WHAT MATHEMATICS SERVES THE MAJORITY OF 21st CENTURY AMERICAN STUDENTS[1]?

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This article has several related goals: to argue for an increased emphasis on applied mathematics in K-12 schools, to argue that the abstract ideas of mathematics can be better absorbed by students *after* that idea has been seen in real world contexts, and to suggest how this balanced approach can be implemented in schools. We hope the article will stimulate an open discussion around what types of middle and high school mathematics experiences best address the needs of all our students, not just those going on to STEM careers. We are not arguing for dropping abstract ideas from the curriculum but for re-examining the *balance* and *sequencing* of pure and applied mathematical topics in the K-12 classroom.

Most of the mathematics students currently encounter is *merely* abstract and if connected to the world in which they live, the connection is weak. This disconnect between school mathematics and the real world leaves many students unprepared to solve many of the mathematical problems they encounter in their future career and home lives. Instead, we believe that K-12 mathematics experiences should be based on equal emphasis of applied and pure mathematics, aiding student transfer of the mathematical knowledge and habits of mind they learn in school into adulthood. Further, this balance of pure and applied topics should be most often ordered, with new mathematical concepts introduced first in an applied setting after which abstract understandings can be built up from these more tangible topics. This is an instance of Guershon Harel's Necessity Principle: *Students are most likely to learn when they see a need for what we intend to teach them…*[2]. And finally, the immediate appeal of solving real life problems can re-engage students who might otherwise tune out in math class.

Our proposals are not made without some understanding of the lay of the land. There are reasons for the current emphasis on abstract mathematics in K-12 mathematics classrooms. For one, the past decade has seen too many teachers pressured by pacing plans and assessments to drag students at 80 mph through standards focused on fluency with abstract procedures. In addition, the mathematical training of teachers has often failed to provide sufficient experiences with applied mathematics, and so teachers are ill-prepared to engage students in real world contexts and problems. Nevertheless, the CCSS-M high school conceptual category, *mathematical modeling*, provides an opportunity for the implementation of many of the proposals in this article. In this article we want to argue for an integration of real life and abstract concepts, one that by *combining* pure and applied mathematical topics will best engage and best prepare *all* our students for 21st century life. We will argue for this with a series of examples

[1] This article is an expanded version of a talk given by DM at The Curtis Center for Mathematics and Teaching, UCLA, on March 2, 2013.

[2] Harel, G. (2007). The DNR System as a Conceptual Framework for Curriculum Development and Instruction, In R. Lesh, J. Kaput, E. Hamilton (Eds.), *Foundations for the Future in Mathematics Education,* Erlbaum, p. 13.

illustrating such combinations. They are a small selection from vast set of ways to teach math through engaging topics.

Finance and Algebra

Finance is one of the areas in which the most students will need the most mathematics in their future lives. Algebra is currently *the* big stumbling block for many students and a formal way of thinking that many adults are glad to forget. We argue here that financial problems provide real world contexts which will motivate and develop student understanding of the concepts and usefulness of formal algebra.

Every citizen makes multiple financial decisions important to their well being. Yet a large percentage are bewildered by “the math” and don’t even try to develop an educated understanding of the trade-offs involved with their financial choices. Instead they rely on professionals – mortgage brokers, tax advisors, bank employees, mutual fund reps – all with their own interests and typically putting their own spin on their advice. *This is crazy!* All American high school students should be given opportunities to access the mathematics necessary for a firm grip on their future financial life.

Here is an example of a complex real world problem that adults face, this one written by Lynn Steen[3] – with figures which are now quite out of date (one thing is certain – all such figures will change again):

- *The rent on your present apartment is $1,200 per month and is likely to increase 5% each year. You have enough saved to put a 25% down payment on a $180,000 townhouse with 50% more space, but those funds are invested in an aggressive mutual fund that has averaged 22% return for the last several years, most of which has been in long-term capital gains (which now have a lower tax rate). Current rates for a 30-year mortgage with 20% down are about 6.75%, with 2 points charged up front; with a 10% down payment the rate increases to 7.00%. The interest on a mortgage is tax deductible on both state and federal returns; in your income bracket, that will provide a 36% tax savings. You expect to stay at your current job for at least 5-7 years, but then may want to leave the area. What should you do?*

Should we expect that bearers of a U.S. high school diploma are able to solve such adult life problems? The business group *Junior Achievement* (JA) has been working to bring such financial problem solving experience to middle and high schools for almost a hundred years and now brings its programs to about four million students a year. Their programs are run by both trained volunteers who come into the classroom and by the teachers themselves. They also bring students to their custom built ‘Finance Parks’ where each of them takes on an adult role with a virtual family, house, budget etc. and must make it work. See their video at http://www.ja.org/video/financepark_students.wmv. Students in 8th grade, for example, work out what kind of car ‘their family’ can afford to buy, what are the benefits of saving and investing and follow this up, some loosing, some gaining money in the stock market. It is

[3] *Beyond eighth grade: Functional mathematics for life and work*, Susan Forman and Lynn Steen, National Center for Research in Vocational Education Graduate School of Education, University of California at Berkeley, p.28.

important to note that JA has classroom ready materials informed by a board of directors composed of senior executives in large corporations, and training programs to introduce teachers to their use.

Finance can be the "bridging context"[4] for introducing algebraic concepts from "*x*" to polynomials to exponential functions. At Crenshaw High School in Los Angeles, mathematics teacher, Michelle Sidwell had students examine the result of incurring a debt with one of the many payday cash advance businesses in the local neighborhood. These local businesses provide cash two weeks in advance of a coming paycheck for a "small finance charge", which is a fixed percent of the amount borrowed. Should the lender need more than two weeks to pay off the loan, an additional finance charge is accrued, at the same percentage but of the total now due. For instance, one company charges $15 for borrowing $100 for the first two-week period, a 15% finance charge. So after two weeks, the borrower owes $115. If unable to pay for another two weeks, the borrower owes $\$115(1.15) = \$100(1.15)^2 \approx \$132.25$.[5]

One key tool which students can use to further examine these types of financial scenarios is the spreadsheet (e.g. *Excel*)[6]. Students can use any cell to record a variable such as the interest rate or the number of two-week periods needed to pay back the loan. For instance, the cell at column C, row 10, is referred to as cell C10, a convention that already introduces a way of referring to a number which can be changed, i.e. a variable. Enter in C10 the number of two-week time periods they might need to borrow cash, and so start by entering 1 there. They can then type in cell D10:

$$\boxed{= \$100 * (1.15)^\wedge(C10)}$$

What appears on the spreadsheet in D10 then is not the equation but $115, the amount needed to pay off the loan. And if students change the number in cell C10 to 2, the product in D10 changes accordingly to $132.25. Thus the idea that C10 is a symbol that can take on any value you choose comes alive. Students will be stunned to find $3,785.68 appear when they change the value in C10 to 26, the number of two week periods in a year.[7] Bearers of a U.S. high school diploma should possess the mathematical skills to avoid this trap.

In the above example, the symbol C10 represented a number as opposed to *x*. Note that the use of *x* itself throughout middle and high school flies in the face of all professional scientific practice. Working scientists, engineers, statisticians or economists almost always use *abbreviations* as opposed to *x, y, z* etc. Thus when Einstein wrote $E=mc^2$, everyone understood that *E* was an abbreviation for energy, *m* for mass and that *c* was the universal convention for the

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[4] "Teaching and Learning Functions", Kalchman and Koedinger, *How Students Learn: Mathematics in the Classroom*, National Research Council, p. 171.

[5] CCSS-M F.LE. 1c: Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

[6] CCSS-M Mathematical Practice 5: Use appropriate tools strategically.

[7] CCSS-M F.LE. 2: Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).★

speed of light. When you introduce the fundamental formula that distance traveled is the product of your speed and the time you travel, you write $d = s \cdot t$ because everyone will remember that d is short for distance, etc., you don't write it $x = y \cdot z$. In a real life situation, variables stand for something real and a formula makes sense if the symbols chosen give cues to their meaning. In practice, *x,y,* and *z* are reserved for coordinates on a plane or in space. In fact, computer programmers are taught to scrupulously avoid using *x* because, when they come to debug or alter computer code, they will forget what *x* is representing.

It is also possible to introduce polynomials in a real world context using examples with money: via mortgages. The payments on a mortgage are calculated so that a constant monthly payment decreases the outstanding balance while also paying interest on this declining balance. At first, most of the payment goes into interest and near the end of the life of the mortgage, almost all the payment goes to pay off the balance. Suppose pay is the monthly payment, loan is the original loan amount and r is the *monthly* interest rate (this will be $(1/12)^{th}$ the annual rate). In a class, you would start with numerical values of loan and r, and the problem would be to solve for pay so that after a given number of months, the balance was zero. But we want to look at the algebra when they are all variables, a topic built on earlier concrete problems. First, the balance after one month's interest increasing the loan and one payment decreasing it is $\text{loan} * (1+r) - \text{pay}$. Replacing loan in this expression by the expression itself, which is the balance after one payment, you get the balance after two payments and it is a quadratic polynomial in 1+r[8]:

$$
\begin{aligned}
&((\text{loan} * (1+r) - \text{pay}) * (1+r) - \text{pay}) \\
&= \text{loan} * (1+r) * (1+r) - \text{pay} * (1+r) - \text{pay} \\
&= \text{loan} * (1+r)^2 - \text{pay} * (1+r)^1 - \text{pay} * (1+r)^0
\end{aligned}
$$

Similarly, one can represent their balance after three payments with:

$$
\begin{aligned}
&((\text{loan} * (1+r) - \text{pay}) * (1+r) - \text{pay}) * (1+r) - \text{pay} \\
&= \text{loan} * (1+r)^3 - \text{pay} * (1+r)^2 - \text{pay} * (1+r)^1 - \text{pay} * (1+r)^0
\end{aligned}
$$

One can now work out what the payment should be so that after three months, the balance is zero[9]:

$$
\begin{aligned}
&0 = \text{loan} * (1+r)^3 - \text{pay} * (1+r)^2 - \text{pay} * (1+r)^1 - \text{pay} * (1+r)^0 \\
&\text{pay} * (1+r)^2 + \text{pay} * (1+r)^1 + \text{pay} * (1+r)^0 = \text{loan} * (1+r)^3 \\
&\text{pay} * [(1+r)^2 + (1+r) + 1] = \text{loan} * (1+r)^3 \\
&\text{pay} = \frac{\text{loan} * (1+r)^3}{(1+r)^2 + (1+r) + 1}
\end{aligned}
$$

[8] CCSS-M A.SSE.2: Use the structure of an expression to identify ways to rewrite it.
[9] CCSS-M A.CED.1: Create equations and inequalities in one variable and use them to solve problems.

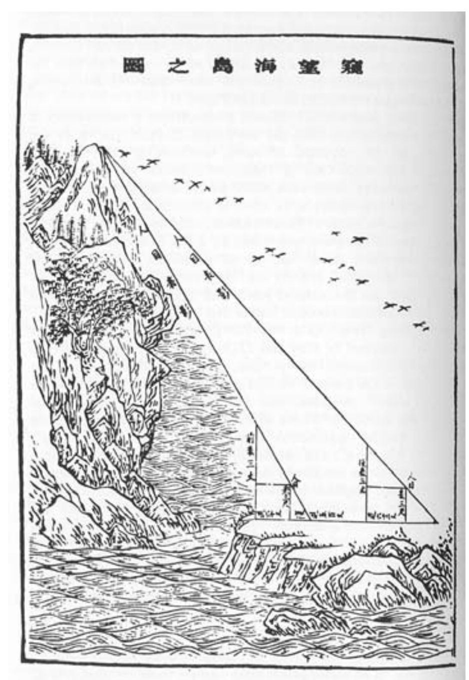
Of course, most people do not pay off a mortgage in only three months, and so students can work out what the payment should be so that after maybe 120 months, the balance is zero. This question creates a need then for the formula for the partial sum of a geometric series[10] as students must compute $\sum_{k=0}^{119}(1+r)^k$.

The mortgage payment problem exemplifies the idea of starting with a real world problem which leads to a more general mathematical problem and thus motivates the derivation of a general formula, a special case of which then solves the initial problem. The financial context helps orient the traditional curriculum (polynomials, equation solving, and geometric series) to the needs of the majority and the real world context helps students understand the abstraction to follow.

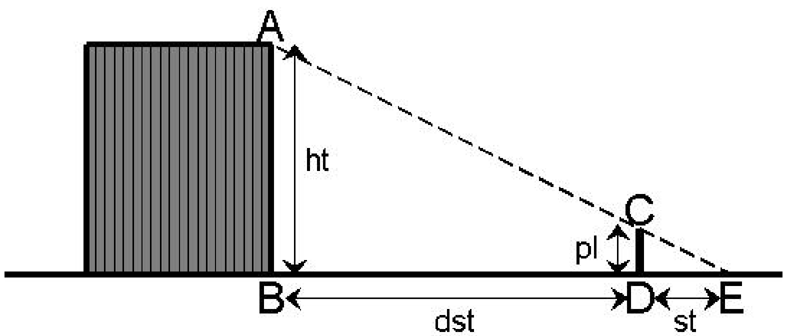
In 11th and 12th grades, students can benefit from learning the basic ideas of economics and how they relate to businesses and governments in "mathenomics" courses, such as the one taught at Winchendon School in Massachusetts. Such a course integrates mathematics and civics, serving to enhance both subjects. The Winchendon course description includes questions such as: "What are financial markets and how do they work? How do supply and demand work together to determine prices? How does the U.S. government raise money and where is it spent?" etc. And with a much broader base, Junior Achievement has written a book *JA Economics*. Students learn about inflation, bankruptcy, economic cycles, rates of exchange and double-entry bookkeeping: real mathematics used in models and statistics in the business world.

Measuring the World via Geometry

A greater emphasis on applied mathematics should also be considered in geometry. The word "geometry" means "measuring the world" and this is how the subject began some four millennia ago. Here is an image from the ancient Chinese book entitled "*The Sea Island Mathematical Manual"*. The figure shows the title problem: how to measure the height of the mountain on an island by using only measurements taken from the mainland? We often ask secondary students to complete a simpler version of this problem when we ask them to measure the height of a building by sighting the top of the building with a pole of a measured height, pl, set a measured distance, dst, from the base of the building. The setup is shown in the diagram below where ht is the unknown height of the structure and st the sighting length that students' measure.



[10] CCSS-M A.SSE.4: Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.

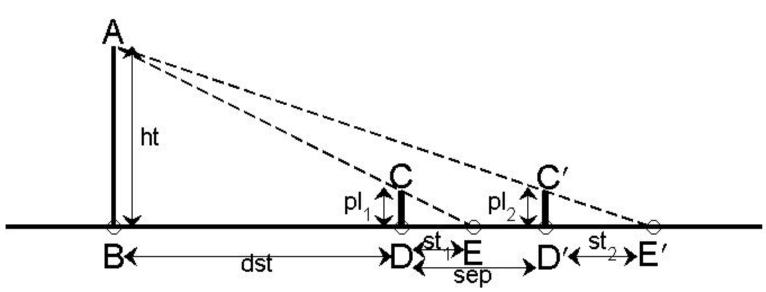


Note that no trigonometry is needed, since by similar triangles ABE and CDE we find

$$\text{ht} = \frac{\text{pl}}{\text{st}}\left(\text{dst} + \text{st}\right)$$

where *pl, st,* and *dst* are known measured distances.[11] This exercise also motivates the later introduction of trigonometry: using trig functions on your calculator (here tangent), you get the height of the building from the angle made by its roof seen from any point E that is a known distance from the base of the building

The Chinese manual poses a more complex question, as *dst* is not known. The manual shows how to measure the height of the mountain on the island using only measurements taken from the mainland by sighting the mountain peak using *two* poles set a known distance apart on the mainland. The set up is shown more clearly in the diagram below, where *ht* is the unknown height of the mountain and *dst* its unknown distance from the first pole. pl_1 and pl_2 are the pole heights, st_1 and st_2 the 'sighting' lengths where the mountain peak aligns with the tip of the pole and sep is the separation of the two poles.



Note that the triangles ABE and CDE are similar, as are the triangles ABE' and C'D'E'. Using these two pairs of triangles, we get the relations:

$$\frac{pl_1}{st_1} = \frac{ht}{dst + st_1}, \quad \frac{pl_2}{st_2} = \frac{ht}{dst + sep + st_2}$$

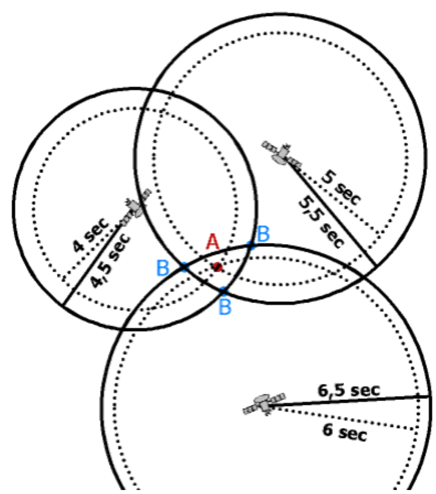
[11] CCSS-M. G.SRT.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Eliminating the unknown *dst*, and solving for *ht*, we get the answer:

$$ht = \frac{st_2 - st_1 + sep}{(st_2/pl_2) - (st_1/pl_1)}$$

Students can engage in the Chinese problem, by completing an outdoor activity like the building height one described above, but where *dist* is not easily measured. Perhaps more startling, they can use a variation of this method to find the altitude of fair weather cumulus clouds. Take two yardsticks and two digital cameras to opposite ends of a playing field and photograph one of the clouds from the ground seen against the yardsticks. On a computer, one can pick out corresponding features of the cloud in the two shots and see how high up on the two yardsticks one such cloud feature is. Then we get a figure like the Chinese one. They know the values of pl_1 and pl_2, st_1 and st_2, and *sep* and can then solve for *ht*. They can then write general directions for finding heights, using appropriate variables for the knowns and unknowns, to create their own "users manual", explaining to another how to estimate any particular height in this manner[12]. For example, the "Met" school in Providence (the Metropolitan and Regional Career and Technical School) immerses students in measuring the world via a related project in which students work in teams measuting some building at Brown University and then drawing up plans for it.

Many related and more difficult problems arise from carpentry – the challenge of cutting boards for a roof with gables and valleys for example. In these experiences students find that they use proportional reasoning, similar triangles, solve simple linear equations as well as systems of linear equations, use Pythagoras's theorem, define variables in a modeling context, generalize a solution method, and practice precision in communicating mathematics[13]. These concrete problems lead to the abstractions of geometry in a totally natural fashion.



In a more advanced class it is especially nice to explain how a GPS works. The figuret illustrates the idea. The signals from three satellites are needed to get one's position assuming you are at sea level (four if not). In the figure, everything is made two dimensional for clarity. The three satellites emit pulses at exactly known times – in this case we assume that all three satellites emit pulses at the same time. The circles show how far the three pulses, travelling at the speed of light, reach at two different times, each circle being labelled by the time when the corresponding signal reaches it.. At point A, the top left signal arrives with a delay of 4 seconds, the top right with a delay of 5 seconds and the bottom (furthest) satellite is heard after 6 seconds. If your clock was perfectly synchronized with the satellites' clocks, you could measure your distance from each satellite by noting when each pulse reached you. But this is not practical. Nonetheless you can use your clock to

[12] CCSS-M MP3: Construct viable arguments and critique the reasoning of others…. They justify their conclusions, communicate them to others, and respond to the arguments of others.

measure the delays *between* the arrival times of the pulses from the three satellites. If you are at point A in the figure, the delays are 1 and 2 seconds. The point is that there is only one place on earth where the three signals would be detected with eaxactly these delays. In the figure each of the points B records one pair with the same delay but not all three. The delays tell you the *difference between your distances* from the corresponding pair of satellites. Knowing where the satellites are, it becomes an interesting math problem to get your position, a problem which was studied by Apollonius and Newton. There are both algebraic and geometric ways of solving it. Using algebra, you have three unknowns: your position (x,y) and your distance r from one satellite. Given r and the delays d_2, d_3 between its signal and the other two, you know your distance from the other satellites and these distances give you three quadratic equations[14] in x,y,r:

$$(x-x_1)^2+(y-y_1)^2=r^2$$
$$(x-x_2)^2+(y-y_2)^2=(r+d_2)^2$$
$$(x-x_3)^2+(y-y_3)^2=(r+d_3)^2$$

Subtracting the first equation from the second and third, you get two linear equations in x,y,r, so you solve for x and y in terms of r and then plug this back into the first equation, getting a quadratic equation for r (see http://en.wikipedia.org/wiki/Problem_of_Apollonius, section 3.3 or http://mathworld.wolfram.com/ApolloniusProblem.html for more details). The geometric approach is a beautiful as well, with students constructing the point (x,y) by straightedge and compass.

Statistics can make a difference

Statistics certainly need not be a dry memorization of formulas for various statistical tests. There are so much data around us that are highly interesting to students: the spread of prices for the used car that you hope to purchase and of course sports stats for your favorite teams and their rivals. In both these cases, it can be quite exciting to calculate whether the price difference or the performance difference of two car models or two players respectively is statistically significant, that is whether the difference could just as easily be a natural fluctuation due to sampling these ads or these games. Formulas for the 'p-value' are so easy to motivate and illustrate with situations all around us.

But not appreciating statistical significance can also be a matter of life and death so it is hugely important to teach when to be skeptical of probabilities (as in Disraeli's quip – 'lies, damned lies and statistics'). A stunning example from our court system is posted on Tim Gowers blog[15]. The mathematics involved is well within the scope of a high school course but was sadly missed by the jury:

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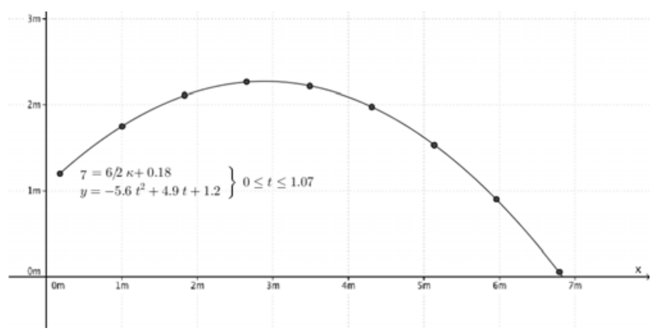
[14] CCSS-M A.CED.3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*★

[15] http://gowers.wordpress.com/2012/06/08/how-should-mathematics-be-taught-to-non-mathematicians/, problem 46.

- In 1972 Diana Sylvester was raped and killed in San Francisco. Despite one or two leads, the police failed to solve the case. However, they kept some DNA, and in 2006 they checked it against a DNA database of 300,000 convicted sex offenders. They discovered that it matched the DNA of John Puckett, who had spent a total of 15 years in jail for two rapes. There was no other evidence linking Puckett to the crime, but the probability that a random person's DNA would match that of the sample was judged to be 1 in 1,000,000. On that basis*, he was found guilty and sentenced to life imprisonment*. How reliable was the conviction?

Here's the problem: suppose none of the 300,000 convicted sex offenders was the actual perpetrator. If that were the case, what is the chance of none of them matching by accident? Each one might match by accident with probability 0.000001 so would not match with probability 0.999999. The probability of none of them matching is then $(0.999999)^{300000}$ which is roughly 0.74[16]. In other words, even if all of them are innocent, the chance of one of them coming up as a match is still approximately 26%. So the odds of making an error by convicting an innocent man now turn out not to be one in a million but merely one in four! Perhaps the jury would still convict him but it is not an open and shut case. This is one of the central lessons in the use of statistics and every adult ought to know something about it.

Linking physics and math



An especial mistake, we believe, in high school curricula is separating math and physics. The physics is most clearly expressed in the language of mathematics and the mathematics comes alive from the models that arise in elementary physical experiments. Physics provides easy examples of functions given by measurable real world quantities whose rates of change are a key part of what's going on and working them out motivates topics from slopes of graphs in middle school to basic calculus later. Topics like the acceleration of a falling object and its corollary, the parabolic trajectory of a ball thrown in the air are traditional fare in the "word problem" section of secondary mathematics texts, but can be brought to life by experiments in the classroom. For instance, students can stand at regular intervals, in a row a few **steps** back from a wall. A ball can be tossed between them and the wall, and they can place a marker on the wall to indicate the height of the ball as it passed. In this manner, a set of vertical vs. horizontal measurements can be gathered, estimating the path of the ball in flight. Students can graph the data using a graphing utility, and choose values of a, b, and c to develop an equation of the form $y = a(x-b)^2 + c$ that reasonably models the path of the ball[17]. As the students work, the trail of

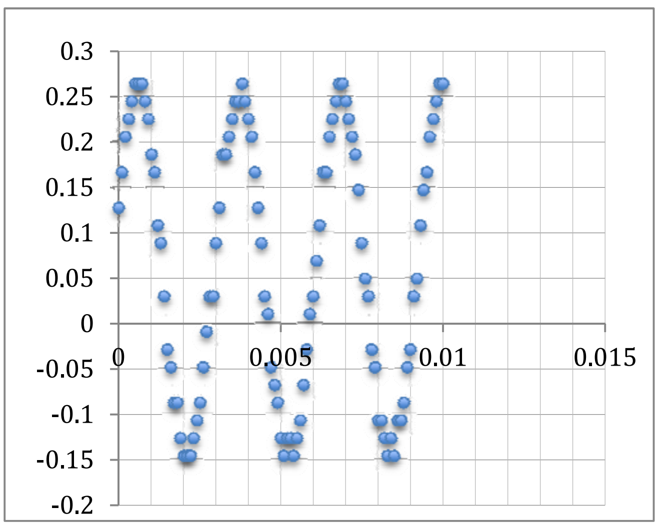
[16] S-CP 5: Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* *

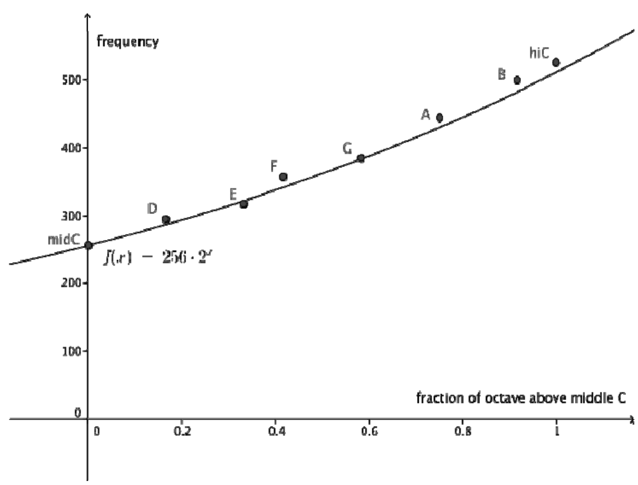
[17] CCSS-M F-LE: Construct and compare linear, quadratic, and exponential models and solve problems.
CCSS-M F.BF.3: Identify the effect on the graph of replacing *f(x)* by *f(x)* + *k*, *kf(x)*, *f(kx)*, and *f(x+k)* for specific values of *k* (both positive

markers on the classroom wall is a concrete, physical reminder of the real world context represented by the points on the graph and the developed equation.

In addition, the ball toss can be videotaped, and the tape can be used to determine the time at which the ball passed each student. This allows students to additionally investigate how the horizontal position of the ball varies with time and how its vertical position varies with time. The graph above shows a model developed in a Los Angeles classroom for a videotaped ball toss. Students can be asked to use their model to estimate when was the ball five feet above the ground or when the ball hit the ground[18]. They can be asked to explain the coefficient of the linear equation in terms of the motion[19] (an estimate of the horizontal velocity of the ball) or to use their model to estimate the peak height of the ball[20].

A less standard connection is via sound. The connection of the math of sines, cosines and periodic functions to the physics of waves and especially pressure waves in air and thus to music is so wonderful. You can use an oscilloscope, a Venier microphone probe, or computer apps such as "WaveSurfer" and "Sonic Visualizer" to gather data related to the pressure wave generated by any sound. For example, students collected the data at left by striking an "E" tuning fork.





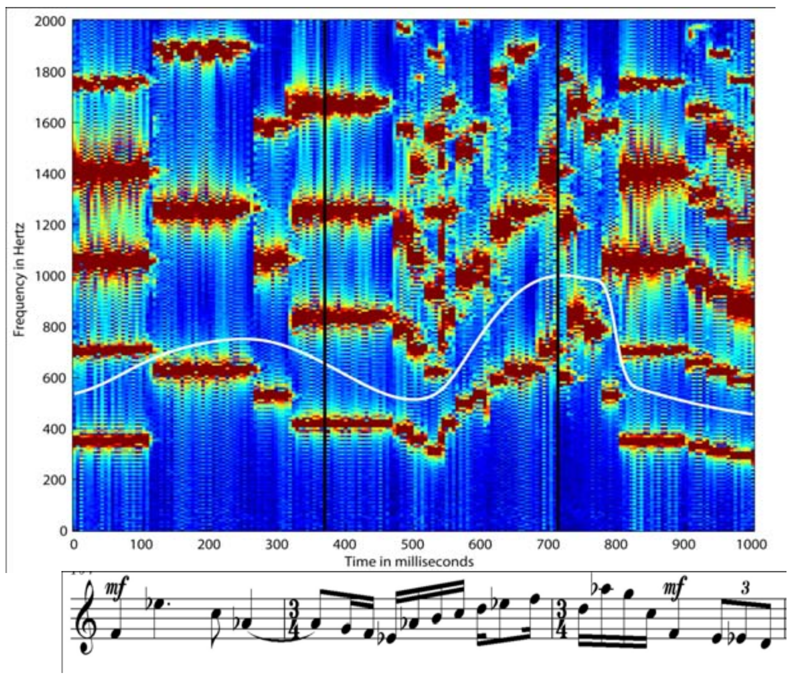
Further, students can use such data to estimate the period in order to come up with a good approximation of the frequency of the note. (The frequency is the reciprocal of the period). If eight groups of students each do this, one for each note in the scale, the class can relate these frequencies to the way they hear music. They will find that the frequencies of notes double from octave to octave and increase by a factor of 3/2 in a major fifth. Looking at the whole major scale, as in the graph to the right, they can develop a model for the exponential relationship

and negative); find the value of *k* given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
CCSS-M S.ID.6: Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.⋆ a) Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.*

[18] CCSS-M A-REI 4: Solve quadratic equations in one variable.
[19] CCSS-M F-LE 5: 5 Interpret the parameters in a linear or exponential function in terms of a context.⋆
[20] CCSS-M A-SSE 3b: Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.

between the frequency of the note and the number of notes it is above middle C.[21] Students can be asked to use their model to estimate at how many steps above middle C a particular frequency is reached[22] or asked to compare the change in frequency resulting from moving up one step in the middle octave on a piano as compared the change in frequency that results from moving up one step in a higher octave.[23]

The ultimate way of understanding music in terms of sines and cosines is what is called the “spectrogram”. This shows for each frequency and time, the size of the sine/cosine term in the pressure wave near that frequency and near that time, so you can actually read off the score from it. In fact, one can record a bit of music and with suitable software, compute “spectrogram” and read off the score! The colorful picture below, done for a Brown class of non-math majors, shows one second of an oboe playing with its spectrogram above. Look at the dark stripes *below* the white line – they trace the score seen below! The lines above are its harmonics (double, triple, etc above the frequency of the basic note) which make the oboe sound so rich (also note the vibrato, clearly visible in the 3rd harmonic of the second note).



The examples given above show clearly how teaching in context provides a reason for students to have to do computations. For instance, asking what number of steps above middle C is associated with a particular frequency motivates the use of logarithms, or asking where the ball hits the ground provides a reason to solve a quadratic equation. The context provides a concrete foundation on which to build their abstract understandings of the mathematics as well as providing a necessity for the abstract mathematics.

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[21] CCSS-M F-LE 1c: Recognizesituationsinwhichaquantitygrowsordecaysbyaconstantpercent rate per unit interval relative to another. F-LE 5: Interpret the parameters in a linear or exponential function in terms of a context.⋆

[22] CCSS-M F-LE 4: For exponential models, express as a logarithm the solution to *ab^(ct)= d* where *a, c,* and *d* are numbers and the base *b* is 2, 10, or *e;* evaluate the logarithm using technology.⋆

[23] CCSS-M F-IF 6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.⋆

21st Century Topics?

We have talked about addressing the needs of all our students by bringing a balance of applied and pure mathematics to algebra, geometry, statistics and pre-calculus, using the applied topics as a foundation for the pure. In addition, and perhaps more controversially, we wish to propose that it may be a good idea to include some 21st century topics replacing some of the traditional topics. In particular we ask the reader to consider the inclusion of computer science and discrete math in K-12 mathematics. What is the argument for including these?

A key component of traditional high school mathematics was the introduction to the concept of ‘proof’ in the context of Euclidean geometry. A proof is correct only if every step is logically justified based on what has been established so far. One unjustified or missing step and the whole proof is invalid. At one time, a central part of every high school geometry class was writing one’s chain of reasoning in a “two-column proof” which displayed this internal coherence and required students to make every step of reasoning explicit. And indeed the CCSS-M incorporates a call both in the content and practice standards for teaching both reasoning and precision[24].

But mathematics has always had two sides: pure math in which proofs are the “gold standard” and applied math in which algorithms are its “gold standard”. While proofs establish what we know to be true, algorithms which actually solve stuff. And unlike proofs, algorithms are everywhere: they are the mathematical version of a kitchen recipe for blackberry pie or IKEA instructions for assembling a desk. In the 21st century, the introduction of computers has transformed applied math and its algorithms are now written in computer code. We would suggest that, taking math as a whole, understanding algorithms is as important as understanding proofs. They are both part of mathematical thinking and, in fact, have a lot in common. Both require breaking down your understanding into explicit steps and both work only if no step is missing. It’s amazing how everyone seems to get both formal proofs and computer code wrong on the first try because we forget to include many things that seem obvious. Both hone one’s understanding of reasoning and precision and are good training for later encounters with legal arguments or political debates.

We propose that an introduction to computer programming should be included in a 21st century high school curriculum. All of us are surrounded by computer technology – in our phones, pads and pods, the internet and our actual computers. Helping students understand what is going on inside these magic boxes exposes them to authentic examples of mathematical thinking, of constructing the chain of tiny steps which result in making the computer useful.

To make clear what this entails, we need to give a simple example. Suppose you want to reduce a fraction n / m to lowest terms. You need to find the greatest common divisor (‘gcd’) of n and m and divide n and m by this. Of course, you can often eye ball this but a standard way on a computer is to use the Euclidean algorithm. The way it works is that you repeatedly subtract

[24] CCSS-M Mathematical Practice Standard 3: Construct viable arguments and critique the reasoning of others.
CCSS-M Mathematical Practice Standard **6:** Attend to precision.

the smaller number from the larger and eventually the smaller becomes zero. Then the other number is the ‘gcd’. For example, say the two numbers are 12 and 9. Then 3 comes out in a few steps:

$$(12,9) \to (3,9) \to (3,6) \to (3,3) \to (3,0)$$

In a sort of universal pseudo-code (immediately translatable into any real coding language), the algorithm can be written:

```
d = gcd(n,m)
        repeat
                if n ≥ m, then n = n — m
                else m = m — n                [repeatedly subtract the smaller from the larger]
        until n = 0 or m =0
        if n = 0, d = m                       [what's left are the numbers 0 and d]
        else d = n
end
```

One can go on to write the algorithm for the least common multiple and for adding two fractions, etc. The similarity to two column proofs seems undeniable. The difference is that what is taught is the idea of an algorithm and especially the two ideas i) of doing different things depending on the situation (if … else) and ii) of repeating a step for a certain number of times. The first is a conditional command, the second a recursion. Recursive rules also arise in many financial settings and thus link this coding concept with algorithms arising in many other contexts.

What is gained by learning to code? First, the teasers: it is a game to write code and see what goes wrong (which it nearly always does on the first few tries) and see if your code is better than that of others. Moreover, there is no reason to limit oneself to writing code for numerical algorithms. One can write code to play games – tic-tac-toe is easy, Sudoku harder and one can even try checkers. The challenge is to do well in the game. Even more of a teaser, I suspect, would be to write code for web sites in ‘html’ or ‘php’. Though it sounds pretty sophisticated, we would argue that it is a small step once the student has mastered simple programs. Everyone wants to make their own web site and by the time they have put some nice fonts, nice images etc. in a web site, they will have acquired a very marketable skill. Incidentally, just as in financial math, there are groups which have been promoting these ideas and have teaching tools all worked out: see for example www.code.org and more specifically www.codehs.com.

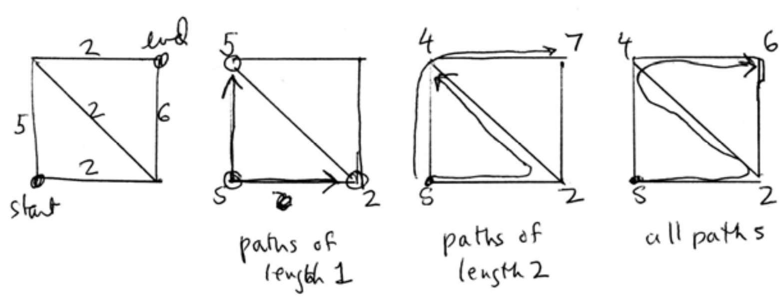
Many mathematical topics can be fit into a course built on the skill of coding. Certainly binary arithmetic is encountered because the computer itself is thinking entirely with binary sequences. One can go on to introduce some algorithms for encryption and thus send secret messages back and forth. Error correction is another natural topic: if some random digits of a message are corrupted, how you can code the message with redundancy so that the true message can be recovered. These are both fundamental areas of algorithms in computer science and electrical engineering. The math can be tied in with statistics by discussing what is a random number generator. What for that matter is a random number anyway? And one can bring in the hot topic of "massive data", how are people going to handle the terabytes of data being recorded when Facebook knows who is whose friend and how an image is stored on a computer.

Another 21st century topic that has been proposed for inclusion in the high school curriculum is discrete math and specifically graph theory. When Solomon Garfunkel made a survey of employees at many firms to see what math they were using in their work, this is what he found:

> "What we observed was operations research at work. We saw instances of *scheduling, inventory, queuing theory, bin packing*. We do not mean to imply that the employees were engaged in using Kruskal's algorithm or constructing minimum-cost spanning trees – only that these ideas were behind the reasoning involved."

Graph theory provides mathematical tools to model many real life situations. Much of discrete math and operations research is all about *optimizing*: finding prices or manufacturing quotas to maximize profits, finding the fastest path to get various places under some constraints. To illustrate this, take the question of how your GPS finds the fastest route from one place to another. As in almost all discrete math problems, there is an underlying graph. In this case take as vertices all highway intersections, adding in your starting and ending point. Connect two by an edge when a highway leads directly from one to another. Assign a weight to each edge given by the time needed to travel at the speed limit between the two points that the edge connects. We seek a path in the graph from the starting point to the ending point such that the sum over the weights of all edges traversed is least. Why is this hard? That's because there are a huge number of such paths.

The solution is a simple case of what is called *dynamic programming*. Beginning at your starting point, first label all its immediate neighbors by the time needed to get there along the corresponding edge. Then go to neighbors of neighbors. There may be more more than one path to them: label them with the minimum of sum of the weights along all two step paths to them. The figure below should clarify what we are doing.



At k^{th} step you have labeled all points accessible in k steps with the least weight path of k or fewer steps. You go on to add one more step and choose the smallest weight path. In the figure, every point can be reached by a path of length at most three and it turns out that the zig-zag path – these must be highways because their weights are so small – is the best way to every point. Note that this figure is not meant to represent distances accurately but just to show the graphical connections of four different places. This is what your GPS is doing and it is fast.

Summarizing

The following chart shows a possible frame for the types of mathematical experiences we have been proposing, experiences that address the needs of all our students not just those going into STEM careers:

<table>
<tr><th>9th grade</th><th>10th grade</th><th>11th grade</th><th>12th grade</th></tr>
<tr><td rowspan="2">Integrated algebra and statistics built on an introduction to the financial world</td><td>Computers, code and algorithms</td><td colspan="2" rowspan="2">A selection of one semester integrated electives:
• Traditional honors math
• Mathenomics
• Math-Physics
• Math-Biology-Environmental studies
• Discrete math with applications
• Engineering and basic machines</td></tr>
<tr><td>Geometry and measuring the world</td></tr>
</table>

These ideas have often been voiced by Lynn Steen who has been lecturing and writing for a long time about modifying K-12 curricula to achieve what he calls *quantitative literacy*, that is to say fluency in the use of the quantitative arts. Recently[25] he described four ways for "*organizing the curriculum to pay greater attention to the goal of transferable knowledge and skills"*. Here are his suggestions:

1. by embedding mathematics in courses focused on applications of mathematics;
2. by team-taught cross-disciplinary courses that blend mathematics with other subjects in which mathematical thinking arises (e.g., genetics, personal finance, medical technology);
3. by project-focused curricula in which all school subjects are submerged into a class group project (e.g., design a solar powered car).
4. by career-focused curricula in which a cohort of students focuses all their school work on particular career areas (e.g., technology, communications, or business).

The first suggestion is what the present article is focused on. We have proposed the second in the context of economics and physics. The 3rd suggestion has been the focus of a lot of controversy. It is one way to flesh the courses in the list above but is certainly not essential. His 4th suggestion is a version of VocEd, an approach that has not had many defenders in recent decades. But it is clearly a very natural way to focus the various electives we have proposed for in the last two

[25] *Reflections on Mathematics and Democracy*, revised text of a paper presented at a joint AMS-MAA Special Session at MathFest 2012 in Madison, Wisconsin.

years of high school. The VocEd approach has always been one of the keystones of quite successful German system of higher education.

It is hard to make room for all this without de-emphasizing some “sacred” topics that are useful only to STEM students. Otherwise, we fall into the trap of teaching students a mile’s worth of topics but at an inch of depth. We list here some topics that we believe are less useful for the majority of 21st century students (but which should be taught in an honors or STEM math track):

- Manipulation of general polynomials, including factoring and its connection to their zeros. General polynomials are rare in applications and their study is important mainly for future mathematics majors. For instance, applied areas typically use splines built on low degree (2 or 3) polynomials to model shape and not on higher polynomials.
- The quadratic formula. Completing the square does come up sometimes, e.g. working on mean and variance or the parabolic trajectories of balls in the air, but memorizing and drilling with “minus *b* plus or minus the square root ….” should not be a goal.
- The introduction of complex numbers and their use solving quadratic equations with no real roots. Complex numbers only become really useful when you treat complex exponentials and their relation to periodic functions – topics for college.
- Lists of forgettable formal rules needed to make algebra into a formal mathematical system.
- And in a senior course with a first introduction to calculus, formal properties of limits. Precise statements of facts that every ordinary person would consider obvious only makes students feel something subtle is going on that they are missing.

These topics should of course still be included in mathematics courses specifically designed to prepare students aiming to study college mathematics or other STEM areas. These courses could be taken at the junior and senior levels, given the sequence shown in the above chart.

Upon reading this, you might initially react in concern that these discussions short-change the area of pure mathematics. Consider this response which came in response to a Huffington Post piece written by the senior author and Sol Garfunkel:

- “You do not study mathematics because it helps you build a bridge. You study mathematics because it is the poetry of the universe. Its beauty transcends mere *things.”*
  Jonathan Farley (Prof of CS, Univ of Maine)

But our point is that you can do *both* pure and applied math and current K-12 mathematics experiences largely do not. Archimedes not only proved some of the most beautiful theorems, but built armaments to hold off the Roman siege of Syracuse. A gem in analysis – Fourier series – was invented to describe the motion of a vibrating string, the motion of the moon and the underground cooling of the soil in winter. *Pure and applied math are inseparable Siamese twins.* It is simply not an either/or situation. The senior author asserts this after a lifetime of mathematical research in both pure and applied mathematics. In addition, you might think that these proposals will result in fewer students choosing to major in pure mathematics. At least for the junior author, this was false. A senior year combined calculus-physics course convinced her to major in pure mathematics. Subsequently, her public school teaching focused students both on *pure and applied mathematics*.

While this article does not present a comprehensive case, we hope it provided at least some examples which inspired you to rethink what mathematical experiences truly help the majority of our students acquire the mathematical understandings and ways of thinking they will need for a rewarding adult life. In the hand wringing over international test scores, this discussion seems too often to be missing. There will always be disagreements but such a discussion seems to be the right basis on which to plan something so important as our children's future.