses this fact on $\mathbf{C}$.

Let ω_1 be a generator of $\Lambda \cap \mathbf{R}$ and $\{\omega_1, \omega_2\}$ a basis of Λ. Then the theta function:

$$\vartheta_\Lambda(s) = \sum_{n \in \mathbf{Z}} e^{\pi i n^2 (\frac{\omega_2}{\omega_1}) + 2\pi i n (\frac{s}{\omega_1})} \tag{9}$$

is a very rapidly converging holomorphic function on $\mathbf{C}$ such that:

a) $\vartheta_\Lambda(s + \omega_1) = \vartheta_\Lambda(s)$

b) $\vartheta_\Lambda(s + \omega_2) = e^{-2\pi i \frac{s}{\omega_1} - \pi i \frac{\omega_2}{\omega_1}} \vartheta_\Lambda(s)$

c) ϑ_Λ has simple zeroes at the points $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 + \Lambda$ and no other zeroes.

(Cf. [13], §1.4).

Let

$$F(s) = c\left(\frac{d}{ds}\log\vartheta_\Lambda(\mathrm{s} - \mathrm{s}_0 + \frac{\omega_1 + \omega_2}{2})\right) - as \tag{10}$$

(a and c to be fixed shortly). The a), b), and c) imply:

a′) $F(s + \omega_1) = F(s)$

b$'$) $F(s+\omega_2) = F(s) - \frac{2\pi i}{\omega_1}c - a\omega_2$

c$'$) F is meromorphic with simple poles at the points $s_0 + \Lambda$ and no other points.

It follows that $\frac{dF}{ds}$ is periodic mod Λ, hence is a meromorphic function on $\mathcal{E}$. It has a *double* pole at P and now choose the ratio a/c so that

$$\frac{dF}{ds}(\bar{P}) = 0,$$

i.e.,

$$\frac{a}{c} = \frac{d^2}{ds^2}\log\vartheta_\Lambda(\bar{\mathrm{s}}_0 - \mathrm{s}_0 + \frac{\omega_1+\omega_2}{2}).$$

Then $\frac{\partial F}{\partial s}$ is zero at $\bar{P}$ and has one remaining zero. Since $2P - \bar{P} \equiv \bar{P}$, this last zero is $\bar{P}$ too, and

$$\frac{dF}{ds} = d\cdot(v(s) + iu(s)^2 - \frac{iR}{2}), \qquad \text{some constant } d.$$

Thus, if $s \in \mathbf{R}, u(s), v(s) \in \mathbf{R}$, so

$$\begin{aligned} \overline{\left(\tfrac{dF}{ds}\right)} &= d\cdot(v(s) - iu(s)^2 + \tfrac{iR}{2}) \\ \therefore \left|\tfrac{dF}{ds}\right|^2 &= |d|^2\cdot(S + \tfrac{R^2}{4}), \qquad \textit{a constant.} \end{aligned}$$

Finally choose a, c so that

$$\left|\frac{dF}{ds}\right| \equiv 1 \qquad \text{on } \mathbf{R}.$$

It now follows that

$$\begin{aligned} \mathbf{R} &\longrightarrow \mathbf{C} \\ s &\longmapsto F(s) \end{aligned}$$

is an arc length parametrization of a plane curve. Finally write

$$\frac{dF}{ds}(s) = e^{i\theta(s)} \qquad \text{or } \theta(s) = \frac{1}{i}\log\frac{\mathrm{dF}}{\mathrm{ds}}$$

and

$$\frac{d\theta}{ds}(s) = \kappa(s) \qquad \text{or } \kappa(s) = \frac{1}{i}\cdot\frac{d^2F/ds^2}{dF/ds}.$$

κ is again a meromorphic function on $\mathcal{E}$, now with simple poles at P and $\bar{P}$ only and using $ds = 2du/v$

$$\begin{aligned}
\frac{dF}{ds} &= d(v + iu^2 - \tfrac{iR}{2}) \\
\frac{d^2F}{ds^2} &= d(\tfrac{dv}{ds} + 2iu\tfrac{du}{ds}) \\
&= d(\tfrac{1}{2}v\tfrac{dv}{du} + 2iu \cdot \tfrac{v}{2}) \\
&= d(\tfrac{1}{4}\tfrac{d}{du}(v^2) + iuv) \\
&= d \cdot (\tfrac{1}{4}(-4u^3 + 2Ru) + iuv) \\
&= iu \cdot d \cdot (iu^2 - \tfrac{iR}{2} + v) \\
&= iu \cdot \tfrac{dF}{ds}.
\end{aligned}$$

Thus $\kappa(s) = u$ and this proves the final result:

Theorem *Elastica are all given in their arc length parametrization by maps:*

$$\begin{aligned}
&\mathbf{R} \longrightarrow \mathbf{C} \\
&s \longmapsto c \cdot \tfrac{d}{ds}\log\vartheta_\Lambda(\mathrm{s} - \eta) - \mathrm{a} \cdot \mathrm{s}
\end{aligned}$$

where $\Lambda \subset \mathbf{C}$ is a lattice such that $\bar{\Lambda} = \Lambda$, η satisfies $\eta - \bar{\eta} \in \frac{1}{2}\Lambda$, a, c are suitable constants defined above, and ϑ_Λ is the theta function defined in (9).

One can continue the analysis and classify the pairs (Λ, η) which give elastica. It turns out that 2 types suffice:

$$\Lambda = \mathbf{Z} + \mathbf{Z} \cdot it, \qquad \eta = -\frac{it}{4},$$

and

$$\Lambda = \mathbf{Z} + \mathbf{Z} \cdot \frac{it+1}{2}, \qquad \eta = 0.$$

31.7 References

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