Algebraic Geometry II
(a penultimate draft)

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Foreword

[from DM] I gave an introductory course in algebraic geometry many times during the 60's and 70's while I was teaching at Harvard. Initially notes to the course were mimeographed and bound and sold by the Harvard math department with a red cover. These old notes were picked up by Springer and are now sold as the "Red Book of Varieites and Schemes". However, every time I taught the course, the content changed and grew. I had aimed to eventually publish more polished notes in three volumes. Volume I, dealing with varieites over the complex numbers appeared in 1976 and roughly 2/3rds of a first draft for volume II was written down at about the same time. This draft covered the material in the Red Book in more depth and added some advanced topics to give it weight. Volume III was intended to be an introduction to moduli problems but this was never started as my interests shifted to other fields in the 80's. To my surprise, however, some students did read the draft for volume II and felt it made some contribution to the growing literature of multiple introductions to algebraic geometry.

[from [TO] I had the good fortune of first getting acquainted with schemes and functorial approaches in algebraic geometry when the first author gave a series of introductory lectures in Tokyo in spring, 1963. Throughout my graduate study at Harvard from October, 1964 through June, 1967, I had many chances to learn further from the first author as my Ph.D. thesis advisor. It is a great honor and privilege to have this opportunity of sharing with as many people as possible the excitement and joy in learning algebraic geometry through the first author's fascinating style.

The Herculean task of preparing the manuscript for publication, improving and fixing it in multiple ways and adding some half a dozen new sections and results is due to the efforts of the second author. Both authors want to thank those who have assisted in this draft that we are posting on the Web, especially Ching-Li Chai, Vikraman Balaji, Frans Oort, Fernando Quadros Gouvêa, Dinesh, Amnon Neeman and Akihiko Yukie. A number of extra sections were added to make the book better. Thanks are due to John Tate for the new proof of the Riemann-Roch theorem, Carlos Simpson for the proof of Belyi's three point theorem and Shigefumi Mori for the proofs of some results of his. The exercises are those found originally in the manuscript plus further exercises kindly provided by Ching-Li Chai who gave a graduate course in algebraic geometry at the University of Pennsylvania using a preliminary version of this book. No systematic attempt was made to produce further exercises.

Special thanks are due to Ching-Li Chai for providing valuable suggestions during the preparation of the manuscript.

Contents

CHAPTER I

Schemes and sheaves: definitions

1. Spec(R)

For any commutative ring R, we seek to represent R as a ring of continuous functions on some topological space. This leads us naturally to Spec(R):

DEFINITION 1.1. $\mathrm{Spec}(R) =$ the set of prime ideals $\mathfrak{p} \subset R$ (here R itself is not considered as a prime ideal, but $\{0\}$, if prime is OK). If $\mathfrak{p}$ is a prime ideal, to avoid confusion we denote the corresponding point of Spec(R) by $[\mathfrak{p}]$.

DEFINITION 1.2. For all $x \in \mathrm{Spec}(R)$, if $x = [\mathfrak{p}]$, let

$$\Bbbk(x) = \text{the quotient field of the integral domain } R/\mathfrak{p}.$$

For all $f \in R$, define the value $f(x)$ of f at x as the image of f via the canonical maps

$$R \twoheadrightarrow R/\mathfrak{p} \to \Bbbk(x).$$

In this way, we have defined a set Spec(R) and associated to each $f \in R$ a function on Spec(R) — with values unfortunately in fields that vary from point to point. The next step is to introduce a topology in Spec(R):

DEFINITION 1.3. For every subset $S \subset R$, let

$$\begin{aligned} V(S) &= \{x \in \mathrm{Spec}(R) \mid f(x) = 0 \text{ for all } f \in S\} \\ &= \{[\mathfrak{p}] \mid \mathfrak{p} \text{ a prime ideal and } \mathfrak{p} \supseteq S\}. \end{aligned}$$

It is easy to verify that V has the properties:

a) If $\mathfrak{a} =$ the ideal generated by S, then $V(S) = V(\mathfrak{a})$,
b) $S_1 \supseteq S_2 \Longrightarrow V(S_1) \subseteq V(S_2)$,
c) $V(S) = \emptyset \Longleftrightarrow$ [1 is in the ideal generated by S].

PROOF. $\Longleftarrow$ is clear; conversely, if $\mathfrak{a} =$ the ideal generated by S and $1 \notin \mathfrak{a}$, then $\mathfrak{a} \subset \mathfrak{m}$, some maximal ideal $\mathfrak{m}$. Then $\mathfrak{m}$ is prime and $[\mathfrak{m}] \in V(S)$. $\square$

d)

$$V(\bigcup_\alpha S_\alpha) = \bigcap_\alpha V(S_\alpha) \text{ for any family of subsets } S_\alpha$$
$$V(\sum_\alpha \mathfrak{a}_\alpha) = \bigcap_\alpha V(\mathfrak{a}_\alpha) \text{ for any family of ideals } \mathfrak{a}_\alpha.$$

e) $V(\mathfrak{a}_1 \cap \mathfrak{a}_2) = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$.

PROOF. The inclusion $\supseteq$ follows from (b). To prove "$\subseteq$", say $\mathfrak{p} \supset \mathfrak{a}_1 \cap \mathfrak{a}_2$ but $\mathfrak{p} \not\supseteq \mathfrak{a}_1$ and $\mathfrak{p} \not\supseteq \mathfrak{a}_2$. Then $\exists f_i \in \mathfrak{a}_i \setminus \mathfrak{p}$, hence $f_1 \cdot f_2 \in \mathfrak{a}_1 \cap \mathfrak{a}_2$ and $f_1 \cdot f_2 \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. This is a contradiction. $\square$

f) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

Because of (d) and (e), we can take the sets $V(\mathfrak{a})$ to be the closed sets of a topology on $\operatorname{Spec}(R)$, known as the *Zariski topology.*

DEFINITION 1.4. For $f \in R$

$$\begin{aligned}\operatorname{Spec}(R)_f &= \{x \in \operatorname{Spec}(R) \mid f(x) \neq 0\}\\ &= \operatorname{Spec}(R) \setminus V(f).\end{aligned}$$

Since $V(f)$ is closed, $\operatorname{Spec}(R)_f$ is open: we call these the *distinguished* open subsets of $\operatorname{Spec}(R)$.

Note that the distinguished open sets form a basis of the topology closed under finite intersections. In fact, every open set U is of the form $\operatorname{Spec}(R) \setminus V(S)$, hence

$$\begin{aligned}U &= \operatorname{Spec} R \setminus V(S)\\ &= \operatorname{Spec} R \setminus \bigcap_{f\in S} V(f)\\ &= \bigcup_{f\in S} \big(\operatorname{Spec} R \setminus V(f)\big)\\ &= \bigcup_{f\in S} \operatorname{Spec}(R)_f\end{aligned}$$

and

$$\bigcap_{i=1}^{n} \big(\operatorname{Spec} R\big)_{f_i} = \big(\operatorname{Spec} R\big)_{f_1\cdots f_n}.$$

DEFINITION 1.5. If $S \subset \operatorname{Spec} R$ is any subset, let

$$I(S) = \{f \in R \mid f(x) = 0,\ \text{all } x \in S\}.$$

We get a Nullstellensatz-like correspondence between subsets of R and of $\operatorname{Spec} R$ given by the operations V and I (cf. Part I [**87**, §1A, (1.5)], Zariski-Samuel [**119**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**27**, Chapter V, §3.3, Proposition 2]):

PROPOSITION 1.6.

(a) *If $\mathfrak{a}$ is any ideal in R, then $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*
(b) *V and I set up isomorphisms inverse to each other between the set of ideals $\mathfrak{a}$ with $\mathfrak{a} = \sqrt{\mathfrak{a}}$, and the set of Zariski-closed subsets of* $\operatorname{Spec} R$.

PROOF. In fact,

$$\begin{aligned}f \in I(V(\mathfrak{a})) &\iff f \in \mathfrak{p} \text{ for every } \mathfrak{p} \text{ with } [\mathfrak{p}] \in V(\mathfrak{a})\\ &\iff f \in \mathfrak{p} \text{ for every } \mathfrak{p} \supseteq \mathfrak{a}\end{aligned}$$

so

$$\begin{aligned}I(V(\mathfrak{a})) &= \bigcap_{\mathfrak{p}\supseteq\mathfrak{a}} \mathfrak{p}\\ &= \sqrt{\mathfrak{a}}\end{aligned}$$

(cf. Zariski-Samuel [**119**, vol. I, p. 151, Note II] or Atiyah-MacDonald [**20**, p. 9]).

(b) is then a straightforward verification. □

The points of Spec(R) *need not be closed*: In fact,

$$\begin{aligned}\overline{\{[\mathfrak{p}]\}} &= \text{smallest set } V(S)\text{, containing } [\mathfrak{p}]\text{, i.e., } S \subseteq \mathfrak{p}\\ &= V(S),\ \text{ with } S \text{ the largest subset of } \mathfrak{p}\\ &= V(\mathfrak{p}),\end{aligned}$$

hence:

$$[\mathfrak{p}'] \in \text{closure of } \{[\mathfrak{p}]\} \Longleftrightarrow \mathfrak{p}' \supseteq \mathfrak{p}.$$

Thus $[\mathfrak{p}]$ is closed if and only if $\mathfrak{p}$ is a maximal ideal. At the other extreme, if R is an integral domain then (0) is a prime ideal contained in every other prime ideal, so the closure of $[(0)]$ is the whole space Spec(R). Such a point is called a generic point of Spec(R).

Definition 1.7. If X is a topological space, a closed subset S is *irreducible* if S is not the union of two properly smaller closed subsets $S_1, S_2 \subsetneqq S$. A point x in a closed subset S is called a *generic point of* S if $S = \overline{\{x\}}$, and will be written η_S.

It is obvious that the closed sets $\overline{\{x\}}$ are irreducible. For Spec(R), we have the converse:

Proposition 1.8. *If* $S \subset \operatorname{Spec}(R)$ *is an irreducible closed subset, then* S *has a unique generic point* η_S.

Proof. I claim S irreducible $\implies$ $I(S)$ prime. In fact, if $f \cdot g \in I(S)$, then for all $x \in S$, $f(x) \cdot g(x) = 0$ in $\Bbbk(x)$, hence $f(x) = 0$ or $g(x) = 0$. Therefore

$$S = [S \cap V(f)] \cup [S \cap V(g)].$$

Since S is irreducible, S equals one of these: say $S = S \cap V(f)$. Then $f \equiv 0$ on S, hence $f \in I(S)$. Thus $I(S)$ is prime and

$$\begin{aligned}S &= V(I(S))\\ &= \text{closure of } [I(S)].\end{aligned}$$

As for uniqueness, if $[\mathfrak{p}_1]$, $[\mathfrak{p}_2]$ were two generic points of S, then $[\mathfrak{p}_1] \in V(\mathfrak{p}_2)$ and $[\mathfrak{p}_2] \in V(\mathfrak{p}_1)$, hence $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \mathfrak{p}_1$. □

Proposition 1.9. *Let* S *be a subset of* R. *Then*

$$\left[\operatorname{Spec}(R) = \bigcup_{f \in S} \operatorname{Spec}(R)_f\right] \Longleftrightarrow \left[1 \in \sum_{f \in S} f \cdot R, \textit{ the ideal generated by } S\right].$$

Proof. In fact,

$$\operatorname{Spec} R \setminus \bigcup_{f \in S} \operatorname{Spec}(R)_f = V\left(\sum_{f \in S} f \cdot R\right)$$

so apply (c) in Definition 1.3. □

Notice that $1 \in \sum_{f \in S} f \cdot R$ if and only if there is a finite set $f_1, \ldots, f_n \in S$ and elements $g_1, \ldots, g_n \in R$ such that

$$1 = \sum g_i \cdot f_i.$$

This equation is the algebraic analog of the partitions of unity which are so useful in differential geometry.

Corollary 1.10. Spec R *is quasi-compact*[1], *i.e., every open covering has a finite subcovering.*

[1] "compact" in the non-Hausdorff space.

PROOF. Because distinguished open sets form a basis, it suffices to check that every covering by distinguished opens has a finite subcover. Because of Proposition 1.9, this follows from the fact that

$$\left[1 \in \sum_{f \in S} f \cdot R\right] \Longrightarrow \left[1 \in \sum_{1=1}^{n} f_i \cdot R, \text{ some finite set } f_1, \ldots, f_n \in S\right].$$

□

When R is noetherian, even more holds:

DEFINITION 1.11. If X is a topological space, the following properties are equivalent:

i) the closed sets satisfy the descending chain condition,
ii) the open sets satisfy the ascending chain condition,
iii) every open set U is quasi-compact.

A space with these properties is called a *noetherian* topological space.

Because of property (b) of V in Definition 1.3, if R is a noetherian ring, then $\operatorname{Spec}(R)$ is a noetherian space and every open is quasi-compact!

The next big step is to "enlarge" the ring R into a whole sheaf of rings on $\operatorname{Spec} R$, written

$$\mathcal{O}_{\operatorname{Spec} R}$$

and called the *structure sheaf* of $\operatorname{Spec} R$. For background on sheaves, cf. Appendix to this chapter. To simplify notation, let $X = \operatorname{Spec} R$. We want to define rings

$$\mathcal{O}_X(U)$$

for every open set $U \subset X$. We do this first for distinguished open sets X_f. Then by Proposition 7 of the Appendix, there is a canonical way to define $\mathcal{O}_X(U)$ for general open sets. The first point is a generalization of Proposition 1.9:

LEMMA 1.12.

$$\left[X_f \subset \bigcup_{i=1}^{n} X_{g_i}\right] \Longleftrightarrow \left[\exists m \geq 1,\ a_i \in R \textit{ such that } f^m = \sum a_i g_i\right].$$

PROOF. The assertion on the left is equivalent to:

$$g_i([\mathfrak{p}]) = 0 \text{ all } i \Longrightarrow f([\mathfrak{p}]) = 0, \text{ for all primes } \mathfrak{p},$$

which is the same as

$$f \in I\Big(V\left(\sum g_i R\right)\Big) = \sqrt{\sum g_i R},$$

which is the assertion on the right. □

We want to define

$$\begin{aligned} \mathcal{O}_X(X_f) &= R_f \\ &= \text{localization of ring } R \text{ with respect to multiplicative system} \\ &\quad \{1, f, f^2, \ldots\};\ \text{or ring of fractions } a/f^n,\ a \in R,\ n \in \mathbb{Z}. \end{aligned}$$

In view of Lemma 1.12, if $X_f \subset X_g$, then $f^m = a \cdot g$ for some $m \geq 1$, $a \in R$, hence there is a canonical map

$$R_g \longrightarrow R_f.$$

(Explicitly, this is the map $b/g^n \mapsto ba^n/(ag)^n = ba^n/f^{nm}$.) In particular, if $X_f = X_g$, there are canonical maps $R_f \to R_g$ and $R_g \to R_f$ which are inverse to each other, so we can identify R_f and R_g. Therefore it is possible to define $\mathcal{O}_X(X_f)$ to be R_f. Furthermore, whenever $X_f \subset X_g$, we take the canonical map $R_g \to R_f$ to be the restriction map. Whenever $X_k \subset X_g \subset X_f$, we get a commutative diagram of canonical maps:

$$\begin{tikzcd} R_f \arrow[rr] \arrow[dr] & & R_k \\ & R_g \arrow[ur] & \end{tikzcd}$$

Thus we have defined a presheaf $\mathcal{O}_X$ on the distinguished open sets. We now verify the sheaf axioms:

Key lemma 1.13. *Assume* $X_f = \bigcup_{i=1}^N X_{g_i}$. *Then*

a) *if* $b/f^k \in R_f$ *maps to* 0 *in each localization* R_{g_i}, *then* $b/f^k = 0$,

b) *if* $b_i/g_i^{k_i} \in R_{g_i}$ *is a set of elements such that* $b_i/g_i^{k_i} = b_j/g_j^{k_j}$ *in* $R_{g_i g_j}$, *then* $\exists\, b/f^k \in R_f$ *which maps to* $b_i/g_i^{k_i}$ *for each* i.

Proof. The hypothesis implies that

$$f^m = \sum a_i g_i$$

for some $m \geq 1$ and $a_i \in R$. Raising this to a high power, one sees that for all n, there exists an m' and a_i' such that

$$f^{m'} = \sum a_i' g_i^n$$

too. To prove (a), if $b/f^k = 0$ in R_{g_i}, then $g_i^n \cdot b = 0$ for all i, if n is large enough. But then

$$f^{m'} \cdot b = \sum a_i'(g_i^n b) = 0$$

hence $b/f^k = 0$ in R_f. To prove (b), note that $b_i/g_i^{k_i} = b_j/g_j^{k_j}$ in $R_{g_i g_j}$ means:

$$(g_i g_j)^{m_{ij}} g_j^{k_j} b_i = (g_i g_j)^{m_{ij}} g_i^{k_i} b_j$$

for some $m_{ij} \geq 1$. If $M = \max m_{ij} + \max k_i$, then

$$\frac{b_i}{g_i^{k_i}} = \frac{\overbrace{b_i g_i^{M-k_i}}^{\text{call this } b_i'}}{g_i^M} \quad \text{in } R_{g_i},$$

and

$$\begin{aligned} g_j^M \cdot b_i' &= (g_j^{M-k_j} g_i^{M-k_i}) \cdot g_j^{k_j} b_i \\ &= (g_j^{M-k_j} g_i^{M-k_i}) \cdot g_i^{k_i} b_j, \quad \text{since } M-k_i \text{ and } M-k_j \text{ are } \geq m_{ij} \\ &= g_i^M \cdot b_j'. \end{aligned}$$

Now choose k and a_i' so that $f^k = \sum a_i' g_i^M$. Let $b = \sum a_j' b_j'$. Then I claim b/f^k equals b_i'/g_i^M in R_{g_i}. In fact,

$$\begin{aligned} g_i^M b &= \sum_j g_i^M a_j' b_j' \\ &= \sum_j g_j^M a_j' b_i' \\ &= f^k \cdot b_i'. \end{aligned}$$

$\square$

This means that $\mathcal{O}_X$ is a sheaf on distinguished open sets, hence by Proposition 7 of the Appendix it extends to a sheaf on all open sets of X. Its stalks can be easily computed:

if $x = [\mathfrak{p}] \in \operatorname{Spec} R$, then

$$\begin{aligned}\mathcal{O}_{x,X} &\underset{\text{def}}{=} \varinjlim_{\substack{\text{open } U\\ x\in U}} \mathcal{O}_X(U)\\ &= \varinjlim_{\substack{\text{dist. open } X_f\\ f(x)\neq 0}} \mathcal{O}_X(X_f)\\ &= \varinjlim_{f\in R\setminus\mathfrak{p}} R_f\\ &= R_{\mathfrak{p}}\end{aligned}$$

where $R_{\mathfrak{p}}$ as usual is the ring of fractions a/f, $a \in R$, $f \in R \setminus \mathfrak{p}$.

Now $R_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}\cdot R_{\mathfrak{p}}$, and residue field:

$$R_{\mathfrak{p}}/(\mathfrak{p}\cdot R_{\mathfrak{p}}) = (\text{quotient field of } R/\mathfrak{p}) = \Bbbk(x).$$

Thus the stalks of our structure sheaf are local rings and the evaluation of functions $f \in R$ defined above is just the map:

$$R = \mathcal{O}_X(X) \longrightarrow \mathcal{O}_{x,X} \longrightarrow \text{residue field } \Bbbk(x).$$

In particular, the evaluation of functions at x extends to all $f \in \mathcal{O}_X(U)$, for any open neighborhood U of x. Knowing the stalks of $\mathcal{O}_X$ we get the following explicit description of $\mathcal{O}_X$ on all open $U \subset X$:

$$\mathcal{O}_X(U) = \left\{ (s_{\mathfrak{p}}) \in \prod_{[\mathfrak{p}]\in U} R_{\mathfrak{p}} \;\middle|\; \begin{array}{l} U \text{ is covered by distinguished} \\ \text{open } X_{f_i}, \text{ and } \exists s_i \in R_{f_i} \\ \text{inducing } s_{\mathfrak{p}} \text{ whenever } f_i \notin \mathfrak{p} \end{array} \right\}.$$

The pairs $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ are called *affine schemes*. We give a name to one of the most important ones:

$$\begin{aligned}\mathbb{A}^n_R &= \left(\operatorname{Spec} R[X_1,\ldots,X_n], \mathcal{O}_{\operatorname{Spec} R[X_1,\ldots,X_n]}\right)\\ &= \text{affine } n\text{-space over } R.\end{aligned}$$

2. $\widetilde{M}$

An important aspect of the construction which defines the structure sheaf $\mathcal{O}_X$ is that it generalizes to a construction which associates a sheaf $\widetilde{M}$ on $\operatorname{Spec}(R)$ to every R-module M. To every distinguished open set X_f, we assign the localized module:

$$\begin{aligned}M_f &\underset{\text{def}}{=} \left\{ \begin{array}{l} \text{set of symbols } m/f^n,\ m \in M,\ n \in \mathbb{Z}, \\ \text{modulo the identification } m_1/f^{n_1} = m_2/f^{n_2} \\ \text{iff } f^{n_2+k}\cdot m_1 = f^{n_1+k}\cdot m_2, \text{ some } k \in \mathbb{Z} \end{array} \right\}\\ &= M \otimes_R R_f.\end{aligned}$$

We check (1) that if $X_f \subset X_g$, then there is a natural map $M_g \to M_f$, (2) that

$$\varinjlim_{[\mathfrak{p}]\in X_f} M_f = M_{\mathfrak{p}}$$

where

$$M_{\mathfrak{p}} \underset{\text{def}}{=} \left\{ \begin{array}{l} \text{set of symbols } m/g,\ m \in M,\ g \in R \setminus \mathfrak{p}, \\ \text{modulo the identification } m_1/g_1 = m_2/g_2 \\ \text{iff } hg_2m_1 = hg_1m_2, \text{ some } h \in R \setminus \mathfrak{p} \end{array} \right\}$$
$$= M \otimes_R R_{\mathfrak{p}},$$

and (3) that $X_f \mapsto M_f$ is a "sheaf on the distinguished open sets", i.e., satisfies Key lemma 1.13. (The proofs are word-for-word the same as the construction of $\mathcal{O}_X$.) We can then extend the map $X_f \mapsto M_f$ to a sheaf $U \mapsto \widetilde{M}(U)$ such that $\widetilde{M}(X_f) = M_f$ as before. Explicitly:

$$\widetilde{M}(U) = \left\{ s \in \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}} \;\middle|\; \text{"}s \text{ given locally by elements of } M_f\text{'s"} \right\}.$$

The sheaf $\widetilde{M}$ that we get is a sheaf of groups. But more than this, it is a sheaf of $\mathcal{O}_X$-modules in the sense of:

Definition 2.1. Let X be a topological space and $\mathcal{O}_X$ a sheaf of rings on X. Then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X is a sheaf $\mathcal{F}$ of abelian groups plus an $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ for all open sets U such that if $U \subset V$, then $\mathrm{res}_{V,U}\colon \mathcal{F}(V) \to \mathcal{F}(U)$ is a module homomorphism with respect to the ring homomorphism $\mathrm{res}_{V,U}\colon \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

In fact check that the restriction of the natural map

$$\prod_{[\mathfrak{p}] \in U} R_{\mathfrak{p}} \times \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}} \longrightarrow \prod_{[\mathfrak{p}] \in U} M_{\mathfrak{p}}$$

maps $\mathcal{O}_X(U) \times \widetilde{M}(U)$ into $\widetilde{M}(U)$, etc.

Moreover, the map $M \mapsto \widetilde{M}$ is a *functor*: given any R-homomorphism of R-modules:

$$\varphi\colon M \longrightarrow N$$

induces by localization:

$$\varphi_f\colon M_f \longrightarrow N_f, \quad \forall f \in R$$

hence

$$\varphi\colon \widetilde{M}(U) \longrightarrow \widetilde{N}(U), \quad \forall \text{ distinguished opens } U.$$

This extends uniquely to a map of sheaves:

$$\widetilde{\varphi}\colon \widetilde{M} \longrightarrow \widetilde{N},$$

which is clearly a homomorphism of these sheaves as $\mathcal{O}_X$-modules.

Proposition 2.2. *Let M, N be R-modules. Then the two maps*

$$\begin{array}{ccc} \mathrm{Hom}_R(M,N) & \rightleftarrows & \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}) \\ \varphi & \longmapsto & \widetilde{\varphi} \\ \left[\begin{array}{l} \psi(X), \text{ the map} \\ \text{on global sections} \end{array} \right] & \longleftarrow & \psi \end{array}$$

are inverse to each other, hence are isomorphisms.

Proof. Immediate. □

Corollary 2.3. *The category of R-modules is equivalent to the category of $\mathcal{O}_X$-modules of the form $\widetilde{M}$.*

This result enables us to translate much of the theory of R-modules into the theory of sheaves on $\operatorname{Spec} R$, and brings various geometric ideas into the theory of modules. (See for instance, Bourbaki [**27**, Chapter IV].)

But there are even stronger categorical relations between R-modules M and the sheaves $\widetilde{M}$: in fact, both the category of R-modules M and the category of sheaves of abelian groups on X are *abelian*, i.e., kernels and cokernels with the usual properties exist in both these categories (cf. Appendix to this chapter). In particular one can define exact sequences, etc. The fact is that $\widetilde{\ }$ preserves these operations too:

PROPOSITION 2.4. *Let $f\colon M \to N$ be a homomorphism of R-modules and let $K = \operatorname{Ker}(f)$, $C = \operatorname{Coker}(f)$. Taking $\widetilde{\ }$'s, we get maps of sheaves:*

$$\widetilde{K} \longrightarrow \widetilde{M} \xrightarrow{\widetilde{f}} \widetilde{N} \longrightarrow \widetilde{C}.$$

Then

(a) $\widetilde{K} = \operatorname{Ker}(\widetilde{f})$, *i.e.,* $\widetilde{K}(U) = \operatorname{Ker}[\widetilde{M}(U) \to \widetilde{N}(U)]$ *for all* U.
(b) $\widetilde{C} = \operatorname{Coker}(\widetilde{f})$*: by definition this means* $\widetilde{C}$ *is the sheafification of* $U \to \widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$*; but in our case, we get the stronger assertion:*

$$\widetilde{C}(X_a) = \operatorname{Coker}\left(\widetilde{M}(X_a) \to \widetilde{N}(X_a)\right), \quad \textit{all distinguished opens } X_a.$$

PROOF. Since $0 \to K \to M \to N \to C \to 0$ is exact, for all $a \in R$ the localized sequence:

$$0 \to K_a \to M_a \to N_a \to C_a \to 0$$

is exact (cf. Bourbaki [**27**, Chapter II, §2.4]; Atiyah-MacDonald [**20**, p. 39]). Therefore

$$0 \to \widetilde{K}(X_a) \to \widetilde{M}(X_a) \to \widetilde{N}(X_a) \to \widetilde{C}(X_a) \to 0$$

is exact for all a. It follows that $\widetilde{K}$ and $\operatorname{Ker}(\widetilde{f})$ are isomorphic on distinguished open sets, hence are isomorphic for all U (cf. Proposition 7 of the Appendix). Moreover it follows that the presheaf $\widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$ is already a sheaf on the distinguished open sets X_a, with values $\widetilde{C}(X_a)$; there is only one sheaf on all open sets U extending this, and this sheaf is on the one hand [sheafification of $U \to \widetilde{N}(U)/\widetilde{f}(\widetilde{M}(U))$] or $\operatorname{Coker}(\widetilde{f})$, (see the Appendix) and on the other hand it is $\widetilde{C}$. □

COROLLARY 2.5. *A sequence*

$$M \longrightarrow N \longrightarrow P$$

of R-modules is exact if and only if the sequence

$$\widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{P}$$

of sheaves is exact.

Moreover in both the category of R-modules and of sheaves of $\mathcal{O}_X$-modules there is an internal Hom: namely if M, N are R-modules, $\operatorname{Hom}_R(M,N)$ has again the structure of an R-module; and if $\mathcal{F}$, $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, there is a sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ whose global sections are $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ (cf. Appendix to this chapter). In some cases Proposition 2.2 can be strengthened:

PROPOSITION 2.6. *Let M, N be R-modules, and assume M is finitely presented, i.e., $\exists$ an exact sequence:*

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0.$$

Then

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N}) \cong \operatorname{Hom}_R(M,N)\widetilde{\ }.$$

PROOF. There is a natural map on all distinguished opens X_f:

$$\begin{aligned}
\mathrm{Hom}_R(M,N)\widetilde{\ }(X_f) &= \mathrm{Hom}_R(M,N)\otimes_R R_f \\
&\to \mathrm{Hom}_{R_f}(M_f,N_f) \\
&\cong \mathrm{Hom}_{\substack{\text{As sheaves of}\\ \mathcal{O}_{X_f}\text{-modules}\\ \text{on } X_f}}(\widetilde{M}|_{X_f},\widetilde{N}|_{X_f}), \text{ by Proposition 2.2} \\
&= \mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})(X_f).
\end{aligned}$$

When M is finitely presented, one checks that the arrow on the second line is an isomorphism using:

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(R^q,N) \longrightarrow \mathrm{Hom}_R(R^p,N)$$

hence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,N)\otimes_R R_f \arrow[r]\arrow[d] & \mathrm{Hom}_R(R^q,N)\otimes_R R_f \arrow[r]\arrow[d, "\approx"'] & \mathrm{Hom}_R(R^p,N)\otimes_R R_f \arrow[d, "\approx"'] \\
0 \arrow[r] & \mathrm{Hom}_{R_f}(M_f,N_f) \arrow[r] & \mathrm{Hom}_{R_f}(R_f^q,N_f) \arrow[r] & \mathrm{Hom}_{R_f}(R_f^p,N_f)
\end{tikzcd}$$

$\square$

Finally, we will need at one point later that $\widetilde{\ }$ commutes with direct sums, even infinite ones (Proposition-Definition 5.1):

PROPOSITION 2.7. *If $\{M_\alpha\}_{\alpha\in S}$ is any collection of R-modules, then*

$$\widetilde{\sum_{\alpha\in S} M_\alpha} = \sum_{\alpha\in S}\widetilde{M}_\alpha.$$

PROOF. Since each open set X_f is quasi-compact,

$$\begin{aligned}
\left(\sum \widetilde{M}_\alpha\right)(X_f) &= \sum\left(\widetilde{M}_\alpha(X_f)\right) \quad \text{cf. remark at the end of Appendix} \\
&= \sum (M_\alpha)_f \\
&= \left(\sum_\alpha M_\alpha\right)_f \\
&= \sum M_\alpha(X_f).
\end{aligned}$$

Therefore these sheaves agree on all open sets. $\square$

3. Schemes

We now proceed to the main definition:

DEFINITION 3.1. An *affine scheme* is a topological space X, plus a sheaf of rings $\mathcal{O}_X$ on X isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R. A *scheme* is a topological space X, plus a sheaf of rings $\mathcal{O}_X$ on X such that there exists an open covering $\{U_\alpha\}$ of X for which each pair $(U_\alpha, \mathcal{O}_X|_{U_\alpha})$ is an affine scheme.

Schemes in general have some of the peculiar topological properties of $\operatorname{Spec} R$. For instance:

PROPOSITION 3.2. *Every irreducible closed subset S of a scheme X is the closure of a unique point $\eta_S \in S$, called its generic point.*

PROOF. Reduce to the affine case, using: U open, $x\in U$, $x\in\overline{\{y\}} \implies y\in U$. $\square$

PROPOSITION 3.3. *If* $(X, \mathcal{O}_X)$ *is a scheme, and* $U \subset X$ *is an open subset, then* $(U, \mathcal{O}_X|_U)$ *is a scheme.*

PROOF. If $\{U_\alpha\}$ is an affine open covering of X, it suffices to show that $U \cap U_\alpha$ is a scheme for all α. But if $U_\alpha = \mathrm{Spec}(R_\alpha)$, then $U \cap U_\alpha$, like any open subset of $\mathrm{Spec}(R_\alpha)$ can be covered by smaller open subsets of the form $\mathrm{Spec}(R_\alpha)_{f_\beta}$, $f_\beta \in R_\alpha$. Therefore we are reduced to proving:

LEMMA 3.4. *For all rings* R *and* $f \in R$,

$$\Big((\mathrm{Spec}\,R)_f, \mathcal{O}_{\mathrm{Spec}\,R}|_{(\mathrm{Spec}\,R)_f}\Big) \cong \Big(\mathrm{Spec}(R_f), \mathcal{O}_{\mathrm{Spec}(R_f)}\Big),$$

hence $(\mathrm{Spec}\,R)_f$ *is itself an affine scheme.*

PROOF OF LEMMA 3.4. Let $i\colon R \to R_f$ be the canonical map. Then if $\mathfrak{p}$ is a prime ideal of R, such that $f \notin \mathfrak{p}$, $i(\mathfrak{p})\cdot R_f$ is a prime ideal of R_f; and if $\mathfrak{p}$ is a prime ideal of R_f, $i^{-1}(\mathfrak{p})$ is a prime ideal of R not containing f. These maps set up a bijection between $\mathrm{Spec}(R)_f$ and $\mathrm{Spec}(R_f)$ (cf. Zariski-Samuel [**119**, vol. I, p. 223]). This is a homeomorphism since the distinguished open sets

$$\mathrm{Spec}(R)_{fg} \subset \mathrm{Spec}(R)_f$$

and

$$\mathrm{Spec}(R_f)_g \subset \mathrm{Spec}(R_f)$$

correspond to each other. But the sections of the structure sheaves $\mathcal{O}_{\mathrm{Spec}(R)}$ and $\mathcal{O}_{\mathrm{Spec}(R_f)}$ on these two open sets are both isomorphic to R_{fg}. Therefore, these rings of sections can be naturally identified with each other and this sets up an isomorphism of (i) the restriction of $\mathcal{O}_{\mathrm{Spec}(R)}$ to $\mathrm{Spec}(R)_f$, and (ii) $\mathcal{O}_{\mathrm{Spec}(R_f)}$ compatible with the homeomorphism of underlying spaces. □

□

Since all schemes are locally isomorphic to a $\mathrm{Spec}(R)$, it follows from §1 that the stalks $\mathcal{O}_{x,X}$ of $\mathcal{O}_X$ are *local* rings. As in §1, define $\Bbbk(x)$ to be the residue field $\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}$ where $\mathfrak{m}_{x,X} =$ maximal ideal, and for all $f \in \Gamma(U, \mathcal{O}_X)$ and $x \in U$, define $f(x) =$ image of f in $\Bbbk(x)$. We can now make the set of schemes into the objects of a category:

DEFINITION 3.5. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two schemes, a *morphism* from X to Y is a continuous map

$$f\colon X \longrightarrow Y$$

plus a collection of homomorphisms:

$$\Gamma(V, \mathcal{O}_Y) \xrightarrow{f_V^*} \Gamma(f^{-1}(V), \mathcal{O}_X)$$

for every open set $V \subset Y$[2], such that

a) whenever $V_1 \subset V_2$ are two open sets in Y, then the diagram:

$$\begin{tikzcd} \Gamma(V_2, \mathcal{O}_Y) \arrow[r, "f_{V_2}^*"] \arrow[d, "\mathrm{res}"] & \Gamma(f^{-1}(V_2), \mathcal{O}_X) \arrow[d, "\mathrm{res}"] \\ \Gamma(V_1, \mathcal{O}_Y) \arrow[r, "f_{V_1}^*"] & \Gamma(f^{-1}(V_1), \mathcal{O}_X) \end{tikzcd}$$

commutes, and

[2]Equivalently, a homomorphism of sheaves

$$\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$$

in the notation introduced at the end of the Appendix to this chapter.

b) because of (a), then f_V^*'s pass in the limit to homomorphisms on the stalks:

$$f_x^* \colon \mathcal{O}_{y,Y} \longrightarrow \mathcal{O}_{x,X}$$

for all $x \in X$ and $y = f(x)$; then we require that f_x^* be a local homomorphism, i.e., if $a \in \mathfrak{m}_{y,Y}$ = the maximal ideal of $\mathcal{O}_{y,Y}$, then $f_x^*(a) \in \mathfrak{m}_{x,X}$ = the maximal ideal of $\mathcal{O}_{x,X}$. Equivalently, if $a(y) = 0$, then $f_x^*(a)(x) = 0$.

To explain this rather elaborate definition, we must contrast the situation among schemes with the situation with differentiable or analytic manifolds. In the case of differentiable or analytic manifolds X, X also carries a "structure sheaf" $\mathcal{O}_X$, i.e.,

$$\mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{ring of real-valued differentiable or} \\ \text{complex-valued analytic functions on } U \end{array} \right\}.$$

Moreover, to define a differentiable or analytic map from X to Y, one can ask for a continuous map $f \colon X \to Y$ with the extra property that:

for all open $V \subset Y$ and all $a \in \mathcal{O}_Y(V)$, the compositie function $a \circ f$ on $f^{-1}(V)$ should be in $\mathcal{O}_X(f^{-1}(V))$.

Then we get a homomorphism:

$$\begin{aligned} \Gamma(V, \mathcal{O}_Y) &\longrightarrow \Gamma(f^{-1}(V), \mathcal{O}_X) \\ a &\longmapsto a \circ f \end{aligned}$$

automatically from the map f on the topological spaces. Note that this homomorphism does have properties (a) and (b) of our definition. (a) is obvious. To check (b), note that the stalks $\mathcal{O}_{x,X}$ of the structure sheaf are the rings of *germs* of differentiable or analytic functions at the point $x \in X$. Moreover, $\mathfrak{m}_{x,X}$ is the ideal of germs a such that $a(x) = 0$, and

$$\begin{aligned} \mathcal{O}_{x,X} &\cong \mathfrak{m}_{x,X} \oplus \mathbb{R} \cdot 1_x \quad \text{(differentiable case)} \\ \mathcal{O}_{x,X} &\cong \mathfrak{m}_{x,X} \oplus \mathbb{C} \cdot 1_x \quad \text{(anallytic case)} \end{aligned}$$

where 1_x represents the germ at x of the constant function $a \equiv 1$ (i.e., every germ a equals $a(x) \cdot 1_x + b$, where $b(x) = 0$). Then given a differentiable or analytic map $f \colon X \to Y$, the induced map on stalks $f_x^* \colon \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ is just the map on germs $a \longmapsto a \circ f$, hence

$$\begin{aligned} a \in \mathfrak{m}_{y,Y} &\iff a(y) = 0 \\ &\iff a \circ f(x) = 0 \\ &\iff f_x^* a \in \mathfrak{m}_{x,X}. \end{aligned}$$

The new feature in the case of schemes is that the structure sheaf $\mathcal{O}_X$ is not equal to a sheaf of functions from X to any field k: it is a sheaf of rings, possibly with nilpotent elements, and whose "values" $a(x)$ lie in different fields $\Bbbk(x)$ as x varies. Therefore the continuous map $f \colon X \to Y$ does not induce a map $f^* \colon \mathcal{O}_Y \to \mathcal{O}_X$ automatically. However property (b) does imply that f^* is compatible with "evaluation" of the elements $a \in \mathcal{O}_Y(U)$, i.e., the homomorphism f_x^* induces one on the residue fields:

$$\Bbbk(y) = \mathcal{O}_{y,Y}/\mathfrak{m}_{y,Y} \xrightarrow{f_x^* \text{ modulo maximal ideals}} \mathcal{O}_{x,X}/\mathfrak{m}_{x,X} = \Bbbk(x).$$

Note that it is injective, (like all maps of fields), and that using it (b) can be strengthened to:

(b′) For all $V \subset Y$, and $x \in f^{-1}(V)$, let $y = f(x)$ and identify $\Bbbk(y)$ with its image in $\Bbbk(x)$ by the above map. Then

$$f^*(a)(x) = a(y)$$

for all $a \in \Gamma(V, \mathcal{O}_Y)$.

Given two morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, we can define their composition $g \circ f \colon X \longrightarrow Z$ in an obvious way. This gives us the *category of schemes.* Also very useful are the related categories of "schemes over S".

DEFINITION 3.6. Fix a scheme S, sometimes referred to as the base scheme. Then a scheme *over* S, written X/S, is a scheme X plus a morphism $p_X \colon X \to S$. If $S = \mathrm{Spec}(R)$, we call this simply a scheme over R or X/R. If X/S and Y/S are two schemes over S, an S-morphism from X/S to Y/S is a morphism $f \colon X \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & S & \end{tikzcd}$$

commutes.

The following theorem is absolutely crucial in tying together these basic concepts:

THEOREM 3.7. *Let X be a scheme and let R be a ring. To every morphism $f \colon X \to \mathrm{Spec}(R)$, associate the homomorphism:*

$$R \cong \Gamma\left(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)}\right) \xrightarrow{f^*} \Gamma(X, \mathcal{O}_X).$$

Then this induces a bijection between $\mathrm{Hom}(X, \mathrm{Spec}(R))$ *in the category of schemes and* $\mathrm{Hom}(R, \Gamma(X, \mathcal{O}_X))$ *in the category of rings.*

PROOF. For all f's, let $A_f \colon R \to \Gamma(X, \mathcal{O}_X)$ denote the induced homomorphism. We first show that f is determined by A_f. We must begin by showing how the map of point sets $X \to \mathrm{Spec}(R)$ is determined by A_f. Suppose $x \in X$. The crucial fact we need is that since $\mathfrak{p} = \{a \in R \mid a([\mathfrak{p}]) = 0\}$, a point of $\mathrm{Spec}(R)$ is determined by the ideal of elements of R vanishing at it. Thus $f(x)$ is determined if we know $\{a \in R \mid a(f(x)) = 0\}$. But this equals $\{a \in R \mid f_x^*(a)(x) = 0\}$, and $f_x^*(a)$ is obtained by restricting $A_f(a)$ to $\mathcal{O}_{x,X}$. Therefore

$$f(x) = [\{a \in R \mid (A_f a)(x) = 0\}]\,.$$

Next we must show that the maps f_U^* are determined by A_f for all open sets $U \subset \mathrm{Spec}(R)$. Since f^* is a map of sheaves, it is enough to show this for a basis of open sets (in fact, if $U = \bigcup U_\alpha$ and $s \in \Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)})$, then $f_U^*(s)$ is determined by its restrictions to the sets $f^{-1}(U_\alpha)$, and these equal $f_{U_\alpha}^*(\mathrm{res}_{U,U_\alpha} s)$). Now let $Y = \mathrm{Spec}(R)$ and consider f^* for the distinguished open set Y_b. It makes the diagram

$$\begin{tikzcd} \Gamma(f^{-1}(Y_b), \mathcal{O}_X) & \Gamma(Y_b, \mathcal{O}_Y) = R_b \arrow[l, "f_{Y_b}^*"'] \\ \Gamma(X, \mathcal{O}_X) \arrow[u, "\mathrm{res}"'] & \Gamma(Y, \mathcal{O}_Y) = R \arrow[l, "A_f"] \arrow[u, "\mathrm{res}"'] \end{tikzcd}$$

commutative. Since these are ring homomorphisms, the map on the ring of fractions R_b is determined by that on R: thus A_f determines everything.

Finally any homomorphism $A \colon R \to \Gamma(X, \mathcal{O}_X)$ comes from some morphism f. To prove this, we first reduce to the case when X is affine. Cover X by open affine sets X_α. Then A induces homomorphisms

$$A_\alpha \colon R \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\mathrm{res}} \Gamma(X_\alpha, \mathcal{O}_{X_\alpha}).$$

Assuming the result in the affine case, there is a morphism $f_\alpha\colon X_\alpha \to \mathrm{Spec}(R)$ such that $A_\alpha = A_{f_\alpha}$. On $X_\alpha \cap X_\beta$, f_α and f_β agree because the homomorphisms

$$\begin{tikzcd}
 & \Gamma(X_\alpha, \mathcal{O}_X) \arrow[dr, "\mathrm{res}"] & \\
R \arrow[ur, "A_\alpha"] \arrow[dr, "A_\beta"'] & & \Gamma(X_\alpha \cap X_\beta, \mathcal{O}_X) \\
 & \Gamma(X_\beta, \mathcal{O}_X) \arrow[ur, "\mathrm{res}"'] &
\end{tikzcd}$$

agree and we know that the morphism is determined by the homomorphism. Hence the f_α patch together to a morphism $f\colon X \to \mathrm{Spec}(R)$, and one checks that A_f is exactly A.

Now let $A\colon R \to B$ be a homomorphism. We want a morphism

$$f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(R).$$

Following our earlier comments, we have no choice in defining f: for all points $[\mathfrak{p}] \in \mathrm{Spec}(B)$,

$$f([\mathfrak{p}]) = [A^{-1}(\mathfrak{p})].$$

This is continuous since for all ideals $\mathfrak{a} \subseteq R$, $f^{-1}(V(\mathfrak{a})) = V(A(\mathfrak{a})\cdot B)$. Moreover if $U = \mathrm{Spec}(R)_a$, then $f^{-1}(U) = \mathrm{Spec}(B)_{A(a)}$, so for f_U^* we need a map $R_a \to B_{A(a)}$. We take the localization of A. These maps are then compatible with restriction, i.e.,

$$\begin{tikzcd}
R_a \arrow[r] \arrow[d] & B_{A(a)} \arrow[d] \\
R_{ab} \arrow[r] & B_{A(a)\cdot A(b)}
\end{tikzcd}$$

commutes. Hence they determine a sheaf map (in fact, if $U = \bigcup U_\alpha$, U_α distinguished, and $s \in \Gamma(U, \mathcal{O}_{\mathrm{Spec}(R)})$ then the elements $f_{U_\alpha}^*(\mathrm{res}_{U,U_\alpha} s)$ patch together to give an element $f_U^*(s)$ in $\Gamma(f^{-1}(U), \mathcal{O}_{\mathrm{Spec}(B)})$). From our definition of f, it follows easily that f^* on $\mathcal{O}_{[A^{-1}\mathfrak{p}]}$ takes the maximal ideal $\mathfrak{m}_{[A^{-1}\mathfrak{p}]}$ into $\mathfrak{m}_{[\mathfrak{p}]}$. □

COROLLARY 3.8. *The category of affine schemes is equivalent to the category of commutative rings with unit, with arrows reversed.*

COROLLARY 3.9. *If X is a scheme and R is a ring, to make X into a scheme over R is the same thing as making the sheaf of rings $\mathcal{O}_X$ into a sheaf of R-algebras. In particular, there is a unique morphism of every scheme to* $\mathrm{Spec}\,\mathbb{Z}$*: "*$\mathrm{Spec}\,\mathbb{Z}$ *is a final object in the category of schemes"!*

Another point of view on schemes over a given ring A is to ask: what is the "raw data" needed to define a scheme X over $\mathrm{Spec}\,A$? It turns out that such an X can be given by a collection of polynomials with coefficients in A and under suitable finiteness conditions (see Definition II.2.6) this is the most *effective* way to construct a scheme. In fact, first cover X by affine open sets U_α (possibly an infinite set) and let $U_\alpha = \mathrm{Spec}\,R_\alpha$. Then each R_α is an A-algebra. Represent R_α as a quotient of a polynomial ring:

$$R_\alpha = A[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, f_\gamma^{(\alpha)}, \ldots)$$

where the $f_\gamma^{(\alpha)}$ are polynomials in the variables $X_\beta^{(\alpha)}$. The scheme X results from glueing a whole lot of isomorphic localizations $(U_{\alpha_1})_{g_{\alpha_1\alpha_2\nu}}$ and $(U_{\alpha_2})_{h_{\alpha_1\alpha_2\nu}}$, and these isomorphisms result from

A-algebra isomorphisms:

$$A\left[\dots, X_\beta^{\alpha_1}, \dots, \frac{1}{g_{\alpha_1\alpha_2\nu}(X_\beta^{(\alpha_1)})}\right] \Big/ (\dots, f_\gamma^{(\alpha_1)}, \dots)$$
$$\cong A\left[\dots, X_\beta^{\alpha_2}, \dots, \frac{1}{h_{\alpha_1\alpha_2\nu}(X_\beta^{(\alpha_2)})}\right] \Big/ (\dots, f_\gamma^{(\alpha_2)}, \dots)$$

given by

$$X_{\beta_2}^{(\alpha_2)} = \frac{\phi_{\alpha_1\alpha_2\nu\beta_2}(\dots, X_\beta^{(\alpha_1)}, \dots)}{(g_{\alpha_1\alpha_2\nu})^{N_{\alpha_1\alpha_2\nu\beta_2}}}$$
$$X_{\beta_1}^{(\alpha_1)} = \frac{\psi_{\alpha_1\alpha_2\nu\beta_1}(\dots, X_\beta^{(\alpha_2)}, \dots)}{(h_{\alpha_1\alpha_2\nu})^{M_{\alpha_1\alpha_2\nu\beta_1}}}.$$

Thus the collection of polynomials f, g, h, ϕ and ψ with coefficients in A explicitly describes X. In reasonable cases, this collection is finite and gives the most effective way of "writing out" the scheme X.

It is much harder to describe explicitly the set of morphisms from $\operatorname{Spec} R$ to X than it is to describe the morphisms from X to $\operatorname{Spec} R$. In one case this can be done however:

PROPOSITION 3.10. *Let R be a local ring with maximal ideal M. Let X be a scheme. To every morphism $f\colon \operatorname{Spec} R \to X$ associate the point $x = f([M])$ and the homomorphism*

$$f_x^*\colon \mathcal{O}_{x,X} \longrightarrow \mathcal{O}_{[M],\operatorname{Spec} R} = R.$$

Then this induces a bijection between $\operatorname{Hom}(\operatorname{Spec} R, X)$ and the set of pairs (x,ϕ), where $x \in X$ and $\phi\colon \mathcal{O}_{x,X} \to R$ is a local homomorphism.

(Proof left to the reader.)

This applies for instance to the case $R = K$ a field, in which case $\operatorname{Spec} K$ consists in only one point $[M] = [(0)]$. A useful example is:

COROLLARY 3.11. *For every $x \in X$, there is a canonical homomorphism*

$$i_x\colon \operatorname{Spec} \Bbbk(x) \longrightarrow X$$

defined by requiring that $\operatorname{Image}(i_x) = x$, and that

$$i_x^*\colon \mathcal{O}_{x,X} \to \mathcal{O}_{[(0)],\operatorname{Spec} \Bbbk(x)} = \Bbbk(x)$$

be the canonial map. For every field k, every morphism

$$f\colon \operatorname{Spec} k \longrightarrow X$$

factors uniquely:

$$\operatorname{Spec} k \xrightarrow{g} \operatorname{Spec} \Bbbk(x) \xrightarrow{i_x} X$$

where $x = \operatorname{Image}(f)$ and g is induced by an inclusion $\Bbbk(x) \to k$.

4. Products

There is one exceedingly important and very elementary existence theorem in the category of schemes. This asserts that arbitrary fibre products exist:

Recall that if morphisms:

$$\begin{tikzcd}
X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\
& S &
\end{tikzcd}$$

are given, a *fibre product* is a commutative diagram

$$\begin{tikzcd}
& X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\
& S &
\end{tikzcd}$$

with the obvious universal property: i.e., given any commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\
& S &
\end{tikzcd}$$

there is a unique morphism $t\colon Z \to X \times_S Y$ such that $q_1 = p_1 \circ t$, $q_2 = p_2 \circ t$. The fibre product is unique up to canonical isomorphism. When S is the final object $\operatorname{Spec}\mathbb{Z}$ in the category of schemes, we drop the S and write $X \times Y$ for the product.

THEOREM 4.1. *If A and B are C-algebras, let the diagram of affine schemes*

$$\begin{tikzcd}
& \operatorname{Spec}(A \otimes_C B) \arrow[dl] \arrow[dr] & \\
\operatorname{Spec}(A) \arrow[dr] & & \operatorname{Spec}(B) \arrow[dl] \\
& \operatorname{Spec}(C) &
\end{tikzcd}$$

be defined by the canonical homomorphisms $C \to A$, $C \to B$, $A \to A \otimes_C B$ $(a \mapsto a \otimes 1)$, $B \to A \otimes_C B$ $(b \mapsto 1 \otimes b)$. This makes $\operatorname{Spec}(A \otimes_C B)$ *a fibre product of* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(B)$ *over* $\operatorname{Spec}(C)$.

THEOREM 4.2. *Given any morphisms $r\colon X \to S$, $s\colon Y \to S$, a fibre product exists.*

PROOF OF THEOREM 4.1. It is well known that in the diagram (of solid arrows):

$$\begin{tikzcd}
& A \arrow[dr] \arrow[drr, dashed] & & \\
C \arrow[ur] \arrow[dr] & & A \otimes_C B \arrow[r] & D \\
& B \arrow[ur] \arrow[urr, dashed] & &
\end{tikzcd}$$

the tensor product has the universal mapping property indicated by dotted arrows, i.e., is the "direct sum" in the category of commutative C-algebras, or the "fibre sum" in the category of commutative rings. Dually, this means that $\operatorname{Spec}(A \otimes_C B)$ is the fibre product in the category of *affine* schemes. But if T is an arbitrary scheme, then by Theorem 3.7, every morphism of T into any affine scheme $\mathrm{Spec}(E)$ factors uniquely through $\mathrm{Spec}(\Gamma(T, \mathcal{O}_T))$:

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & \mathrm{Spec}(E) \\ & \mathrm{Spec}(\Gamma(T, \mathcal{O}_T)) \arrow[ur, dashed] & \end{tikzcd}$$

Using this, it follows immediately that $\mathrm{Spec}(A \otimes_C B)$ is the fibre product in the category of all schemes. $\square$

Theorem 4.1 implies for instance that:

$$\mathbb{A}^n_R \cong \mathbb{A}^n_{\mathbb{Z}} \times \mathrm{Spec}\, R.$$

Proof of Theorem 4.2. There are two approaches to this. The first is a *patching argument* that seems quite straightforward and "mechanical", but whose details are really remarkably difficult. The second involves the direct construction of $X \times_S Y$ as a local ringed space and then the verification that locally it is indeed the same product as that given by Theorem 4.1. We will sketch both. For the first, the main point to notice is this: suppose

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\ & S & \end{tikzcd}$$

is some fibre product and suppose that $X_\circ \subset X$, $Y_\circ \subset Y$ and $S_\circ \subset S$ are open subsets. Assume that $r(X_\circ) \subset S_\circ$ and $s(Y_\circ) \subset S_\circ$. Then the open subset

$$p_1^{-1}(X_\circ) \cap p_2^{-1}(Y_\circ) \subset X \times_S Y$$

is always the fibre product of $X_\circ$ and $Y_\circ$ over $S_\circ$. This being so, it is clear how we must set about constructing a fibre product: first cover S by open affines:

$$\mathrm{Spec}(C_k) = W_k \subset S.$$

Next, cover $r^{-1}(W_k)$ and $s^{-1}(W_k)$ by open affines:

$$\mathrm{Spec}(A_{k,i}) = U_{k,i} \subset X,$$
$$\mathrm{Spec}(B_{k,j}) = V_{k,j} \subset Y.$$

Then the affine schemes:

$$\mathrm{Spec}(A_{k,i} \otimes_{C_k} B_{k,j}) = \Phi_{k,i,j}$$

must make an open affine covering of $X \times_S Y$ if it exists at all. To patch together $\Phi_{k,i,j}$ and $\Phi_{k',i',j'}$, let p_1, p_2, and p_1', p_2' stand for the canonical projections of $\Phi_{k,i,j}$ and $\Phi_{k',i',j'}$ onto its factors. Then one must next check that the open subsets:

$$p_1^{-1}(U_{k,i} \cap U_{k',i'}) \cap p_2^{-1}(V_{k,j} \cap V_{k',j'}) \subset \Phi_{k,i,j}$$

and

$$(p_1')^{-1}(U_{k',i'} \cap U_{k,i}) \cap (p_2')^{-1}(V_{k',j'} \cap V_{k,j}) \subset \Phi_{k',i',j'}$$

are both fibre products of $U_{k,i} \cap U_{k',i'}$ and $V_{k,j} \cap V_{k',j'}$ over S. Hence they are canonically isomorphic and can be patched. Then you have to check that everything is consistent at triple overlaps. Finally you have to check the universal mapping property. All this is in some sense obvious but remarkably confusing unless one takes a sufficiently categorial point of view. For details, cf. EGA [**1**, Chapter I, pp. 106–107].

The second proof involves explicitly constructing $X \times_S Y$ as a local ringed space. To motivate the construction note that if $z \in X \times_S Y$ lies over $x \in X$, $y \in Y$ and $s \in S$, then the residue fields of the four points lie in a diagram:

$$\begin{tikzcd} & \mathbb{k}(z) & \\ \mathbb{k}(x) \arrow[ur, hook, "p_1^*"] & & \mathbb{k}(y) \arrow[ul, hook', "p_2^*"'] \\ & \mathbb{k}(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

From Theorem 4.1, one sees that the local rings of $X \times_S Y$ are generated by tensor product of the local rings of X and Y and this implies that in the above diagram $\mathbb{k}(z)$ is the quotient field of its subring $\mathbb{k}(x) \cdot \mathbb{k}(y)$, i.e., $\mathbb{k}(z)$ is a compositum of $\mathbb{k}(x)$ and $\mathbb{k}(y)$ over $\mathbb{k}(s)$. We may reverse these conclusions and use them as a basis of a definition of $X \times_S Y$;

i) As a point set, $X \times_S Y$ is the set of 5-tuples (x, y, L, α, β) where

$$x \in X, \quad y \in Y,$$

lie over the same point $s \in S$ and

$$L = \text{a field extension of } \mathbb{k}(s)$$
$$\alpha,\ \beta \text{ are homomorphisms:}$$

$$\begin{tikzcd} & L & \\ \mathbb{k}(x) \arrow[ur, hook, "\alpha"] & & \mathbb{k}(y) \arrow[ul, hook', "\beta"'] \\ & \mathbb{k}(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

such that

$$L = \text{quotient field of } \mathbb{k}(x) \cdot \mathbb{k}(y).$$

Two such points are equal if the points x, y on X and Y are equal and the corresponding diagrams of fields are isomorphic.

ii) As a topological space, a basis of open sets is given by the distinguished open sets

$$\mathcal{U}(V, W, \{f_l\}, \{g_l\})$$

where

$$\begin{aligned} &V \subset X \text{ is affine open} \\ &W \subset Y \text{ is affine open} \\ &f_l \in \mathcal{O}_X(V) \\ &g_l \in \mathcal{O}_Y(W) \\ \mathcal{U} = \{&(x, y, L, \alpha, \beta) \mid x \in V, y \in W, \\ &\sum_l \alpha(f_l) \cdot \beta(g_l) \neq 0 \text{ (this sum taken in } L)\}. \end{aligned}$$

iii) The structure sheaf $\mathcal{O}_{X \times_S Y}$ is defined as a certain sheaf of maps from open sets in $X \times_S Y$ to:

$$\coprod_{x,y,L,\alpha,\beta} \mathcal{O}_{(x,y,L,\alpha,\beta)}$$

where

$$\mathcal{O}_{(x,y,L,\alpha,\beta)} = \left[\begin{array}{l} \text{localization of } \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{s,S}} \mathcal{O}_{y,Y} \\ \text{at } \mathfrak{p} = \operatorname{Ker}(\mathcal{O}_x \otimes_{\mathcal{O}_s} \mathcal{O}_y \xrightarrow[\alpha\otimes\beta]{} L) \end{array} \right]$$

(i.e., the elements of the sheaf will map points $(x, y, L, \alpha, \beta) \in X \times_S Y$ to elements of the corresponding ring $\mathcal{O}_{(x,y,L,\alpha,\beta)}$.) The sheaf is defined to be those maps which locally are given by expressions

$$\begin{aligned} &\frac{\sum f_l \otimes g_l}{\sum f'_l \otimes g'_l} \\ &f_l, f'_l \in \mathcal{O}_X(V) \\ &g_l, g'_l \in \mathcal{O}_Y(W) \\ &\text{on open sets } \mathcal{U}(V, W, \{f'_l\}, \{g'_l\}). \end{aligned}$$

This certainly gives us a local ringed space, but it must be proven to be a scheme and to be the fibre product. We will not give details. For the first, one notes that the construction is local on X and Y and hence it suffices to prove that if $X = \operatorname{Spec} R'$, $Y = \operatorname{Spec} S'$ and $S = \operatorname{Spec} A$, then the local ringed space $X \times_S Y$ constructed above is simply $\operatorname{Spec}(R' \otimes_A S')$. The first step then is to verify:

LEMMA 4.3. *The set of prime ideals of $R \otimes_A S$ is in one-to-one correspondence with the set of 5-tuples $(\mathfrak{p}_R, \mathfrak{p}_S, L, \alpha, \beta)$ where $\mathfrak{p}_R \subset R$ and $\mathfrak{p}_S \subset S$ are prime ideals with the same inverse image $\mathfrak{p}_A \subset A$ and (L, α, β) is a compositum of the quotient fields of $R/\mathfrak{p}_R$, $S/\mathfrak{p}_S$ over $A/\mathfrak{p}_A$.*

The proof is straightforward.

COROLLARY 4.4 (of proof). *As a point set, $X \times_S Y$ is the set of pairs of points $x \in X$, $y \in Y$ lying over the same point of S, plus a choice of compositum of their residue fields up to isomorphisms:*

$$\begin{tikzcd} & L & \\ \Bbbk(x) \arrow[ur, hook, "\alpha"] & & \Bbbk(y) \arrow[ul, hook', "\beta"'] \\ & \Bbbk(s) \arrow[ul, hook', "r^*"] \arrow[ur, hook, "s^*"'] & \end{tikzcd}$$

□

Summarizing the above proof, we can give in a special case the following "explicit" idea of what fibre product means: Suppose we are in the situation

$$\begin{tikzcd} X \arrow[dr, "r"'] & & \operatorname{Spec}(B) \arrow[dl, "s"] \\ & \operatorname{Spec}(A) & \end{tikzcd}$$

and that $X = \bigcup U_\alpha$, U_α affine. Then each U_α is $\operatorname{Spec} R_\alpha$ and via r^*,

$$R_\alpha = A[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, f_\gamma^{(\alpha)}, \ldots)$$

as in §3, where the $f_\gamma^{(\alpha)}$ are polynomials in the variables $X_\beta^{(\alpha)}$. Represent the glueing between the U_α's by a set of polynomials $g_{\alpha_1,\alpha_2,\nu}$, $h_{\alpha_1,\alpha_2,\nu}$, $\phi_{\alpha_1,\alpha_2,\nu,\beta_2}$ and $\psi_{\alpha_1,\alpha_2,\nu,\beta_1}$ as in §3 again. Let

s correspond to a homomorphism $\sigma\colon A \to B$. If f is a polynomial over A, let ${}^{\sigma}f$ denote the polynomial over B gotten by applying σ to its coefficients. Then

$$\begin{aligned} X \times_{\operatorname{Spec} A} \operatorname{Spec} B &\cong \bigcup_{\alpha} U_\alpha \times_{\operatorname{Spec} A} \operatorname{Spec} B \\ &\cong \bigcup_{\alpha} \operatorname{Spec}\left[\left(A[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, f_\gamma^{(\alpha)}, \ldots)\right) \otimes_A B\right] \\ &\cong \bigcup_{\alpha} \operatorname{Spec}\left[B[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, {}^{\sigma}f_\gamma^{(\alpha)}, \ldots)\right]. \end{aligned}$$

In other words, the new scheme $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ is gotten by glueing corresponding affines, each defined by the new equations in the same variables gotten by pushing their coefficients from A to B via σ. Moreover, it is easy to see that the identification on $(U_\alpha \times_{\operatorname{Spec} A} \operatorname{Spec} B) \cap (U_\beta \times_{\operatorname{Spec} A} \operatorname{Spec} B)$ is gotten by glueing the distinguished opens ${}^{\sigma}g_{\alpha_1,\alpha_2,\nu} \neq 0$ and ${}^{\sigma}h_{\alpha_1,\alpha_2,\nu} \neq 0$ by isomorphisms given by the polynomials ${}^{\sigma}\phi$ and ${}^{\sigma}\psi$. Or we may simply say that the collection of polynomials ${}^{\sigma}f$, ${}^{\sigma}g$, ${}^{\sigma}h$, ${}^{\sigma}\phi$, ${}^{\sigma}\psi$ with coefficients in B explicitly describes $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ by the same recipe used for X.

We can illustrate this further by a very important special case of fibre products: suppose $f\colon X \to Y$ is any morphism and $y \in Y$. Consider the fibre product:

$$\begin{tikzcd} X \times_Y \operatorname{Spec} \Bbbk(y) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} \Bbbk(y) \arrow[r, "i_Y"] & Y \end{tikzcd}$$

Definition 4.5. Denote $X \times_Y \operatorname{Spec} \Bbbk(y)$ by $f^{-1}(y)$ and call it the *fibre of f over y*.

To describe $f^{-1}(y)$ explicitly, let $U \subset Y$ be an affine neighborhood of y, let $U = \operatorname{Spec}(R)$, and $y = [\mathfrak{p}]$. It is immediate that the fibre product $X \times_Y U$ is just the open subscheme $f^{-1}(U)$ of X, and by associativity of fibre products, $f^{-1}(y) \cong f^{-1}(U) \times_U \operatorname{Spec} \Bbbk(y)$. Now let $f^{-1}(U)$ be covered by affines:

$$\begin{aligned} V_\alpha &= \operatorname{Spec}(S_\alpha) \\ S_\alpha &\cong R[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, f_\gamma^{(\alpha)}, \ldots). \end{aligned}$$

Then $f^{-1}(y)$ is covered by affines

$$\begin{aligned} V_\alpha \cap f^{-1}(y) &= \operatorname{Spec}(S_\alpha \otimes_R \Bbbk(y)) \\ &= \operatorname{Spec}\left[\Bbbk(y)[\ldots, X_\beta^{(\alpha)}, \ldots]/(\ldots, \overline{f}_\gamma^{(\alpha)}, \ldots)\right] \end{aligned}$$

($\overline{f} =$ polynomial gotten from f via coefficient homomorphism $R \to \Bbbk(y)$). Notice that the underlying topological space of $f^{-1}(y)$ is just the subspace $f^{-1}(y)$ of X. In fact via the ring homomorphism

$$S_\alpha \xrightarrow{\phi} (S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))} \cong S_\alpha \otimes_R \Bbbk(y)$$

the usual maps

$$\begin{aligned} \mathfrak{q} &\longmapsto \phi(\mathfrak{q}) \cdot (S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))} \\ \phi^{-1}(\mathfrak{q}) &\longleftarrow\!\shortmid \mathfrak{q} \end{aligned}$$

set up a bijection between all the prime ideals of $(S_\alpha/\mathfrak{p}S_\alpha)_{(R/\mathfrak{p}\setminus(0))}$ and the prime ideals $\mathfrak{q} \subset S_\alpha$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, and it is easily seen to preserve the topology. This justifies the notation $f^{-1}(y)$.

5. Quasi-coherent sheaves

For background on kernels and cokernels in the category of sheaves of abelian groups, see the Appendix to this chapter. If $(X, \mathcal{O}_X)$ is a scheme, the sheaves of interest to us are the sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules (Definition 2.1). These form an abelian category too, if we consider $\mathcal{O}_X$-linear homomorphisms as the maps. (In fact, given $\alpha\colon \mathcal{F} \to \mathcal{G}$, the sheaf $U \mapsto \operatorname{Ker}(\alpha\colon \mathcal{F}(U) \to \mathcal{G}(U))$ is again a sheaf of $\mathcal{O}_X$-modules; and the sheafification of $U \mapsto \mathcal{G}(U)/\alpha\mathcal{F}(U)$ has a canonical $\mathcal{O}_X$-module structure on it.) The most important of these sheaves are the quasi-coherent ones, which are the ones locally isomorphic to the sheaves $\widetilde{M}$ defined in §2:

PROPOSITION-DEFINITION 5.1. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The following are equivalent:*

i) *for all $U \subset X$, affine and open, $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M,*
ii) *$\exists$ an affine open covering $\{U_\alpha\}$ of X such that $\mathcal{F}|_{U_\alpha} \cong \widetilde{M}_\alpha$ for some $\Gamma(U_\alpha, \mathcal{O}_X)$-module M_α,*
iii) *for all $x \in X$, there is a neighborhood U of x and an exact sequence of sheaves on U:*

$$(\mathcal{O}_X|_U)^I \to (\mathcal{O}_X|_U)^J \to \mathcal{F}|_U \to 0$$

(where the exponents I, J denote direct sums, possibly infinite).

If $\mathcal{F}$ has these properties, we call it quasi-coherent.

PROOF. It is clear that (i) $\implies$ (ii). Conversely, to prove (ii) $\implies$ (i), notice first that if U is an open affine set such that $\mathcal{F}|_U \cong \widetilde{M}$ for some $\Gamma(U, \mathcal{O}_X)$-module M, then for all $f \in \Gamma(U, \mathcal{O}_X)$, $\mathcal{F}|_{U_f} \cong \widetilde{M_f}$. Therefore, starting with condition (ii), we deduce that there is a *basis* $\{U_i\}$ for the topology of X consisting of open affines such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. Now if U is any open affine set and $R = \Gamma(U, \mathcal{O}_X)$, we can cover U by a finite number of these U_i's. Furthermore, we can cover each of these U_i's by smaller open affines of the type U_g, $g \in R$. Since $U_g = (U_i)_g$, $\mathcal{F}|_{U_g}$ is isomorphic to $\widetilde{(M_i)_g}$. In other words, we get a finite covering of U by affines U_{g_i} such that $\mathcal{F}|_{U_{g_i}} \cong \widetilde{N_i}$, N_i an R_{g_i}-module.

For every open set $V \subset U$, the sequence

$$0 \longrightarrow \Gamma(V, \mathcal{F}) \longrightarrow \prod_i \Gamma(V \cap U_{g_i}, \mathcal{F}) \longrightarrow \prod_{i,j} \Gamma(V \cap U_{g_i} \cap U_{g_j}, \mathcal{F})$$

is exact. Define new sheaves $\mathcal{F}_i^*$ and $\mathcal{F}_{i,j}^*$ by:

$$\begin{aligned} \Gamma(V, \mathcal{F}_i^*) &= \Gamma(V \cap U_{g_i}, \mathcal{F}) \\ \Gamma(V, \mathcal{F}_{i,j}^*) &= \Gamma(V \cap U_{g_i} \cap U_{g_j}, \mathcal{F}). \end{aligned}$$

Then the sequence of sheaves:

$$0 \longrightarrow \mathcal{F} \longrightarrow \prod_i \mathcal{F}_i^* \longrightarrow \prod_{i,j} \mathcal{F}_{i,j}^*$$

is exact, so to prove that $\mathcal{F}$ is of the form $\widetilde{M}$, it suffices to prove this for $\mathcal{F}_i^*$ and $\mathcal{F}_{i,j}^*$. But if M_i° is M_i viewed as an R-module, then $\mathcal{F}_i^* \cong \widetilde{M_i^\circ}$. In fact, for all distinguished open sets U_g,

$$\begin{aligned} \Gamma(U_g, \mathcal{F}_i^*) &= \Gamma(U_g \cap U_{g_i}, \mathcal{F}) \\ &= \Gamma((U_{g_i})_g, \mathcal{F}|_{U_{g_i}}) \\ &= (M_i)_g \\ &= \Gamma(U_g, \widetilde{M_i^\circ}). \end{aligned}$$

The same argument works for the $\mathcal{F}_{i,j}^*$'s.

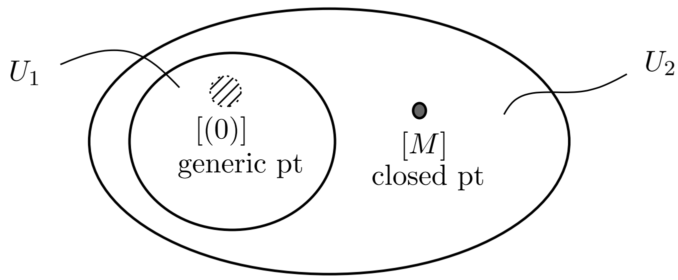

FIGURE I.1. The Spectrum of a discrete valuation ring

Next, (ii) $\implies$ (iii) because if $\mathcal{F}|_{U_\alpha} \cong \widetilde{M_\alpha}$, write $\widetilde{M_\alpha}$ by generators and relations:

$$R_\alpha^I \longrightarrow R_\alpha^J \longrightarrow M_\alpha \longrightarrow 0$$

where $R_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$. By Corollary 2.5

$$\widetilde{(R_\alpha^I)} \longrightarrow \widetilde{(R_\alpha^J)} \longrightarrow \widetilde{M_\alpha} \longrightarrow 0$$

is exact. But $\widetilde{R_\alpha} \cong \mathcal{O}_X|_{U_\alpha}$ since U_α is affine and $\widetilde{\quad}$ commutes with direct sums (even infinite ones by Proposition 2.7) so we get the required presentation of $\mathcal{F}|_{U_\alpha}$.

Finally (iii) $\implies$ (ii). Starting with (iii), we can pass to smaller neighborhoods so as to obtain an *affine* open covering $\{U_\alpha\}$ of X in which presentations exist:

$$\begin{array}{ccccc} (\mathcal{O}_X|_{U_\alpha})^I & \xrightarrow{h} & (\mathcal{O}_X|_{U_\alpha})^J & \longrightarrow \mathcal{F}|_{U_\alpha} \longrightarrow 0 \\ \| & & \| & \\ (R_\alpha^I)^\sim & & (R_\alpha^J)^\sim . & \end{array}$$

By Proposition 2.2, h is induced by an R_α-homomorphism $k\colon R_\alpha^I \to R_\alpha^J$. Let $M_\alpha = \operatorname{Coker}(k)$. Then by Proposition 2.4, $\widetilde{M_\alpha} \cong \mathcal{F}|_{U_\alpha}$. □

COROLLARY 5.2. *If $\alpha\colon \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-homomorphism of quasi-coherent sheaves, then* $\operatorname{Ker}(\alpha)$ *and* $\operatorname{Coker}(\alpha)$ *are quasi-coherent.*

PROOF. Use characterization (i) of quasi-coherent and Proposition 2.4. □

We can illustrate the concept of quasi-coherent quite clearly on $\operatorname{Spec} R$, R a discrete valuation ring. R has only two prime ideals, (0) and M the maximal ideal. Thus $\operatorname{Spec} R$ has two points, one in the closure of the other as in Figure I.1: and only two non-empty sets: U_1 consisting of $[(0)]$ alone, and U_2 consisting of the whole space. M is principal and if π is a generator, then U_1 is the distinguished open set $(\operatorname{Spec} R)_\pi$. Thus:

a) the structure sheaf is:

$$\begin{aligned} \mathcal{O}_{\operatorname{Spec} R}(U_2) &= R, \\ \mathcal{O}_{\operatorname{Spec} R}(U_1) &= R\left[\frac{1}{\pi}\right] \\ &= \text{quotient field } K \text{ of } R \end{aligned}$$

b) general sheaf of abelian groups is a pair of abelian groups

$$\mathcal{F}(U_1),\ \mathcal{F}(U_2) \text{ plus a homomorphism } \operatorname{res}\colon \mathcal{F}(U_2) \to \mathcal{F}(U_1),$$

c) general sheaf of $\mathcal{O}_{\operatorname{Spec} R}$-modules is an R-module $\mathcal{F}(U_2)$, a K-vector space $\mathcal{F}(U_1)$ plus an R-linear homomorphism $\operatorname{res}\colon \mathcal{F}(U_2) \to \mathcal{F}(U_1)$,

d) quasi-coherence means that $\mathcal{F} = \widetilde{\mathcal{F}(U_2)}$, i.e., res factors through an isomorphism:

$$\mathcal{F}(U_2) \longrightarrow \mathcal{F}(U_2) \otimes_R K \xrightarrow{\approx} \mathcal{F}(U_1).$$

The next definition gives the basic finiteness properties of quasi-coherent sheaves:

DEFINITION 5.3. A quasi-coherent sheaf $\mathcal{F}$ is *locally of finite type* if every $x \in X$ has a neighborhood U in which there is a surjective $\mathcal{O}_X$-homomorphism:

$$(\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

some $n \geq 1$. $\mathcal{F}$ is *locally of finite presentation*, or *coherent*[3] if every $x \in X$ has a neighborhood U in which there is an exact sequence:

$$(\mathcal{O}_X|_U)^m \longrightarrow (\mathcal{O}_X|_U)^n \longrightarrow \mathcal{F}|_U \longrightarrow 0.$$

$\mathcal{F}$ is *locally free* (of finite rank) if every $x \in X$ has a neighborhood U in which there is an isomorphism

$$(\mathcal{O}_X|_U)^n \xrightarrow{\approx} \mathcal{F}|_U.$$

The techniques used in the proof of Proposition 5.1 show easily that if $U \subset X$ is affine and open and $\mathcal{F}$ is locally of finite type (resp. coherent), then $\mathcal{F}|_U = \widetilde{M}$ where M is finitely generated (resp. finitely presented) as module over $\Gamma(U, \mathcal{O}_X)$.

REMARK. (Added in publication) Although the notion of "locally of finite presentation" coincides with that of "coherent" for X locally noetherian, the standard definition of the latter is slightly different for general X. A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *coherent*, if

- $\mathcal{F}$ is locally of finite type over $\mathcal{O}_X$, and
- for every affine open $U \subset X$ and every $\mathcal{O}_U$-linear homomorphism $h\colon (\mathcal{O}_U)^n \to \mathcal{F}|_U$, the kernel of h is of finite type.

Note that if X is covered by a finite number of affine opens U_i such that the above property holds for each $(U_i, \mathcal{F}|_{U_i})$, then $\mathcal{F}$ is coherent.

Here are the basic properties of $\mathcal{O}_X$-modules that are locally of finite presentation or coherent:

(1) If $\mathcal{H}$ is an $\mathcal{O}_X$-module that is locally of finite presentation, then for every $\mathcal{O}_X$-module $\mathcal{G}$ and every $x \in X$, the natural map

$$\left(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{H}, \mathcal{G})\right)_x \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{H}_x, \mathcal{G}_x)$$

is an isomorphism.

(2) If $\phi\colon \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_X$-linear homomorphism between coherent $\mathcal{O}_X$-modules, then $\mathrm{Ker}(\phi)$, $\mathrm{Coker}(\phi)$, $\mathrm{Image}(\phi)$ and $\mathrm{Coimage}(\phi)$ are all coherent $\mathcal{O}_X$-modules.

(3) If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence of quasi-coherent $\mathcal{O}_X$-modules such that $\mathcal{F}_1$ and $\mathcal{F}_3$ are coherent, then $\mathcal{F}_2$ is coherent. (Actually, the statement remains valid if we only assume $\mathcal{F}_2$ to be an $\mathcal{O}_X$-module instead of a quasi-coherent $\mathcal{O}_X$-module.)

(4) If $\mathcal{F}$ and $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $\mathcal{F} \otimes_{\mathcal{O}_X} G$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ are coherent $\mathcal{O}_X$-modules.

(5) $\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module if and only if X is locally noetherian.

[3]If X is locally noetherian, i.e., X is covered by $\mathrm{Spec}\, R$'s with R noetherian (see §II.2), then it is immediate that a quasi-coherent $\mathcal{F}$ locally of finite type is also coherent; and that sub- and quotient-sheaves of coherent $\mathcal{F}$'s are automatically coherent. The notion of coherent will not be used except on noetherian X's. (What about §IV.4?)

(6) If X is locally noetherian, then a quasi-coherent $\mathcal{O}_X$-module is coherent if and only if it is locally of finite type over $\mathcal{O}_X$.

The proof is left to the reader.

Here is an unpleasant example: Let k be a field. For each integer $n \geq 1$, let

$$R_n := k[x_0, x_1, \ldots, x_n]/(x_0^2, x_0x_1, x_0x_2, \ldots, x_0x_n)$$

for variables $x_0, x_1, \ldots, x_n$. Let $R := \prod_{n\geq 1} R_n$, and let $u \in R$ be the element whose n-th component is the image of x_0 in R_n. Let $X = \operatorname{Spec} R$, and let $\phi\colon \mathcal{O}_X \to \mathcal{O}_X$ be the $\mathcal{O}_X$-linear homomorphism given by the multiplication by $u \in R$. Then $\operatorname{Ker}(\phi)$ is the quasi-coherent $\mathcal{O}_X$-ideal associated to the ideal $I := \prod_{n\geq 1} I_n$ of R, where I_n is the ideal of R_n generated by the images of $x_0, x_1, \ldots, x_n$ in R_n. It is easy to see that I is not a finitely generated ideal of R, hence $\mathcal{O}_X$ is not a coherent $\mathcal{O}_X$-module.

Definition 5.4. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. Then for all $x \in X$, in addition to the stalk of $\mathcal{F}$ at x, we get a vector space over $\mathbb{k}(x)$ the residue field:

$$\mathcal{F}(x) = \mathcal{F}_x \otimes \mathbb{k}(x)$$
$$\operatorname{rk}_x \mathcal{F} = \dim_{\mathbb{k}(x)} \mathcal{F}(x).$$

A very important technique for quasi-coherent sheaves locally of finite type is Nakayama's[4] lemma:

Proposition 5.5 (Nakayama). *Let $\mathcal{F}$ be a quasi-coherent sheaf locally of finite type on a scheme X. Then*

i) *if $x \in X$ and if the images of $s_1, \ldots, s_n \in \mathcal{F}_x$ in $\mathcal{F}(x)$ span the vector space $\mathcal{F}(x)$, then the s_i extend to a neighborhood of x on which they define a surjective homomorphism*

$$(\mathcal{O}_X|_U)^n \xrightarrow{(s_1,\ldots,s_n)} \mathcal{F}|_U \longrightarrow 0$$

on U. When this holds, we say that $s_1, \ldots, s_n$ generate $\mathcal{F}$ over U.

ii) *if $\operatorname{rk}_x \mathcal{F} = 0$, then x has a neighborhood U such that $\mathcal{F}|_U = \{0\}$.*

iii) $\operatorname{rk}\colon x \mapsto \operatorname{rk}_x \mathcal{F}$ *is upper semi-continuous, i.e., for all $k \geq 0$, $\{x \in X \mid \operatorname{rk}_x \mathcal{F} \leq k\}$ is open.*

iv) *(Added in publication) (cf. Mumford* [**86**, Souped-up version II, Chap. III, §2, p. 213]*) Suppose X is noetherian and reduced. Then* rk *is locally constant if and only if $\mathcal{F}$ is locally free.*

Proof. (i) is the geometric form of the usual Nakayama lemma. Because of its importance, we recall the proof. (i) reduces immediately to the affine case where it says this:

> R any commutative ring, $\mathfrak{p}$ a prime ideal, M an R-module, generated by $m_1, \ldots, m_k$. If $n_1, \ldots, n_l \in M$ satisfy
>
> $$\overline{n_1}, \ldots, \overline{n_l} \text{ generate } M_{\mathfrak{p}} \otimes \mathbb{k}(\mathfrak{p}) \text{ over } \mathbb{k}(\mathfrak{p})$$
>
> then $\exists f \in R \setminus \mathfrak{p}$ such that
>
> $$n_1, \ldots, n_l \text{ generate } M_f \text{ over } R_f.$$

[4](Added in publication) According to Nakayama himself, this lemma should be attributed to Krull-Azumaya-Nakayama, or, NAK.

But the hypothesis gives us immediately:

$$a_i m_i = \sum_{j=1}^{l} b_{ij} n_j + \sum_{j=1}^{k} c_{ij} m_j, \quad 1 \le i \le k$$

for some $a_i \in R \setminus \mathfrak{p}$, $b_{ij} \in R$, $c_{ij} \in \mathfrak{p}$. Solving these k equations for the m_i by Cramer's rule, we get

$$\left(\det_{p,q}(a_p \delta_{pq} - c_{pq})\right) \cdot m_i = \sum_{j=1}^{l} b'_{ij} n_j.$$

Let f be this determinant. Then $f \notin \mathfrak{p}$ and $n_1, \ldots, n_l$ generate M_f over R_f.

(ii) and (iii) are immediate consequences of (i). □

The following Corollary is often useful:

Corollary 5.6. *Let X be a quasi-compact scheme, $\mathcal{F}$ a quasi-coherent sheaf of $\mathcal{O}_X$-modules locally of finite type. Suppose that for each $x \in X$, there exists a finite number of global sections of $\mathcal{F}$ which generate $\mathcal{F}(x)$. Then there exists a finite number of global sections of $\mathcal{F}$ that generate $\mathcal{F}$ everywhere.*

An important construction is the tensor product of quasi-coherent sheaves. The most general setting for this is when we have

$$\begin{tikzcd}
 & X \times_S Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X \arrow[dr, "r"'] & & Y \arrow[dl, "s"] \\
 & S &
\end{tikzcd}$$

$\mathcal{F}$ quasi-coherent on X

$\mathcal{G}$ quasi-coherent on Y.

Then we can construct a quasi-coherent sheaf $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ on $X \times_S Y$ analogously to our definition and construction of $X \times_S Y$ itself—viz.[5]

Step I: characterize $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ by a universal mapping property: consider all quisi-coherent[6] sheaves of $\mathcal{O}_{X \times_S Y}$-modules $\mathcal{H}$ plus collections of maps:

$$\mathcal{F}(U) \times \mathcal{G}(V) \to \mathcal{H}(p_1^{-1}U \cap p_2^{-1}V)$$

($U \subset X$ and $V \subset Y$ open) which are $\mathcal{O}_X(U)$-linear in the first variable and $\mathcal{O}_Y(V)$-linear in the second and which commute with restriction. $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ is to be the universal one.

Step II: Show that when $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} C$, $\mathcal{F} = \widetilde{M}$, $\mathcal{G} = \widetilde{N}$, then $(M \otimes_C N)^{\sim}$ on $\operatorname{Spec}(A \otimes_C B)$ has the required property.

Step III: "Glue" these local solutions $(M_\alpha \otimes_{C_\alpha} N_\alpha)^{\sim}$ together to form a sheaf $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$.

We omit the details. Notice that the stalks of $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ are given by:

If $z \in X \times_S Y$ has images $x \in X$, $y \in Y$ and $s \in S$,

$$(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G})_z \cong \left\{ \begin{array}{l} \text{localization of the } \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{s,S}} \mathcal{O}_{y,Y}\text{-module} \\ \mathcal{F}_x \otimes_{\mathcal{O}_{s,S}} \mathcal{G}_y \text{ with respect to the} \\ \text{prime ideal } \mathfrak{m}_{x,X} \otimes \mathcal{O}_{y,Y} + \mathcal{O}_{x,X} \otimes \mathfrak{m}_{y,Y} \end{array} \right\}.$$

(Use the description of $\otimes$ in the affine case.) Two cases of this construction are most important:

[5](Added in publication) $\mathcal{F} \boxtimes_{\mathcal{O}_S} \mathcal{G}$ is the accepted notation nowadays.

[6]In fact, $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ is universal for non-quasi-coherent $\mathcal{H}$'s too.

i) $X = Y = S$: Given two quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, we get a third one $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ with stalks $\mathcal{F}_x \otimes_{\mathcal{O}_{x,X}} \mathcal{G}_x$. On affines, it is given by:

$$\widetilde{M} \otimes_{\mathrm{Spec}\, R} \widetilde{N} \cong (M \otimes_R N)^{\widetilde{\ }}.$$

ii) $Y = S$, $\mathcal{F} = \mathcal{O}_X$: Given a morphism $r \colon X \to Y$ and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, we get a quasi-coherent $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{G}$. This is usually written $r^*(\mathcal{G})$ and has stalks $(r^*\mathcal{G})_x = \mathcal{O}_{x,X} \otimes_{\mathcal{O}_{y,Y}} \mathcal{G}_y$ $(y = r(x))$. If X and Y are affine, say $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$, then it is given by:

$$r^*(\widetilde{M}) \cong (M \otimes_S R)^{\widetilde{\ }}.$$

The general case can be reduced to these special cases by formula:

$$\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G} \cong p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X \times_S Y}} p_2^*\mathcal{G}.$$

Also iterating (i), we define $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{F}_k$; symmetrizing or skew-symmetrizing, we get $\mathrm{Symm}^k \mathcal{F}$ and $\bigwedge^k \mathcal{F}$ just like the operations $\mathrm{Symm}^k M$, $\bigwedge^k M$ on modules.

We list a series of properties of quasi-coherent sheaves whose proofs are straightforward using the techniques already developed. These are just a sample from the long list to be found in EGA [**1**].

5.7. *If $\mathcal{F}$ is a quasi-coherent sheaf on X and $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals, then the sheaf*

$$\mathcal{I} \cdot \mathcal{F} \underset{def}{=} \left[\begin{array}{l} \textit{subsheaf of } \mathcal{F} \textit{ generated by} \\ \textit{the submodules } \mathcal{I}(U) \cdot \mathcal{F}(U) \end{array} \right]$$

is quasi-coherent and for U affine

$$\mathcal{I} \cdot \mathcal{F}(U) = \mathcal{I}(U) \cdot \mathcal{F}(U).$$

5.8. *If $\mathcal{F}$ is quasi-coherent and $U \subset V \subset X$ are two affines, then*

$$\mathcal{F}(U) \cong \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U).$$

5.9. *Let X be a scheme and let*

$$U \longmapsto \mathcal{F}(U)$$

be a presheaf. Suppose that for all affine U and all $f \in R = \Gamma(U, \mathcal{O}_X)$, the map

$$\mathcal{F}(U) \otimes_R R_f \longrightarrow \mathcal{F}(U_f)$$

is an isomorphism. Then the sheafification $\mathrm{sh}(\mathcal{F})$ *of $\mathcal{F}$ is quasi-coherent and*

$$\mathrm{sh}(\mathcal{F})(U) \cong \mathcal{F}(U)$$

for all affine U.

5.10. *If $\mathcal{F}$ is coherent and $\mathcal{G}$ is quasi-coherent, then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent, with a canonical homomorphism*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{G}.$$

(cf. Appendix to this chapter and Proposition 2.6.*)*

5.11. *Let $f \colon X \to Y$ be a morphism of schemes, $\mathcal{F}$ a quasi-coherent sheaf on X and $\mathcal{G}$ a quasi-coherent sheaf on Y. Then*

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

(See (ii) above for the definition of $f^*\mathcal{G}$.)

5.12. *Let R be an S-algebra and let $f\colon \operatorname{Spec} R \to \operatorname{Spec} S$ be the corresponding morphism of affine schemes. Let M be an R-module. Then M can be considered as an S-module too and we can form $\widetilde{M}_R$, $\widetilde{M}_S$ the corresponding sheaves on* $\operatorname{Spec} R$ *and* $\operatorname{Spec} S$. *Then*

$$f_*(\widetilde{M}_R) = \widetilde{M}_S.$$

(cf. Appendix to this chapter for the definition of f_*.)

6. The functor of points

We have had several indications that the underlying point set of a scheme is peculiar from a geometric point of view. Non-closed points are odd for one thing. Another peculiarity is that the point set of a fibre product $X \times_S Y$ does not map injectively into the set-theoretic product of X and Y. The explanation of these confusing facts is that there are really two concepts of "point" in the language of schemes. To see this in its proper setting, look at some examples in other categories:

EXAMPLE. Let $\mathcal{C}$ = category of differentiable manifolds. Let $\mathbf{z}$ be the manifold with *one* point. Then for any manifold X,

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X) \cong X \quad \text{as a point set.}$$

EXAMPLE. Let $\mathcal{C}$ = category of groups. Let $\mathbf{z} = \mathbb{Z}$. Then for any group G

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, G) \cong G \quad \text{as a point set.}$$

EXAMPLE. Let $\mathcal{C}$ = category of rings with 1 (and homomorphisms f such that $f(1) = 1$). Let $\mathbf{z} = \mathbb{Z}[X]$. Then for any ring R,

$$\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, R) \cong R \quad \text{as a point set.}$$

This indicates that if $\mathcal{C}$ is any category, whose objects may not be point sets to begin with, and $\mathbf{z}$ is an object, one can try to conceive of $\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ as the underlying set of points of the object X. In fact:

$$X \longmapsto \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$$

extends to a functor from the category $\mathcal{C}$ to the category (Sets), of sets. But, it is not satisfactory to call $\operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ the set of points of X unless this functor is *faithful*, i.e., unless a morphism f from X_1 to X_2 is determined by the map of sets:

$$\tilde{f}\colon \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X_1) \longrightarrow \operatorname{Mor}_{\mathcal{C}}(\mathbf{z}, X_2).$$

EXAMPLE. Let (Hot) be the category of CW-complexes, where

$$\operatorname{Mor}(X, Y)$$

is the set of homotopy-classes of continuous maps from X to Y. If $\mathbf{z}$ = the 1 point complex, then

$$\operatorname{Mor}_{(\mathrm{Hot})}(\mathbf{z}, X) = \pi_0(X), \quad \text{(the set of components of } X)$$

and this does *not* give a faithful functor.

EXAMPLE. Let $\mathcal{C}$ = category of schemes. Take for instance $\mathbf{z}$ to be the *final* object of the category $\mathcal{C}$: $\mathbf{z} = \operatorname{Spec}(\mathbb{Z})$. Now

$$\operatorname{Mor}_{\mathcal{C}}(\operatorname{Spec}(\mathbb{Z}), X)$$

is absurdly small, and does not give a faithful functor.

Grothendieck's ingenious idea is to remedy this defect by considering (for arbitrary categories $\mathcal{C}$) not *one* $\mathbf{z}$, but *all* $\mathbf{z}$: attach to X the whole set:

$$\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X).$$

In a natural way, this always gives a faithful functor from the category $\mathcal{C}$ to the category (Sets). Even more than that, the "extra structure" on the set $\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ which characterizes the object X, can be determined. It consists in:

i) the decomposition of $\bigcup_{\mathbf{z}} \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ into subsets $S_{\mathbf{z}} = \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$, one for each $\mathbf{z}$.

ii) the natural maps from one set $S_{\mathbf{z}}$ to another $S_{\mathbf{z}'}$, given for each morphism $g\colon \mathbf{z}' \to \mathbf{z}$ in the category.

Putting this formally, it comes out like this:

Attach to each X in $\mathcal{C}$, the *functor* h_X (contravariant, from $\mathcal{C}$ itself to (Sets)) via

($*$) $h_X(\mathbf{z}) = \mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$, $\mathbf{z}$ an object in $\mathcal{C}$.

($**$) $h_X(g) = $ [induced map from $\mathrm{Mor}_{\mathcal{C}}(\mathbf{z}, X)$ to $\mathrm{Mor}_{\mathcal{C}}(\mathbf{z}', X)$], $g\colon \mathbf{z}' \to \mathbf{z}$ a morphism in $\mathcal{C}$.

Now the functor h_X is an object in a category too: viz.

$$\mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})),$$

(where Funct stands for functors, $\mathcal{C}^\circ$ stands for $\mathcal{C}$ with arrows reversed). It is also clear that if $g\colon X_1 \to X_2$ is a morphism in $\mathcal{C}$, then one obtains a morphism of functors $h_g\colon h_{X_1} \to h_{X_2}$. All this amounts to one big functor:

$$h\colon \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})).$$

PROPOSITION 6.1. *h is fully faithful, i.e., if X_1, X_2 are objects of $\mathcal{C}$, then, under h,*

$$\mathrm{Mor}_{\mathcal{C}}(X_1, X_2) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Funct}}(h_{X_1}, h_{X_2}).$$

PROOF. Easy. □

The conclusion, heuristically, is that an object X of $\mathcal{C}$ can be *identified* with the functor h_X, which is basically just a structured set.

Return to algebraic geometry! What we have said motivates I hope:

DEFINITION 6.2. If X and K are schemes, a *K-valued point* of X is a morphism $f\colon K \to X$; if $K = \mathrm{Spec}(R)$, we call this an R-valued point of X. If X and K are schemes over a third scheme S, i.e., we are given morphisms $p_X\colon X \to S$, $p_K\colon K \to S$, then f is a K-valued point of X/S if $p_X \circ f = p_K$; if $K = \mathrm{Spec}(R)$, we call this an R-valued point of X/S. The set of all R-valued points of a scheme X, or of X/S, is denoted $X(R)$.

Proposition 3.10, translated into our new terminology states that if R is a local ring, there is a bijection between the set of R-valued points of X and the set of pairs (x, ϕ), where $x \in X$ and $\phi\colon \mathcal{O}_{x,X} \to R$ is a local homomorphism. Corollary 3.11 states that for every point $x \in X$ in the usual sense, there is a canonical $\Bbbk(x)$-valued point i_x of X in our new sense. In particular, suppose X is a scheme over $\mathrm{Spec}\, k$ for a field k: then there is a bijection

$$\left\{ \begin{array}{l} \text{set of } k\text{-valued points} \\ \text{of } X/\operatorname{Spec} k \end{array} \right\} \cong \left\{ \begin{array}{l} \text{set of points } x \in X \text{ such that} \\ \text{the natural map } k \to \Bbbk(x) \\ \text{is surjective} \end{array} \right\}$$

given by associating i_x to x. Points $x \in X$ with $k \xrightarrow{\approx} \Bbbk(x)$ are called *k-rational points* of X.

K-valued points of a scheme are compatible with products. In fact, if K, X, Y are schemes over S, then the set of K-valued points of $(X \times_S Y)/S$ is just the (set-theoretic) product of the set of K-valued points of X/S and the set of K-valued points of Y/S. This is the definition of the fibre product.

The concept of an R-valued point generalizes the notion of a solution of a set of diophantine equations in the ring R. In fact, let:

$$f_1, \dots, f_m \in \mathbb{Z}[X_1, \dots, X_n]$$
$$X = \operatorname{Spec}(\mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m)).$$

I claim an R-valued point of X is the "same thing" as an n-tuple $a_1, \dots, a_n \in R$ such that

$$f_1(a_1, \dots, a_n) = \cdots = f_m(a_1, \dots, a_n) = 0.$$

But in fact a morphism

$$\operatorname{Spec}(R) \xrightarrow{g} \operatorname{Spec}(\mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m))$$

is determined by the n-tuple $a_i = g^*(X_i)$, $1 \leq i \leq n$, and those n-tuples that occur are exactly those such that $h \mapsto h(a_1, \dots, a_n)$ defines a homomorphism

$$R \xleftarrow{g^*} \mathbb{Z}[X_1, \dots, X_n]/(f_1, \dots, f_m),$$

i.e., solutions of $f_1, \dots, f_m$.

An interesting point is that a scheme is actually determined by the functor of its R-valued points as well as by the larger functor of its K-valued points. To state this precisely, let X be a scheme, and let $h_X^{(\circ)}$ be the *covariant* functor from the category (Rings) of commutative rings with 1 to the category (Sets) defined by:

$$h_X^{(\circ)}(R) = h_X(\operatorname{Spec}(R)) = \operatorname{Mor}(\operatorname{Spec}(R), X).$$

Regarding $h_X^{(\circ)}$ as a functor in X in a natural way, one has:

PROPOSITION 6.3. *For any two schemes X_1, X_2,*

$$\operatorname{Mor}(X_1, X_2) \xrightarrow{\sim} \operatorname{Mor}(h_{X_1}^{(\circ)}, h_{X_2}^{(\circ)}).$$

Hence $h^{(\circ)}$ is a fully faithful functor from the category of schemes to

$$\operatorname{Funct}((\text{Rings}), (\text{Sets})).$$

This result is more readily checked privately than proven formally, but it may be instructive to sketch how a morphism $F\colon h_{X_1}^{(\circ)} \to h_{X_2}^{(\circ)}$ will induce a morphism $f\colon X_1 \to X_2$. One chooses an affine open covering $U_i \cong \operatorname{Spec}(A_i)$ of X_1; let

$$I_i\colon \operatorname{Spec}(A_i) \cong U_i \to X_1$$

be the inclusion. Then I_i is an A_i-valued point of X_1. Therefore $F(I_i) = f_i$ is an A_i-valued point of X_2, i.e., f_i defines

$$U_i \cong \operatorname{Spec}(A_i) \to X_2.$$

Modulo a verification that these f_i patch together on $U_i \cap U_j$, these f_i give the morphism f via

$$\begin{tikzcd} U_i \arrow[r, "f_i"] \arrow[d, phantom, "\cap"] & X_2. \\ X_1 \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition 6.3 suggests a whole new approach to the foundations of the theory of schemes. Instead of defining a scheme as a space X plus a sheaf of rings $\mathcal{O}_X$ on X, why not define a scheme as a covariant functor F from (Rings) to (Sets) which satisfies certain axioms strong enough to show that it is isomorphic to a functor $h_X^{(\circ)}$ for some scheme in the usual sense? More precisely:

DEFINITION 6.4. A covariant functor $F\colon (\text{Rings}) \to (\text{Sets})$ is a *sheaf in the Zariski topology* if for all rings R and for all equations

$$1 = \sum_{i=1}^{n} f_i g_i,$$

then

a) the natural map $F(R) \to \prod_{i=1}^{n} F(R_{f_i})$ is injective
b) for all collections $s_i \in F(R_{f_i})$ such that s_i and s_j have the same image in $F(R_{f_i f_j})$, there is an $s \in F(R)$ mapping onto the s_i's.

If F is a functor and $\xi \in F(R)$, we get a morphism of functors:

$$\phi_\xi\colon h_R \longrightarrow F$$

i.e., a set of maps

$$\phi_{\xi,S}\colon h_R(S) \underset{\text{def}}{=} \operatorname{Hom}(R,S) \to F(S)$$

given by:

$$\begin{aligned} &\forall R \xrightarrow{\alpha} S \\ &\phi_{\xi,S}(\alpha) = F(\alpha)(\xi). \end{aligned}$$

If $\mathfrak{a} \subset R$ is an ideal, define the subfunctor

$$h_R^{\mathfrak{a}} \subset h_R$$

by

$$h_R^{\mathfrak{a}}(S) = \left\{ \begin{array}{l} \text{set of all homomorphisms } \alpha\colon R \to S \\ \text{such that } \alpha(\mathfrak{a}) \cdot S = S \end{array} \right\}.$$

DEFINITION 6.5. Let $F\colon (\text{Rings}) \to (\text{Sets})$ be a functor. An element $\xi \in F(R)$ is an *open subset* if

a) $\phi_\xi\colon h_R \to F$ is injective
b) for all rings S and all $\eta \in F(S)$, consider the diagram:

$$\begin{tikzcd} h_R \arrow[r, hook, "\phi_\xi"'] & F \\ & h_S \arrow[u, "\phi_\eta"'] \end{tikzcd}$$

Then there is an ideal $\mathfrak{a} \subset S$ such that $\phi_\eta^{-1}(h_R) = \text{subfunctor } h_S^{\mathfrak{a}}$ of h_S.

DEFINITION 6.6. A functor $F\colon (\text{Rings}) \to (\text{Sets})$ is a *scheme functor* if

a) it is a sheaf in the Zariski-topology,
b) there exist open subsets $\xi_\alpha \in F(R_\alpha)$ such that for all fields k,

$$F(k) = \bigcup_{\alpha} \phi_{\xi_\alpha} h_{R_\alpha}(k).$$

We leave it to the reader now to check that the scheme-functors F are precisely those given by

$$F(R) = \mathrm{Mor}(\operatorname{Spec} R, X)$$

for some scheme X. This point of view is worked out in detail in Demazure-Gabriel [**35**]. It is moreover essential in a very important generalization of the concept of scheme which arose as follows. One of the principal goals in Grothendieck's work on schemes was to find a characterization of scheme-functors by weak general properties that could often be checked in practice and so lead to many existence theorems in algebraic geometry (like Brown's theorem[7] in (Hot)). It seemed at first that this program would fail completely and that scheme-functors were really quite special[8];but then Artin discovered an extraordinary approximation theorem which showed that there was a category of functors F only a "little" larger than the scheme-functors which can indeed be characterized by weak general properties. Geometrically speaking, his functors F are like spaces gotten by dividing affines by étale equivalence relations (cf. Chapter V) and then glueing. He called these *algebraic spaces* (after algebraic functions, i.e., meromorphic functions on $\mathbb{C}$ satisfying a polynomial equation; see Artin [**16**], [**17**], [**18**], [**19**], Knutson [**71**])[9].

7. Relativization

The goal of this section is to extend the concept of Spec in a technical but very important way. Instead of starting with a *ring* R and defining a *scheme* $\operatorname{Spec} R$, we want to start with a *sheaf of rings* $\mathcal{R}$ on an arbitrary scheme X and define a *scheme over* X, $\pi\colon \operatorname{Spec}_X \mathcal{R} \to X$. More precisely, $\mathcal{R}$ must be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. We may approach the definition of $\operatorname{Spec}_X \mathcal{R}$ by a universal mapping property as follows:

THEOREM-DEFINITION 7.1. *Let X be a scheme and let $\mathcal{R}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Then there is a scheme over X:*

$$\pi\colon \operatorname{Spec}_X \mathcal{R} \to X$$

and an isomorphism of $\mathcal{O}_X$-algebras:

$$\mathcal{R} \xrightarrow{\approx} \pi_*(\mathcal{O}_{\operatorname{Spec}_X \mathcal{R}})$$

uniquely characterized by the property:

For all morphisms

$$f\colon Y \to X$$

plus homomorphisms of $\mathcal{O}_X$-algebras

$$\alpha\colon \mathcal{R} \to f_*(\mathcal{O}_Y)$$

there is a unique factorization:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "f"'] & & \operatorname{Spec}_X \mathcal{R} \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

for which α is given by g^*:

$$\mathcal{R} \xrightarrow{\approx} \pi_*(\mathcal{O}_{\operatorname{Spec}_X \mathcal{R}}) \xrightarrow{g^*} f_*(\mathcal{O}_Y).$$

[7] See Spanier [**109**, Chapter 7, §7].

[8] See for instance Hironaka [**60**] and Mumford [**83**, p. 83].

[9] (Added in publication) For more details and later developments see, §8 below and, e.g., FAG [**3**].

The situation is remarkably similar to the construction of fibre products:

Firstly, if X is affine, then this existence theorem has an immediate solution:

$$\operatorname{Spec}_X \mathcal{R} = \operatorname{Spec}(\mathcal{R}(X)).$$

The universal mapping property is just a rephrasing of Theorem 3.7 and (5.12).

Secondly, we can use the solution in the affine case to prove the general existence theorem modulo a patching argument. In fact, let U_α be an affine open covering of X. Then the open subset

$$\pi^{-1}(U_\alpha) \subset \operatorname{Spec}_X(\mathcal{R})$$

will have to be

$$\operatorname{Spec}_{U_\alpha}(\mathcal{R}|_{U_\alpha})$$

(just restrict the universal mapping property to those morphisms $f\colon Y \to X$ which factor through U_α). Therefore $\operatorname{Spec}_X(\mathcal{R})$ must be the union of affine open pieces $\operatorname{Spec}(\mathcal{R}(U_\alpha))$. To use this observation as a construction for all α, β, we must identify the open subsets below:

$$\begin{tikzcd}
\operatorname{Spec}(\mathcal{R}(U_\alpha)) \supset \arrow[dr, "\pi_\alpha"'] & \pi_\alpha^{-1}(U_\alpha\cap U_\beta) \arrow[rr, equal, "\overset{??}{\sim}"] \arrow[dr] & & \pi_\beta^{-1}(U_\alpha\cap U_\beta) \arrow[dl] & \subset \operatorname{Spec}(\mathcal{R}(U_\beta)) \arrow[dl, "\pi_\beta"] \\
& U_\alpha \supset & U_\alpha \cap U_\beta & \subset U_\beta &
\end{tikzcd}$$

Note that

$$\pi_{\alpha,*}(\mathcal{O}_{\operatorname{Spec}\mathcal{R}(U_\alpha)}) \cong \mathcal{R}|_{U_\alpha}$$

by (5.12) hence

$$\Gamma(\pi_\alpha^{-1}(U_\alpha \cap U_\beta), \mathcal{O}_{\operatorname{Spec}\mathcal{R}(U_\alpha)}) \cong \mathcal{R}(U_\alpha \cap U_\beta).$$

Composing this with

$$\mathcal{R}(U_\beta) \underset{\text{res}}{\longrightarrow} \mathcal{R}(U_\alpha \cap U_\beta)$$

and using Theorem 3.7, we get a morphism

$$\pi_\alpha^{-1}(U_\alpha \cap U_\beta) \to \operatorname{Spec}\mathcal{R}(U_\beta)$$

that factors through $\pi_\beta^{-1}(U_\alpha \cap U_\beta)$. Interchanging α and β, we see that we have an isomorphism.

Thirdly, we can also give a totally explicit construction of $\operatorname{Spec}_X \mathcal{R}$ as follows:

i) as a point set, $\operatorname{Spec}_X \mathcal{R}$ is the set of pairs $(x, \mathfrak{p})$, where $x \in X$ and $\mathfrak{p} \subset \mathcal{R}_x$ is a prime ideal such that if

$$i\colon \mathcal{O}_x \to \mathcal{R}_x$$

is the given map, then

$$i^{-1}(\mathfrak{p}) = \mathfrak{m}_x$$

ii) as a topological space, we get a basis of open sets:

$$\{\mathcal{U}(V, f) \mid V \subset X \text{ open affine, } f \in \mathcal{R}(V)\}$$

where

$$\mathcal{U}(V, f) = \{(x, \mathfrak{p}) \mid x \in V,\ f \notin \mathfrak{p}\}.$$

iii) the structure sheaf is a certain sheaf of functions from open sets in $\operatorname{Spec}_X \mathcal{R}$ to

$$\coprod_{x,\mathfrak{p}} (\mathcal{R}_x)_{\mathfrak{p}},$$

namely the functions which are locally given by f/f', $f, f' \in \mathcal{R}(V)$, on $\mathcal{U}(V, f')$.

Corollary 7.2 (of proof). *π has the property that for all affine open sets $U \subset X$, $\pi^{-1}(U)$ is affine.*

In fact, we can formulate the situation as follows:

PROPOSITION-DEFINITION 7.3. *Let $f\colon Y \to X$ be a morphism of schemes. Then the following are equivalent:*

i) *for all affine open $U \subset X$, $f^{-1}(U)$ is affine,*
ii) *there is an affine open covering $\{U_\alpha\}$ of X such that $f^{-1}(U_\alpha)$ is affine,*
iii) *there is a quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{R}$ such that*

$$Y \cong \mathrm{Spec}_X(\mathcal{R}).$$

Such an f is called an affine morphism.

PROOF. (i) $\Longrightarrow$ (ii) is obvious.
(iii) $\Longrightarrow$ (i) has just been proven.
(ii) $\Longrightarrow$ (iii): let $\mathcal{R} = f_*\mathcal{O}_Y$. Note that if $V_\alpha = f^{-1}(U_\alpha)$ and f_α is the restriction of f to

$$f_\alpha\colon V_\alpha \longrightarrow U_\alpha,$$

then $f_*\mathcal{O}_{V_\alpha}$ is quasi-coherent by (5.10). But $\mathcal{R}|_{U_\alpha} = f_*\mathcal{O}_{V_\alpha}$, so $\mathcal{R}$ is quasi-coherent. Now compare Y and $\mathrm{Spec}_X \mathcal{R}$. Using the isomorphism

$$f_*\mathcal{O}_Y = \mathcal{R} = \pi_*(\mathcal{O}_{\mathrm{Spec}_X \mathcal{R}})$$

the universal mapping property for $\mathrm{Spec}_X \mathcal{R}$ gives us a morphism ϕ

$$\begin{tikzcd} Y \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \mathrm{Spec}_X \mathcal{R} \arrow[dl, "\pi"] \\ & X. & \end{tikzcd}$$

But $f^{-1}(U_\alpha)$ is affine, so

$$\begin{aligned} f^{-1}(U_\alpha) &\cong \mathrm{Spec}_{U_\alpha}(f_*\mathcal{O}_Y|_{U_\alpha}) \\ &\cong \mathrm{Spec}_{U_\alpha}(\mathcal{R}|_{U_\alpha}) \\ &\cong \pi^{-1}(U_\alpha) \end{aligned}$$

hence ϕ is an isomorphism. □

8. Defining schemes as functors

(Added in publication)

To illustrate the power of Grothendieck's idea (cf. FGA [**2**]) referred to in §6, we show examples of schemes defined as functors.

For any category $\mathcal{C}$ we defined in §6 a fully faithful functor

$$h : \mathcal{C} \longrightarrow \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets})).$$

Here is a result slightly more general than Proposition 6.1:

PROPOSITION 8.1 (Yoneda's lemma). *For any $X \in \mathcal{C}$ and any $F \in \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets}))$, we have a natural bijection*

$$F(X) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{Funct}}(h_X, F).$$

The proof is again easy, and can be found in EGA [**1**, Chapter 0 revised, Proposition (1.1.4)]. From this we easily get the following:

PROPOSITION-DEFINITION 8.2. $F \in \mathrm{Funct}(\mathcal{C}^\circ, (\mathrm{Sets}))$ *is said to be* representable *if it is isomorphic to* h_X *for some* $X \in \mathcal{C}$. *This is the case if and only if there exists* $X \in \mathcal{C}$ *and* $u \in F(X)$, *called the* universal element, *such that*

$$\mathrm{Mor}(Z, X) \ni \varphi \longmapsto F(\varphi)(u) \in F(Z)$$

is a bijection for all $Z \in \mathcal{C}$. *The pair* (X, u) *is determined by* F *up to unique isomorphism.*

Let us now fix a scheme S and restrict ourselves to the case

$$\mathcal{C} = (\mathrm{Sch}/S) = \text{the category of schemes over } S \text{ and } S\text{-morphisms.}$$

For schemes X and Y over S, denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms. (cf. Definition 3.6.)

A representable $F \in \mathrm{Funct}((\mathrm{Sch}/S), (\mathrm{Sets}))$ thus defines a scheme over S.

Suppose F is represented by X. Then for any open covering $\{U_i\}_{i \in I}$ of Z, the sequence

$$F(Z) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j)$$

is an exact sequence of sets, that is, for any $(f_i)_{i \in I} \in \prod_{i \in I} F(U_i)$ such that the images of f_i and f_j in $F(U_i \cap U_j)$ coincide for all $i, j \in I$, there exists a unique $f \in F(Z)$ whose image in $F(U_i)$ coincides with f_i for all $i \in I$. This is because a morphism $f \in F(Z) = \mathrm{Hom}_S(Z, X)$ is obtained uniquely by glueing morphisms $f_i \in F(U_i) = \mathrm{Hom}_S(U_i, X)$ satisfying the compatibility condition $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. Another way of looking at this condition is that F is a *sheaf of sets* (cf. Definition 3 in the Appendix below).

Actually, a representable functor satisfies a stronger necessary condition: it is a sheaf of sets in the "faithfully flat quasi-compact topology". (See §IV.2 for related topics. See also FAG [**3**].)

EXAMPLE 8.3. Let X and Y be schemes over S. The functor

$$\begin{aligned} F(Z) &= \mathrm{Hom}_S(Z, X) \times \mathrm{Hom}_S(Z, Y) \\ &= \{(q_1, q_2) \mid q_1 \colon Z \to X,\ q_s \colon Z \to Y \text{ are } S\text{-morphisms}\}, \end{aligned}$$

with obvious maps $F(f) \colon F(Z) \to F(Z')$ for S-morphisms $f \colon Z' \to Z$, is represented by the fibre product $X \times_S Y$ by Theorem 4.2. The universal element is $(p_1, p_2) \in F(X \times_S Y)$, where $p_1 \colon X \times_S Y \to X$ and $p_2 \colon X \times_S Y \to Y$ are projections.

EXAMPLE 8.4. The functor

$$\begin{aligned} F(Z) &= \Gamma(Z, \mathcal{O}_Z), \quad \text{for } Z \in (\mathrm{Sch}/S) \\ F(f) &= f^* \colon \Gamma(Z, \mathcal{O}_Z) \to \Gamma(Z', \mathcal{O}_{Z'}), \quad \text{for } f \in \mathrm{Hom}_S(Z', Z) \end{aligned}$$

is represented by the relatively affine S-scheme $\mathbb{G}_{a,S} := \mathrm{Spec}_S(\mathcal{O}_S[T])$ by Theorem-Definition 7.1, where $\mathcal{O}_S[T]$ is the polynomial algebra over $\mathcal{O}_S$ in one variable T. The universal element is $T \in \Gamma(S, \mathcal{O}_S[T])$. This S-scheme $\mathbb{G}_{a,S}$ is a commutative group scheme over S in the sense to be defined in §VI.1.

More generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.4.9)]):

EXAMPLE 8.5. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on S. Then the relatively affine S-scheme

$$\mathrm{Spec}_S(\mathrm{Symm}(\mathcal{F})),$$

where $\mathrm{Symm}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_S$, represents the functor F defined as follows: For any S-scheme $\varphi \colon Z \to S$, denote by $\varphi^*\mathcal{F} = \mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}$ the inverse image of $\mathcal{F}$ by the morphism $\varphi \colon Z \to S$ (cf. §5).

$$F(Z) = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_Z), \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with the obvious map

$$F(f) = f^* : \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{F}_Z, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_{Z'}}(\mathcal{O}_{Z'} \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_{Z'}) = \mathrm{Hom}_{\mathcal{O}_{Z'}}(f^*(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}), f^*\mathcal{O}_Z)$$

for $f \in \mathrm{Hom}_S(Z', Z)$. If we denote by $\pi\colon X = \mathrm{Spec}_S(\mathrm{Symm}(\mathcal{F})) \to S$ the canonical projection, then the universal element is $\pi^*\mathcal{F} \to \mathcal{O}_X$ corresponding to the canonical injection $\mathcal{F} \to \pi_*\mathcal{O}_X = \mathrm{Symm}(\mathcal{F})$. This S-scheme is a commutative group scheme over S in the sense defined in §VI.1.

Similarly to Example 8.4, we have:

Example 8.6. The functor

$$F(Z) = \Gamma(Z, \mathcal{O}_Z)^*, \quad \text{for } Z \in (\mathrm{Sch}/S)$$
$$F(f) = f^*\colon \Gamma(Z, \mathcal{O}_Z)^* \to \Gamma(Z, \mathcal{O}_{Z'})^*, \quad \text{for } f \in \mathrm{Hom}_S(Z', Z),$$

where the asterisk denotes the set of invertible elements, is represented by the relatively affine S-scheme

$$\mathbb{G}_{m,S} := \mathrm{Spec}_S(\mathcal{O}_S[T, T^{-1}]).$$

The universal element is again $T \in \Gamma(S, \mathcal{O}_S[T, T^{-1}])$. This S-scheme $\mathbb{G}_{m,S}$ is a commutative group scheme over S in the sense to be defined in §VI.1.

More generally:

Example 8.7. Let n be a positive integer. The relatively affine S-scheme defined by

$$\mathrm{GL}_{n,S} = \mathrm{Spec}_S\left(\mathcal{O}_S\left[T_{11}, \ldots, T_{nn}, \frac{1}{\det(T)}\right]\right),$$

where $T = (T_{ij})$ is the $n \times n$-matrix with indeterminates T_{ij} as entries, represents the functor

$$F(Z) = \mathrm{GL}_n(\Gamma(Z, \mathcal{O}_Z)), \qquad \text{for } Z \in (\mathrm{Sch}/S),$$

the set of invertible $n \times n$-matrices with entries in $\Gamma(Z, \mathcal{O}_Z)$, with obvious maps corresponding to S-morphisms. This S-scheme is a group scheme over S in the sense defined in §VI.1.

Even more generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)]):

Example 8.8. Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank (cf. Definition 5.3). The functor F defined by

$$F(Z) = \mathrm{Aut}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E}) \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with obvious maps corresponding to S-morphisms is represented by a relatively affine S-scheme $\mathrm{GL}(\mathcal{E})$. (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)].) This S-scheme is a group scheme over S in the sense defined in §VI.1. Example 8.7 is a special case with

$$\mathrm{GL}_{n,S} = \mathrm{GL}(\mathcal{O}_S^{\oplus n}).$$

Example 8.9. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module, and r a positive integer. For each S-scheme Z, exact sequences of $\mathcal{O}_Z$-modules

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$
$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{E}$ and $\mathcal{E}'$ are locally free $\mathcal{O}_Z$-modules of rank r, are said to be *equivalent* if there exists an $\mathcal{O}_Z$-isomorphism $\alpha\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ so that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_Z \otimes \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "\alpha"'] & 0 \\
\mathcal{O}_Z \otimes \mathcal{F} \arrow[r] & \mathcal{E}' \arrow[r] & 0.
\end{tikzcd}$$

For each S-scheme Z, let

$$F(Z) = \{\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{E} \to 0 \mid \text{ exact with } \mathcal{E} \text{ locally free } \mathcal{O}_Z\text{-module of rank } r\}/\sim$$

($\sim$ denotes the set of equivalence classes). For each S-morphism $f\colon Z' \to Z$ and an exact sequence $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{E} \to 0$, the inverse image by f

$$\mathcal{O}_{Z'} \otimes_{\mathcal{O}_S} \mathcal{F} = f^*(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}) \longrightarrow f^*\mathcal{E} \longrightarrow 0$$

defines an element of $F(Z')$, since the inverse image preserves surjective homomorphisms and local freeness. Thus we have a functor $F\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$. This functor turns out to be representable. The proof can be found in EGA [**1**, Chapter I, revised, Proposition (9.7.4)]. The S-scheme representing it is denoted by $\pi\colon \mathrm{Grass}^r(\mathcal{F}) \to S$ and is called the *Grassmannian* scheme over S. The universal element is given by an exact sequence

$$\pi^*\mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with a locally free $\mathcal{O}_{\mathrm{Grass}^r(\mathcal{F})}$-module $\mathcal{Q}$ of rank r called the *universal quotient.*

Locally free $\mathcal{O}_S$-modules of rank one are called *invertible* $\mathcal{O}_S$-modules. (cf. Definition III.1.1.) As a special case for $r = 1$ we have the following:

Example 8.10. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. The functor

$$F(Z) = \{\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{L} \to 0 \mid \text{ exact with } \mathcal{L} \text{ invertible } \mathcal{O}_Z\text{-module}\}/\sim$$

with the map $F(f)\colon F(Z) \to F(Z')$ defined by the inverse image by each $f\colon Z' \to Z$ is represented by an S-scheme

$$\pi\colon \mathbb{P}(\mathcal{F}) = \mathrm{Proj}_S(\mathrm{Symm}(\mathcal{F})) \longrightarrow S$$

with the universal element given by the universal quotient invertible sheaf

$$\pi^*\mathcal{F} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \longrightarrow 0.$$

(cf. Definition II.5.7, Theorem III.2.8.)

When $S = \mathrm{Spec}(k)$ with k an algebraically closed field, the set of k-rational points of the Grassmann variety $\mathrm{Grass}^r(k^{\oplus n})$ over k parametrizes the r-dimenensional quotient spaces of $k^{\oplus n}$, hence parametrizes $(n-r)$-dimensional subspaces of $k^{\oplus n}$ that are the kernels of the quotient maps. In particular the set of k-rational points of the $(n-1)$-dimensional projective space $\mathbb{P}(k^{\oplus n})$ parametrizes the *one-dimensional quotient spaces* of $k^{\oplus n}$ hence $(n-1)$-dimensional subspaces. To have a functor in the general setting, however, it is crucial to take the quotient approach instead of the subspace approach, since tensor product is not left exact.

S-morphisms between representable functors can be defined as morphisms of functors by Proposition 6.1. Here are examples:

Example 8.11. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Then the *Plücker S-morphism*

$$\mathrm{Grass}^r(\mathcal{F}) \longrightarrow \mathbb{P}(\bigwedge^r \mathcal{F})$$

is defined in terms of the functors they represent as follows: For any S-Scheme Z and

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \quad \text{exact with locally free } \mathcal{O}_Z\text{-module } \mathcal{E} \text{ of rank } r,$$

the r-th exterior product gives rise to an exact sequence

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \bigwedge^r \mathcal{F} \longrightarrow \bigwedge^r \mathcal{E} \longrightarrow 0,$$

with $\bigwedge^r \mathcal{E}$ an invertible $\mathcal{O}_Z$-module, hence a morphism $Z \to \mathbb{P}(\bigwedge^r \mathcal{F})$. EGA [**1**, Chapter I, revised, §9.8] shows that the Plücker S-morphism is a closed immersion (cf. Definition II.3.2).

For quasi-coherent $\mathcal{O}_S$-modules $\mathcal{F}$ and $\mathcal{F}'$, the *Segre S-morphism*

$$\mathbb{P}(\mathcal{F}) \times_S \mathbb{P}(\mathcal{F}') \longrightarrow \mathbb{P}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}')$$

is defined in terms of the functors they represent as follows: For any S-scheme Z and exact sequences

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow 0 \quad \text{and} \quad \mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}' \longrightarrow \mathcal{L}' \longrightarrow 0,$$

with invertible $\mathcal{O}_Z$-modules $\mathcal{L}$ and $\mathcal{L}'$, the tensor product gives rise to an exact sequence

$$\mathcal{O}_Z \otimes_{\mathcal{O}_S} (\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}') \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{L}' \longrightarrow 0,$$

with $\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{L}'$ an invertible $\mathcal{O}_Z$-module, hence a morphism $Z \to \mathbb{P}(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{F}')$. The Segre S-morphism also turns out to be a closed immersion (cf. EGA [**1**, Chapter I, revised, §9.8]).

Some of the important properties of schemes and morphisms can be checked in terms of the functors and morphisms of functors representing them: for instance, valuative criterion for properness (cf. Proposition II.6.8) and criterion for smoothness (cf. Criterion V.4.10).

In some cases, the tangent space of a scheme over a field at a point can be defined in terms of the funtor representing it (cf. §V.1).

Example 8.12. The Picard group $\mathrm{Pic}(X)$ of a scheme X is the set of isomorphism classes of invertible $\mathcal{O}_X$-modules forming a commutative group under tensor product (cf. Definition III.1.2). The inverse image by each morphism $f\colon X' \to X$ gives rise to a homomorphism $f^*\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X')$. The contravariant functor thus obtained is far from being representable. Here is a better formulation: For each S-scheme X define a functor $\mathrm{Pic}_{X/S}\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$ by

$$\mathrm{Pic}_{X/S}(Z) = \mathrm{Coker}[\varphi^*\colon \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(X \times_S Z)], \quad \text{for each } S\text{-scheme } \varphi\colon Z \to S.$$

The inverse image by each S-morphism $f\colon Z' \to Z$ gives rise to the map $f^*\colon \mathrm{Pic}_{X/S}(Z) \to \mathrm{Pic}_{X/S}(Z')$. The representability of (modified versions of) the relative Picard functor $\mathrm{Pic}_{X/S}$ has been one of the important issues in algebraic geometry. The reader is referred to FGA [**2**] as well as Kleiman's account on the interesting history (before and after FGA [**2**]) in FAG [**3**, Chapter 9]. When representable, the S-scheme $\mathrm{Pic}_{X/S}$ representing it is called the *relative Picard scheme* of X/S and the universal invertible sheaf on $X \times_S \mathrm{Pic}_{X/S}$ is called the *Poincaré* invertible sheaf. It is a commutative group scheme over S in the sense defined in §VI.1.

Example 8.13. Using the notion of flatness to be defined in Definition IV.2.10 and §IV.4, the *Hilbert functor* for an S-scheme X, is defined by

$$\mathrm{Hilb}_{X/S}(Z) = \{Y \subset X \times_S Z \mid \text{closed subschemes flat over } Z\}$$

with the maps induced by the inverse image by S-morphisms.

Giving a closed subscheme $Y \subset X \times_S Z$ is the same as giving a surjective homomomorphism

$$\mathcal{O}_{X \times_S Z} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

of $\mathcal{O}_{X \times_S Z}$-modules. Thus the Hilbert functor is a special case of the more general functor defined for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$ on an S-scheme X by

$$\mathrm{Quot}_{\mathcal{E}/X/S}(Z) = \{\mathcal{O}_{X \times_S Z} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{F} \to 0 \mid \text{ with } \mathcal{F} \text{ flat over } \mathcal{O}_Z\}/\sim$$

with the maps induced by the inverse image by S-morphisms.

The representability of $\mathrm{Hilb}_{X/S}$ and $\mathrm{Quot}_{\mathcal{E}/X/S}$ has been another major issues. See, for instance, FGA [**2**] and Nitsure's account in FAG [**3**, Chapters 5 and 7].

There are many other important schemes that could be defined as functors such as $\mathrm{Aut}_S(X)$ for an S-scheme, $\mathrm{Hom}_S(X,Y)$ for S-schemes X and Y, moduli spaces, etc. introduced in FGA [**2**]. For later purposes, we list here the basic representability theorem for $\mathrm{Hom}_S(X,Y)$.

Let S be a scheme. For S-schemes X and Y, the functor

$$\mathbf{Hom}_S(X,Y)\colon (\mathrm{Sch}/S)^\circ \longrightarrow (\mathrm{Sets})$$

is defined as follows: For each S-scheme T,

$$\mathbf{Hom}_S(X,Y)(T) = \mathrm{Hom}_T(X\times_S T, Y\times_S T)$$

and for each S-morphism $\phi\colon T'\to T$,

$$\phi^*\colon\ \mathbf{Hom}_S(X,Y)(T) \longrightarrow \mathbf{Hom}_S(X,Y)(T')$$

sends $f\colon X\times_S T\to Y\times_S T$ to

$$\phi^*(f) = (f\circ p_1, p_2)\colon X\times_S T' = (X\times_S T)\times_T T' \longrightarrow (Y\times_S T)\times_T T',$$

where $p_1\colon (X\times_S T)\times_T T'\to X\times_S T$ and $p_2\colon (X\times_S T)\times_T T'\to T'$ are projections.

THEOREM 8.14 (Grothendieck). (cf. FGA [**2**, exposé 221, p. 20], FAG [**3**, Theorem 5.23]) *Let S be a locally noetherian scheme. Let X be an S-scheme that is projective and flat over S, while Y is an S-scheme that is quasi-projective over S. (For "projective" and "quasi-projective", see Definition* II.5.8, *while for "flat" see Definition* IV.2.10 *and* §IV.4.*) Then the functor*

$$\mathbf{Hom}_S(X,Y)\colon (\text{locally noetherian } \mathrm{Sch}/S)^\circ \longrightarrow (\mathrm{Sets})$$

is representable. In other words, there exists a locally noetherian S-scheme $\mathbf{Hom}_S(X,Y)$ and a universal $\mathbf{Hom}_S(X,Y)$-morphism

$$u\colon X\times_S \mathbf{Hom}_S(X,Y) \longrightarrow Y\times_S \mathbf{Hom}_S(X,Y)$$

such that for any locally noetherian S-scheme T, and a T-morphism $f\colon X\times_S T\to Y\times_S T$, there exists a unique S-morphism $\phi\colon T\to \mathbf{Hom}_S(X,Y)$ such that $f=\phi^*(u)$.

Appendix: Theory of sheaves

DEFINITION 1. Let X be a topological space. A presheaf $\mathcal{F}$ on X consists in:

a) for all open sets $U\subset X$, a set $\mathcal{F}(U)$,
b) whenever $U\subset V\subset X$, a map

$$\mathrm{res}_{V,U}\colon \mathcal{F}(V)\longrightarrow \mathcal{F}(U)$$

called the restriction map,

such that

c) $\mathrm{res}_{U,U} = \text{identity}$
d) if $U\subset V\subset W$, then $\mathrm{res}_{V,U}\circ \mathrm{res}_{W,V} = \mathrm{res}_{W,U}$.

DEFINITION 2. If $\mathcal{F}$, $\mathcal{G}$ are presheaves on X, a map $\alpha\colon \mathcal{F}\to\mathcal{G}$ is a set of maps

$$\alpha(U)\colon \mathcal{F}(U)\longrightarrow \mathcal{G}(U)$$

one for each open $U\subset X$, such that for all $U\subset V\subset X$,

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\alpha(V)} & \mathcal{G}(V) \\ {\scriptstyle \mathrm{res}_{V,U}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_{V,U}} \\ \mathcal{F}(U) & \xrightarrow{\alpha(U)} & \mathcal{G}(U) \end{array}$$

commutes.

DEFINITION 3. A presheaf $\mathcal{F}$ is a sheaf if for all open $V \subset X$ and all open coverings $\{U_\alpha\}_{\alpha \in S}$ of V the two properties hold:

a) if $s_1, s_2 \in \mathcal{F}(V)$ and $\mathrm{res}_{V,U_\alpha}(s_1) = \mathrm{res}_{V,U_\alpha}(s_2)$ in each set $\mathcal{F}(U_\alpha)$, then $s_1 = s_2$.
b) if $s_\alpha \in \mathcal{F}(U_\alpha)$ is a set of elements such that for all $\alpha, \beta \in S$,

$$\mathrm{res}_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \mathrm{res}_{U_\beta, U_\alpha \cap U_\beta}(s_\beta) \text{ in } \mathcal{F}(U_\alpha \cap U_\beta),$$

then there exists an $s \in \mathcal{F}(V)$ such that $\mathrm{res}_{V,U_\alpha}(s) = s_\alpha$ for all α.

(Thus $\mathcal{F}(V)$ can be reconstructed from the local values $\mathcal{F}(U_\alpha)$, $\mathcal{F}(U_\alpha \cap U_\beta)$ of the sheaf.) If $\mathcal{F}$ is a sheaf, we will sometimes write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$ and call it the set of sections of $\mathcal{F}$ over U.

DEFINITION 4. If $\mathcal{F}$ is a sheaf on X and $x \in X$, then with respect to the restriction maps, one can form

$$\mathcal{F}_x = \varinjlim_{\substack{\text{all open } U \\ \text{with } x \in U}} \mathcal{F}(U).$$

$\mathcal{F}_x$ is called the stalk of $\mathcal{F}$ at x.

Thus $\mathcal{F}_x$ is the set of germs of sections of $\mathcal{F}$ at x — explicitly, $\mathcal{F}_x$ is the set of all $s \in \Gamma(U, \mathcal{F})$, for all neighborhoods U of x, modulo the equivalence relation:

$$s_1 \sim s_2 \quad \text{if} \quad \mathrm{res}_{U_1, U_1 \cap U_2}(s_1) = \mathrm{res}_{U_2, U_1 \cap U_2}(s_2).$$

The usefulness of stalks is due to the proposition:

PROPOSITION 5.

i) *For all sheaves $\mathcal{F}$ and open sets U, if $s_1, s_2 \in \mathcal{F}(U)$, then $s_1 = s_2$ if and only if the images of s_1, s_2 in $\mathcal{F}_x$ are equal for all $x \in U$.*
ii) *Let $\alpha\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then $\alpha(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all U (resp. bijective for all U), if and only if the induced map on stalks $\alpha_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$ (resp. bijective for all $x \in X$).*

(Proof left to the reader.)

DEFINITION 6. A sheaf $\mathcal{F}$ is a sheaf of groups, rings, etc., if its values $\mathcal{F}(U)$ are groups, rings, etc., and its restriction maps are homomorphisms.

A typical example of a sheaf is the following: Let X and Y be topological spaces and define, for all open $U \subset X$:

$$\mathcal{F}(U) = \{\text{continuous maps from } U \text{ to } Y\}.$$

If $Y = \mathbb{R}$, $\mathcal{F}$ is a sheaf of rings whose stalks $\mathcal{F}_x$ are the rings of germs of continuous real functions at x.

In our applications to schemes, we encounter the situation where we are given a *basis* $\mathfrak{B} = \{U_\alpha\}$ for the open sets of a topological space X, closed under intersection, and a "sheaf" only on $\mathfrak{B}$, i.e., satisfying the properties in Definition 3 for open sets and coverings of $\mathfrak{B}$ — call this a $\mathfrak{B}$-sheaf. In such a situation, we have the facts:

PROPOSITION 7. *Every $\mathfrak{B}$-sheaf extends canonically to a sheaf on all open sets. If $\mathcal{F}$ and $\mathcal{G}$ are two sheaves, every collection of maps*

$$\phi(U_\alpha)\colon \mathcal{F}(U_\alpha) \to \mathcal{G}(U_\alpha) \quad \textit{for all } U_\alpha \in \mathfrak{B}$$

commuting with restriction extends uniquely to a map $\phi\colon \mathcal{F} \to \mathcal{G}$ of sheaves.

IDEA OF PROOF. Given $\mathcal{F}(U_\alpha)$ for $U_\alpha \in \mathfrak{B}$, define stalks

$$\mathcal{F}_x = \varinjlim_{\substack{U_\alpha \in \mathfrak{B} \\ x \in U_\alpha}} \mathcal{F}(U_\alpha)$$

as before. Then for all open U, set

$$\mathcal{F}(U) = \left\{ (s_x) \in \prod_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_{\alpha_i}\} \text{ of } U,\ U_{\alpha_i} \in \mathfrak{B}, \\ \text{and } s_i \in \mathcal{F}(U_{\alpha_i}) \text{ such that} \\ s_x = \operatorname{res} s_i \text{ whenever } x \in U_{\alpha_i} \end{array} \right\}.$$

□

If $\mathcal{F}$ is a presheaf, we can define several associated presheaves:

a) $\forall U$, $\forall s_1, s_2 \in \mathcal{F}(U)$, say

$$s_1 \sim s_2 \quad \begin{array}{l} \text{if } \exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such that} \\ \operatorname{res}_{U,U_\alpha}(s_1) = \operatorname{res}_{U,U_\alpha}(s_2), \text{ for all } \alpha. \end{array}$$

This is an equivalence relation, so we may set

$$\mathcal{F}^{(\mathrm{a})}(U) = \mathcal{F}(U)/(\text{the above equivalence relation } \sim).$$

Then $\mathcal{F}^{(\mathrm{a})}$ is a presheaf satisfying (a) in Definition 3 of sheaves.

b) $\forall U$, consider sets $\{U_\alpha, s_\alpha\}$ where $\{U_\alpha\}$ is a covering of U and $s_\alpha \in \mathcal{F}^{(\mathrm{a})}(U_\alpha)$ satisfy

$$\operatorname{res}_{U_\alpha, U_\alpha \cap U_\beta}(s_\alpha) = \operatorname{res}_{U_\beta, U_\alpha \cap U_\beta}(s_\beta), \quad \text{all } \alpha, \beta.$$

Say

$$\{U_\alpha, s_\alpha\} \sim \{V_\alpha, t_\alpha\} \quad \text{if } \operatorname{res}_{U_\alpha, U_\alpha \cap V_\beta}(s_\alpha) = \operatorname{res}_{V_\beta, U_\alpha \cap V_\beta}(t_\beta), \text{ all } \alpha, \beta.$$

Let

$$\operatorname{sh}(\mathcal{F})(U) = \left\{ \begin{array}{l} \text{the set of sets } \{U_\alpha, s_\alpha\} \text{ modulo} \\ \text{the above equivalence relation} \end{array} \right\}.$$

Then $\operatorname{sh}(\mathcal{F})$ is in fact a sheaf.

DEFINITION 8. $\operatorname{sh}(\mathcal{F})$ is the sheafification of $\mathcal{F}$.

It is trivial to check that the canonical map

$$\mathcal{F} \longmapsto \operatorname{sh}(\mathcal{F})$$

is universal with respect to maps of $\mathcal{F}$ to sheaves, i.e., $\forall \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$, $\mathcal{G}$ a sheaf, $\exists \beta\colon \operatorname{sh}(\mathcal{F}) \to \mathcal{G}$ such that

$$\begin{tikzcd} & \operatorname{sh}(\mathcal{F}) \arrow[d, "\beta"] \\ \mathcal{F} \arrow[ur] \arrow[r, "\alpha"'] & \mathcal{G} \end{tikzcd}$$

commutes. A useful connection between these concepts is:

PROPOSITION 9. *Let $\mathfrak{B}$ be a basis of open sets and $\mathcal{F}$ a presheaf defined on all open sets, but which is already a sheaf on $\mathfrak{B}$. Then the unique sheaf that extends the restriction to $\mathfrak{B}$ of $\mathcal{F}$ is the sheafification of the full $\mathcal{F}$.*

(Proof left to the reader)

The set of all sheaves of abelian groups on a fixed topological space X forms an *abelian category* (cf., e.g., Bass [**21**, p. 21]). In fact

a) the set of maps $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ from one sheaf $\mathcal{F}$ to another $\mathcal{G}$ is clearly an abelian group because we can add two maps; and composition of maps is bilinear.
b) the 0-sheaf, $0(U) = \{0\}$ for all U, is a 0-object (i.e., $\mathrm{Hom}(0,\mathcal{F}) = \mathrm{Hom}(\mathcal{F},0) = \{0\}$, for all $\mathcal{F}$),
c) sums exist, i.e., if $\mathcal{F}$, $\mathcal{G}$ are two sheaves, define $(\mathcal{F}\oplus\mathcal{G})(U) = \mathcal{F}(U)\oplus\mathcal{G}(U)$. This is a sheaf which is categorically both a sum and a product (i.e., $\mathrm{Hom}(\mathcal{H},\mathcal{F}\oplus\mathcal{G}) = \mathrm{Hom}(\mathcal{H},\mathcal{F})\oplus\mathrm{Hom}(\mathcal{H},\mathcal{G})$ and $\mathrm{Hom}(\mathcal{F}\oplus\mathcal{G},\mathcal{H}) = \mathrm{Hom}(\mathcal{F},\mathcal{H})\oplus\mathrm{Hom}(\mathcal{G},\mathcal{H})$).

(This means we have an *additive* category.)

d) Kernels exist: if $\alpha\colon \mathcal{F}\to\mathcal{G}$ is any homomorphism, define
$$\mathrm{Ker}(\alpha)(U) = \{s\in\mathcal{F}(U) \mid \alpha(s) = 0 \text{ in } \mathcal{G}(U)\}.$$
Then one checks immediately that $\mathrm{Ker}(\alpha)$ is a sheaf and is a categorical kernel, i.e.,
$$\mathrm{Hom}(\mathcal{H},\mathrm{Ker}(\alpha)) = \{\beta\in\mathrm{Hom}(\mathcal{H},\mathcal{F}) \mid \alpha\circ\beta = 0\}.$$
e) Cokernels exist: if $\alpha\colon \mathcal{F}\to\mathcal{G}$ is any homomorphism, look first at the *presheaf*:
$$\text{Pre-Coker}(\alpha)(U) = \text{quotient of } \mathcal{G}(U) \text{ by } \alpha(\mathcal{F}(U)).$$
This is not usually a sheaf, but set
$$\mathrm{Coker}(\alpha) = \text{sheafification of Pre-Coker}(\alpha).$$
One checks that this is a categorical cokernel, i.e.,
$$\mathrm{Hom}(\mathrm{Coker}(\alpha),\mathcal{H}) = \{\beta\in\mathrm{Hom}(\mathcal{G},\mathcal{H}) \mid \beta\circ\alpha = 0\}.$$
f) Finally, the main axiom: given $\alpha\colon \mathcal{F}\to\mathcal{G}$, then
$$\mathrm{Ker}(\mathcal{G}\to\mathrm{Coker}\,\alpha) \cong \mathrm{Coker}(\mathrm{Ker}\,\alpha\to\mathcal{F}).$$

PROOF. By definition
$$\begin{aligned}&\mathrm{Coker}(\mathrm{Ker}\,\alpha\to\mathcal{F})\\ &\quad= \text{sheafification of } \{U\mapsto \mathcal{F}(U)/\mathrm{Ker}(\alpha)(U)\}\\ &\quad= \text{sheafification of } \{U\mapsto \text{Image of } \mathcal{F}(U) \text{ in } \mathcal{G}(U)\}.\end{aligned}$$

Since the presheaf $U\mapsto\alpha\mathcal{F}(U)$ satisfies the first condition for a sheaf, and is contained in a sheaf $\mathcal{G}$, its sheafification is simply described as:
$$\begin{aligned}&\mathrm{Coker}(\mathrm{Ker}\,\alpha\to\mathcal{F})(U)\\ &\quad= \left\{ s\in\mathcal{G}(U) \;\middle|\; \begin{array}{l}\exists \text{ a covering } \{U_\alpha\} \text{ of } U\\ \text{such that } \mathrm{res}_{U,U_\alpha}(s)\in\alpha\mathcal{F}(U)\end{array}\right\}.\end{aligned}$$
But
$$\begin{aligned}&\mathrm{Ker}(\mathcal{G}\to\mathrm{Coker}\,\alpha)(U)\\ &\quad= \{s\in\mathcal{G}(U) \mid s\mapsto 0 \text{ in } \mathrm{Coker}(\alpha)(U)\}\\ &\quad= \left\{ s\in\mathcal{G}(U) \;\middle|\; \begin{array}{l}\text{image of } s \text{ in the presheaf}\\ U\mapsto\mathcal{G}(U)/\alpha\mathcal{F}(U) \text{ is killed by}\\ \text{process (a) of sheafification}\end{array}\right\}\\ &\quad= \left\{ s\in\mathcal{G}(U) \;\middle|\; \begin{array}{l}\exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such}\\ \text{that } s\mapsto 0 \text{ in } \mathcal{G}(U_\alpha)/\alpha\mathcal{F}(U_\alpha)\end{array}\right\}\\ &\quad= \left\{ s\in\mathcal{G}(U) \;\middle|\; \begin{array}{l}\exists \text{ a covering } \{U_\alpha\} \text{ of } U \text{ such}\\ \text{that } \mathrm{res}_{U,U_\alpha}(s)\in\alpha\mathcal{F}(U_\alpha)\end{array}\right\}.\end{aligned}$$

□

The essential twist in the theory of abelian sheaves is that if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence, then:

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U) \quad \text{is exact}$$

but

$$\mathcal{G}(U) \longrightarrow \mathcal{H}(U) \quad \text{is } \textit{not in general surjective}.$$

In fact, to test the surjectivity of a sheaf homomorphism $\alpha \colon \mathcal{G} \to \mathcal{H}$, one must see whether the presheaf $U \longmapsto \mathcal{H}(U)/\mathcal{G}(U)$ dies when it is sheafified, i.e.,

$$[\alpha \colon \mathcal{G} \to \mathcal{H} \text{ surjective}] \Longleftrightarrow \left[\begin{array}{l} \forall s \in \mathcal{H}(U), \exists \text{ covering} \\ \{U_\alpha\} \text{ of } U \text{ such that} \\ \mathrm{res}_{U,U_\alpha}(s) \in \text{Image of } \mathcal{G}(U_\alpha) \end{array} \right].$$

As one easily checks, this is equivalent to the induced map on stalks $\mathcal{G}_x \to \mathcal{H}_x$ being surjective for all $x \in X$.

The category of abelian sheaves also has *infinite* sums and products but one must be a little careful: if $\{\mathcal{F}_\alpha\}_{\alpha \in S}$ is any set of sheaves, then

$$U \longmapsto \prod_{\alpha \in S} \mathcal{F}_\alpha(U)$$

is again a sheaf, and it is categorically the product of the $\mathcal{F}_\alpha$'s but

$$U \longmapsto \sum_{\alpha \in S} \mathcal{F}_\alpha(U)$$

need not be a sheaf. It has property (a) but not always property (b), so we must define the sheaf $\sum \mathcal{F}_\alpha$ to be its sheafification, i.e.,

$$\sum_{\alpha \in S} \mathcal{F}_\alpha(U) = \left\{ s \in \prod_{\alpha \in S} \mathcal{F}_\alpha(U) \;\middle|\; \begin{array}{l} \exists \text{ a covering } \{U_\beta\} \text{ of } U \text{ such that} \\ \text{for all } \beta, \, \mathrm{res}_{U,U_\beta}(s) \text{ has only a} \\ \text{finite number of non-zero components} \end{array} \right\}.$$

This $\sum_{\alpha \in S} \mathcal{F}_\alpha$ is a categorical sum. But note that if U is quasi-compact, i.e., all open coverings have finite subcoverings, then clearly

$$\sum_{\alpha \in S} \mathcal{F}_\alpha(U) = \sum_{\alpha \in S} (\mathcal{F}_\alpha(U)).$$

There are several more basic constructions that we will use:

a) given $\mathcal{F}$, $\mathcal{G}$ abelian sheaves on X, we get a new abelian sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \{\text{homomorphisms over } U \text{ from } \mathcal{F}|_U \text{ to } \mathcal{G}|_U\}.$$

b) given a continuous map $f \colon X \to Y$ of topological spaces and a sheaf $\mathcal{F}$ on X, we get a sheaf $f_*\mathcal{F}$ on Y by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

It is trivial to check that both of these are indeed sheaves.

Exercise

(1) Let $\mathcal{F}$ be a presheaf of sets on a topological space X. Show that there is a sheaf $a\mathcal{F}$ on X and a map $\alpha\colon \mathcal{F} \to a\mathcal{F}$ of presheaves such that α induces a bijection

$$\mathrm{Hom}_{\mathrm{sheaves}}(a\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{presheaves}}(\mathcal{F}, \mathcal{G})$$

for every sheaf of sets $\mathcal{G}$ on X. Here the first Hom means maps in the category of sheaves, while the second Hom means maps in the category of presheaves.
Hint: Try to use a direct limit construction to force the sheaf property to hold. You probably will need to apply the same procedure twice, because when applied for the first time you are likely to get only a *separated presheaf*, i.e., for every open covering U_i of an open U, the map $\mathcal{G}(U) \to \prod_i \mathcal{G}(U_i)$ is injective. Repeating the process, you get the exactness of

$$\mathcal{G}(U) \to \prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j).$$

(2) Let $f\colon X \to Y$ be a continuous map of topological spaces.
   (i) Show that the functor f_* from the category of presheaves on X to the category of presheaves on Y has a left adjoint $f^\sharp$.
   Hint: Let $\mathcal{F}$ be a sheaf of sets on Y. For any open subset $U \subset X$, let

$$f^\sharp(U) = \varinjlim_{U \hookrightarrow f^{-1}(V)} \mathcal{F}(V),$$

   where the indexing set of the direct limit is the set of all open subsets $V \subset Y$ such that $f(U) \subset V$.
   (ii) Show that the functor f_* from the category of sheaves on X to the category of sheaves on Y has a left adjoint $f^\bullet$.
   Hint: Let $f^\bullet\mathcal{F}$ be the sheafification $af^\sharp(\mathcal{F})$ of the presheaf $f^\sharp(\mathcal{F})$.
   (iii) When $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$, f is given by a ring homomorphism from S to R, and $\mathcal{F} = \widetilde{M}$ is the quasi-coherent $\mathcal{O}_Y$-module attached to an S-module M, check that $\mathcal{O}_X \otimes_{f^\bullet\mathcal{O}_Y} f^*\mathcal{F}$ is naturally isomorphic to the quasi-coherent $\mathcal{O}_X$-module attached to $R \otimes_S M$.

(3) Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_Y$-module. Verify that $f^*\mathcal{F} := \mathcal{O}_X \otimes_{f^\bullet\mathcal{O}_Y} f^\bullet\mathcal{F}$ is canonically isomorphic to the pull-back of quasi-coherent modules explained after Corollary 5.6 and before (5.7). Similarly, suppose that $r\colon X \to S$ and $s\colon Y \to S$ are S-schemes, and $\mathcal{F}$ (resp. $\mathcal{G}$) is a quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_Y$-module). Verify that $p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_S Y}} p_2^*\mathcal{G}$ is canonically isomorphic to the quasi-coherent $\mathcal{O}_{X\times_S Y}$-module "$\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_Y$" after Corollary 5.6 and before (5.7).

(4) Verify that for any commutative ring R with 1, the set of all R-valued points of $\mathrm{GL}_{n,\mathbb{Z}}$ is in bijection with the set of all units of the algebra $M_n(R)$ of $n \times n$-matrices with entries in R.

(5) Denote by $\mathbb{A}_{\mathbb{Q}}$ the ring of all $\mathbb{Q}$-adèles, defined to be the subset of $\mathbb{R} \times \prod_p \mathbb{Q}_p$, consisting of all sequences $(x_i)_{i\in\Sigma}$, where the indexing set Σ consists of ∞ and the set of all prime numbers, $x_\infty \in \mathbb{R}$, $x_p \in \mathbb{Q}_p$ for all p, and $x_p \in \mathbb{Z}_p$ for all but a finite number of p's. Describe explicitly the set of all $\mathbb{A}_{\mathbb{Q}}$-points of $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[T, 1/T])$, GL_n and $\mathbb{A}^1 \setminus \{0, 1\}$.

(6) Give an example of a sheaf on $\mathrm{Spec}(\mathbb{Z}[T])$ that is not quasi-coherent.

(7) Let X be a scheme. Do infinite products exists in the category of all quasi-coherent $\mathcal{O}_X$-modules? Either give a proof or a counterexample.

(8) Let k be a field. Are $\operatorname{Spec}(k[x,y,z]/(x^2-y^2-z^4))$ and $\operatorname{Spec}(k[x,y,z]/(x^2-y^2-z^2))$ isomorphic as k-schemes? Either give a proof or a counterexample.

(9) Let k be a field of characteristic $p>0$. Let $\sigma\colon \operatorname{Spec}(k)\to\operatorname{Spec}(k)$ be the morphism such that σ^* is the Frobenius homomorphism $u\mapsto u^p$ for $u\in k$. For any k-scheme X, denote by $X^{(p)}$ the fibre product $X\times_{\operatorname{Spec}(k)}(\sigma,\operatorname{Spec}(k))$. Give an example in which the scheme $X^{(p)}$ is *not* isomorphic to X.

(10) Give an example of an additive category which is not an abelian category.

(11) (Weil's retriction of scalars) Let $T\to S$ and $X\to T$ be schemes. The Weil restriction $\mathrm{R}_{T/S}(X)$ is the contravariant functor from the category of S-schemes to the category of sets such that

$$\mathrm{R}_{T/S}(X)(Z)=\operatorname{Hom}_T(T\times_S Z,X)$$

for every S-scheme $Z\to S$. If $T=\operatorname{Spec}(R)$ and $S=\operatorname{Spec}(A)$, one often shortens the notation to $\mathrm{R}_{R/A}(X)$. Prove that $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)\cong\operatorname{Spec}(\mathbb{R}[x,y,(x^2+y^2)^{-1}])$.

Note: Here is a more intrinsic way to think about the ring $\mathbb{R}[x,y,(x^2+y^2)^{-1}]$. Let $B=\operatorname{Symm}^*(\mathbb{C}^\vee)$ be the $\mathbb{R}$-algebra of polynomial functions on $\mathbb{C}$, where $\mathbb{C}^\vee=\operatorname{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{R})$. Let Tr and Nm be the elements in B corresponding to $\operatorname{Tr}_{\mathbb{C}/\mathbb{R}}$ and $\operatorname{Nm}_{\mathbb{C}/\mathbb{R}}$, respectively. Then the localization $B[(\mathrm{Nm})^{-1}]$ of B represents $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$.

(12) (Continuation on the Weil restriction) Let A be a ring, and let R be finitely generated A-algebra which is a projective A-module. Let $R^\vee=\operatorname{Hom}_A(R,A)$ be the A-module dual to R. Denote by δ the element of $R\otimes_A R^\vee$ corresponding to the identity map $\mathrm{id}\in\operatorname{End}_A(R)$ under the natural isomorphism $\operatorname{End}_A(R)=R\otimes_A R^\vee$.

(i) Let B be an R-algebra. For any $f\in B$ and any $u\in R^\vee$, let $u_\flat(f)\in R^\vee\otimes_A B$ be the image of the element $\delta\otimes f\in R\otimes_A R^\vee\otimes_A B$ under the map

$$u\otimes\mathrm{id}_{R^\vee\otimes B}\colon R\otimes_A R^\vee\otimes_A B\longrightarrow R^\vee\otimes_A B.$$

For any $f_1,f_2\in B$ and any element $u\in R^\vee$, let $u(f_1,f_2)$ be the image of $(\delta\otimes f_1)\otimes(\delta\otimes f_2)$ under the composition of the following maps

$$\begin{aligned}&(R\otimes_A R^\vee\otimes_A B)\otimes_A(R\otimes_A R^\vee_A\otimes B)\\ \xrightarrow{\sim}\quad&(R\otimes_A R)\otimes_A(R^\vee\otimes_A B)\otimes_A(R^\vee\otimes_A B)\\ \xrightarrow{\mu\otimes\mathrm{can}}\quad&R\otimes_A\operatorname{Symm}^2(R^\vee\otimes_A B)\\ \xrightarrow{u\otimes\mathrm{id}}\quad&\operatorname{Symm}^2(R^\vee\otimes_A B),\end{aligned}$$

where the arrow $\mu\otimes\mathrm{can}$ is induced by the multiplication $\mu\colon R\otimes R\to R$ of R and the natural surjection $\mathrm{can}\colon(R^\vee\otimes_A B)\otimes_A(R^\vee\otimes_A B)\to\operatorname{Symm}^2(R^\vee\otimes_A B)$. Let $F=\operatorname{Symm}^*_A(R^\vee\otimes_A B)$ be the symmetric algebra of the A-module $R^\vee\otimes_A B$. Denote by $\mathrm{R}_{R/A}(B)$ the quotient ring of F with respect to the ideal I generated by all elements of the form

$$u(f_1,f_2)-u_\flat(f_1\cdot f_2),\qquad u\in R^\vee,\ f_1,f_2\in B.$$

Show that $\operatorname{Spec}(\mathrm{R}_{R/A}(B))$ represents $\mathrm{R}_{R/A}\operatorname{Spec}(B)$.

(ii) Show that for any R-scheme X, the functor $\mathrm{R}_{R/A}(X)$ is representable in the category of S-schemes.

CHAPTER II

Exploring the world of schemes

1. Classical varieties as schemes

Having now defined the category of schemes, we would like to see how the principal objects of classical geometry—complex projective varieties—fit into the picture. In fact a variety is essentially a very special kind of scheme and a regular correspondence between two varieties is a morphism. I would like first to show very carefully how a variety is made into a scheme, and secondly to analyze step by step what special properties these schemes have and how we can characterize varieties among all schemes.

I want to change notation slightly to bring it in line with that of the last chapter and write $\mathbb{P}^n(\mathbb{C})$ for complex projective n-space, the set of non-zero $(n+1)$-tuples $(a_0, \ldots, a_n)$ of complex numbers modulo $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for $\lambda \in \mathbb{C}^*$. Let

$$X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$$

be a complex projective variety, i.e., the set of zeroes of the homogeneous equations $f \in \mathfrak{p}$, $\mathfrak{p} \subset \mathbb{C}[X_0, \ldots, X_n]$ being a homogeneous prime ideal. Next for every irreducible subvariety:

$$W(\mathbb{C}) \subset X(\mathbb{C}), \qquad \dim W(\mathbb{C}) \geq 1$$

let η_W be a new point. Define X to be the union of $X(\mathbb{C})$ and the set of these new points $\{\ldots, \eta_W, \ldots\}$. This will be the underlying point set of a scheme with $X(\mathbb{C})$ as its closed points and the η_W's as the non-closed points. Extend the topology from $X(\mathbb{C})$ to X as follows:

$$\begin{aligned}&\text{for all Zariski open } U(\mathbb{C}) \subset X(\mathbb{C}),\\ &\text{let } U = U(\mathbb{C}) \cup \{\eta_W \mid W(\mathbb{C}) \cap U(\mathbb{C}) \neq \emptyset\}\,.\end{aligned}$$

One sees easily that the map $U(\mathbb{C}) \mapsto U$ preserves arbitrary unions and finite intersections, hence it defines a topology on X. Moreover, in this topology:

a) $\forall x \in X(\mathbb{C})$, $x \in \overline{\{\eta_W\}} \Longleftrightarrow x \in W(\mathbb{C})$
b) $\forall V(\mathbb{C}) \subset X(\mathbb{C})$, $\eta_V \in \overline{\{\eta_W\}} \Longleftrightarrow V(\mathbb{C}) \subset W(\mathbb{C})$,

hence $\overline{\{\eta_W\}}$ is just W, i.e., η_W is a generic point of W. You can picture $\mathbb{P}^2$ for instance, something like that in Figure II.1.

To put a sheaf on X, we can proceed in two ways:

Method (1). Recall that we have defined in Part I [**87**, Chapter 2], a function field $\mathbb{C}(X)$ and for every $x \in X(\mathbb{C})$, a local ring $\mathcal{O}_{x,X}$ with quotient field $\mathbb{C}(X)$. Now for every open set $U \subset X$, define

$$\mathcal{O}_X(U) = \bigcap_{x \in U(\mathbb{C})} \mathcal{O}_{x,X}$$

and whenever $U_1 \subset U_2$, note that $\mathcal{O}_X(U_2)$ is a subring of $\mathcal{O}_X(U_1)$: let

$$\mathrm{res}_{U_2,U_1} : \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1)$$

be the inclusion map. In this way we obviously get a sheaf; in fact a subsheaf of the constant sheaf with value $\mathbb{C}(X)$ on every U.

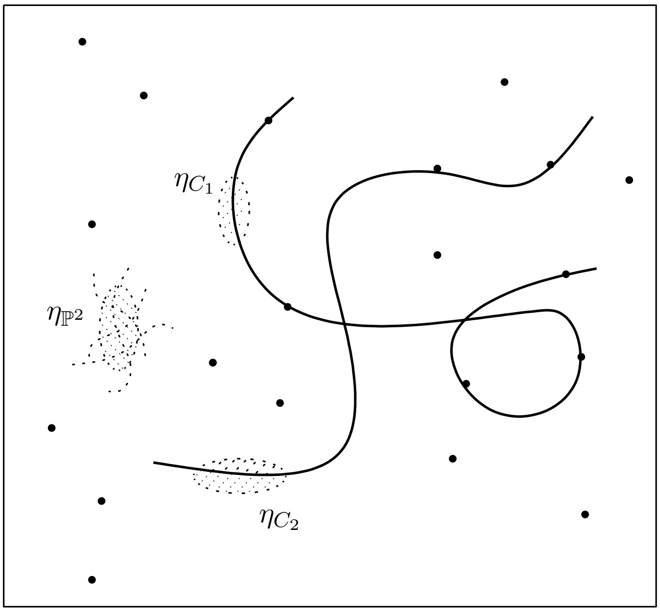

FIGURE II.1. $\mathbb{P}^2$

METHOD (2). Instead of working inside $\mathbb{C}(X)$, we can work inside the sheaf of functions from the closed points of X to $\mathbb{C}$:

$$\mathbb{C}^X(U) = \{\text{set of functions } f\colon U(\mathbb{C}) \longrightarrow \mathbb{C}\}$$

restriction now being just restriction of functions. Then define

$$\mathcal{O}_X(U) = \left\{ \begin{array}{l} \text{subset of } \mathbb{C}^X(U) \text{ of functions } f \text{ such that for every} \\ x \in U(\mathbb{C}), \text{ there is a neighborhood } U_x \text{ of } x \text{ in } U \text{ and a} \\ \text{rational function } a(x_0, \ldots, x_n)/b(x_0, \ldots, x_n),\ a \text{ and } b \\ \text{homogeneous of the same degree, such that} \\ f(y_0, \ldots, y_n) = \dfrac{a(y_0, \ldots, y_n)}{b(y_0, \ldots, y_n)}, \quad b(y_0, \ldots, y_n) \neq 0 \\ \text{for every } y \in U_x \end{array} \right\}.$$

This is clearly a subsheaf of $\mathbb{C}^X$. To see that we have found the same sheaf twice, call these two sheaves $\mathcal{O}_X^{\mathrm{I}}$, $\mathcal{O}_X^{\mathrm{II}}$ for a minute and observe that we have maps:

$$\mathcal{O}_X^{\mathrm{I}}(U) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathcal{O}_X^{\mathrm{II}}(U)$$

$$\alpha(f) = \left\{ \begin{array}{l} \text{the function } x \mapsto f(x) \text{ (OK since } f(x) \text{ is defined)} \\ \text{whenever } f \in \mathcal{O}_{x,X} \end{array} \right\}$$

$$\beta(f) = \left\{ \begin{array}{l} \text{the element of } \mathbb{C}(X) \text{ represented by any of the} \\ \text{rational functions } a(x_0, \ldots, x_n)/b(x_0, \ldots, x_n) \\ \text{which equal } f \text{ in a Zariski open subset of } U. \\ \text{(OK since if } a/b \text{ and } c/d \text{ have the same values in a} \\ \text{non-empty Zariski-open } U \cap V, \text{ then } ad - bc \equiv 0 \\ \text{on } X, \text{ hence } a/b = c/d \text{ in } \mathbb{C}(X).) \end{array} \right\}$$

From now on, we identify these two sheaves and consider the structure sheaf $\mathcal{O}_X$ either as a subsheaf of the constant sheaf $\mathbb{C}(X)$ or of $\mathbb{C}^X$, whichever is appropriate. The main point now is that $(X, \mathcal{O}_X)$ is indeed a scheme. To see this it is easiest first to note that we can make all the above definitions starting with a complex affine variety $Y(\mathbb{C}) \subset \mathbb{C}^n$ instead of with a projective variety. And moreover, just as $X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ is covered by affine varieties

$$Y_i(\mathbb{C}) = X(\mathbb{C}) \setminus X(\mathbb{C}) \cap V(X_i)$$

so too the pair $(X, \mathcal{O}_X)$ is locally isomorphic at every point to $(Y_i, \mathcal{O}_{Y_i})$ for some i. Therefore it is enough to show that $(Y_i, \mathcal{O}_{Y_i})$ is a scheme. But if the affine $Y(\mathbb{C})$ equals $V(\mathfrak{p})$, $\mathfrak{p} \subset \mathbb{C}[X_1, \ldots, X_n]$ a prime ideal, then I claim:

$$(Y, \mathcal{O}_Y) \cong (\operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}, \mathcal{O}_{\operatorname{Spec} \mathbb{C}[X]/\mathfrak{p}}). \tag{1.1}$$

Proof. The prime ideals $\overline{\mathfrak{q}} \subset \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ are in one-to-one correspondence with the prime ideals $\mathfrak{q}$:

$$\mathfrak{p} \subset \mathfrak{q} \subset \mathbb{C}[X_1, \ldots, X_n],$$

and these are in one-to-one correspondence with the set of irreducible closed subsets of $V(\mathfrak{p})$, i.e., to the points of $Y(\mathbb{C})$ plus the positive dimensional subvarieties of $Y(\mathbb{C})$. Therefore there is a canonical bijection:

$$Y \cong \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$$

via

$$\eta_{V(\mathfrak{q})} \longleftrightarrow [\overline{\mathfrak{q}}] \text{ for } \overline{\mathfrak{q}} \text{ not maximal}$$
$$Y(\mathbb{C}) \ni a \longleftrightarrow [(X_1 - a_1, \ldots, X_n - a_n) \mod \mathfrak{p}]$$

[Recall that the maximal ideals of $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$ are the ideals $I(a)$ of all functions vanishing at a point $a \in X(\mathbb{C})$, i.e., the ideals $(X_1 - a_1, \ldots, X_n - a_n)/\mathfrak{p}$.] It is seen immediately that this bijection is a homeomorphism. To identify the sheaves, note that for all $f \in \mathbb{C}[X_1, \ldots, X_n]$,

$$\mathcal{O}_Y(Y_f) \underset{\text{def}}{=} \bigcap_{a \in Y_f(\mathbb{C})} \mathcal{O}_{a,Y}$$

$$= \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\text{localization of } \quad \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p} \quad \text{at} \quad \overbrace{(X_1 - a_1, \ldots, X_n - a_n)/\mathfrak{p}}^{I(a)})$$

while

$$\mathcal{O}_{\operatorname{Spec} \mathbb{C}[X]/\mathfrak{p}}(Y_f) \underset{\text{def}}{=} \text{localization } (\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p})_f.$$

These are both subrings of $\mathbb{C}(Y)$, the quotient field of $\mathbb{C}[X_1, \ldots, X_n]/\mathfrak{p}$. Now since $f(a) \neq 0 \Longrightarrow f \in (\mathbb{C}[X]/\mathfrak{p}) \setminus I(a)$, we see that

$$(\mathbb{C}[X]/\mathfrak{p})_f \subset \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\mathbb{C}[X]/\mathfrak{p})_{I(a)}.$$

And if

$$g \in \bigcap_{\substack{a \in Y(\mathbb{C}) \\ f(a) \neq 0}} (\mathbb{C}[X]/\mathfrak{p})_{I(a)},$$

let

$$\mathfrak{a} = \{h \in \mathbb{C}[X]/\mathfrak{p} \mid gh \in \mathbb{C}[X]/\mathfrak{p}\}.$$

If $f(a) \neq 0$, then $\exists g_a, h_a \in \mathbb{C}[X]/\mathfrak{p}$ and $h_a \notin I(a)$ such that $g = g_a/h_a$, hence $h_a \in \mathfrak{a}$. Thus $a \notin V(\mathfrak{a})$. Since this holds for all $a \in Y(\mathbb{C})_f$, we see that $V(\mathfrak{a}) \subset V(f)$, hence by the Nullstellensatz (cf. Part I [**87**, §1A, (1.5)], Zariski-Samuel [**119**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**27**, Chapter V, §3.3, Proposition 2]) $f^N \in \mathfrak{a}$ for some $N \geq 1$. This means precisely that $g \in (\mathbb{C}[X]/\mathfrak{p})_f$. Thus the sheaves are the same too. $\square$

To simplify terminology, we will now call the scheme X attached to $X(\mathbb{C})$ a complex projective variety too. Next, if

$$\begin{aligned} X(\mathbb{C}) &\subset \mathbb{P}^n(\mathbb{C}) \\ Y(\mathbb{C}) &\subset \mathbb{P}^n(\mathbb{C}) \end{aligned}$$

are two complex projective varieties and if

$$Z(\mathbb{C}) \subset X(\mathbb{C}) \times Y(\mathbb{C})$$

is a regular correspondence from X to Y, we get a canonical morphism

$$f_Z \colon X \longrightarrow Y.$$

In fact, as a map of sets, define the following.

$$\text{If } x \in X(\mathbb{C})\text{: } f_Z(x) = \text{the unique } y \in Y(\mathbb{C}) \text{ such that } (x,y) \in Z(\mathbb{C})$$

$$\text{If } W(\mathbb{C}) \subset X(\mathbb{C})\text{: } f_Z(\eta_W) = \begin{cases} \eta_V & \text{if } \dim V(\mathbb{C}) \geq 1 \\ v & \text{if } V(\mathbb{C}) = \{v\} \end{cases}$$
$$\text{where} \quad V(\mathbb{C}) = p_2\left[(W(\mathbb{C}) \times Y(\mathbb{C})) \cap Z(\mathbb{C})\right].$$

One checks immediately that this map is continuous. To define the map backwards on sheaves, proceed in either of two ways:

METHOD (1). Recall that Z defined a map $Z^* \colon \mathbb{C}(Y) \to \mathbb{C}(X)$ and the fact that Z is regular implies that for all $x \in X(\mathbb{C})$, if $y = f_Z(x)$, then

$$Z^*(\mathcal{O}_{y,Y}) \subset \mathcal{O}_{x,X}.$$

Therefore, for every open set $U \subset Y$,

$$\begin{aligned} Z^*(\mathcal{O}_Y(U)) &= Z^*\left(\bigcap_{y \in U(\mathbb{C})} \mathcal{O}_{y,Y}\right) \\ &\subset \bigcap_{x \in f_Z^{-1}U(\mathbb{C})} \mathcal{O}_{x,X} \\ &= \mathcal{O}_X(f_Z^{-1}U) \end{aligned}$$

giving a map of sheaves.

METHOD (2). Define a map

$$f_Z^* \colon \mathbb{C}^Y(U) \longrightarrow \mathbb{C}^X(f_Z^{-1}U)$$

by composition with f_Z, i.e., if $\alpha \colon U(\mathbb{C}) \to \mathbb{C}$ is a function, then $\alpha \circ f_Z$ is a function $f_Z^{-1}U(\mathbb{C}) \to \mathbb{C}$. One checks immediately using the regularity of Z that f_Z^* maps functions α in the subring $\mathcal{O}_Y(U)$ to functions $\alpha \circ f_Z \in \mathcal{O}_X(f_Z^{-1}(U))$.

There is one final point in this direction which we will just sketch. That is:

PROPOSITION 1.2. *Let $X(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ and $Y(\mathbb{C}) \subset \mathbb{P}^m(\mathbb{C})$ be complex projective varieties. Let $Z(\mathbb{C}) \subset \mathbb{P}^{nm+n+m}(\mathbb{C})$ be their set-theoretic product, embedded by the Segre embedding as third complex projective variety (cf. Part I [**87**, Chapter 2]). Then the scheme Z is canonically isomorphic to the fibre product $X \times_{\mathrm{Spec}(\mathbb{C})} Y$ of the schemes X and Y.*

Idea of proof. Let $X_0, \dots, X_n, Y_0, \dots, Y_m$ and Z_{ij} $(0 \le i \le n, 0 \le j \le m)$ be homogeneous coordinates in $\mathbb{P}^n(\mathbb{C})$, $\mathbb{P}^m(\mathbb{C})$ and $\mathbb{P}^{nm+n+m}(\mathbb{C})$. Then by definition $Z(\mathbb{C})$ is covered by affine pieces $Z_{i_0j_0} \neq 0$ which are set-theoretically the product of the affine $X_{i_0} \neq 0$ in $X(\mathbb{C})$ and $Y_{j_0} \neq 0$ in $Y(\mathbb{C})$. The Segre embedding is given in this piece by

$$\frac{Z_{ij}}{Z_{i_0j_0}} = \frac{X_i}{X_{i_0}} \cdot \frac{Y_j}{Y_{j_0}}$$

so the affine ring of Z comes out:

$$\begin{aligned}\mathbb{C}[\dots, &\frac{Z_{ij}}{Z_{i_0j_0}}, \dots]/\{\text{functions } 0 \text{ on } Z(\mathbb{C})\} \\ &= \mathbb{C}[\dots, \frac{X_i}{X_{i_0}}, \dots, \frac{Y_j}{Y_{j_0}}, \dots]/\{\text{functions } 0 \text{ on } X(\mathbb{C}) \times Y(\mathbb{C})\}.\end{aligned}$$

To see that this is the tensor product of the affine rings of X and Y:

$$\left(\mathbb{C}[\dots, \frac{X_i}{X_{i_0}}, \dots]/\{\text{functions } 0 \text{ on } X(\mathbb{C})\}\right) \otimes_{\mathbb{C}} \left(\mathbb{C}[\dots, \frac{Y_j}{Y_{j_0}}, \dots]/\{\text{functions } 0 \text{ on } Y(\mathbb{C})\}\right)$$

one uses the ordinary Nullstellensatz (cf. Part I [**87**, §1A, (1.5)], Zariski-Samuel [**119**, vol. II, Chapter VII, §3, Theorem 14] and Bourbaki [**27**, Chapter V, §3.3, Proposition 2]) plus:

Lemma 1.3. *If R and S are k-algebras with no nilpotents, k a perfect field, then $R \otimes_k S$ has no nilpotent elements.*

(cf. §IV.2 below.) □

Corollary 1.4. *Let $X(\mathbb{C})$, $Y(\mathbb{C})$ be complex projective varieties. Then the set of regular correspondences from $X(\mathbb{C})$ to $Y(\mathbb{C})$ and the set of $\mathbb{C}$-morphisms from the scheme X to the scheme Y are the same.*

Idea of proof. Starting from $f : X \to Y$, we get a morphism

$$f \times 1_Y : X \times_{\mathrm{Spec}(\mathbb{C})} Y \longrightarrow Y \times_{\mathrm{Spec}(\mathbb{C})} Y.$$

If $\Delta(\mathbb{C}) \subset Y(\mathbb{C}) \times Y(\mathbb{C})$ is the diagonal, which is easily checked to be closed, define $\Gamma = (f \times 1_Y)^{-1}(\Delta)$, then $\Gamma(\mathbb{C})$ is closed in $X(\mathbb{C}) \times Y(\mathbb{C})$ and is the graph of $\mathrm{res}(f)$. Therefore $\Gamma(\mathbb{C})$ is a single-valued correspondence and a local computation shows that it is regular. □

2. The properties: reduced, irreducible and finite type

The goal of this section is to analyze some of the properties that make classical varieties special in the category of schemes. We shall do two things:

a) Define for general schemes, and analyze the first consequences, of three basic properties of classical varieties: being *irreducible*, *reduced*, and *of finite type over a field k*. A scheme with these properties will be defined to be a *variety* over k.
b) Show that for reduced schemes X of finite type over any algebraically closed field k, the structure sheaf $\mathcal{O}_X$ can be considered as a sheaf of k-valued functions and a morphism is determined by its map of points. Thus varieties over algebraically closed k's form a truly geometric category which is quite parallel to differentiable manifolds/analytic spaces/classical varieties.

Property 1. *A complex projective variety X is irreducible, or equivalently has a generic point η_X.*

This is obvious from the definition. To put this property in its setting, we can prove that every scheme has a unique irredundant decomposition into irreducible components. In fact:

DEFINITION 2.1. A scheme X is locally *noetherian* if every $x \in X$ has an affine neighborhood U which is $\mathrm{Spec}(R)$, R noetherian. A scheme is noetherian if it is locally noetherian and quasi-compact; or equivalently, if it has a finite covering by Spec's of noetherian rings.

PROPOSITION 2.2. *Every scheme X has a unique decomposition*

$$X = \bigcup_\alpha Z_\alpha, \quad Z_\alpha \text{ irreducible closed, } Z_\alpha \not\subset Z_\beta \quad \text{if } \alpha \neq \beta.$$

If X is locally noetherian, this decomposition is locally finite. If X is noetherian, then the decomposition is finite.

PROOF. The general case is immediate, and the noetherian cases from the fact that in a noetherian ring R, $\sqrt{(0)}$ is a finite intersection of prime ideals. □

An important point concerning the definition of locally noetherian is:

PROPOSITION 2.3. *If X is locally noetherian, then for every affine open* $\mathrm{Spec}(R) \subset X$, *$R$ is noetherian.*

Without this proposition, "locally noetherian" would be an awkward artificial concept. This proposition is the archetype of a large class of propositions that "justify" a definition by showing that if some property is checked for a covering family of open affines, then it holds for all open affines.

PROOF OF PROPOSITION 2.3. Let $U_\alpha = \mathrm{Spec}(R_\alpha)$ be an open cover of X with R_α noetherian. Then $\mathrm{Spec}(R)$ is covered by distinguished open subsets of the U_α, and each of these is of the form $\mathrm{Spec}((R_\alpha)_{f_\alpha})$, i.e., Spec of another noetherian ring. But now when $f \in R$ is such that:

$$\mathrm{Spec}(R_f) \subset \mathrm{Spec}(R_\alpha)_{f_\alpha}$$

then

$$\mathrm{Spec}(R_f) \cong \mathrm{Spec}\left(((R_\alpha)_{f_\alpha})_{\mathrm{res}\, f}\right), \quad \text{via res}\colon R \to (R_\alpha)_{f_\alpha},$$

hence

$$R_f \cong ((R_\alpha)_{f_\alpha})_{\mathrm{res}\, f}$$

hence R_f is noetherian. Therefore we can cover $\mathrm{Spec}(R)$ by distinguished opens $\mathrm{Spec}(R_{f_i})$ with R_{f_i} noetherian. Since $\mathrm{Spec}(R)$ is quasi-compact, we can take this covering finite. This implies that if $\mathfrak{a}_\alpha$ is an ascending chain of ideals in R, $\mathfrak{a}_\alpha \cdot R_{f_i}$ is stationary for all i if α is large enough, and then

$$\mathfrak{a}_{\alpha+1} = \bigcap_{i=1}^n \mathfrak{a}_{\alpha+1} R_{f_i} = \bigcap_{i=1}^n \mathfrak{a}_\alpha R_{f_i} = \mathfrak{a}_\alpha.$$

□

PROPERTY 2. *A complex projective variety X is reduced, in the sense of:*

DEFINITION 2.4. A scheme X is *reduced* if all its local rings $\mathcal{O}_{x,X}$ have no non-zero nilpotent elements.

It is easy to check that a ring R has non-zero nilpotents if and only if at least one of its localizations $R_\mathfrak{p}$ has nilpotents: therefore a scheme X is reduced if and only if it has an affine covering U_α such that $\mathcal{O}_X(U_\alpha)$ has no non-zero nilpotents, or if and only if this holds for all affine $U \subset X$. Moreover, it is obvious that a complex projective variety is reduced.

Reduced and irreducible schemes in general begin to look a lot like classical varieties. In fact:

PROPOSITION 2.5. *Let X be a reduced and irreducible scheme with generic point η. Then the stalk $\mathcal{O}_{\eta,X}$ is a field which we will denote $\mathbf{R}(X)$, the function field of X. Then*

i) *for all affine open $U \subset X$, (resp. all points $x \in X$), $\mathcal{O}_X(U)$ (resp. $\mathcal{O}_{x,X}$) is an integral domain with quotient field $\mathbf{R}(X)$,*
ii) *for all open $U \subset X$,*
$$\mathcal{O}_X(U) = \bigcap_{x \in X} \mathcal{O}_{x,X}$$
(the intersection being taken inside $\mathbf{R}(X)$) and if $U_1 \subset U_2$, then $\mathrm{res}_{U_2,U_1}\colon \mathcal{O}_X(U_2) \to \mathcal{O}_X(U_1)$ is the inclusion map between subrings of $\mathbf{R}(X)$.

PROOF. If $U = \operatorname{Spec} R$ is an affine open of X and $\eta = [\mathfrak{p}]$, $\mathfrak{p}$ a prime ideal of R, then $\overline{\{\eta\}} \supset U$ implies that $\mathfrak{p}$ is contained in all prime ideals of R, hence $\mathfrak{p} = \sqrt{(0)}$ in R. But R has no nilpotents so $\mathfrak{p} = (0)$, i.e., R is an integral domain. Moreover $\mathcal{O}_{\eta,X} = \mathcal{O}_{[\mathfrak{p}],\operatorname{Spec} R} = R_{\mathfrak{p}} =$ quotient field of R. Thus $\mathcal{O}_{\eta,X} \underset{\text{def}}{=} \mathbf{R}(X)$ is a field and is the common quotient field both of the affine rings R of X and of all localizations R_S of these such as the local rings $R_{\mathfrak{q}} = \mathcal{O}_{[\mathfrak{q}],X}$ ($\mathfrak{q} \subset R$ any prime ideal). This proves (i). Now if $U \subset X$ is any open set, consider
$$\mathrm{res}\colon \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{\eta,X} = \mathbf{R}(X).$$
For all $s \in \mathcal{O}_X(U)$, $s \neq 0$, there is an affine $U' = \operatorname{Spec} R' \subset U$ such that $\mathrm{res}_{U,U'}(s)$ is not 0 in R'. Since $R' \subset \mathbf{R}(X)$, $\mathrm{res}(s) \in \mathbf{R}(X)$ is not 0. Thus res is injective. Since it factors through $\mathcal{O}_{x,X}$ for all $x \in U$, this shows that
$$\mathcal{O}_X(U) \subset \bigcap_{x \in U} \mathcal{O}_{x,X}.$$
Conversely, if $s \in \bigcap_{x \in X} \mathcal{O}_{x,X}$, then there is an open covering $\{U_\alpha\}$ of U and $s_\alpha \in \mathcal{O}_X(U_\alpha)$ mapping to s in $\mathbf{R}(X)$. Then $s_\alpha - s_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)$ goes to 0 in $\mathbf{R}(X)$, so it is 0. Since $\mathcal{O}_X$ is a sheaf, then s_α's patch together to an $s \in \mathcal{O}_X(U)$. This proves (ii). □

PROPERTY 3. *A complex projective variety X is a scheme of finite type over $\mathbb{C}$, meaning:*

DEFINITION 2.6. A morphism $f\colon X \to Y$ is locally of finite type (resp. locally finitely presented) if X has an affine covering $\{U_\alpha\}$ such that $f(U_\alpha) \subset V_\alpha$, V_α an affine of Y, and the ring $\mathcal{O}_X(U_\alpha)$ is isomorphic to $\mathcal{O}_Y(V_\alpha)[t_1, \ldots, t_n]/\mathfrak{a}$ (resp. same with finitely generated $\mathfrak{a}$). f is *quasi-compact* if there exists an affine covering $\{V_\alpha\}$ of Y such that each $f^{-1}(V_\alpha)$ has a finite affine covering; f is of finite type (resp. finitely presented) if it is locally of finite type (resp. locally finitely presented) and quasi-compact.

It is clear that the canonical morphism of a complex projective variety to $\operatorname{Spec}(\mathbb{C})$ has all these properties. As above with the concept of noetherian, these definitions should be "justified" by checking:

PROPOSITION 2.7. *If f is locally of finite type, then for every pair of affine opens $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, $\mathcal{O}_X(U)$ is a finitely generated $\mathcal{O}_Y(V)$-algebra; if f is quasi-compact, then for every quasi-compact open subset $S \subset Y$, $f^{-1}(S)$ is quasi-compact. (Analogous results hold for the concept "locally finitely presented".)*

PROOF. The proof of the first assertion parallels that of Proposition 2.3. We are given U_α's, V_α's with $\mathcal{O}_X(U_\alpha)$ finitely generated over $\mathcal{O}_Y(V_\alpha)$. Using the fact that $R_f \cong R[x]/(1 - xf)$, hence is finitely generated over R, we can replace U_α, V_α by distinguished opens to get new

U_β's, V_β's such that $\mathcal{O}_X(U_\beta)$ is still finitely generated over $\mathcal{O}_Y(V_\beta)$, but now $U_\beta \subset U$, $V_\beta \subset V$ and $U = \bigcup U_\beta$. Next make another reduction until U_γ (resp. V_γ) is a distinguished open in U (resp. V). Then since $\mathcal{O}_X(U_\gamma)$ is finitely generated over $\mathcal{O}_Y(V_\gamma)$ and $\mathcal{O}_Y(V_\gamma) \cong \mathcal{O}_Y(V)_{f_\gamma}$ is finitely generated over $\mathcal{O}_Y(V)$, we may replace V_α by V. We come down to the purely algebraic lemma:

$$\left.\begin{array}{l} S \textit{ is an } R\textit{-algebra} \\ 1 = \displaystyle\sum_{i=1}^n f_i g_i, \quad f_i, g_i \in S \\ S_{f_i} \textit{ finitely generated over } R \end{array}\right\} \Longrightarrow S \textit{ finitely generated over } R.$$

Proof. Take a finite set of elements x_λ of S including the f_i's, g_i's and elements whose images in S_{f_i} plus $1/f_i$ generate S_{f_i} over R. These generate S, because if $k \in S$, then

$$k = \frac{P_i(x_\lambda)}{f_i^N} \quad \text{in} \quad S_{f_i}$$
$$P_i = \text{polynomial over } R.$$

Thus $f_i^{N+M} k = f_i^M P_i(x_\lambda)$ in S. But

$$\begin{aligned} 1 &= \left(\sum f_i g_i\right)^{n(N+M)} \\ &= \sum_{i=1}^n Q_i(f,g) \cdot f_i^{N+M} \end{aligned}$$

hence

$$k = \sum_{i=1}^n Q_i(f,g) f_i^M P_i(x_\lambda).$$

□

We leave the proof of the second half of Proposition 2.7 to the reader. □

A morphism of finite type has good topological properties generalizing those we found in Part I [**87**, (2.31)]. To state these, we must first define:

Definition 2.8. If X is a scheme, a constructible subset $S \subset X$ is an element of the Boolean algebra of subsets generated by the open sets: in other words,

$$S = S_1 \cup \cdots \cup S_t$$

where S_i is locally closed, meaning it is an intersection of an open set and a closed subset.

Theorem 2.9 (Chevalley's Nullstellensatz). *Let $f\colon X \to Y$ be a morphism of finite type and Y a noetherian scheme. Then for every constructible $S \subset X$, $f(S) \subset Y$ is constructible.*

Proof. First of all, we can reduce the theorem to the special case where X and Y are affine: in fact there are finite affine covering $\{U_i\}$ of X and $\{V_i\}$ of Y such that $f(U_i) \subset V_i$. Let $f_i = \operatorname{res} f\colon U_i \to V_i$. Then for every $S \subset X$ constructible, $f(S) = \bigcup f_i(S \cap U_i)$ so if $f_i(S \cap U_i)$ is constructible, so is $f(S)$. Secondly if $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} S$, we can reduce the theorem to the case $R = S[x]$. In fact, if $R = S[x_1, \ldots, x_n]$, we can factor f:

$$\begin{gathered} X = \operatorname{Spec} S[x_1, \ldots, x_n] \to \operatorname{Spec} S[x_1, \ldots, x_{n-1}] \to \cdots \\ \cdots \to \operatorname{Spec} S[x_1] \to \operatorname{Spec} S = Y. \end{gathered}$$

Now a basic fact is that every closed subset $V(\mathfrak{a})$ of an affine scheme $\operatorname{Spec}(R)$ is homeomorphic to the affine scheme $\operatorname{Spec}(R/\mathfrak{a})$. In fact there is a bijection between the set of prime ideals $\overline{\mathfrak{q}} \subset R/\mathfrak{a}$ and the set of prime ideals $\mathfrak{q} \subset R$ such that $\mathfrak{q} \supset \mathfrak{a}$ and this is readily seen to be a homeomorphism (we will generalize this in §3). Also, since $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$, $V(\mathfrak{a})$ is homeomorphic to the reduced scheme $\operatorname{Spec}(R/\sqrt{\mathfrak{a}})$ too. We use this first to make a third reduction to the case

$$f\colon \operatorname{Spec} S[X] \longrightarrow \operatorname{Spec} S.$$

In fact, if R is generated over S by one element, then $R \cong S[X]/\mathfrak{a}$ and via the diagram:

$$\begin{tikzcd} \operatorname{Spec} R \arrow[dr, "f'"'] & \cong \; V(\mathfrak{a}) \; \subset & S[X] \arrow[dl, "f"] \\ & \operatorname{Spec} S & \end{tikzcd}$$

the theorem for f implies the theorem for f'. Fourthly, we make a so-called "noetherian induction": since the closed subsets $V(\mathfrak{a}) \subset \operatorname{Spec} S$ satisfy the descending chain condition, if the theorem is false, there will be a minimal $V(\mathfrak{a}) \subset \operatorname{Spec}(S)$ such that

$$\operatorname{res} f\colon f^{-1}(V(\mathfrak{a})) \longrightarrow V(\mathfrak{a})$$

does not take constructibles to constructibles. Since $f^{-1}(V(\mathfrak{a})) = V(\mathfrak{a} \cdot S[X])$, we can replace $\operatorname{Spec} S$ by $\operatorname{Spec} S/\mathfrak{a}$ and $\operatorname{Spec} S[X]$ by $\operatorname{Spec}(S/\mathfrak{a})[X]$ and reduce to the case:

$(*)$ for all constructible sets $C \subset \operatorname{Spec} S[X]$, if $\overline{f(C)} \subsetneqq \operatorname{Spec}(S)$, then $f(C)$ is constructible.

Of course we can assume in this reduction that $\mathfrak{a} = \sqrt{\mathfrak{a}}$, so that the new S has no nilpotents. $\operatorname{Spec} S$ in fact must be irreducible too: if not,

$$\operatorname{Spec} S = Z_1 \cup Z_2, \qquad Z_i \subsetneqq \operatorname{Spec} S, \quad Z_i \text{ closed.}$$

Then if $C \subset \operatorname{Spec} S[X]$ is constructible, so are $C \cap f^{-1}(Z_i)$, hence by $(*)$ so are $f(C \cap f^{-1}(Z_i)) = f(C) \cap Z_i$; hence $f(C) = (f(C) \cap Z_1) \cup (f(C) \cap Z_2)$ is constructible. Thus S is an integral domain. In view of $(*)$, it is clear that the whole theorem is finally reduced to:

LEMMA 2.10. *Let S be an integral domain and let $\eta \in \operatorname{Spec} S$ be its generic point. Let $C \subset \operatorname{Spec} S[X]$ be an irreducible closed set and $C_0 \subset C$ an open subset. Consider the morphism:*

$$f\colon \operatorname{Spec} S[X] \longrightarrow \operatorname{Spec} S.$$

Then there is a non-empty open set $U \subset \operatorname{Spec} S$ such that either $U \subset f(C_0)$ or $U \cap f(C_0) = \emptyset$.

PROOF OF LEMMA 2.10. Let K be the quotient field of S. Note that $f^{-1}(\eta) \cong \operatorname{Spec} K[X] = \mathbb{A}^1_K$, which consists only of a generic point η^* and its closed points. $C \cap f^{-1}(\eta)$ is a closed irreducible subset of $f^{-1}(\eta)$, hence there are three possibilities:

Case i) $C \supset f^{-1}(\eta)$, so $C = \operatorname{Spec} S[X]$,
Case ii) $C \cap f^{-1}(\eta) = \{\zeta\}$, ζ a closed point of $f^{-1}(\eta)$, and
Case iii) $C \cap f^{-1}(\eta) = \emptyset$.

In case (i), C_0 contains some distinguished open $\operatorname{Spec} S[X]_g$, where $g = a_0X^n + a_1X^{n-1} + \cdots + a_n$, $a_0 \neq 0$. Let $U = \operatorname{Spec} S_{a_0}$. For all $x \in \operatorname{Spec} S$, $f^{-1}(x) \cong \operatorname{Spec} \Bbbk(x)[X] = \mathbb{A}^1_{\Bbbk(x)}$ and:

$$C_0 \cap f^{-1}(x) \supset \left\{ y \in \mathbb{A}^1_{\Bbbk(x)} \;\middle|\; \begin{array}{l} \overline{g}(y) \neq 0, \text{ where } \overline{g} = \overline{a}_0X^n + \cdots + \overline{a}_n \\ \text{and } \overline{a}_i = \text{image of } a_i \text{ in } \Bbbk(x) \end{array} \right\}.$$

So if $x \in U$, $\overline{a}_0 \neq 0$, hence $\overline{g} \not\equiv 0$, hence the generic point of $f^{-1}(x)$ is in $C_0 \cap f^{-1}(x)$, hence $x \in f(C_0)$. In case (ii), let $C = V(\mathfrak{p})$. Then

$$\mathfrak{p} \cdot K[X] = g \cdot K[X], \qquad g \text{ irreducible.}$$

We may assume that $g = a_0X^n + \cdots + a_n$ is in $\mathfrak{p}$, hence $a_i \in S$. Then

$$V(g) \supset C \supset C_0,$$

but all three sets intersect the generic fibre $f^{-1}(\eta)$ in only one point ζ. Thus $V(g) \setminus C_0$ is a constructible set disjoint from $f^{-1}(\eta)$. Let:

$$\overline{V(g) \setminus C_0} = W_1 \cup \cdots \cup W_t, \quad W_i \text{ irreducible with generic points } w_i \notin f^{-1}(\eta).$$

Then $f(W_i) \subset \overline{\{f(w_i)\}}$ and $\overline{\{f(w_i)\}}$ is a closed proper subset of $\operatorname{Spec} S$. Thus

$$f(V(q) \setminus C_0) \subset \bigcup_{i=1}^{t} \overline{\{f(w_i)\}} \subset \text{some subset } V(\alpha) \text{ of } \operatorname{Spec} S$$

($\alpha \in S$, $\alpha \neq 0$). Now let $U = \operatorname{Spec} S_{a_0\alpha}$. Then if $x \in U$,

$$\begin{aligned} \alpha(x) \neq 0 \Longrightarrow f^{-1}(x) \cap C_0 &= f^{-1}(x) \cap V(g) \\ &= \left\{ y \in \mathbb{A}^1_{\Bbbk(x)} \;\middle|\; \overline{g}(y) = 0 \right\}. \end{aligned}$$

Since $a_0(x) \neq 0$, $\overline{g} \not\equiv 0$, hence $\overline{g}$ has an irreducible factor $\overline{g}_1$ and the prime ideal $\overline{g}_1 \cdot \Bbbk(x)[X]$ defines a point of $f^{-1}(x)$ where g is zero. Thus $x \in f(C_0)$, which proves $U \subset f(C_0)$. In case (iii), let ζ be the generic point of C. Then

$$f(C) \subset \overline{\{f(\zeta)\}}$$

hence $U = \operatorname{Spec} S \setminus \overline{\{f(\zeta)\}}$ is an open set disjoint from $f(C_0)$. □

□

COROLLARY 2.11. *Let k be a field and X a scheme of finite type over k. If $x \in X$ then*

$$[x \textit{ is closed}] \Longleftrightarrow \left[\begin{array}{l} x \textit{ is an algebraic point} \\ \textit{i.e., } \Bbbk(x) \textit{ is an algebraic extension of } k. \end{array} \right]$$

PROOF. First assume x closed and let $U = \operatorname{Spec} R$ be an affine neighborhood of x. Then x is closed in U and hence $\{x\}$ is a constructible subset of U. Let $R \cong k[X_1, \ldots, X_n]/\mathfrak{a}$. Each X_i defines a morphism $p_i \colon U \to \mathbb{A}^1_k$ by Theorem I.3.7. p_i is clearly of finite type so by Theorem 2.9 $p_i(x)$ is a constructible point of $\mathbb{A}^1_k$. Now apply:

LEMMA 2.12 (Euclid). *For any field k, $\mathbb{A}^1_k$ contains an infinite number of closed points.*

PROOF. $\mathbb{A}^1_k = \operatorname{Spec} k[X]$ and its closed points are of the form $[(f)]$, f monic and irreducible. If $f_1, \ldots, f_N$ is any finite set of such irreducible polynomials, then an irreducible factor g of $\prod_{i=1}^N f_i + 1$ cannot divide any of the f_i, hence $[(g)] \neq [(f_i)]$ for any i. □

It follows that the generic point of $\mathbb{A}^1_k$ is not a constructible set! Thus $k(p_i(x))$ is algebraic over k. Since the residue field $\Bbbk(x)$ is generated over k by the values of the coordinates X_i, i.e., by the subfields $k(p_i(x))$, $\Bbbk(x)$ is algebraic over k. Conversely, if x is algebraic but not closed, let $y \in \overline{\{x\}}$, $y \neq x$. Let $U = \operatorname{Spec} R$ be an affine neighborhood of y. Then $x \in U$ too, so x is not closed in U. Let $x = [\mathfrak{p}]$ and use the fact that if ξ is algebraic over k, then $k[\xi]$ is already a field. Since $\Bbbk(x) \supset R/\mathfrak{p} \supset k$, all elements of $R/\mathfrak{p}$ are algebraic over k, hence $R/\mathfrak{p}$ is already a field. Therefore $\mathfrak{p}$ is maximal and x must be closed in U — contradiction. □

COROLLARY 2.13. *Let k be a field and X a scheme of finite type over k. Then:*

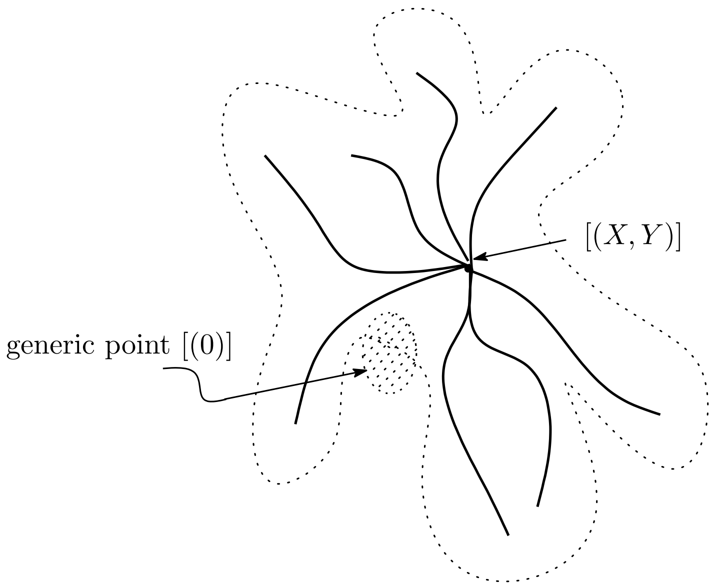

FIGURE II.2. "Parody of $\mathbb{P}^1_k$"

a) *If* $U \subset X$ *is open, and* $x \in U$*, then* x *is closed in* U *if and only if* x *is closed in* X*.*
b) *For all closed subsets* $S \subset X$*, the closed points of* S *are dense in* S*.*
c) *If* $\mathrm{Max}(X)$ *is the set of closed points of* X *in its induced topology, then there is a natural bijection beween* X *and the set of irreducible closed subsets of* $\mathrm{Max}(X)$ *(i.e.,* X *can be reconstructed from* $\mathrm{Max}(X)$ *as schemes were from classical varieties).*

PROOF. (a) is obvious by Corollary 2.11. To prove (b), we show that for every affine open $U \subset X$, if $U \cap S \neq \emptyset$, then $U \cap S$ contains a point closed in X. But if $U = \operatorname{Spec} R$, and $U \cap S = V(\mathfrak{p})$, then in the ring R, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Then $[\mathfrak{m}]$ is a closed point of U in $U \cap S$. By (a), $[\mathfrak{m}]$ is closed in X. Finally (c) is a formal consequence of (b) which we leave to the reader. $\square$

To illustrate what might go wrong here, contrast the situation with the case

$$X = \mathrm{Spec}(\mathcal{O}), \quad \mathcal{O} \text{ local noetherian, maximal ideal } \mathfrak{m}.$$

If

$$U = X \setminus [\mathfrak{m}],$$

then U satisfies the descending chain condition for closed sets so it has lots of closed points. But none of them can be closed in X, since $[\mathfrak{m}]$ is the only closed point of X. Take the case $\mathcal{O} = k[X,Y]_{(X,Y)}$: its prime ideals are $\mathfrak{m} = (X,Y)$, principal prime ideals f (with f irreducible) and (0). In this case, U has only closed points and one generic point and is a kind of parody of $\mathbb{P}^1_k$ as in Figure II.2 (cf. §5 below).

We have now seen that any scheme of finite type over a field shares many properties with classical projective varieties and when it is reduced and irreducible the resemblance is even closer. We canonize this similarity with a very important definition:

DEFINITION 2.14. Let k be a field. A variety X over k is a reduced and irreducible scheme X plus a morphism $p\colon X \to \operatorname{Spec} k$ making it of finite type over k. The dimension of X over k is $\operatorname{trdeg}_k \mathbf{R}(X)$.

We want to finish this section by showing that when k is algebraically closed, the situation is even more classical.

PROPOSITION 2.15. *Let k be an algebraically closed field and let X be a scheme of finite type over k. Then:*

a) *For all $x \in X$*

$$[x \text{ is closed}] \iff [x \text{ is rational, i.e., } \Bbbk(x) \cong k].$$

Let $X(k)$ denote the set of such points.

b) *Evaluation of functions define a homomorphism of sheaves:*

$$\mathcal{O}_X \longrightarrow k^{X(k)}$$

where

$$k^{X(k)}(U) = \text{ring of } k\text{-valued functions on } U(k).$$

If X is reduced, this is injective.

Now let X and Y be two schemes of finite type over k and $f: X \to Y$ a k-morphism. Then:

c) *$f(X(k)) \subset Y(k)$.*

d) *If X is reduced, f is uniquely determined by the induced map $X(k) \to Y(k)$, hence by its graph*

$$\{(x, f(x)) \mid x \in X(k)\} \subset X(k) \times Y(k).$$

PROOF. (a) is just Corollary 2.11 in the case k algebraically closed. To check (b), let $U = \operatorname{Spec} R$ be an affine. If $f \in R$ is 0 at all closed points of U, then $U \setminus V(f)$ has no closed points in it, hence is empty. Thus

$$f \in \bigcap_{\mathfrak{p} \text{ prime of } R} \mathfrak{p} = \sqrt{(0)}$$

and if X is reduced, $f = 0$. (c) follows immediately from (a) since for all $x \in X$, we get inclusions of fields:

$$\Bbbk(x) \hookleftarrow \Bbbk(f(x)) \hookleftarrow k.$$

As for (d), it follows immediately from the density of $X(k)$ in X, plus (b). □

3. Closed subschemes and primary decompositions

The deeper properties of complex projective varieties come from the fact that they are closed subschemes of projective space. To make this precise, in the next two sections we will discuss two things—closed subschemes and a construction called Proj. At the same time that we make the definitions necessary for characterizing complex projective varieties, we want to study the more general classes of schemes that naturally arise.

DEFINITION 3.1. Let X be a scheme. A *closed subscheme* $(Y, \mathcal{I})$ consists in two things:

a) a closed subset $Y \subset X$

b) a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that

$$\mathcal{I}_x \subsetneq \mathcal{O}_{X,x} \quad \text{iff} \quad x \in Y$$

and such that Y, plus the sheaf of rings $\mathcal{O}_X/\mathcal{I}$ supported by Y is a scheme.

DEFINITION 3.2. Let $f: Y \to X$ be a morphism of schemes. Then f is a *closed immersion* if

a) f is an injective closed map,

b) the induced homomorphisms

$$f_y^*\colon \mathcal{O}_{X,f(y)} \longrightarrow \mathcal{O}_{Y,y}$$

are surjective, for every $y \in Y$.

It is clear that

a) if you start from a closed subscheme $(Y, \mathcal{I})$, then the morphism $(Y, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$ defined by the inclusion of Y in X and the surjection of $\mathcal{O}_X$ to $\mathcal{O}_X/\mathcal{I}$ is a closed immersion;

b) conversely if you start with a closed immersion $f\colon Y \to X$, then the closed subset $f(Y)$ and the sheaf $\mathcal{I}$:

$$\mathcal{I}(U) = \operatorname{Ker}\big(\mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}U)\big)$$

is a closed subscheme.

Thus these two concepts are essentially equivalent. A *locally closed subscheme* or simply *subscheme* (resp. *immersion*) in general is defined to be a closed subscheme of an open set $U \subset X$ (resp. a morphism f such that $f(Y) \subset U$ open and $\operatorname{res} f\colon Y \to U$ is a closed immersion). The simplest example of a closed immersion is the morphism

$$f\colon \operatorname{Spec}(R/\mathfrak{a}) \longrightarrow \operatorname{Spec}(R)$$

where $\mathfrak{a}$ is any ideal in R. In fact, as noted in the proof of Theorem 2.9 above, f maps $\operatorname{Spec}(R/\mathfrak{a})$ homeomorphically onto the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec}(R)$. And if $\overline{\mathfrak{q}} \subset R/\mathfrak{a}$ is a prime ideal, $\overline{\mathfrak{q}} = \mathfrak{q}/\mathfrak{a}$, then the induced map on local rings is clearly surjective:

$$\begin{array}{ccccc} (R/\mathfrak{a})_{\mathfrak{q}} & \cong & R_{\mathfrak{q}}/\mathfrak{a}\cdot R_{\mathfrak{q}} & \twoheadleftarrow & R_{\mathfrak{q}} \\ \| & & & & \| \\ \mathcal{O}_{\operatorname{Spec}(R/\mathfrak{a}),[\overline{\mathfrak{q}}]} & & & & \mathcal{O}_{\operatorname{Spec}(R),[\mathfrak{q}]} \end{array}$$

We will often say for short, "consider the closed subscheme $\operatorname{Spec}(R/\mathfrak{a})$ of $\operatorname{Spec}(R)$". What we want to check is that these are the only closed subschemes of $\operatorname{Spec} R$.

We prove first:

PROPOSITION 3.3. *If $(Y, \mathcal{I})$ is a closed subscheme of X, then $\mathcal{I}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules.*

PROOF. On the open set $X \setminus Y$, $\mathcal{I} \cong \mathcal{O}_X$ so it is quasi-coherent. If $x \in Y$, we begin by finding an affine neighborhood $U \subset X$ of x such that $U \cap Y$ is affine in Y. To find U, start with any affine neighborhood U_1 and let $V_1 \subset U_1 \cap Y$ be an affine neighborhood of x in Y. Then choose some $\alpha \in \Gamma(U_1, \mathcal{O}_X)$ such that $\alpha \equiv 0$ on $U_1 \cap Y \setminus V_1$, while $\alpha(x) \neq 0$. Let $U = (U_1)_\alpha$. Since $U \cap Y = (U_1 \cap Y)_{\operatorname{res}\alpha} = (V_1)_{\operatorname{res}\alpha}$, $U \cap Y$ is affine in Y too. Next, suppose that $U = \operatorname{Spec} R$, $U \cap Y = \operatorname{Spec} S$ and let the inclusion of $U \cap Y$ into U correspond to $\phi\colon R \to S$. Let $I = \operatorname{Ker}(\phi)$: I claim then that

$$\mathcal{I}|_U \cong \widetilde{I}$$

hence $\mathcal{I}$ is quasi-coherent. But for all $\beta \in \Gamma(U, \mathcal{O}_X)$,

$$\begin{aligned} \widetilde{I}(U_\beta) &= I_\beta \\ &\cong \operatorname{Ker}(R_\beta \to S_\beta) \\ &\cong \operatorname{Ker}\left(\mathcal{O}_X(U_\beta) \to \mathcal{O}_Y(Y \cap U_\beta)\right) \\ &= \mathcal{I}(U_\beta) \end{aligned}$$

hence $\widetilde{I} \cong \mathcal{I}|_U$. $\square$

COROLLARY 3.4. *If* $(Y, \mathcal{I})$ *is a closed subscheme of* X*, then for all affine open* $U \subset X$*,* $U \cap Y$ *is affine in* Y *and if* $U = \operatorname{Spec} R$*, then* $U \cap Y \cong \operatorname{Spec}(R/\mathfrak{a})$ *for some ideal* $\mathfrak{a} \subset R$*, i.e.,* $Y \cong \operatorname{Spec}_X(\mathcal{O}_X/\mathcal{I})$*.*

PROOF. Since $\mathcal{I}$ is quasi-coherent, $\mathcal{I}|_U = \widetilde{\mathfrak{a}}$ for some ideal $\mathfrak{a} \subset R$. But then

$$\begin{aligned} \mathcal{O}_Y|_U &= \operatorname{Coker}(\mathcal{I}|_U \to \mathcal{O}_X|_U) \\ &= \operatorname{Coker}(\widetilde{\mathfrak{a}} \to \widetilde{R}) \\ &= \widetilde{R/\mathfrak{a}} \end{aligned}$$

hence

$$(Y, \mathcal{O}_Y) = (V(\mathfrak{a}), \widetilde{R/\mathfrak{a}}) \cong (\operatorname{Spec}(R/\mathfrak{a}), \mathcal{O}_{\operatorname{Spec} R/\mathfrak{a}}).$$

□

COROLLARY 3.5. *Let* $f\colon Y \to X$ *be a morphism. Then* f *is a closed immersion if and only if:*

$$(*) \qquad \begin{array}{l} \exists \textit{ an affine covering } \{U_i\} \textit{ of } X \textit{ such that } f^{-1}(U_i) \textit{ is affine} \\ \textit{and } \Gamma(U_i, \mathcal{O}_X) \to \Gamma(f^{-1}(U_i), \mathcal{O}_Y) \textit{ is surjective.} \end{array}$$

PROOF. Immediate. □

We want to give some examples of closed subschemes and particularly of how one can have many closed subschemes attached to the same underlying subset.

EXAMPLE 3.6. *Closed subschemes of* $\operatorname{Spec}(k[t])$, k algebraically closed. Since $k[t]$ is a PID, all non-zero ideals are of the form

$$\mathfrak{a} = \left(\prod_{i=1}^{n} (t - a_i)^{r_i} \right).$$

The corresponding subscheme Y of $\mathbb{A}^1_k = \operatorname{Spec}(k[t])$ is supported by the n points $a_1, \ldots, a_n$, and at a_i its structure sheaf is

$$\mathcal{O}_{a_i, Y} = \mathcal{O}_{a_i, \mathbb{A}^1_k} / \mathfrak{m}_i^{r_i},$$

where $\mathfrak{m}_i = \mathfrak{m}_{a_i, \mathbb{A}^1} = (t - a_i)$. Y is the union of the a_i's "with multiplicity r_i". The real significance of the multiplicity is that if you restrict a function f on $\mathbb{A}^1_k$ to this subscheme, the restriction can tell you not only the value $f(a_i)$ but the first $(r_i - 1)$-derivatives:

$$\frac{d^l f}{dt^l}(a_i), \qquad l \leq r_i - 1.$$

In other words, Y contains the $(r_i - 1)$st-order normal neighborhood of $\{a_i\}$ in $\mathbb{A}^1_k$.

Consider all possible subschemes supported by $\{0\}$. These are the subschemes

$$Y_n = \operatorname{Spec}\left(k[t]/(t^n)\right).$$

Y_1 is just the point as a reduced scheme, but the rest are not reduced. Corresponding to the fact that the defining ideals are included in each other:

$$(t) \supset (t^2) \supset (t^3) \supset \cdots \supset (t^n) \supset \cdots \supset (0),$$

the various schemes are subschemes of each other:

$$Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_n \subset \cdots \subset \mathbb{A}^1_k.$$

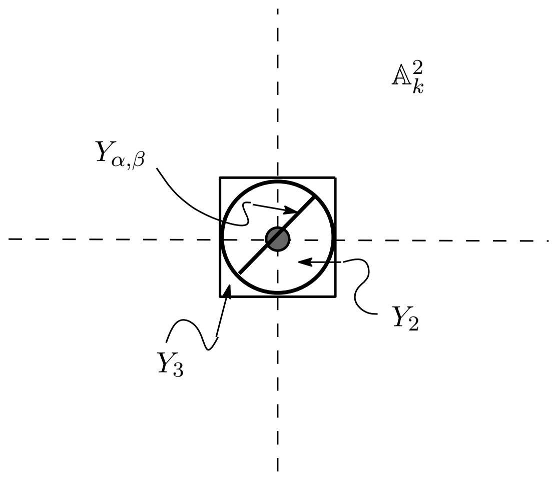

FIGURE II.3. 0-dimensional subschemes of $\mathbb{A}^2_k$

EXAMPLE 3.7. *Closed subschemes of* $\mathrm{Spec}(k[x,y])$, k algebraically closed. Every ideal $\mathfrak{a} \subset k[x,y]$ is of the form:

$$(f) \cap Q$$

for some $f \in k[x,y]$ and Q of finite codimension (to check this use noetherian decomposition and the fact that prime ideals are either maximal or principal). Let $Y = \mathrm{Spec}(k[x,y]/\mathfrak{a})$ be the corresponding subscheme of $\mathbb{A}^2_k$. First, suppose $\mathfrak{a} = (f)$. If $f = \prod_{i=1}^n f_i^{r_i}$, with f_i irreducible, then the subscheme Y is the union of the irreducible curves $f_i = 0$, "with multiplicity r_i". As before, if g is a function on $\mathbb{A}^2_k$, then one can compute solely from the restriction of g to Y the first $r_i - 1$ normal derivatives of g to the curve $f_i = 0$. Second, look at the case $\mathfrak{a}$ of finite codimension. Then

$$\mathfrak{a} = Q_1 \cap \cdots \cap Q_t$$

where $\sqrt{Q_i}$ is the maximal ideal $(x - a_i, y - b_i)$. Therefore, the support of Y is the finite set of points (a_i, b_i), and the stalk of Y at (a_i, b_i) is the finite dimensional algebra $k[x,y]/Q_i$. For simplicity, look at the case $\mathfrak{a} = Q_1$, $\sqrt{Q_1} = (x,y)$. The lattice of such ideals $\mathfrak{a}$ is much more complicated than in the one-dimensional case. Consider, for example, the ideals:

$$(x,y) \supset (\alpha x + \beta y, x^2, xy, y^2) \supset (x^2, xy, y^2) \supset (x^2, y^2) \supset (0).$$

These define subschemes:

$$\{(0,0) \text{ with reduced structure}\} \subset Y_{\alpha,\beta} \subset Y_2 \subset Y_3 \subset \mathbb{A}^2_k.$$

Since $(\alpha x + \beta y, x^2, xy, y^2) \supset (\alpha x + \beta y)$, $Y_{\alpha,\beta}$ is a subscheme of the reduced line $\ell_{\alpha,\beta}$ defined by $\alpha x + \beta y = 0$: $Y_{\alpha,\beta}$ is the point and *one* normal direction. But Y_2 is not a subscheme of any reduced line: it is the full double point and is invariant under rotations. Y_3 is even bigger, is *not* invariant under rotations, but still does not contain the second order neighborhood of $(0,0)$ along any line. If g is a function on $\mathbb{A}^2_k$, $g|_{Y_{\alpha,\beta}}$ determines *one* directional derivative of g at $(0,0)$, $g|_{Y_2}$ determines both partial derivatives of g at $(0,0)$ and $g|_{Y_3}$ even determines the mixed partial $\frac{\partial^2 g}{\partial x \partial y}(0,0)$ (cf. Figure II.3). As an example of the general case, look at $\mathfrak{a} = (x^2, xy)$. Then $\mathfrak{a} = (x) \cap (x^2, xy, y^2)$. Since $\sqrt{\mathfrak{a}} = (x)$, the support of Y is y-axis. The stalk $\mathcal{O}_{z,Y}$ has no nilpotents in it except when $z = (0,0)$. This is an "embedded point", and if a function g on $\mathbb{A}^2_k$ is cut down to Y, the restriction determines both partials of g at $(0,0)$, but only $\frac{\partial}{\partial y}$ at other points (cf. Figure II.4):

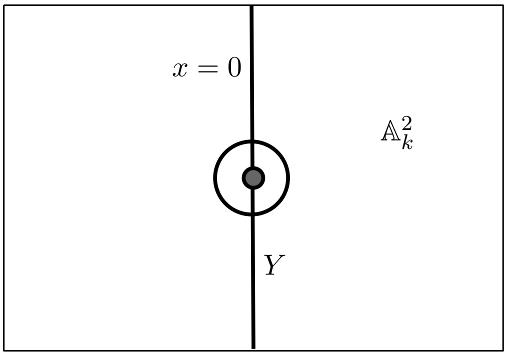

FIGURE II.4. Subschemes of $\mathbb{A}_k^2$

EXAMPLE 3.8. The theory of the primary decomposition of an ideal is an attempt to describe more "geometrically" a general closed subscheme of $\operatorname{Spec} R$, when R is noetherian. In fact, if

$$Z = \operatorname{Spec} R/\mathfrak{a} \subset \operatorname{Spec} R$$

is a closed subscheme, then the theory states that we can write:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

where $\mathfrak{q}_i$ is primary with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ prime. Then geometrically:

$$\begin{aligned} Z =& \text{scheme-theoretic union of (i.e., smallest closed} \\ & \text{subscheme containing) } W_1, \dots, W_n \\ \text{where} \quad W_i =& \operatorname{Spec} R/\mathfrak{q}_i \\ =& \text{set-theoretically } V(\mathfrak{p}_i)\text{, the closure of } [\mathfrak{p}_i] \\ & \text{but with some infinitesimal thickening.} \end{aligned}$$

The property which distinguishes the W_i's is described as follows:

$$\begin{aligned} \mathfrak{q} \text{ is } \mathfrak{p}\text{-primary} \iff & \mathfrak{p} = \sqrt{\mathfrak{q}} \text{ and } \mathfrak{q} = R \cap \mathfrak{q} \cdot (R_{\mathfrak{p}}) \\ \iff & \text{set-theoretically } \operatorname{Spec} R/\mathfrak{q} \text{ is } V(\mathfrak{p}) \text{ and the map} \\ & \Gamma(\mathcal{O}_{\operatorname{Spec} R/\mathfrak{q}}) \to \big(\text{the generic stalk } \mathcal{O}_{\operatorname{Spec} R/\mathfrak{q},[\mathfrak{p}]}\big) \\ & \text{is injective.} \end{aligned}$$

(In other words, a "function" $f \in R$ is to have 0 restriction everywhere to $\operatorname{Spec} R/\mathfrak{q}$ if it restricts to 0 at the generic point of $\operatorname{Spec} R/\mathfrak{q}$.) The unfortunate thing about the primary decomposition is that it is not unique: if W_i is an "embedded component", i.e., set-theoretically $W_i \subsetneqq W_j$, then the scheme structure on W_i is not unique. However the subsets W_i are uniquely determined by Z. By far the clearest treatment of this is in Bourbaki [**27**, Chapter 4] who considers the problem module-theoretically rather than ideal-theoretically. His theory globalizes immediately to give:

THEOREM 3.9. *Let X be a noetherian scheme, $\mathcal{F}$ a coherent sheaf on X. Then there is a finite set of points $x_1, \dots, x_t \in X$ such that*

i) $\forall U \subset X$, $\forall s \in \mathcal{F}(U)$, $\exists I \subset \{1, \dots, t\}$ *such that:*

$$\operatorname{Supp}(s) \underset{def}{=} \{x \in U \mid \textit{the image } s_x \in \mathcal{F}_x \textit{ is not } 0\} = \bigcup_{i \in I} \overline{\{x_i\}} \cap U$$

ii) *if U is affine, then any subset of U of the form $\bigcup_{i \in I} \overline{\{x_i\}} \cap U$ occurs as the support of some $s \in \mathcal{F}(U)$.*

These x_i are called the associated points of $\mathcal{F}$, or $\mathrm{Ass}(\mathcal{F})$.

PROOF. Note that if $U = \operatorname{Spec} R$, $\mathcal{F}|_U = \widetilde{M}$, $s \in M$, and $\mathrm{Ann}(s) = \{a \in R \mid as = 0\}$, then

$$\begin{aligned} s_x \neq 0 \text{ in } \mathcal{F}_x &\iff s \not\mapsto 0 \quad \text{in } M_{\mathfrak{p}} \\ \text{where } x = [\mathfrak{p}] & \\ &\iff \forall a \in R \setminus \mathfrak{p},\ a \cdot s \neq 0 \\ &\iff \mathrm{Ann}(s) \subset \mathfrak{p} \\ &\iff x \in V(\mathrm{Ann}(s)) \end{aligned}$$

so that $\mathrm{Supp}(s) = V(\mathrm{Ann}(s))$. It follows from the results in Bourbaki [**27**, Chapter 4, §1] that in this case his set of points $\mathrm{Ass}(M) \subset \operatorname{Spec} R$ has our two required properties[1]. Moreover, he proves in [**27**, §1.3] that $\mathrm{Ass}(M_f) = \mathrm{Ass}(M) \cap \operatorname{Spec} R_f$: hence the finite subsets $\mathrm{Ass}(M)$ all come from one set $\mathrm{Ass}(\mathcal{F})$ by $\mathrm{Ass}(M) = \mathrm{Ass}(\mathcal{F}) \cap \operatorname{Spec} R$. □

Note that $\mathrm{Ass}(\mathcal{F})$ must include the generic points of $\mathrm{Supp}(\mathcal{F})$ but may also include in addition embedded associated points.

COROLLARY 3.10. *If $Z \subset \operatorname{Spec} R$ is a closed subscheme and*

$$Z = W_1 \cup \cdots \cup W_t$$

is a primary decomposition, then

$$\mathrm{Ass}(\mathcal{O}_Z) = \{w_1, \ldots, w_t\},$$

where w_i = generic point of W_i.

PROOF. Let $Z = \operatorname{Spec} R/\mathfrak{a}$, $W_i = \operatorname{Spec} R/\mathfrak{q}_i$, so that $\mathfrak{a} = \bigcap \mathfrak{q}_i$. A primary decomposition is assumed irredundant, i.e., $\forall i$,

$$\mathfrak{q}_i \not\supset \bigcap_{j \neq i} \mathfrak{q}_j.$$

This means $\exists f \in \bigcap_{j \neq i} \mathfrak{q}_j \setminus \mathfrak{q}_i$, i.e., the "function" f is identically 0 on the *subschemes* W_j, $j \neq i$, but it is not 0 at the generic point of W_i, i.e., in $\mathcal{O}_{w_i, W_i}$. Therefore as a section of $\mathcal{O}_Z$, $\mathrm{Supp}(f) = W_i$. On the other hand, we get natural maps:

$$R/\mathfrak{a} \hookrightarrow \bigoplus_{i=1}^{t} R/\mathfrak{q}_i \hookrightarrow \bigoplus_{i=1}^{t} R_{\mathfrak{p}_i}/\mathfrak{q}_i R_{\mathfrak{p}_i}$$

hence

$$\mathcal{O}_Z \hookrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_{W_i} \hookrightarrow \bigoplus_{i=1}^{t} (\text{constant sheaf on } W_i \text{ with value } \mathcal{O}_{w_i, W_i})$$

from which it follows readily that the support of any section of $\mathcal{O}_Z$ is a union of various W_i's. □

For instance, in the example $R = k[x, y]$, $\mathfrak{a} = (x^2, xy)$,

$$(\text{Supp in } R/\mathfrak{a})(y) = \text{whole subset } V(\mathfrak{a})$$
$$(\text{Supp in } R/\mathfrak{a})(x) = \text{embedded point} V(x, y).$$

In order to globalize the theory of primary decompositions, or to analyze the uniqueness properties that it has, the following result is very useful:

[1] In fact, if $s \in M$, then $R/\mathrm{Ann}(s) \hookrightarrow M$ by multiplication by s, hence $\mathrm{Ass}(R/\mathrm{Ann}(s)) \subset \mathrm{Ass}(M)$; if $\mathrm{Supp}(s) = S_1 \cup \cdots \cup S_k$, S_i irreducible and $S_i \not\subset S_j$, then $S_i = V(\mathfrak{p}_i)$, and $\mathfrak{p}_i$ are the minimal primes in $\mathrm{Supp}(R/\mathrm{Ann}(s))$, hence by his [**27**, Chapter 4, §1, Proposition 7], are in $\mathrm{Ass}(R/\mathrm{Ann}(s))$. This gives our assertion (i). Conversely, for all $\mathfrak{p} \in \mathrm{Ass}(M)$, there is an $s \in M$ with $\mathrm{Ann}(s) = \mathfrak{p}$, hence $\mathrm{Supp}(s) = V(\mathfrak{p})$. Adding these, we get our assertion (ii).

PROPOSITION 3.11. *If X is locally noetherian*[2] *and $Y \subset X$ is a locally closed subscheme, then there is a smallest closed subscheme $\overline{Y} \subset X$ containing Y as an open subscheme, called the* scheme-theoretic closure *of Y. The ideal sheaf $\mathcal{I}$ defining $\overline{Y}$ is given by:*

$$\mathcal{I}(U) = \mathrm{Ker}[\mathcal{O}_X(U) \longrightarrow \mathcal{O}_Y(Y \cap U)],$$

and the underlying point set of $\overline{Y}$ is the topological closure of Y. $\overline{Y}$ can be characterized as the unique closed subscheme of X containing Y as an open subscheme such that $\mathrm{Ass}(\mathcal{O}_{\overline{Y}}) = \mathrm{Ass}(\mathcal{O}_Y)$.

PROOF. Everything is easy except the fact that $\mathcal{I}$ is quasi-coherent. To check this, it suffices to show that if $U = \mathrm{Spec}\, R$ is an affine in X and $U_f = \mathrm{Spec}\, R_f$ is a distinguished affine subset, then:

$$\mathrm{Ker}(R \to \mathcal{O}_Y(Y \cap U)) \cdot R_f = \mathrm{Ker}(R_f \to \mathcal{O}_Y(Y \cap U_f))$$

because then $\widetilde{\mathrm{Ker}}(R \to \mathcal{O}_Y(Y \cap U))$ agrees with $\mathcal{I}$ on all U_f's, hence agrees with $\mathcal{I}$ on U. Since " $\subset$ " is obvious, we mush check that if $a \in R$ and $a/f^n = 0$ in $\mathcal{O}_Y(Y \cap U_f)$, then $\exists m$, $f^m(a/f^n) = 0$ in $\mathcal{O}_Y(Y \cap U)$. Now U is noetherian so $Y \cap U$ is quasi-compact, hence is covered by a *finite* number of affines V_i. For each i,

$$a/f^n = 0 \text{ in } \mathcal{O}_Y((V_i)_f) \Longrightarrow \exists m_i,\ f^{m_i}(a/f^n) = 0 \text{ in } \mathcal{O}_Y(V_i)$$

and taking $m = \max(m_i)$

$$\Longrightarrow \exists m,\ f^m(a/f^n) = 0 \text{ in } \mathcal{O}_Y(Y \cap U).$$

□

REMARK. (Added in publication) As noted in the footnote to Proposition 3.11, if $f\colon X \to Y$ is a *quasi-compact* morphism of schemes (cf. Definition 4.9 below), then $\mathrm{Ker}\,(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_Y$. This ideal defines a closed subscheme of Y, which is called the *scheme-theoretic closure* of the image of f.

Here is a sketch of the proof: We may assume Y to be affine. Let $\{U_i \mid i \in I\}$ be an open affine cover of X indexed by a finite set I. Let $\iota_i\colon U_i \hookrightarrow X$ be the inclusion morphism. Applying f_* to the injection

$$\mathcal{O}_X \longrightarrow \prod_{i \in I} \iota_{i*}\iota_i^*\mathcal{O}_X,$$

we get an injection

$$f_*\mathcal{O}_X \longrightarrow \prod_{i \in I} (f \circ \iota_i)_*\mathcal{O}_{U_i}.$$

Hence

$$\mathrm{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X) = \mathrm{Ker}\left(\mathcal{O}_Y \to \prod_{i \in I} (f \circ \iota_i)_*\mathcal{O}_{U_i}\right).$$

Note that $\prod_{i \in I} (f \circ \iota_i)_*\mathcal{O}_{U_i}$ is a quasi-coherent $\mathcal{O}_Y$-module since each U_i is affine, hence the kernel of the above map is quasi-coherent.

We can apply Proposition 3.11 to globalize Example 3.8:

THEOREM 3.12. *Let X be a noetherian scheme, let Z be a subscheme and let $\mathrm{Ass}(\mathcal{O}_Z) = \{w_1, \ldots, w_t\}$. Then there exist closed subschemes $W_1, \ldots, W_t \subset Z$ such that*

[2] Actually all we need here is that the inclusion of Y in X is a quasi-compact morphism. (cf. Definition 4.9 below.)

a) *W_i is irreducible with generic point w_i and for all open $U_i \subset W_i$,*

$$\mathcal{O}_{W_i}(U_i) \longrightarrow \mathcal{O}_{w_i,W_i}$$

is injective (i.e., $\mathrm{Ass}(\mathcal{O}_{W_i}) = \{w_i\}$*).*

b) *Z is the scheme-theoretic union of the W_i's, i.e., set-theoretically $Z = W_1 \cup \cdots \cup W_t$ and*

$$\mathcal{O}_Z \longrightarrow \bigoplus_{i=1}^{t} \mathcal{O}_{W_i}$$

is surjective.

PROOF. For each i, let $U_i = \operatorname{Spec} R_i$ be an affine neighborhood of w_i, let $Z \cap U_i = \operatorname{Spec} R_i/\mathfrak{a}_i$, let $w_i = [\mathfrak{p}_i]$ and let $\mathfrak{q}_i$ be a $\mathfrak{p}_i$-primary component of $\mathfrak{a}_i$. Let

$$W_i = \text{scheme-theoretic closure of } \operatorname{Spec} R_i/\mathfrak{q}_i \text{ in } X.$$

(a) and (b) are easily checked. □

Proposition 3.11 can also be used to strip off various associated points from a subscheme. For instance, returning to Example 3.8:

$$\operatorname{Spec} R \supset Z = W_1 \cup \cdots \cup W_t, \qquad \text{a primary decomposition,}$$

and applying the proposition with $X = \operatorname{Spec} R$, $Y = Z \cap U$ where U is an open subset of $\operatorname{Spec} R$, we get

$$\overline{Z \cap U} = \bigcup_{\substack{i \text{ such that} \\ W_i \cap U \neq \emptyset}} W_i,$$

and hence these unions of the W_i's are schemes independent of the primary decomposition chosen.

Two last results are often handy:

PROPOSITION 3.13. *Let X be a scheme and $Z \subset X$ a closed subset. Then among all closed subschemes of X with support Z, there is a unique one $(Z, \mathcal{O}_X/\mathcal{I})$ which is reduced. It is a subscheme of any other subscheme $(Z, \mathcal{O}_X/\mathcal{I}')$ with support Z, i.e., $\mathcal{I} \supset \mathcal{I}'$.*

PROOF. In fact define $\mathcal{I}$ by

$$\mathcal{I}(U) = \{s \in \mathcal{O}_X(U) \mid s(x) = 0,\ \forall x \in U \cap Z\}.$$

The rest of the proof is left to the reader. □

PROPOSITION 3.14. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. If $g \circ f$ is an immersion, then f is an immersion.*

PROOF. The morphism f is the composite of the graph $\Gamma_f\colon X \to X \times_Z Y$ and the second projection $p_2\colon X \times_Z Y \to Y$. We know that Γ_f is an immersion for every morphism f (cf. Proposition 4.1 below), while p_2 is an immersion since it is a base extension of the immersion $g \circ f\colon X \to Z$. □

(Added in publication) The corresponding statement for *closed immersions* is false as was pointed out by Chai. As an example, let $Y = U_1 \cup U_2$ be "$\mathbb{A}^1$ with duplicated origin" as in Example 4.4 below. Let $X = U_1 = \operatorname{Spec} k[T_1]$ and $Z = \operatorname{Spec} k[T]$, with f the inclusion of U_1 to Y and with $g\colon Y \to Z$ the natural projection. Clearly, $g \circ f$ is a closed immersion (in fact an isomorphism), but f is not a closed immersion, since U_1 is not closed in Y.

4. Separated schemes

In the theory of topological spaces, the concept of a Hausdorff space plays an important role. Recall that a topological space X is called Hausdorff if for any two distinct points $x, y \in X$, there are disjoint open sets $U, V \subset X$ such that $x \in U$, $y \in V$. This very rarely holds in the Zariski topology so it might seem as if the Hausdorff axiom has no relevance among schemes. But if the product topology is given to the set-theoretic product $X \times X$, then the Hausdorff axiom for X is equivalent to the diagonal $\Delta \subset X \times X$ being closed. In the category of schemes, the product scheme $X \times X$ is neither set-theoretically nor topologically the simple Cartesian product of X by itself so the closedness of the diagonal gives a way to interpret the Hausdorff property for schemes. The most striking way to introduce this property is by proving a theorem that asserts the equivalence of a large number of properties of X, one of them being that the diagonal Δ is closed in $X \times X$.

Before giving this theorem, we need some preliminaries. We first introduce the concept of the *graph of a morphism*. Say we have an S-morphism f of two schemes X, Y over S, i.e., a diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

Then f induces a section of the projection:

$$\begin{tikzcd} X \times_S Y \arrow[d, "p_1"', shift right] \\ X \arrow[u, bend right, "\Gamma_f"'] \end{tikzcd}$$

defined by $\Gamma_f = (1_X, f)$. I claim that Γ_f is an immersion. In fact, choose affine coverings $\{U_i\}$ of X, $\{V_i\}$ of Y and $\{W_i\}$ of S such that $f(U_i) \subset V_i$ and $q(V_i) \subset W_i$. Then

$$\Gamma_f^{-1}(U_i \times_S V_i) = U_i$$

and if $U_i = \operatorname{Spec} R_i$, $V_i = \operatorname{Spec} S_i$, $W_i = \operatorname{Spec} T_i$, then

$$\operatorname{res} \Gamma_f \colon U_i \longrightarrow U_i \times_S V_i$$

corresponds to the ring map

$$\begin{aligned} R_i \otimes_{T_i} S_i &\longrightarrow R_i \\ a \otimes b &\longmapsto a \cdot f^* b \end{aligned}$$

which is surjective. Therefore if $\mathcal{U} = \bigcup_i (U_i \times_S V_i)$, then Γ_f factors

$$X \underset{\substack{\text{closed}\\ \text{immersion}}}{\hookrightarrow} \mathcal{U} \underset{\substack{\text{open}\\ \text{subscheme}}}{\subset} X \times_S Y.$$

This proves:

PROPOSITION 4.1. *If X and Y are schemes over S and $f \colon X \to Y$ is an S-morphism, then $\Gamma_f = (1_X, f) \colon X \to X \times_S Y$ is an immersion.*

The simplest example of Γ_f arises when $X = Y$ and $f = 1_X$. Taking $S = \operatorname{Spec} \mathbb{Z}$, we get the *diagonal map*

$$\delta = (1_X, 1_X) \colon X \longrightarrow X \times X.$$

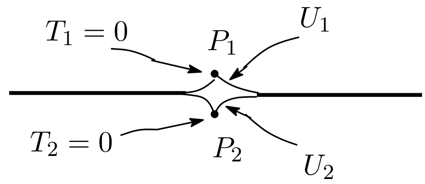

FIGURE II.5. $\mathbb{A}^1$ with "duplicate origin"

We have proven that if $\{U_i\}$ is an open cover of X, then δ is an isomorphism of X with a closed subscheme $\delta(X)$ of $\mathcal{U} \subset X \times X$, where

$$\mathcal{U} = \bigcup_i (U_i \times U_i).$$

But is $\delta(X)$ closed in $X \times X$? This leads to:

PROPOSITION 4.2. *Let X be a scheme. The following properties are equivalent:*

i) *$\delta(X)$ is closed in $X \times X$.*
ii) *There is an open affine covering $\{U_i\}$ of X such that for all i, j, $U_i \cap U_j$ is affine and $\mathcal{O}_X(U_i)$, $\mathcal{O}_X(U_j)$ generate $\mathcal{O}_X(U_i \cap U_j)$.*
iii) *For all open affines $U, V \subset X$, $U \cap V$ is affine and $\mathcal{O}_X(U)$, $\mathcal{O}_X(V)$ generate $\mathcal{O}_X(U \cap V)$.*

PROOF. (i) $\implies$ (iii): Given open affines U, V, note that $U \times V$ is an open affine subset of $X \times X$ such that $\mathcal{O}_{X \times X}(U \times V)$ is $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V)$. If $\delta(X)$ is closed in $X \times X$, δ is a closed immersion. Therefore $\delta^{-1}(U \times V)$ is affine and its ring is generated by $\mathcal{O}_{X \times X}(U \times V)$. But $\delta^{-1}(U \times V) = U \cap V$ so this proves (iii).

(iii) $\implies$ (ii) is obvious.

(ii) $\implies$ (i): Note that if $\{U_i\}$ is an open affine covering of X, then $\{U_i \times U_j\}$ is an open affine covering of $X \times X$. Since $\delta^{-1}(U_i \times U_j) = U_i \cap U_j$, (ii) is exactly the hypothesis $(*)$ of Corollary 3.5. The corollary says that then δ is a closed immersion, hence (i) holds. □

DEFINITION 4.3. X is a *separated* scheme if the equivalent properties of Proposition 4.2 hold.

Here's the simplest example of a non-separated scheme X:

EXAMPLE 4.4. Take $X = U_1 \cup U_2$ where

$$U_1 = \operatorname{Spec} k[T_1]$$
$$U_2 = \operatorname{Spec} k[T_2]$$

and where U_1 and U_2 are identified along the open sets:

$$(U_1)_{T_1} = \operatorname{Spec} k[T_1, T_1^{-1}]$$
$$(U_2)_{T_2} = \operatorname{Spec} k[T_2, T_2^{-1}]$$

by the isomorphism

$$i\colon \operatorname{Spec} k[T_1, T_1^{-1}] \xrightarrow{\sim} \operatorname{Spec} k[T_2, T_2^{-1}]$$
$$i(T_1) = T_2.$$

This "looks" like Figure II.5, i.e., it is $\mathbb{A}^1_k$ except that the origin occurs twice!

The same construction with the real line gives a simple non-Hausdorff one-dimensional manifold. It is easy to see $\delta(X)$ is not closed in $U_1 \times U_2$ or $U_2 \times U_1$ because $(P_1, P_2) \in U_1 \times U_2$ and $(P_2, P_1) \in U_2 \times U_1$ will be in its closure.

Once a scheme is known to be separated, many other intuitively reasonable things follow. For example:

PROPOSITION 4.5. *Let $f\colon X \to Y$ be a morphism and assume Y is separated. Then*

$$\Gamma_f\colon X \longrightarrow X \times Y$$

is a closed immersion. Hence for all $U \subset X$, $V \subset Y$ affine, $U \cap f^{-1}(V)$ is affine and its ring is generated by $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$.

PROOF. Consider the diagram:

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times Y \arrow[d, "(f\times 1_Y)"] \\ Y \arrow[r, "\delta_Y"'] & Y \times Y \end{tikzcd}$$

It is easy to see that this diagram makes X into the fibre product of Y and $X \times Y$ over $Y \times Y$, so the proposition follows from the following useful result: $\square$

PROPOSITION 4.6. *If $X \to S$ is a closed immersion and $Y \to S$ is any morphism, then $X \times_S Y \to Y$ is a closed immersion.*

PROOF. Follows from Corollary 3.5 and (using the definition of fibre product) the fact that $(A/I) \otimes_A B \cong B/I \cdot B$. $\square$

Before giving another useful consequence of separation, recall from §I.6, that two morphisms

$$\operatorname{Spec} k \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$$

are equal if and only if $f_1(\operatorname{Spec} k) = f_2(\operatorname{Spec} k)$ — call this point x — and the induced maps

$$f_1^*\colon \Bbbk(x) \longrightarrow k$$
$$f_2^*\colon \Bbbk(x) \longrightarrow k$$

are equal. Now given two morphisms

$$Z \overset{f_1}{\underset{f_2}{\rightrightarrows}} X$$

one can consider the "subset of Z where $f_1 = f_2$": the way to define this is:

$$\operatorname{Eq}(f_1, f_2) = \left\{ z \in Z \;\middle|\; \begin{array}{l} f_1(z) = f_2(z) \text{ and the induced maps} \\ f_1^*, f_2^*\colon \Bbbk(f_1(z)) \to \Bbbk(z) \text{ are equal} \end{array} \right\}.$$

Using this concept, we have:

PROPOSITION 4.7. *Given two morphisms $f_1, f_2\colon Z \to X$ where X is separated,* $\operatorname{Eq}(f_1, f_2)$ *is a closed subset of Z.*

PROOF. f_1 and f_2 define

$$(f_1, f_2)\colon Z \longrightarrow X \times X$$

and it is straightforward to check that $\operatorname{Eq}(f_1, f_2) = (f_1, f_2)^{-1}(\delta(X))$. $\square$

Looking at reduced and irreducible separated schemes, another useful perspective is that such schemes are characterized by the set of their affine rings, i.e., the glueing need not be given explicitly. The precise statement is this:

PROPOSITION 4.8. *Let X and Y be two reduced and irreducible separated schemes with the same function field $K = \mathbf{R}(X) = \mathbf{R}(Y)$. Suppose $\{U_i\}$ and $\{V_i\}$ are affine open coverings of X and Y such that for all i, $\mathcal{O}_X(U_i) = \mathcal{O}_Y(V_i)$ as subrings of K. Then $X \cong Y$.*

PROOF. Left to the reader. □

Another important consequence of separation is the quasi-coherence of direct images. More precisely:

DEFINITION 4.9. A morphism $f\colon X \to Y$ of schemes is *quasi-compact* if for all $U \subset Y$ quasi-compact, $f^{-1}U$ is quasi-compact. Equivalently, for all affine open $U \subset Y$, $f^{-1}U$ is covered by a finite set of affine open subsets of X.

PROPOSITION 4.10. *Let $f\colon X \to Y$ be a quasi-compact morphism of separated schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $f_*\mathcal{F}$ is quasi-coherent.*

PROOF. The assertion is local on Y so we may assume $Y = \operatorname{Spec} R$. Let $\{U_i\}$ be a finite affine open cover of X and let $f_i\colon U_i \to Y$ be the restriction of f to U_i. Since X is separated, $U_i \cap U_j$ is also affine. Let $f_{ij}\colon U_i \cap U_j \to Y$ be the restriction of f to $U_i \cap U_j$. Then consider the homomorphisms:

$$0 \longrightarrow f_*\mathcal{F} \xrightarrow{\alpha} \prod_i f_{i,*}\mathcal{F} \xrightarrow{\beta} \prod_{j,k} f_{jk,*}\mathcal{F}$$

where α is just restriction and β is the difference of restrictions, i.e.,

$$\beta(\{s_i\})_{jk} = \operatorname{res}(s_j) - \operatorname{res}(s_k).$$

By the sheaf property of $\mathcal{F}$ and the definition of direct images, this sequence is exact! But f_i and f_{jk} are affine morphisms by Proposition 4.5 and the products are *finite* so the second and third sheaves are quasi-coherent by Lemma (I.5.12). Therefore $f_*\mathcal{F}$ is quasi-coherent. □

REMARK. (Added in publication) A morphism $f\colon X \to Y$ of schemes is said to be *separated* (resp. *quasi-separated*) if the diagonal morphism

$$\Delta_{X/Y}\colon X \longrightarrow X \times_Y X$$

is a closed immersion (resp. quasi-compact).

Proposition 4.10 above remains valid in the following form:

> Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of schemes and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $f_*\mathcal{F}$ is quasi-coherent.

The proof is essentially the same.

From this point on a blancket assumption is made that all schemes are separated over $\operatorname{Spec}\mathbb{Z}$, which implies that all morphisms are separated. Without this blancket assumption, some adjustment may be needed in subsequent materials. For instance, in Lemma III,4.1 below, (i) holds for any quasi-compact scheme X, but in (ii) one needs to assume that X is quasi-separated over $\operatorname{Spec}\mathbb{Z}$.

> (∗∗) **In the rest of this book, we will always assume that all our schemes are separated, hence all morphisms are separated.** (∗∗)

5. Proj R

The essential idea behind the construction of $\mathbb{P}^n$ can be neatly generalized. Let

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

be any graded ring (i.e., $R_i \cdot R_j \subset R_{i+j}$), and let

$$R_+ = \bigoplus_{i=1}^{\infty} R_i$$

be the ideal of elements of positive degree. We define a scheme $\operatorname{Proj} R$ as follows:

(I) As a point set:

$$\operatorname{Proj} R = \left\{ \mathfrak{p} \subset R \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ a homogeneous prime ideal} \\ (\text{i.e., } \mathfrak{p} = \bigoplus_{i=0}^{\infty} \mathfrak{p} \cap R_i) \text{ and } \mathfrak{p} \not\supset R_+ \end{array} \right\}.$$

(II) As a topological space:

$$\text{for all subsets } S \subset R, \text{ let } \quad V(S) = \{[\mathfrak{p}] \in \operatorname{Proj} R \mid S \subset \mathfrak{p}\}.$$

If $\mathfrak{a}$ is the homogeneous ideal generated by the homogeneous parts of all $f \in S$, then

$$V(S) = V(\mathfrak{a}).$$

It follows easily that the $V(S)$ are the closed sets of a topology and that the "distinguished open subsets"

$$(\operatorname{Proj} R)_f = \{[\mathfrak{p}] \in \operatorname{Proj} R \mid f \notin \mathfrak{p}\}, \quad \text{where } f \in R_k, \text{ some } k \geq 1$$

form a basis of open sets.
[Problem for the reader: check that if $f \in R_0$, then

$$\{[\mathfrak{p}] \in \operatorname{Proj} R \mid f \notin \mathfrak{p}\} = \bigcup_{k \geq 1} \bigcup_{g \in R_k} (\operatorname{Proj} R)_{fg}.]$$

(III) The structure sheaf:

$$(*) \qquad \text{for all } f \in R_k,\ k \geq 1, \text{ let } \quad \mathcal{O}_{\operatorname{Proj} R}((\operatorname{Proj} R)_f) = (R_f)_0,$$

where $(R_f)_0$ = degree 0 component of the localization R_f. This definition is justified in a manner quite parallel to the construction of Spec, resting in this case however on:

PROPOSITION 5.1. *Let f, $\{g_i\}_{i \in S}$ be homogeneous elements of R, with* $\deg f > 0$. *Then*

$$\left[(\operatorname{Proj} R)_f = \bigcup_{i \in S} (\operatorname{Proj} R)_{g_i} \right] \Longleftrightarrow \left[f^n = \sum a_i g_i, \ \text{some } n \geq 1, \ \text{some } a_i \in R \right].$$

PROOF. The left hand side means

$$\forall [\mathfrak{p}] \in \operatorname{Proj} R,\ f \notin \mathfrak{p} \Longrightarrow \exists i \text{ such that } g_i \notin \mathfrak{p}$$

which is equivalent to saying

$$f \in \left\{ \bigcap \mathfrak{p} \;\middle|\; \begin{array}{l} \mathfrak{p} \text{ homogeneous prime ideal such that} \\ \mathfrak{p} \supset \sum g_i R \text{ but } \mathfrak{p} \not\supset R_+ \end{array} \right\}.$$

Since $\mathfrak{p} \supset R_+$ implies $f \in \mathfrak{p}$, we can ignore the second restriction on $\mathfrak{p}$ in the braces. and what we need is:

LEMMA 5.2. *If* $\mathfrak{a} \subset R$ *is a homogeneous ideal, then*

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \text{ homogeneous} \\ \mathfrak{p} \supset \mathfrak{a}}} \mathfrak{p}.$$

PROOF OF LEMMA 5.2. Standard, i.e., if $f \notin \sqrt{\mathfrak{a}}$, choose a homogeneous ideal $\mathfrak{q} \supset \mathfrak{a}$ maximal among those such that $f^n \notin \mathfrak{q}$, all $n \geq 1$. Check that $\mathfrak{q}$ is prime. □

□

COROLLARY 5.3. *If* $\deg f, \deg g > 0$, *then*

$$\begin{aligned} (\operatorname{Proj} R)_f \subset (\operatorname{Proj} R)_g &\Longrightarrow f^n = a \cdot g, \ \textit{some } n,\ a \\ &\Longrightarrow \exists \textit{ canonical map } (R_g)_0 \to (R_f)_0. \end{aligned}$$

COROLLARY 5.4. *If* $\deg g_i > 0$, $\forall i \in S$, *then*

$$\left[\operatorname{Proj} R = \bigcup_{i \in S} (\operatorname{Proj} R)_{g_i}\right] \Longleftrightarrow \left[R_+ = \sqrt{\sum g_i R}.\right]$$

We leave to the reader the details in checking that there is a unique sheaf $\mathcal{O}_{\operatorname{Proj} R}$ satisfying $(*)$ and with restriction maps coming from Corollary 5.3. The fact that we get a scheme in this way is a consequence of:

PROPOSITION 5.5. *Let* $f \in R_k$, $k \geq 1$. *Then there is a canonical isomorphism:*

$$((\operatorname{Proj} R)_f, \operatorname{res} \mathcal{O}_{\operatorname{Proj} R}) \cong \left(\operatorname{Spec}\left((R_f)_0\right), \mathcal{O}_{\operatorname{Spec}\left((R_f)_0\right)}\right).$$

PROOF. For all homogeneous primes $\mathfrak{p} \subset R$ such that $f \notin \mathfrak{p}$, let

$$\mathfrak{p}' = \{a/f^n \mid a \in \mathfrak{p} \cap R_{nk}\} = \mathfrak{p} \cdot R_f \cap (R_f)_0.$$

This is a prime ideal in $(R_f)_0$. Conversely, if $\mathfrak{p}' \subset (R_f)_0$ is prime, let

$$\mathfrak{p} = \bigoplus_{n=0}^{\infty} \left\{a \in R_n \;\middle|\; a^k/f^n \in \mathfrak{p}'\right\}.$$

It follows readily that there are inverse maps which set up the set-theoretic isomorphism $(\operatorname{Proj} R)_f \cong \operatorname{Spec}(R_f)_0$. It is straightforward to check that it is a homeomorphism and that the two structure sheaves are canonically isomorphic on corresponding distinguished open sets. □

Moreover, just as with Spec, the construction of the structure sheaf carries over to modules too. In this case, for every *graded* R-module M, we can define a quasi-coherent sheaf of $\mathcal{O}_{\operatorname{Proj}(R)}$-modules $\widetilde{M}$ by the requirement:

$$\widetilde{M}\left((\operatorname{Proj} R)_f\right) = (M_f)_0.$$

We give next a list of fairly straightforward properties of the operations Proj and of $\widetilde{\ }$:

a) The homomorphisms $R_0 \to (R_f)_0$ for all $f \in R_k$, all $k \geq 1$ induce a morphism
$$\operatorname{Proj}(R) \longrightarrow \operatorname{Spec}(R_0).$$
b) If R is a finitely generated R_0-algebra, then $\operatorname{Proj}(R)$ is of finite type over $\operatorname{Spec}(R_0)$.
c) If S_0 is an R_0-algebra, then
$$\operatorname{Proj}(R \otimes_{R_0} S_0) \cong \operatorname{Proj}(R) \times_{\operatorname{Spec} R_0} \operatorname{Spec} S_0.$$
d) If $d \geq 1$ and $R\langle d\rangle = \bigoplus_{k=0}^{\infty} R_{dk}$, then $\operatorname{Proj} R \cong \operatorname{Proj} R\langle d\rangle$.
[Check that for all $f \in R_k$, $k \geq 1$, the rings $(R_f)_0$ and $(R\langle d\rangle_{f^d})_0$ are canonically isomorphic; this induces isomorphisms $(\operatorname{Proj} R)_f \cong (\operatorname{Proj} R\langle d\rangle)_{f^d}, \ldots$.]

e) Because of (c), it is possible to *globalize* Proj just as Spec was globalized in §I.7. If X is a scheme, and

$$\mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots$$

is a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras where each $\mathcal{R}_i$ is quasi-coherent, then one can construct a scheme over X:

$$\pi\colon \operatorname{Proj}_X(\bigoplus \mathcal{R}_i) \longrightarrow X$$

as follows: for all $U \subset X$ open affine, take the scheme $\operatorname{Proj}(\bigoplus \mathcal{R}_i(U))$, which lies over $\operatorname{Spec} \mathcal{O}_X(U)$, i.e., U. For any two open affines $U_1, U_2 \subset X$ construct an isomorphism ϕ_{12}:

$$\begin{tikzcd}
\operatorname{Proj}(\bigoplus \mathcal{R}_i(U)) \arrow[r, "\pi_1"] & U_1 \\
\pi_1^{-1}(U_1 \cap U_2) \arrow[u, phantom, "\cup"] \arrow[dd, "\phi_{12}"', "\approx"] \arrow[dr] & \\
& U_1 \cap U_2 \arrow[uu, phantom, "\cup"] \arrow[dd, phantom, "\cap"] \\
\pi_2^{-1}(U_1 \cap U_2) \arrow[ur] \arrow[d, phantom, "\cap"] & \\
\operatorname{Proj}(\bigoplus \mathcal{R}_i(U_2)) \arrow[r, "\pi_2"] & U_2
\end{tikzcd}$$

by covering $U_1 \cap U_2$ by open affine U_3, and noting that:

$$\begin{aligned}
\pi_1^{-1}(U_3) &\cong \operatorname{Proj}\left(\bigoplus \mathcal{R}_i(U_1) \otimes_{\mathcal{O}_X(U_1)} \mathcal{O}_X(U_3)\right) \\
&\cong \operatorname{Proj}\left(\bigoplus \mathcal{R}_i(U_3)\right) \\
&\cong \operatorname{Proj}\left(\bigoplus \mathcal{R}_i(U_2) \otimes_{\mathcal{O}_X(U_2)} \mathcal{O}_X(U_3)\right) \\
&\cong \pi_2^{-1}(U_3).
\end{aligned}$$

f) If $\mathfrak{a} \subset R$ is a homogeneous ideal, then there is a canonical closed immersion:

$$\operatorname{Proj} R/\mathfrak{a} \hookrightarrow \operatorname{Proj} R.$$

A somewhat harder result is the converse in the case when R is finitely generated over R_0: that every closed subscheme Z of $\operatorname{Proj} R$ is isomorphic to $\operatorname{Proj} R/\mathfrak{a}$ for some $\mathfrak{a}$. The proof uses the remark that if $f_1, f_2, g \in R_n$ and g/f_1 vanishes on $Z \cap (\operatorname{Proj} R)_{f_1}$ then for some k, gf_1^k/f_2^{k+1} vanishes on $Z \cap (\operatorname{Proj} R)_{f_2}$.

g) $\operatorname{Proj} R$ is separated.

Proof. Use Criterion (ii) of Proposition 4.2, applying it to a covering of $\operatorname{Proj} R$ by distinguished affines. $\square$

h) The map taking R to $\operatorname{Proj} R$ is not a functor but it does have a partial functoriality. To be precise, let R and R' be two graded rings and let

$$\phi\colon R \longrightarrow R'$$

be a homomorphism such that for some $d > 0$,

$$\phi(R_n) \subset R'_{nd}, \quad \text{all } n.$$

(Usually $d = 1$ but this isn't necessary.) Let

$$R_+ = \sum_{n>0} R_n$$
$$\mathfrak{a} = \phi(R_+) \cdot R'.$$

Then ϕ induces a natural map:

$$f\colon \operatorname{Proj} R' \setminus V(\mathfrak{a}) \longrightarrow \operatorname{Proj} R.$$

In fact,

$$\operatorname{Proj} R' \setminus V(\mathfrak{a}) = \bigcup_{n \geq 1} \bigcup_{a \in R_n} (\operatorname{Proj} R')_{\phi a}.$$

Define the restriction of f to $(\operatorname{Proj} R')_{\phi a}$ to be the morphism from $(\operatorname{Proj} R')_{\phi a}$ to $(\operatorname{Proj} R)_a$ induced by the ring homomorphism

$$\phi\colon (R_a)_0 \longrightarrow (R'_{\phi a})_0$$
$$\phi\left(\frac{b}{a^l}\right) = \frac{\phi(b)}{\phi(a)^l}, \quad b \in R_{nl}.$$

It is easy to check that these morphisms agree on intersections hence glue together to the morphism f.

i) If R and R' are two graded rings with the same degree 0 piece: $R_0 = R'_0$, then

$$\operatorname{Proj} R \times_{\operatorname{Spec} R_0} \operatorname{Proj} R' = \operatorname{Proj} R''$$

where

$$R'' = \bigoplus_{n=0}^{\infty} R_n \otimes_{R_0} R'_n.$$

PROOF. This follows easily from noting that for all $f \in R_n$, $f' \in R'_n$,

$$(R_f)_0 \otimes_{R_0} (R'_{f'})_0 \cong (R''_{f \otimes f'})_0$$

hence

$$(\operatorname{Proj} R)_f \times_{\operatorname{Spec} R_0} (\operatorname{Proj} R')_{f'} \cong (\operatorname{Proj} R'')_{f \otimes f'}$$

and glueing. □

j) $M \longmapsto \widetilde{M}$ is an exact functor; more precisely $\forall \phi\colon M \to N$ preserving degrees, we get an $\mathcal{O}_{\operatorname{Proj} R}$-homomorphism $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$ and if

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\psi} P \longrightarrow 0$$

is a sequence with $\psi \circ \phi = 0$ and such that

$$0 \longrightarrow M_k \longrightarrow N_k \longrightarrow P_k \longrightarrow 0$$

is exact if $k \gg 0$, then

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{N} \longrightarrow \widetilde{P} \longrightarrow 0$$

is exact.

k) There is a natural map:

$$M_0 \longrightarrow \Gamma(\operatorname{Proj} R, \widetilde{M})$$

given by

$$m \longmapsto \text{element } m/1 \in (M_f)_0 = \widetilde{M}((\operatorname{Proj} R)_f).$$

There is a natural relationship between Spec and Proj which generalizes the fact that ordinary complex projective space $\mathbb{P}^n$ is the quotient of $\mathbb{C}^{n+1} \setminus (0)$ by homotheties. If R is any graded ring, let

$$R_+ = \sum_{n>0} R_n.$$

Then there is a canonical morphism

$$\pi\colon \operatorname{Spec} R \setminus V(R_+) \longrightarrow \operatorname{Proj} R.$$

In fact, for all $n \geq 1$, $a \in R_n$, the restriction of π to $(\operatorname{Spec} R)_a$ will be the morphism

$$\begin{array}{ccc} (\operatorname{Spec} R)_a & \longrightarrow & (\operatorname{Proj} R)_a \\ \wr\| & & \wr\| \\ \operatorname{Spec} R_a & & \operatorname{Spec}(R_a)_0 \end{array}$$

given by the inclusion of $(R_a)_0$ in R_n.

The most important Proj is:

DEFINITION 5.6. $\mathbb{P}^n_R = \operatorname{Proj} R[X_0, \ldots, X_n]$.

Note that since $X_0, \ldots, X_n$ generate the ideal of elements of positive degree, this Proj is covered by the distinguished affines $(\operatorname{Proj} R[X_0, \ldots, X_n])_{X_i}$, i.e., by the $n+1$ copies of $\mathbb{A}^n_R$:

$$U_i = \operatorname{Spec} R\left[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}\right], \quad 0 \leq i \leq n$$

glued in the usual way. Moreover if $R = \bigoplus_{i=0}^{\infty} R_i$ is any graded ring generated over R_0 by elements of R_1 and with R_1 finitely generated as R_0-module, then R is a quotient of $R_0[X_0, \ldots, X_n]$ for some n: just choose generators $a_0, \ldots, a_n$ of R_1 and define

$$\begin{array}{c} R_0[X_0, \ldots, X_n] \longrightarrow R \\ \text{by} \qquad X_i \longmapsto a_i. \end{array}$$

Therefore by (f), $\operatorname{Proj} R$ is a closed subscheme of $\mathbb{P}^n_{R_0}$.

More generally, let X be any scheme and let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf of $\mathcal{O}_X$-modules. Then we can construct symmetric powers $\operatorname{Symm}^n(\mathcal{F})$ by

$$\operatorname{Symm}^n(\mathcal{F})(U) = \operatorname{Symm}^n(\mathcal{F}(U)), \quad \text{all affine open } U$$

and hence a quasi-coherent graded $\mathcal{O}_X$-algebra:

$$\operatorname{Symm}^* \mathcal{F} = \mathcal{O}_X \oplus (\mathcal{F}) \oplus (\operatorname{Symm}^2 \mathcal{F}) \oplus (\operatorname{Symm}^3 \mathcal{F}) \oplus \cdots.$$

DEFINITION 5.7. $\mathbb{P}_X(\mathcal{F}) = \operatorname{Proj}_X(\operatorname{Symm}^* \mathcal{F})$.

Note that by (f) above, if $\mathcal{R}$ is any quasi-coherent graded $\mathcal{O}_X$-algebra with

$$\begin{array}{l} \mathcal{R}_0 = \mathcal{O}_X \\ \mathcal{R}_1 \text{ finitely generated} \\ \mathcal{R}_n \text{ generated by } \mathcal{R}_1,\ n \geq 2, \end{array}$$

then we get a surjection

$$\operatorname{Symm}^* \mathcal{R}_1 \twoheadrightarrow \mathcal{R}$$

hence a closed immersion

$$\operatorname{Proj}_X(\mathcal{R}) \hookrightarrow \mathbb{P}_X(\mathcal{R}_1).$$

This motivates:

DEFINITION 5.8. Let $f\colon X \to Y$ be a morphism of schemes.

a) f is projective if $X \cong \mathrm{Proj}_Y(\bigoplus \mathcal{R}_i)$, some quasi-coherent graded $\mathcal{O}_Y$-algebra $\bigoplus \mathcal{R}_i$ but such that $\mathcal{R}_0 = \mathcal{O}_Y$, $\mathcal{R}_1$ finitely generated as $\mathcal{O}_Y$-module and $\mathcal{R}_1$, multiplied by itself n times generates $\mathcal{R}_n$, $n \geq 2$. Equivalently $\exists$ a diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "\substack{\text{closed}\\ \text{immersion}}"] \arrow[dr, "f"'] & & \mathbb{P}_Y(\mathcal{F}) \arrow[dl] \\ & Y & \end{tikzcd}$$

where $\mathcal{F}$ is quasi-coherent, finitely generated.

b) f is quasi-projective[3] if $\exists$ a diagram:

$$\begin{tikzcd} X \arrow[rr, hook, "\text{open}"'] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

where f' is projective.

Note that if $Y = \mathrm{Spec}\, R$, say, then

$$\begin{aligned} &f \text{ projective} \Longleftrightarrow X \text{ is a closed subscheme of } \mathbb{P}^n_R, \text{ some } n \\ &f \text{ quasi-projective} \Longleftrightarrow X \text{ is a subscheme of } \mathbb{P}^n_R, \text{ some } n. \end{aligned}$$

We can now make the final link between classical geometry and the theory of schemes: when $R = \mathbb{C}$ it is clear that $\mathbb{P}^n_R$ becomes the scheme that we associated earlier to the classical variety $\mathbb{P}^n(\mathbb{C})$. Moreover the reduced and irreducible closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ are precisely the schemes $\mathrm{Proj}(\mathbb{C}[X_0, \ldots, X_n]/\mathfrak{p})$, which are the schemes that we associated earlier to the classical varieties $V(\mathfrak{p}) \subset \mathbb{P}^n(\mathbb{C})$. In short, "complex projective varieties" as in Part I [**87**] define "complex projective varieties" in the sense of Definition 5.8, and, up to isomorphism, they all arise in this way.

Note too that for $\mathbb{P}^n_R$, the realization of $\mathbb{P}^n_R \times_{\mathrm{Spec}\, R} \mathbb{P}^m_R$ as a Proj in (i) above is identical to the Segre embedding studied in Part I [**87**]. In fact, the construction of (i) shows:

$$\begin{aligned} \mathbb{P}^n_R \times_{\mathrm{Spec}\, R} \mathbb{P}^m_R &= \mathrm{Proj}\, R[X_0, \ldots, X_n] \times_{\mathrm{Spec}\, R} \mathrm{Proj}\, R[Y_0, \ldots, Y_m] \\ &\cong \mathrm{Proj} \left[\begin{array}{l} \text{subring of } R[X] \otimes_R R[Y] \text{ generated by} \\ \text{polynomials of degrees } (k,k), \text{ some } k \end{array} \right] \\ &\cong \mathrm{Proj} \left[\begin{array}{l} \text{subring of } R[X_0, \ldots, X_n, Y_0, \ldots, Y_m] \\ \text{generated by elements } X_i Y_j \end{array} \right]. \end{aligned}$$

Let U_{ij}, $0 \leq i \leq n$, $0 \leq j \leq m$, be new indeterminates. Then for some homogeneous prime ideal $\mathfrak{p} \subset R[U]$,

$$R[U_{00}, \ldots, U_{nm}]/\mathfrak{p} \cong \left[\begin{array}{l} \text{subring of } R[X_0, \ldots, Y_m] \\ \text{generated by elements } X_i Y_j \end{array} \right]$$
$$\text{via} \quad U_{ij} \longmapsto X_i Y_j.$$

Thus $\mathbb{P}^n_R \times_{\mathrm{Spec}\, R} \mathbb{P}^m_R$ is isomorphic to a closed subscheme of $\mathbb{P}^{nm+n+m}_R$. This is clearly the Segre embedding from a new angle:

The really important property of Proj is that the fundamental theorem of elimination theory (Part I [**87**, Chapter 2]) can be generalized to it.

---

[3] Grothendieck's definition agrees with this only when Y is quasi-compact. I made the above definition only to avoid complications and have no idea which works better over non-quasi-compact bases.

THEOREM 5.9 (Elimination theory for Proj). *If R is a finitely generated R_0-algebra, then the map*

$$\pi\colon \operatorname{Proj} R \longrightarrow \operatorname{Spec} R_0$$

is closed.

PROOF. Every closed subset of $\operatorname{Proj} R$ is isomorphic to $V(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \subset R_+$. But $V(\mathfrak{a}) \cong \operatorname{Proj} R/\mathfrak{a}$, so to show that $\pi(V(\mathfrak{a}))$ is closed in $\operatorname{Spec} R_0$, we may as well replace R by $R/\mathfrak{a}$ to start with and reduce the theorem to simply showing that $\operatorname{Image}\pi$ is closed. Also, we may reduce the theorem to the case when R is generated over R_0 by elements of degree 1. This follows because of $\operatorname{Proj} R \cong \operatorname{Proj} R\langle d\rangle$ and the amusing exercise:

LEMMA 5.10. *Let R be a graded ring, finitely generated over R_0. Then for some d, $R\langle d\rangle$ is generated over R_0 by $R\langle d\rangle_1 = R_d$.*

(Proof left to the reader).
After these reductions, take $\mathfrak{p}_0 \subset R_0$ a prime ideal. Then

$$[\mathfrak{p}_0] \notin \operatorname{Image}\pi \iff \begin{bmatrix} \nexists \text{ homogeneous prime } \mathfrak{p} \subset R \text{ such} \\ \text{that } \mathfrak{p} \cap R_0 = \mathfrak{p}_0,\ \mathfrak{p} \not\supset R_+ \end{bmatrix}.$$

Let $R' = R \otimes_{R_0} (R_0)_{\mathfrak{p}_0}$. Then homogeneous primes $\mathfrak{p}$ in R such that $\mathfrak{p} \cap R_0 = \mathfrak{p}_0$ are in one-to-one correspondence with homogeneous primes $\mathfrak{p}'$ in R' such that $\mathfrak{p}' \supset \mathfrak{p}_0 \cdot R'$. Therefore

$$\begin{aligned} [\mathfrak{p}_0] \notin \operatorname{Image}\pi &\iff \begin{bmatrix} \nexists \text{ homogeneous prime } \mathfrak{p}' \subset R' \text{ such} \\ \text{that } \mathfrak{p}' \supset \mathfrak{p}_0 \cdot R' \text{ and } \mathfrak{p}' \not\supset R'_+ \end{bmatrix} \\ &\iff \sqrt{\mathfrak{p}_0 \cdot R'} \supset R'_+ \\ &\iff \exists n,\ \mathfrak{p}_0 \cdot R' \supset (R'_+)^n \text{ (since } R_+ \text{ is a finitely generated ideal)} \\ &\iff \exists n,\ \mathfrak{p}_0 \cdot R'_n \supset R'_n \text{ (since } R_+ \text{ is generated by } R_1) \\ &\iff \begin{bmatrix} \exists n,\ R'_n = (0) \\ \text{(by Nakayama's lemma since } R'_n \\ \text{is a finitely generated } R'\text{-module)} \end{bmatrix}. \end{aligned}$$

Now for any finitely generated R_0-module M,

$$M_{\mathfrak{p}_0} = (0) \Longrightarrow M_f = (0), \quad \text{some } f \in R_0 \setminus \mathfrak{p}_0,$$

hence

$$\{[\mathfrak{p}_0] \in \operatorname{Spec} R_0 \mid M_{\mathfrak{p}_0} = (0)\}$$

is the maximal open set of $\operatorname{Spec} R_0$ on which $\widetilde{M}$ is trivial, i.e.,

$$\operatorname{Supp} \widetilde{M} = \{[\mathfrak{p}_0] \in \operatorname{Spec} R_0 \mid M_{\mathfrak{p}_0} \neq (0)\}$$

and this is a *closed* set. What we have proven is:

$$\begin{aligned} [\mathfrak{p}_0] \in \operatorname{Image}\pi &\iff \forall n,\ R_n \otimes_{R_0} (R_0)_{\mathfrak{p}_0} \neq (0) \\ &\iff [\mathfrak{p}_0] \in \bigcap_{n=1}^{\infty} \operatorname{Supp} \widetilde{R_n}. \end{aligned}$$

Thus $\operatorname{Image}\pi$ is closed. ∎

6. Proper morphisms

Theorem 5.9 motivates one of the main non-trivial definitions in scheme-theory:

DEFINITION 6.1. Let $f\colon X \to Y$ be a morphism of schemes. Then f is *proper*[4] if

a) f is of finite type,
b) for all $Y' \to Y$, the canonical map

$$X \times_Y Y' \longrightarrow Y'$$

is closed.

When $Y = \operatorname{Spec} k$, X is *complete* over k if $f\colon X \to \operatorname{Spec}(k)$ is proper.

Since "proper" is defined by such an elementary requirement, it is easy to deduce several general properties:

Suppose we are given $X \xrightarrow{f} Y \xrightarrow{g} Z$. Then

i) f, g proper $\implies$ $g \circ f$ proper
ii) $g \circ f$ proper $\implies$ f proper
iii)

$$\left.\begin{array}{l} g \circ f \text{ proper} \\ f \text{ surjective} \\ g \text{ of finite type} \end{array}\right\} \implies g \text{ proper}$$

iv) Proper morphisms are "maximal" in the following sense: given

$$\begin{tikzcd} X \arrow[dr, "f"'] & \subset & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

where X is open and dense in X',

$$f \text{ proper} \implies X = X'.$$

For instance, take (ii) which is perhaps subtler. One notes that f can be gotten as a composition:

$$\begin{tikzcd} X \arrow[r, hook, "(1,f)"] & X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "g\circ f"'] & Z \end{tikzcd}$$

where $(1, f)$ is a closed immersion.

Using the concept proper, the Elimination Theorem (Theorem 5.9) now reads:

COROLLARY 6.2. *A projective morphism $f\colon X \to Y$ is proper.*

PROOF. Note that $f\colon X \to Y$ is closed if there exists an open cover $\{U_i\}$ of Y such that $f^{-1}U_i \to U_i$ is closed. Therefore Corollary 6.2 follows from Theorem 5.9, the definition of Proj_Y and Property (c) of Proj. $\square$

[4](Added in publication) According to the standard definition, a morphism $f\colon X \to Y$ is *proper* if it is separated, (a) of finite type and (b) universally closed. Here both X and Y are assumed to be separated over $\mathbb{Z}$ by the convention at the end of §4. Hence f is automatically separated.

On the other hand, "projective" is the kind of explicit constructive property that gives one a very powerful hold on such morphisms, whereas "proper" is just an abstraction of the main qualitative property that projective morphisms possess. Now there exist varieties over k that are complete but not projective—even non-singular complex varieties—so proper is certainly weaker than projective. But what makes proper a workable concept is that it is not too much weaker than projective because of the following:

THEOREM 6.3 ("Chow's lemma"). *Let $f\colon X \to Y$ be a morphism of finite type between noetherian schemes. Then there exists*

a) *a surjective projective morphism $\pi\colon X' \to X$, "birational" in the sense that there is an open set U such that:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, phantom, "\underset{dense}{\subset}"] \arrow[d, "\approx", "isomorphism"'] & X' \arrow[d, "\pi"] \\ U \arrow[r, phantom, "\underset{dense}{\subset}"] & X, \end{tikzcd}$$

b) *a factorization of $f \circ \pi$:*

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X' \arrow[ur, "\pi"] \arrow[dr, "i"'] & & Y \\ & \mathbb{P}^n \times Y \arrow[ur, "p_2"'] & \end{tikzcd}$$

where i is an immersion, so that $f \circ \pi$ is quasi-projective.

If f is proper, then i is a closed immersion, so we have π and $f \circ \pi$ projective, i.e., f is a "factor" of projective morphisms!

PROOF. We do this in several steps:

STEP (I). $\exists$ a finite affine covering $\{U_i\}$ of X such that $\bigcap U_i$ is dense in X.

PROOF. Let $X = X_1 \cup \cdots \cup X_t$ be the components of X and let $\{V_i\}$ be any finite affine covering of X. For all s, $1 \leq s \leq t$, let X_s° be an affine open subset of X_s such that

a) $X_s^\circ \cap X_r = \emptyset$ if $r \neq s$
b) $X_s^\circ \subset V_i$ whenever $X_s \cap V_i \neq \emptyset$.

Then define U_i to be the union of V_i and those X_s° such that $V_i \cap X_s = \emptyset$. Since V_i and all these X_s° are disjoint, U_i is affine too. Moreover $\bigcap U_i \supset \bigcup X_s^\circ$, hence is dense in X. $\square$

STEP (II). For each i, $\operatorname{res} f\colon U_i \to Y$ can be factored

$$U_i \underset{I_i}{\hookrightarrow} \mathbb{A}^{\nu_i} \times Y \underset{p_2}{\longrightarrow} Y$$

where I_i is a closed immersion.

PROOF. Let $\{V_j\}$ be an affine covering of Y. Then $U_i \cap f^{-1}(V_j)$ is affine and its ring is generated by $\mathcal{O}_X(U_i) \otimes \mathcal{O}_Y(V_j)$. Let $f_1, \ldots, f_{\nu_i} \in \mathcal{O}_X(U_i)$ be enough elements to generate the affine rings of $U_i \cap f^{-1}(V_j)$ over $\mathcal{O}_Y(V_j)$ for all j. Define $I_{i,1}\colon U_i \to \mathbb{A}^{\nu_i}$ by $I_{i,1}^*(X_k) = f_k$ and define $I_i = (I_{i,1}, \operatorname{res} f)$. One sees easily that I_i is a closed immersion. $\square$

STEP (III). Consider the immersions:

$$I_i'\colon U_i \hookrightarrow \mathbb{P}^{\nu_i} \times Y$$

gotten by composing I_i with the usual inclusion of $\mathbb{A}^{\nu_i}$ in $\mathbb{P}^{\nu_i}$. Let $\overline{U_i}$ be the scheme-theoretic closure:

$$U_i \underset{\substack{\text{open}\\ \text{dense}}}{\subset} \overline{U_i} \underset{\substack{\text{closed}\\ \text{immersion}}}{\longrightarrow} \mathbb{P}^{\nu_i} \times Y.$$

Let $U = \bigcap_{i=1}^{N} U_i$. Consider the immersion:

$$U \hookrightarrow \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_N} \times Y$$

given by $(I'_1, \ldots, I'_N)$ and the inclusion of U in each U_i. Let $\overline{U}$ be the scheme-theoretic closure of the image here. Via the Segre embedding, we get:

$$\begin{tikzcd}
U \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[dr, "f"'] & \overline{U} \arrow[r, hook, "\substack{\text{closed}\\ \text{immersion}}"'] \arrow[d] & \mathbb{P}^n \times Y \arrow[dl, "p_2"] \\
 & Y &
\end{tikzcd}$$

Note that by projecting $\mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_N}$ on its i-th factor, we get morphisms:

$$\begin{tikzcd}
U \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[d, phantom, "\cap"] & \overline{U} \arrow[d, "p_i"] \\
U_i \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] & \overline{U_i}
\end{tikzcd}$$

Define X' to be the open subscheme of $\overline{U}$ which is the union of the open subschemes $p_i^{-1}(U_i)$. Finally, define $\pi\colon X' \to X$ by:

$$\begin{tikzcd}
X' \arrow[r, dashed, "\pi"] & X \\
p_i^{-1}U_i \arrow[u, phantom, "\cup"] \arrow[r, "p_i"] & U_i \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Note that this is OK because on the open set $U \subset \bigcap p_i^{-1}U_i$, all these morphisms p_i equal the inclusion morphism $U \hookrightarrow X$, and hence $p_i = p_j$ on $p_i^{-1}U_i \cap p_j^{-1}U_j$ since U is scheme-theoretically dense in $p_i^{-1}U_i \cap p_j^{-1}U_j$.

STEP (IV). $\pi\colon X' \to X$ is projective. In fact note that $p_i\colon \overline{U} \to \overline{U_i}$ is the restriction of the projection $\mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_n} \times Y \to \mathbb{P}^{\nu_i} \times Y$ to $\overline{U}$, hence it is projective, hence it is proper. Therefore $\operatorname{res} p_i\colon p_i^{-1}(U_i) \to U_i$ is proper. We are now in the abstract situation:

LEMMA 6.4. *If $\pi\colon X \to Y$ is a morphism, $U_i \subset X$, $V_i \subset Y$ open dense sets covering X and Y such that $\pi(U_i) \subset V_i$, $\operatorname{res}\pi\colon U_i \to V_i$ proper, then $\pi^{-1}(V_i) = U_i$ and π is proper.*

(Proof left to the reader.)

But now consider the morphism

$$j\colon X' \longrightarrow \mathbb{P}^n \times X$$

induced by a) $X' \subset \overline{U} \hookrightarrow \mathbb{P}^n \times Y \xrightarrow{p_1} \mathbb{P}^n$ and b) $\pi\colon X' \to X$. It is an immersion since the composite $X' \xrightarrow{j} \mathbb{P}^n \times X \xrightarrow{1\times f} \mathbb{P}^n \times Y$ is an immersion. Since $\pi\colon X' \to X$ is proper, j is proper too, hence $j(X')$ is closed, hence j is a closed immersion. Thus π is projective. Finally, if f is proper too, then $f \circ \pi\colon X' \to Y$ is proper, hence the immersion $X' \to \mathbb{P}^n \times Y$ is proper, hence it is a *closed* immersion, hence $f \circ \pi$ is projective. $\square$

Interestingly, when this result first appeared in the context of varieties, it was considered quite clear and straightforward. It is one example of an idea which got much harder when transported to the language of schemes.

Proper morphisms arise in another common situation besides Proj:

PROPOSITION 6.5. *Let $\phi\colon A \to B$ be a homomorphism of rings where B is a finite A-module (equivalently, B is a finitely generated A-algebra and B is integrally dependent on A). Then the induced morphism f*

$$f\colon \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

is proper.

PROOF. This is simply the "going-up" theorem (Zariski-Samuel [**119**, vol. I, Chapter V, §2, Theorem 3]). It suffices to show f is closed. Let $Z = V(\mathfrak{a}) \subset \operatorname{Spec} B$ be a closed set. I claim

$$f(Z) = V(\phi^{-1}(\mathfrak{a})).$$

We must show that if $\mathfrak{p}$ is a prime ideal:

$$\phi^{-1}(\mathfrak{a}) \subset \mathfrak{p} \subset A,$$

then there is a prime ideal $\mathfrak{q}$:

$$\mathfrak{a} \subset \mathfrak{q} \subset B, \qquad \phi^{-1}(\mathfrak{q}) = \mathfrak{p}.$$

Apply the going-up theorem to $\mathfrak{p}/\phi^{-1}(\mathfrak{a})$ and the inclusion:

$$A/\phi^{-1}(\mathfrak{a}) \subset B/\mathfrak{a}.$$

□

One globalizes this situation via a definition:

DEFINITION 6.6. A morphism $f\colon X \to Y$ is called *finite* if $X \cong \operatorname{Spec}_Y \mathcal{R}$ where $\mathcal{R}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras which is finitely generated as $\mathcal{O}_Y$-modules.

COROLLARY 6.7. *A finite morphism is proper.*

There is a very important criterion for properness known as the "valuative criterion":

PROPOSITION 6.8. *Let $f\colon X \to Y$ be a morphism of finite type. Then f is proper if and only if the "valuative criterion" holds:*

> *For all valuation rings R, with quotient field K, every K-valued point α of X extends to an R-valued point if the K-valued point $f \circ \alpha$ of Y extends, i.e., given the solid arrows:*
>
> $$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$
>
> *the dotted arrow exists.*

PROOF. It's obvious that the criterion is necessary: just make the base change by the extended morphism $\beta\colon \operatorname{Spec} R \to Y$:

$$\begin{tikzcd} & X' = X \times_Y \operatorname{Spec} R \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} K \arrow[r, hook] \arrow[ur, dashed, "\alpha'=(\alpha,i)"] & \operatorname{Spec} R \arrow[r, "\beta"] & Y \end{tikzcd}$$

Then α defines a morphism $\alpha' = (\alpha, i)$ from $\operatorname{Spec} K$ to $X \times_Y \operatorname{Spec} R$, i.e., a section of f' over $\operatorname{Spec} K$. Let $z \in X \times_Y \operatorname{Spec} R$ be the image of α' and let $Z = \overline{\{z\}}$. Then if f is proper:

$$f'(Z) = \overline{f'(Z)} = \overline{\{f'(z)\}} = \operatorname{Spec} R.$$

Let $w \in Z$ lie over the closed point of $\operatorname{Spec} R$. Then we get homomorphisms

$$\begin{tikzcd} & \mathcal{O}_{w,Z} \arrow[dl, hook, "\alpha^*"'] \\ K & R \arrow[l, phantom, "\supset"] \arrow[u, "(f')^*"'] \end{tikzcd}$$

Since R is a valuation ring and $(f')^*$ is a local homomorphism, this can only hold if $R = \mathcal{O}_{w,Z}$ (a valuation ring is a maximal subring of its quotient field with respect to local homomorphisms: Zariski-Samuel [**119**, vol. II, Chapter VI, §2]). Then

$$\operatorname{Spec} \mathcal{O}_{w,Z} \longrightarrow Z$$

defines the required extension:

$$\operatorname{Spec} R \longrightarrow Z \subset X' \longrightarrow X$$

of α.

The converse is only a bit harder. Assume f satisfies the criterion. Then so does $p_2 \colon X \times_Y Y' \to Y'$ after every base change $Y' \to Y$, so replacing f by p_2, it suffices to check that f itself is closed. Everything is local over Y so we may also assume Y is affine: say $Y = \operatorname{Spec} S$. Since f is of finite type, X is the union of finitely many affines X_α: say $X_\alpha = \operatorname{Spec} R_\alpha$. Now let $Z \subset X$ be closed. Then

$$Z = \bigcup_\alpha \overline{(Z \cap X_\alpha)}$$

so if $f(\overline{Z \cap X_\alpha})$ is closed for every α, so is $f(Z)$. We can therefore also replace Z by $\overline{Z \cap X_\alpha}$ for some α, i.e., we can assume $Z \cap X_\alpha$ dense in Z for some α. There are two steps:

a) for every irreducible component W of $\overline{f(Z)}$, the generic point η_W equals $f(z)$, some $z \in Z$,
b) for every $z \in Z$ and $y \in \overline{\{f(z)\}}$, there is a point $x \in \overline{\{z\}}$ such that $f(x) = y$.

Together, these prove that $f(Z)$ is closed.

Proof of (a). The affine morphism

$$Z \cap X_\alpha \longrightarrow \overline{f(Z)} = \overline{f(Z \cap X_\alpha)}$$

corresponds to an injective ring homomorphism

$$R_\alpha/\mathfrak{b}_\alpha \xleftarrow{\;f^*\;} S/\mathfrak{a}$$

between rings without nilpotents. η_W corresponds to a minimal prime ideal $\mathfrak{p} \subset S/\mathfrak{a}$. Localizing with respect to $M = ((S/\mathfrak{a}) \setminus \mathfrak{p})$, we still get an injection $(\operatorname{res} f)^*$ in the diagram

$$\begin{tikzcd} (R_\alpha/\mathfrak{b}_\alpha)_M & (S/\mathfrak{a})_M \arrow[l, hook', "(\operatorname{res} f)^*"'] \\ R_\alpha/\mathfrak{b}_\alpha \arrow[u, "j"] & S/\mathfrak{a} \arrow[l, hook', "f^*"] \arrow[u] \end{tikzcd}$$

(Zariski-Samuel [**119**, vol. I, Chapter IV, §9] and Bourbaki [**27**, Chapter II, §2.4, Theorem 1]). But $(S/\mathfrak{a})_M$ is the field $\Bbbk(\eta_W)$, so if $\mathfrak{q} \subset (R_\alpha/\mathfrak{b}_\alpha)_M$ is any prime ideal, $((\operatorname{res} f)^*)^{-1}(\mathfrak{q}) = (0)$. Then $j^{-1}(\mathfrak{q})$ defines a point $z = [j^{-1}(\mathfrak{q})] \in Z \cap X_\alpha$ such that $f(z) = \eta_W$. $\square$

PROOF OF (b). In the notation of (b), let $W = \overline{\{f(z)\}}$. Then we have a diagram of rings

$$\mathcal{O}_{y,W} \hookrightarrow \text{its quotient field } \Bbbk(f(z)) \subset \Bbbk(z).$$

We use the fundamental valuation existence theorem (Zariski-Samuel [**119**, vol. II, Chapter VI, Theorem 5]) which states that there is a valuation ring $R \subset \Bbbk(z)$ with quotient field $K = \Bbbk(z)$ such that $\mathcal{O}_{y,W} \to R$ is a local homomorphism. This gives us maps:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \overline{\{z\}} \arrow[d, "\operatorname{res} f"] & \subset & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & W & \subset & Y \end{tikzcd}$$

By the criterion, a lifting $\operatorname{Spec} R \to X$ exists, and this must factor through $\overline{\{z\}}$ (since $\operatorname{Spec} K$ is dense in $\operatorname{Spec} R$). Then x, the image under this lifting of the closed point of $\operatorname{Spec} R$, is the required point of $\overline{\{z\}}$. □

□

An amusing exercise that shows one way the definition of properness can be used is:

PROPOSITION 6.9. *Let k be a field and let X be a scheme proper over* $\operatorname{Spec} k$. *Then the algebra $\Gamma(X, \mathcal{O}_X)$ is integrally dependent on k.*

PROOF. Let $a \in \Gamma(X, \mathcal{O}_X)$. Define a morphism

$$f_a \colon X \longrightarrow \mathbb{A}^1_k$$

by the homomorphism

$$\begin{aligned} k[T] &\longrightarrow \Gamma(X, \mathcal{O}_X) \\ T &\longmapsto a. \end{aligned}$$

Let $i \colon \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ be the inclusion. Consider the diagram

$$\begin{tikzcd} X \arrow[rr, "i \circ f_a"] \arrow[dr, "\pi_1"'] & & \mathbb{P}^1_k \arrow[dl, "\pi_2"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

where π_1, π_2 are the canonical maps. Since π_1 is proper, so is $i \circ f_a$ (cf. remarks following Definition 6.1). Therefore the image of $i \circ f_a$ is closed. But $\infty \notin \operatorname{Image}(i \circ f_a)$, so the image must be a proper subscheme of $\mathbb{A}^1_k$. Since $k[T]$ is a principal ideal domain, the image is contained in $V(p)$, some monic polynomial $p(T)$. Therefore the function

$$p(a) \in \Gamma(X, \mathcal{O}_X)$$

is everywhere zero on X. On each affine, such a function is nilpotent (an element in every prime ideal of a ring is nilpotent) and X is covered by a finite number of affines. Thus

$$p(a)^N = 0$$

some $N \geq 1$, and a is integral over k. □

COROLLARY 6.10. *Let k be an algebraically closed field and let X be a complete k-variety. Then $\Gamma(X, \mathcal{O}_X) = k$.*

The following result, given in a slightly stronger form in EGA [**1**, Chapter III, §3.1], will be needed in the proof of Snapper's theorem (Theorem VII.11.1) in the proper but non-projective case.

DEFINITION 6.11. Let $\mathbf{K}$ be an abelian category, and denote by $\mathrm{Ob}(\mathbf{K})$ the set of its objects. A subset $\mathbf{K}' \subset \mathrm{Ob}(\mathbf{K})$ is said to be *exact* if $0 \in \mathbf{K}'$ and if the following is satisfied: In an exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathbf{K}$, if two among A, A' and A'' belong to $\mathbf{K}'$, then the third belongs to $\mathbf{K}'$.

THEOREM 6.12 ("Lemma of dévissage"). *Let $\mathbf{K}$ be the abelian category of coherent $\mathcal{O}_X$-modules on a noetherian scheme X, and $\mathbf{K}' \subset \mathrm{Ob}(\mathbf{K})$ an exact subset. We have $\mathbf{K}' = \mathrm{Ob}(\mathbf{K})$, if for any closed irreducible subset $Y \subset X$ with generic point y there exists $\mathcal{G} \in \mathbf{K}'$ with support Y such that $\mathcal{G}_y$ is a one-dimensional $\Bbbk(y)$-vector space.*

PROOF. For simplicity, a closed subset $Y \subset X$ is said to have property $\mathbf{P}(Y)$ if any $\mathcal{S} \in \mathrm{Ob}(\mathbf{K})$ with $\mathrm{Supp}(\mathcal{S}) \subset Y$ satisfies $\mathcal{S} \in \mathbf{K}'$.

We need to show that X has property $\mathbf{P}(X)$.

By noetherian induction, it suffices to show that a closed subset $Y \subset X$ has property $\mathbf{P}(Y)$ if any closed subset $Y' \subsetneqq Y$ has property $\mathbf{P}(Y')$.

Thus we now show $\mathcal{F} \in \mathrm{Ob}(\mathbf{K})$ satisfies $\mathcal{F} \in \mathbf{K}'$ if $\mathrm{Supp}(\mathcal{F}) \subset Y$. Endow Y with the unique structure of closed reduced subscheme of X with the ideal sheaf $\mathcal{J}$. Since $\mathcal{J} \supset \mathrm{Ann}(\mathcal{F})$, there exists $n > 0$ such that $\mathcal{J}^n\mathcal{F} = (0)$. Looking at successive quotients in the filtration

$$\mathcal{F} \supset \mathcal{J}\mathcal{F} \supset \mathcal{J}^2\mathcal{F} \supset \cdots \supset \mathcal{J}^{n-1}\mathcal{F} \supset \mathcal{J}^n\mathcal{F} = (0),$$

we may assume $n = 1$, that is, $\mathcal{J}\mathcal{F} = (0)$, in view of the exactness of $\mathbf{K}'$. Let $j\colon Y \to X$ be the closed immersion so that $\mathcal{F} = j_*j^*\mathcal{F}$.

Suppose Y is reducible and $Y = Y' \cup Y''$ with closed reduced subschemes $Y', Y'' \subsetneqq Y$. Let $\mathcal{J}'$ and $\mathcal{J}''$ be the ideal sheaves of $\mathcal{O}_X$ defining Y' and Y'', respectively, so that $\mathcal{J} = \mathcal{J}' \cap \mathcal{J}''$. Let $\mathcal{F}' = \mathcal{F} \otimes (\mathcal{O}_X/\mathcal{J}')$ and $\mathcal{F}'' = \mathcal{F} \otimes (\mathcal{O}_X/\mathcal{J}'')$, both of which belong to $\mathbf{K}'$ by assumption. Regarding the canonical $\mathcal{O}_X$-homomorphism

$$u\colon \mathcal{F} \longrightarrow \mathcal{F}' \oplus \mathcal{F}'',$$

we have $\mathcal{F}' \oplus \mathcal{F}'' \in \mathbf{K}'$ by exactness, while $\mathrm{Ker}(u), \mathrm{Coker}(u) \in \mathbf{K}'$ by assumption, since the induced homomorphism of the stalks at each $z \notin Y' \cap Y''$ is obviously bijective. Hence we have $\mathcal{F} \in \mathbf{K}'$ by exactness.

It remains to deal with the case Y irreducible. Endowing Y with the unique integral scheme structure, let y be the generic point of Y. Since $\mathcal{O}_{y,Y} = \Bbbk(y)$ and $j^*\mathcal{F}$ is $\mathcal{O}_Y$-coherent, $\mathcal{F}_y = (j^*\mathcal{F})_y$ is a $\Bbbk(y)$-vector space of finite dimension m, say. By assumption there exists $\mathcal{G} \in \mathbf{K}'$ with $\mathrm{Supp}(\mathcal{G}) = Y$ and $\dim_{\Bbbk(y)} \mathcal{G}_y = 1$. Hence there is a $\Bbbk(y)$-isomorphism $(\mathcal{G}_y)^{\oplus m} \to \mathcal{F}_y$ which extends to an $\mathcal{O}_Y$-isomorphism in a neighborhood W in X of y. Let $\mathcal{H}$ be the graph in $(\mathcal{G}^{\oplus m} \oplus \mathcal{F})|_W$ of the $\mathcal{O}_Y|_W$-isomorphism $\mathcal{G}^{\oplus m}|_W \to \mathcal{F}|_W$. The projections from $\mathcal{H}$ to $\mathcal{G}^{\oplus m}|_W$ and $\mathcal{F}|_W$ are isomorphisms. Hence there exists a coherent $\mathcal{O}_X$-submodule $\mathcal{H}_0 \subset \mathcal{G}^{\oplus m} \oplus \mathcal{F}$ such that $\mathcal{H}_0|_W = \mathcal{H}$ and that $\mathcal{H}_0|_{X\setminus Y} = (0)$, since $\mathrm{Supp}(\mathcal{G}), \mathrm{Supp}(\mathcal{F}) \subset Y$. The projections from $\mathcal{H}_0$ to $\mathcal{G}^{\oplus m}$ and $\mathcal{F}$ are homomorphisms of $\mathcal{O}_X$-modules which are isomorphisms on W and $X \setminus Y$. Thus their kernels and cokernels have support in $Y \setminus (Y \cap W) \subsetneqq Y$, hence belong to $\mathbf{K}'$. Since $\mathcal{G} \in \mathbf{K}'$, we thus have $\mathcal{H}_0 \in \mathbf{K}'$, hence $\mathcal{F} \in \mathbf{K}'$. $\square$

Exercise

(1) Let $f\colon X \to Y$ be a finite morphism. If the fibre $f^{-1}(y)$ over one point $y \in Y$ is isomorphic to $\mathrm{Spec}\,\Bbbk(y)$, show that $\mathrm{res}\, f\colon f^{-1}(U) \to U$ is a closed immersion for some neighborhood U of y.

(2) (Referred to in the proof of Lang's Theorem VI.2.1.) Let $f\colon X \to Y$ be a morphism of finite type with Y noetherian such that $f^{-1}(y)$ is finite for all $y \in Y$. Show that $\exists$ an open dense $U \subset Y$ such that
$$\operatorname{res} f\colon f^{-1}(U) \longrightarrow U$$
is finite.

(3) (Complement to Proposition 3.11) Let $f\colon X \to Y$ be a quasi-compact morphism of schemes. Show that $\mathcal{I} := \operatorname{Ker}(\mathcal{O}_Y \to f_*\mathcal{O}_X)$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. (The closed subscheme of X defined by $\mathcal{I}$ is called the *scheme-theoretic closure* of the image of X in Y.)

(4) Let $f\colon X \to Y$ be a quasi-compact and quasi-separated morphism of schemes, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module. (Recall as in Remark at the end of §4 that f is defined to be quasi-separated if the diagonal morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ is quasi-compact.)

(5) Give an example of a scheme X with two affine open subsets U and V such that $U \cap V$ is not affine.

CHAPTER III

Elementary global study of Proj R

1. Intertible sheaves and twists

DEFINITION 1.1. Let X be a scheme. A sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules is called *invertible* if $\mathcal{L}$ is locally free of rank one. This means that each point has an open neighborhood U such that

$$\mathcal{L}|_U \approx \mathcal{O}_X|_U;$$

or equivalently, there exists an open covering $\{U_\alpha\}$ of X such that for each α,

$$\mathcal{L}|_{U_\alpha} \approx \mathcal{O}_X|_{U_\alpha}.$$

The reason why invertible sheaves are called invertible is that their isomorphism classes form a group under the tensor product over $\mathcal{O}_X$ for multiplication, as we shall now see.

(a) If $\mathcal{L}$, $\mathcal{L}'$ are invertible, so is $\mathcal{L} \otimes \mathcal{L}'$.

PROOF. For each point we can find an open neighborhood U such that both $\mathcal{L}$, $\mathcal{L}'$ are isomorphic to $\mathcal{O}_X$ when restricted to U, so $\mathcal{L} \otimes \mathcal{L}'$ is isomorphic to $\mathcal{O}_X \otimes \mathcal{O}_X = \mathcal{O}_X$ when restricted to U. $\square$

(b) It is clear that $\mathcal{L} \otimes \mathcal{O}_X \approx \mathcal{L} \approx \mathcal{O}_X \otimes \mathcal{L}$, so $\mathcal{O}_X$ is a unit element for the multiplication, up to isomorphism.

(c) Let $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{L}^\vee$ is invertible.

PROOF. Restricting to a suitable open set U we may assume that $\mathcal{L} \approx \mathcal{O}_X$, in which case

$$\mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \approx \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \approx \mathcal{O}_X.$$

$\square$

(d) The natural map

$$\mathcal{L} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$$

is an isomorphism.

PROOF. Again restricting to an appropriate open set U, we are reduced to proving the statement when $\mathcal{L} = \mathcal{O}_X$, in which case the assertion is immediate. $\square$

Thus $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$, which is call the *dual sheaf*, is an inverse for $\mathcal{L}$ up to isomorphism. This proves that isomorphism classes of invertible sheaves over $\mathcal{O}_X$ form a group.

We also have the property:

(e) Let $f\colon X \to Y$ be a morphism and $\mathcal{L}$ an invertible sheaf on Y. Then $f^*\mathcal{L}$ is an invertible sheaf on X.

DEFINITION 1.2. Let X be a scheme. We let $\mathrm{Pic}(X)$, the *Picard group*, be the group of all isomorphism classes of invertible sheaves.

Invertible sheaves and Proj are closely related because under a certain hypothesis, Proj R carries a canonical invertible sheaf, known as $\mathcal{O}_{\operatorname{Proj} R}(1)$.

Let R be a graded ring,

$$R = \bigoplus_{n \geq 0} R_n.$$

Then R is an algebra over R_0. *The hypothesis that allows us to define $\mathcal{O}_{\operatorname{Proj} R}(1)$ is that R is generated by R_1 over R_0, that is*

$$R = R_0[R_1]$$

(cf. Proposition II.5.1). *We shall make this hypothesis throughout this section.*

EXAMPLE 1.3. The most basic ring of this type is obtained as in Definition II.5.6 as follows. Let A be any commutative ring, and let

$$R = A[T_0, \dots, T_r]$$

be the polynomial ring in $r+1$ variables. Then $R_0 = A$, and R_n consists of the homogeneous polynomials of degree n with coefficients in A. Furthermore R_1 is the free module over A, with basis $T_0, \dots, T_r$.

For simplicity, we abbreviate

$$\mathbb{P} = \operatorname{Proj} R.$$

To define $\mathcal{O}_{\mathbb{P}}(1)$, start with any graded module M. Then for all integer $d \in \mathbb{Z}$ we may define the d-twist $M(d)$ of M, which is the module M but with the new grading

$$M(d)_n = M_{d+n}.$$

Then we define

$$\mathcal{O}_{\mathbb{P}}(1) = \widetilde{R(1)}$$

where the $\widetilde{\ }$ is the projective $\widetilde{\ }$. If $f \in R$ is a homogeneous element, we abbreviate the open subset

$$(\operatorname{Proj} R)_f = \mathbb{P}_f \text{ or } U_f.$$

PROPOSITION 1.4. *The sheaf $\mathcal{O}_{\mathbb{P}}(1)$ is invertible on* Proj R. *In fact: Given $f \in R_1$, the multiplication by f*

$$m_f \colon R \longrightarrow R(1)$$

is a graded homomorphism of degree 0, *whose induced sheaf homomorphism*

$$\widetilde{m_f} \colon \widetilde{R} = \mathcal{O}_{\mathbb{P}} \longrightarrow \widetilde{R(1)} = \mathcal{O}_{\mathbb{P}}(1)$$

restricts to an isomorphism on U_f. Let $\varphi_f = \widetilde{m_f}$. For $f, g \in R_1$, the sheaf map $\varphi_f^{-1} \circ \varphi_g$ is multiplication by g/f on $U_f \cap U_g$.

PROOF. By definition

$$\mathcal{O}_{\mathbb{P}}(1)|_{U_f} = (\widetilde{R(1)_f})_0,$$

and we have an isomorphism

$$\text{multiplication by } f \colon R_f \longrightarrow R(1)_f.$$

This induces an isomorphism on the parts of degree 0, whence taking the affine $\widetilde{\ }$, it induces the isomorphism

$$\mathcal{O}_{\mathbb{P}}|_{U_f} \longrightarrow \mathcal{O}_{\mathbb{P}}(1)|_{U_f}.$$

In fact, the module associated with $\mathcal{O}_{\mathbb{P}}(1)$ on U_f is just given by

$$(R_f)_0 \cdot f,$$

and is consequently free of rank 1 over the affine coordinate ring of $\operatorname{Spec}(R_f)_0$. Since R is generated by R_1, the U_f's cover $\operatorname{Proj} R$, and this shows that $\mathcal{O}_{\mathbb{P}}(1)$ is invertible. $\square$

PROPOSITION 1.5. *Let M be a graded R-module. Then the isomorphism*

$$M \otimes_R R(1) \longrightarrow M(1)$$

induces an isomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(1) \longrightarrow \widetilde{M(1)}.$$

PROOF. Let $f \in R_1$. On $\mathbb{P}_f$ the isomorphism of graded modules induces the corresponding isomorphism of $(R_f)_0$-modules

$$(M_f)_0 \otimes (R(1)_f)_0 \longrightarrow (M(1)_f)_0,$$

where the tensor product is taken over $(R_f)_0$. Taking the affine tilde yields the desired sheaf isomorphism. $\square$

DEFINITION 1.6. For every integer d we define

$$\mathcal{O}_{\mathbb{P}}(d) = \widetilde{R(d)},$$

and for any sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}}$-modules, we define

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(d).$$

PROPOSITION 1.7.

(i) *For $d, m \in \mathbb{Z}$ we have $\mathcal{F}(d+m) \approx \mathcal{F}(d) \otimes \mathcal{O}_{\mathbb{P}}(m)$.*

(ii) *For d positive,*

$$\mathcal{O}_{\mathbb{P}}(d) \approx \mathcal{O}_{\mathbb{P}}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}}(1) \quad (\textit{product taken } d \textit{ times}).$$

(iii) *For $d \in \mathbb{Z}$ the natural pairing*

$$\mathcal{O}_{\mathbb{P}}(d) \otimes \mathcal{O}_{\mathbb{P}}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}}$$

identifies $\mathcal{O}_{\mathbb{P}}(-d)$ with the dual sheaf $\mathcal{O}_{\mathbb{P}}(d)^\vee$.

(iv) *For a graded module M, we have $\widetilde{M(d)} \approx \widetilde{M}(d)$.*

PROOF. The first assertion follows from the formula

$$(\widetilde{M \otimes_R N}) \approx \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}}} \widetilde{N}$$

for any two graded R-modules M and N, because R is generated by R_1. Indeed, for $f \in R_1$ we have

$$(M \otimes_R N)_f = M_f \otimes_{R_f} N_f.$$

The other assertions are immediate. $\square$

The collection of sheaves $\widetilde{M}(d)$ attached to M allows us to interpret globally each graded piece of the module M. In fact, for each d, we get a canonical homomorphism (cf. §II.5)

$$M_d = M(d)_0 \longrightarrow \Gamma(\mathbb{P}, \widetilde{M(d)}) \xrightarrow{\approx} \Gamma(\mathbb{P}, \widetilde{M}(d)).$$

For any sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}}$-modules, we define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{m \in \mathbb{Z}} \Gamma(\mathbb{P}, \mathcal{F}(m)).$$

Then we obtain a canonical homomorphism

$$M \longrightarrow \Gamma_*(\widetilde{M}).$$

In particular, when $M = R$, we get a *ring* homomorphism

$$R \longrightarrow \bigoplus_{d=0}^{\infty} \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) = \Gamma_*(\widetilde{R}) = \Gamma_*(\mathcal{O}_{\mathbb{P}}),$$

where multiplication on the right hand side is defined by the tensor product.

We also note that $\Gamma_*(\mathcal{F})$ is a graded R-module as follows. We have the inclusion of R_d in $\Gamma(\mathbb{P}, \widetilde{R(d)})$, and the product of R_d on $\Gamma(\mathbb{P}, \mathcal{F}_{\mathbb{P}}(m))$ is induced by the tensor product

$$\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes \Gamma(\mathbb{P}, \mathcal{F}(m)) \longrightarrow \Gamma(\mathbb{P}, \mathcal{F}(m+d)).$$

It is not always the case that there is an isomorphism

$$\Gamma_*(\mathcal{O}_{\mathbb{P}}) \approx R,$$

so for some positive integer d, it may happen that the module of sections $\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$ is larger than R_d. We now give an example when these are equal.

PROPOSITION 1.8. *Let A be a ring and $R = A[T_0, \dots, T_r]$, $r \geq 1$. Let $\mathbb{P} = \operatorname{Proj} R = \mathbb{P}^r_A$. Then for all integers $d \in \mathbb{Z}$ we have*

$$R_d \approx \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \quad \textit{so} \quad R \approx \Gamma_*(\mathcal{O}_{\mathbb{P}}).$$

PROOF. For $i = 0, \dots, r$ let $U_i = U_{T_i}$, so U_i is the usual affine open subscheme of Proj R, complement of the hyperplanes $T_i = 0$. A section $s \in \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ is the same as a family of sections $s_i \in \mathcal{O}_{\mathbb{P}}(n)(U_i)$ for all i, such that $s_i = s_j$ on $U_i \cap U_j$ for all i, j. But a section in $\mathcal{O}_{\mathbb{P}}(n)(U_i)$ is simply an element

$$s_i = \frac{f_i(T)}{T_i^{k(i)}}$$

where $k(i)$ is an integer and $f_i(T)$ is a homogeneous polynomial of degree $k(i)+n$. The restriction to $U_i \cap U_j$ is the image of that element in the localization $R_{T_iT_j}$. Since the elements $T_0, \dots, T_r$ are not zero-divisors in R, the natural maps

$$R \longrightarrow R_{T_i} \quad \text{and} \quad R_{T_i} \longrightarrow R_{T_iT_j}$$

are injective, and all such localized rings can be viewed as subrings of $R_{T_0 \cdots T_r}$. Hence $\Gamma_*(\mathcal{O}_{\mathbb{P}})$ is the intersection $\bigcap R_{T_i}$ taken inside $R_{T_0 \cdots T_r}$. Any homogeneous element of $R_{T_0 \cdots T_r}$ can be written in the form

$$f(T_0, \dots, T_r) T_0^{k(0)} \cdots T_r^{k(r)}$$

where $f(T_0, \dots, T_r)$ is a homogeneous polynomial not divisible by any T_i ($i = 0, \dots, r$) and $k(0), \dots, k(r) \in \mathbb{Z}$. Such an element lies in R_{T_i} if and only if $k(j) \geq 0$ for all $j \neq i$. Hence the intersection of all the R_{T_i} for $i = 0, \dots, r$ is equal to R. This proves the proposition. □

The proposition both proves a result and gives an example of the previous constructions. In particular, we see that the elements $T_0, \dots, T_r$ form a basis of R_1 over A, and can be viewed as a basis of the A-module of sections $\Gamma(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}}(1))$.

Next we look at the functoriality of twists with respect to graded ring homomorphisms. As in §II.5 we let R' be a graded ring which we now assume generated by R'_1 over R'_0. Let

$$\varphi\colon R \longrightarrow R'$$

be a graded homomorphism of degree 0. Let V be the subset of Proj R' consisting of those primes $\mathfrak{p}'$ such that $\mathfrak{p}' \not\supset \varphi(R_+)$. Then we saw that V is open in Proj R', and that the inverse image map on prime ideals

$$f\colon V \longrightarrow \operatorname{Proj} R = \mathbb{P}$$

defines a morphism of schemes.

PROPOSITION 1.9. *Let* $\mathbb{P}' = \operatorname{Proj} R'$. *Then*

$$f^*\mathcal{O}_{\mathbb{P}}(d) = \mathcal{O}_{\mathbb{P}'}(d)|_V \quad \textit{and} \quad f_*(\mathcal{O}_{\mathbb{P}'}(d)|_V) = (f_*\mathcal{O}_V)(d).$$

PROOF. These assertions about the twists hold more generally for any graded R-module M, because

$$f^*(\widetilde{M}) \approx (\widetilde{M \otimes_R R'})|_V$$

and for any graded R'-module N, we have

$$f_*(\widetilde{N}|_V) \approx \widetilde{(N_R)},$$

where N_R is N viewed as R-module via φ. The proof is routine and left to the reader. $\square$

To conclude this section we note that everything we have said extends to the global Proj without change. Instead of $\operatorname{Proj} R$, we can consider $\operatorname{Proj}_X \mathcal{R}$ where $\mathcal{R}$ is a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras. We need to make the hypothesis that $\mathcal{R}_n$ is generated by $\mathcal{R}_1$ over $\mathcal{R}_0$, i.e., the multiplication map

$$\operatorname{Symm}^n_{\mathcal{R}_0} \mathcal{R}_1 \longrightarrow \mathcal{R}_n$$

is surjective. Let $\mathbb{P} = \operatorname{Proj}_X \mathcal{R}$. Then if $\mathcal{M}$ is a quasi-coherent graded sheaf of $\mathcal{R}$-modules, we define $\mathcal{M}(d)$ by

$$\mathcal{M}(d)_n = \mathcal{M}_{d+n}.$$

Then let

$$\mathcal{O}_{\mathbb{P}}(d) = \widetilde{\mathcal{R}(d)}$$

and for every quasi-coherent $\mathcal{F}$ on $\mathbb{P}$, let

$$\mathcal{F}(d) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(d).$$

As before, $\mathcal{O}_{\mathbb{P}}(1)$ is invertible, with powers $\mathcal{O}_{\mathbb{P}}(d)$ and

$$\widetilde{\mathcal{M}(d)} = (\widetilde{\mathcal{M}})(d).$$

The extension of the definition of $\Gamma_*(\mathcal{F})$ to the global case is:

$$\pi_*^{\mathrm{gr}}\mathcal{F} = \bigoplus_{m \in \mathbb{Z}} \pi_*\mathcal{F}(m)$$

where π is the projection of $\operatorname{Proj}_X \mathcal{R}$ to X. This is quasi-coherent provided that $\mathcal{R}_1$ is finitely generated as $\mathcal{R}_0$-modules, since this implies that π is quasi-compact, hence Proposition II.4.10 applies. As above, we have a natural graded homomorphism

$$\mathcal{M} \longrightarrow \pi_*^{\mathrm{gr}}(\widetilde{\mathcal{M}}).$$

Finally Proposition 1.8 globalizes immediately to:

PROPOSITION 1.10. *Let* $\mathcal{E}$ *be a locally free sheaf of* $\mathcal{O}_X$*-module and consider* $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}_X(\operatorname{Symm}^* \mathcal{E})$. *Then the natural homomorphism*

$$\operatorname{Symm}^d \mathcal{E} \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$$

is an isomorphism. In particular, $\operatorname{Symm}^* \mathcal{E} \approx \pi_*^{\mathrm{gr}}\mathcal{O}_{\mathbb{P}(\mathcal{E})}$.

2. The functor of $\operatorname{Proj} R$

Throughout this section we let R be a graded ring, generated by R_1 over R_0. We let $S = \operatorname{Spec}(R_0)$, $\mathbb{P} = \operatorname{Proj} R$ and let $\pi\colon \mathbb{P} \to S$ be the canonical projection.

An important example of a graded ring R as above is $\operatorname{Symm}_{R_0}(R_1)$, namely the symmetric algebra, but we shall meet other cases, so we do not restrict our attention to this special case.

We are interested in schemes X over S, and in morphisms of X into $\operatorname{Proj} R$ over S:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{P} = \operatorname{Proj} R \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

In the simplest case, $\mathbb{P} = \mathbb{P}^r_{R_0}$ and f becomes a morphism of X into projective space.

Given such a morphism $f\colon X \to \mathbb{P}$, we can take the inverse image $f^*\mathcal{O}_{\mathbb{P}}(1)$, which is an invertible sheaf on X. By the general formalism of inverse images of sheaves, this induces a natural map on global sections

$$f^*\colon \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \longrightarrow \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)),$$

and in light of the natural map $R_1 \to \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ induces a homomorphism

$$\varphi_f = \varphi\colon R_1 \longrightarrow \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)).$$

Thus to each morphism $f\colon X \to \mathbb{P}$ we have associated a pair $(\mathcal{L}, \varphi)$ consisting of an invertible sheaf $\mathcal{L}$ (in this case $f^*\mathcal{O}_{\mathbb{P}}(1)$) and a homomorphism

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L}).$$

To describe an additional important property of this homomorphism, we need a definition.

Definition 2.1. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $\{s_i\}$ be a family of sections. We say that this family *generates* $\mathcal{F}$ if any one of the following conditions is satisfied:

(1) For every point $x \in X$ the family of images $\{(s_i)_x\}$ generates $\mathcal{F}_x$ as an $\mathcal{O}_x$-module, or equivalently (by Nakayama's lemma Proposition I.5.5) $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$.

(2) For each point $x \in X$ there exists some open neighborhood U of x such that the sections $\{s_i|_U\}$ generate $\mathcal{F}(U)$ over $\mathcal{O}_X(U)$.

Note that by Proposition 1.4, if $g \in R_1$, then over the open set $(\operatorname{Proj} R)_g$ of $\mathbb{P} = \operatorname{Proj} R$ the section $g \in \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ generates the sheaf $\mathcal{O}_{\mathbb{P}}(1)$. Since these open sets cover the scheme $\mathbb{P}$, it follows that the collection of global sections R_1 of $\mathcal{O}_{\mathbb{P}}(1)$ generates $\mathcal{O}_{\mathbb{P}}(1)$ everywhere (see Nakayama's lemma Proposition I.5.5), or equivalently that

$$\pi^* R_1 \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$$

is surjective.

From the definition of the inverse image f^*, which is locally given by the tensor product, it follows that the inverse image f^*R_1 generates $f^*\mathcal{O}_{\mathbb{P}}(1)$.

Thus finally, to each morphism $f\colon X \to \operatorname{Proj} R$ we have associated a pair $(\mathcal{L}, \varphi)$ consisting of an invertible sheaf $\mathcal{L}$ on X and a homomorphism

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L})$$

such that $\varphi(R_1)$ generates $\mathcal{L}$, or equivalently, the homomorphism

$$f^* R_1 \longrightarrow \mathcal{L}$$

is surjective.

THEOREM 2.2. *Let $\mathbb{P} = \operatorname{Proj} R$. Assume that $R = \operatorname{Symm}_{R_0}(R_1)$. Let $S = \operatorname{Spec}(R_0)$. Let $p\colon X \to S$ be a scheme over S and let $(\mathcal{L}, \varphi)$ be a pair consisting of an invertible sheaf $\mathcal{L}$ on X and a homomorphism*

$$\varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L})$$

which generates $\mathcal{L}$. Then there exists a unique pair (f, ψ) consisting of a morphism $f\colon X \to \operatorname{Proj} R$ over S and a homomorphism $\psi\colon f^*\mathcal{O}_{\mathbb{P}}(1) \to \mathcal{L}$ making the following diagram commutative:

$$\begin{tikzcd} R_1 \arrow[r, "\varphi"] \arrow[d] & \Gamma(X, \mathcal{L}) \\ \Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \arrow[r, "f^*"'] & \Gamma(X, f^*\mathcal{O}_{\mathbb{P}}(1)) \arrow[u, "\Gamma(\psi)"'] \end{tikzcd}$$

Furthermore, the homomorphism ψ is an isomorphism.

Before giving the proof, we make some comments. An important special case occurs when R_1 is a free module of finite rank $r+1$ over R_0. Then $\mathbb{P} = \mathbb{P}^r_{R_0}$. The R_0-module R_1 then has a basis $T_0, \dots, T_r$. Let $s_0, \dots, s_r$ be sections of $\mathcal{L}$ which generate $\mathcal{L}$. There is a unique homomorphism $\varphi\colon R_1 \to \Gamma(X, \mathcal{L})$ such that $\varphi(T_i) = s_i$. The theorem asserts that there is a unique morphism $f\colon X \to \mathbb{P}^r_{R_0}$ such that $f^*\mathcal{O}_{\mathbb{P}}(1)$ is isomorphic to $\mathcal{L}$, and the sections s_i correspond to f^*T_i under this isomorphism. This is the formulation of the theorem in terms of the homogeneous coordinates $T_0, \dots, T_r$.

The proof of Theorem 2.2 will require some lemmas. We first consider the uniqueness, and for this the hypothesis that $R = \operatorname{Symm}_{R_0}(R_1)$ will *not* be used.

Let s be a section of an invertible sheaf $\mathcal{L}$ over the scheme X. Let s_x be the value of the section in $\mathcal{L}_x$, and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_x$. Then s_x generates L_x if and only if $s_x \notin \mathfrak{m}_x\mathcal{L}_x$.

LEMMA 2.3. *Let $\mathcal{L}$ be an invertible sheaf on the scheme X. Let $s \in \Gamma(X, \mathcal{L})$ be a global section of $\mathcal{L}$. Then the set of points $x \in X$ such that s_x generates $\mathcal{L}_x$ is an open set which we denote by X_s. Multiplication by s, that is,*

$$m_s\colon \mathcal{O}_X|_U \longrightarrow \mathcal{L}|_U$$

is an isomorphism on this open set.

PROOF. We may suppose that $X = \operatorname{Spec}(A)$, and $\mathcal{L} = \mathcal{O}_X$ since the conclusions of the lemma are local. Then $s \in A$. The first assertion is then obvious from the definition of $\operatorname{Spec}(A)$. As to the second, s is a unit in A_s so multiplication by s induces an isomorphism on the sheaf on the open subset $\operatorname{Spec}(A_s)$. This proves the lemma. (No big deal.) □

To show uniqueness, we suppose given the pair

$$f\colon X \longrightarrow \operatorname{Proj} R \quad \text{and} \quad \varphi\colon R_1 \longrightarrow \Gamma(X, \mathcal{L}),$$

and investigate the extent to which f is determined by φ. Note that for all $a \in R_1$ the map f restricts to a morphism

$$X_{\varphi(a)} = f^{-1}((\operatorname{Proj} R)_a) \longrightarrow (\operatorname{Proj} R)_a$$

and

$$(\operatorname{Proj} R)_a = \operatorname{Spec}(R_a)_0.$$

If $b \in R_1$, then the map φ sends b, a to $\varphi(b)$, $\varphi(a)$ respectively, and so

$$f^* \colon \frac{b}{a} \longmapsto \frac{\varphi(b)}{\varphi(a)} = m_a^{-1}(\varphi(b)).$$

But the set of elements b/a with $b \in R_1$ generates $(R_a)_0$. Consequently the ring homomorphism

$$(R_a)_0 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$$

is uniquely determined by φ. This proves the uniqueness.

Next we wish to show existence.

LEMMA 2.4. *Let R be a graded ring generated by R_1 over R_0. Let $a \in R_1$. Then there is a unique (not graded) ring homomorphism*

$$R/(a-1) \xrightarrow{\approx} (R_a)_0$$

such that for $b \in R_1$ we have

$$b \longmapsto \frac{b}{a}$$

PROOF OF LEMMA 2.4. The map $b \mapsto b/a$ defines an additive homomorphism of R_1 into $(R_a)_0$. Consequently, this additive map extends uniquely to a ring homomorphism

$$h \colon R \longrightarrow (R_a)_0,$$

because of the assumption $R = \operatorname{Symm}_{R_0}(R_1)$, and $a - 1$ is in the kernel. Since a becomes invertible under the map $R \to R/(a-1)$, we can factor h as follows:

$$R \longrightarrow R_a \longrightarrow R/(a-1) \longrightarrow (R_a)_0.$$

The first map is the natural map of R into the localization of R by a. Since R_1 generates R, any element of the homogeneous component R_n can be written as a sum of elements in the form $b_1 \cdots b_n$ for some $b_i \in R_1$, so an element of $(R_a)_0$ is a sum of elements of the form

$$\frac{b_1 \cdots b_n}{a^n} = \left(\frac{b_1}{a}\right) \cdots \left(\frac{b_n}{a}\right).$$

Since $(R_a)_0$ is contained in R_a, it follows that the composite map

$$(R_a)_0 \xrightarrow{\text{inclusion}} R_a \longrightarrow R/(a-1) \longrightarrow (R_a)_0$$

is the identity. Furthermore given an element in $R/(a-1)$ represented by a product $b_1 \cdots b_n$ with $b_i \in R_1$, it is the image of an element in $(R_a)_0$ since $a \equiv 1 \mod (a-1)$. Hence the map

$$(R_a)_0 \longrightarrow R/(a-1)$$

is an isomorphism. This concludes the proof of Lemma 2.4. □

We revert to the existence part of Theorem 2.2. Given the data $(\mathcal{L}, \varphi)$ we wish to construct the morphism

$$f \colon X \longrightarrow \operatorname{Proj} R.$$

For each $a \in R_1$ we let $X_{\varphi(a)}$ be the open set of points $x \in X$ such that $\varphi(a)(x) \neq 0$ (we are using Lemma 2.3). Since $\varphi(R_1)$ generates $\mathcal{L}$, it follows that the sets $X_{\varphi(a)}$ cover X for $a \in R_1$. On the other hand,

$$\operatorname{Proj} R = \bigcup_{a \in R_1} \operatorname{Spec}(R_a)_0.$$

It will suffice to construct for each $a \in R_1$ a morphism

$$X_{\varphi(a)} \longrightarrow \operatorname{Spec}(R_a)_0 \subset \operatorname{Proj} R$$

such that this family is compatible on the intersections of the open sets $X_{\varphi(a)}$. The construction is done for the corresponding rings of global sections. By restriction from X to $X_{\varphi(a)}$ the map φ gives rise to a map

$$\varphi_a\colon R_1 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{L}).$$

Composing with the multiplication m_a^{-1} as in Lemma 2.3, we obtain a homomorphism $R_1 \to \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$ as in the following triangle:

$$\begin{tikzcd} R_1 \arrow[rr] \arrow[dr] & & \Gamma(X_{\varphi(a)}, \mathcal{L}) \arrow[dl, "m_a^{-1}"] \\ & \Gamma(X_{\varphi(a)}, \mathcal{O}_X) & \end{tikzcd}$$

But m_a^{-1} sends $\varphi(a)$ to the section represented by 1. By the assumption that $R = \mathrm{Symm}_{R_0}(R_1)$, the additive R_0-homomorphism

$$R_1 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X)$$

induces a ring homomorphism

$$\psi_a\colon R/(a-1) = (R_a)_0 \longrightarrow \Gamma(X_{\varphi(a)}, \mathcal{O}_X).$$

This is the homomorphism of global sections that we wanted. Then ψ_a induces a morphism

$$f_a\colon X_{\varphi(a)} \longrightarrow \mathrm{Spec}(R_a)_0.$$

We now leave to the reader the verification that these morphisms are compatible on the intersections of two open subschemes $X_{\varphi(a)} \cap X_{\varphi(b)}$. From the construction, it is also easy to verify that the morphism

$$f\colon X \longrightarrow \mathrm{Proj}\, R$$

obtained by glueing the morphisms f_a together has the property that

$$f^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{L},$$

and that the original map φ is induced by f^*. This proves the existence.

Finally, the fact that ψ is an isomorphism results from the following lemma.

LEMMA 2.5. *Let $\psi\colon \mathcal{L}' \to \mathcal{L}$ be a surjective homomorphism of invertible sheaves. Then ψ is an isomorphism.*

PROOF. The proof is immediate and will be left to the reader. □

We used the assumption that $R = \mathrm{Symm}_{R_0}(R_1)$ only once in the proof. In important applications, like those in the next section, we deal with a ring R which is not $\mathrm{Symm}_{R_0}(R_1)$, and so we give another stronger version of the result with a weaker, but slightly more complicated hypothesis.

The symmetric algebra had the property that a module homomorphism on R_1 induces a ring homomorphism on R. We need a property similar to this one. We have the graded ring

$$\Gamma_*(\mathcal{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n),$$

where $\mathcal{L}^n = \mathcal{L}^{\otimes n}$ is the tensor product of $\mathcal{L}$ with itself n times. The R_0-homomorphism $\varphi\colon R_1 \to \Gamma(\mathcal{L})$ induces a graded algebra homomorphism

$$\mathrm{Symm}(\varphi)\colon\ \mathrm{Symm}_{R_0}(R_1) \longrightarrow \Gamma_*(\mathcal{L}).$$

We say that $\mathrm{Symm}(\varphi)$ *factors through* R if there is a commutative diagram of graded algebras

$$\begin{tikzcd} \mathrm{Symm}_{R_0}(R_1) \arrow[rr] \arrow[dr] & & \Gamma_*(\mathcal{L}) \\ & R \arrow[ur] & \end{tikzcd}$$

so for each n we have a commutative diagram:

$$\begin{tikzcd} \mathrm{Symm}^n_{R_0}(R_1) \arrow[rr, "\mathrm{Symm}^n(\varphi)"] \arrow[dr] & & \Gamma(\mathcal{L}^n) \\ & R_n \arrow[ur] & \end{tikzcd}$$

THEOREM 2.6. *Theorem 2.2 is valid without change except that instead of assuming* $R = \mathrm{Symm}_{R_0}(R_1)$ *we need only assume that* $\mathrm{Symm}(\varphi)$ *factors through* R.

PROOF. The proof is the same, since the hypothesis that $\mathrm{Symm}(\varphi)$ factors through R can be used instead of $R = \mathrm{Symm}_{R_0}(R_1)$. $\square$

COROLLARY 2.7. *Let* $\mathcal{E}$ *be a locally free sheaf on the scheme* X. *Then sections* $s\colon X \to \mathbb{P}_X(\mathcal{E}) = \mathrm{Proj}_X(\mathrm{Symm}_{\mathcal{O}_X}(\mathcal{E}))$ *are in bijection with surjective homomorphisms*

$$\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

of $\mathcal{E}$ *onto invertible sheaves over* X.

PROOF. Take $X = S$ in Theorem 2.2. $\square$

Let $\mathcal{R}$ be a quasi-coherent graded sheaf of $\mathcal{O}_X$-algebras, and let $\mathbb{P} = \mathrm{Proj}_X \mathcal{R}$. We have a canonical homomorphism

$$\mathcal{R}_1 \longrightarrow \pi_*\mathcal{O}_{\mathbb{P}}(1)$$

or equivalently (cf. Lemma (I.5.11))

$$\pi^*\mathcal{R}_1 \longrightarrow \mathcal{O}_{\mathbb{P}}(1)$$

which is surjective. This leads to the following generalization of Theorem 2.2:

THEOREM 2.8. *Let* $p\colon Z \to X$ *be a scheme over* X *and let* $\mathcal{L}$ *be an invertible sheaf on* Z. *Let*

$$h\colon p^*\mathcal{R}_1 \longrightarrow \mathcal{L}$$

be a surjective homomorphism. Assume in addition that $\mathcal{R} = \mathrm{Symm}_{\mathcal{R}_0}(\mathcal{R}_1)$ *or that* $\mathrm{Symm}(h)$ *factors through* $\mathcal{R}$. *Then there exists a unique pair* (f, ψ) *consisting of a morphism*

$$f\colon Z \longrightarrow \mathrm{Proj}_X(\mathcal{R}) = \mathbb{P}$$

over X *and a homomorphism*

$$\psi\colon f^*\mathcal{O}_{\mathbb{P}}(1) \longrightarrow \mathcal{L}$$

making the following diagram commutative:

$$\begin{tikzcd} f^*\pi^*(\mathcal{R}_1) = p^*(\mathcal{R}_1) \arrow[rr, "f^*(\text{canonical})"] \arrow[dr, "h"'] & & f^*\mathcal{O}_{\mathbb{P}}(1) \arrow[dl, "\psi"] \\ & \mathcal{L} & \end{tikzcd}$$

In other words, $h\colon p^*(\mathcal{R}_1) \to \mathcal{L}$ *is obtained from* $\pi^*(\mathcal{R}_1) \to \mathcal{O}_{\mathbb{P}}(1)$ *by applying* f^* *and composing with* ψ. *Furthermore, this homomorphism* ψ *is an isomorphism.*

3. Blow ups

This section provides examples for Proj of some graded rings, in one of the major contexts of algebraic geometry.

Throughout this section, we let X be a scheme.

Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals of $\mathcal{O}_X$. We may then form the sheaf of graded algebras

$$\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{I}^n$$

where by definition $\mathcal{I}^0 = \mathcal{O}_X$. Then $\mathcal{R}$ satisfies the hypotheses stated at the beginning of §2, so the results of §2 apply to such $\mathcal{R}$. The sheaf of ideals $\mathcal{I}$ defines a closed subscheme Y whose structure sheaf is

$$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}.$$

We define the *blow up of X along Y*, or *with respect to $\mathcal{I}$*, to be:

$$\mathrm{Bl}_Y(X) = \mathrm{Proj}_X \mathcal{R}.$$

Let

$$\pi\colon \mathrm{Bl}_Y(X) \longrightarrow X$$

be the structural morphism.

Let

$$f\colon X' \longrightarrow X$$

be a morphism. Let $\mathcal{I}$ be a sheaf of ideals of $\mathcal{O}_X$. Then we have homomorphism

$$f^*\mathcal{I} \longrightarrow f^*\mathcal{O}_X = \mathcal{O}_{X'}$$

(cf. §I.5). We let

$$f^{-1}(\mathcal{I})\mathcal{O}_{X'} \quad \text{or also} \quad \mathcal{I}\mathcal{O}_{X'}$$

to be the image of this homomorphism. Then $\mathcal{I}\mathcal{O}_{X'}$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_{X'}$.

THEOREM 3.1. *Let $X' = \mathrm{Bl}_Y(X)$ be the blow up of X along Y, where Y is the closed subscheme defined by a sheaf of ideals $\mathcal{I}$, and let $\pi\colon X' \to X$ be the structural morphism.*

i) *The morphism π gives an isomorphism*

$$X' \setminus \pi^{-1}(Y) \xrightarrow{\approx} X \setminus Y.$$

ii) *The inverse image sheaf $\mathcal{I}\mathcal{O}_{X'}$ is invertible, and in fact*

$$\mathcal{I}\mathcal{O}_{X'} = \mathcal{O}_{X'}(1).$$

PROOF. The first assertion is immediate since $\mathcal{I} = \mathcal{O}_X$ on the complement of Y by definition. So if we put $U = X \setminus Y$, then

$$\pi^{-1}(U) = \mathrm{Proj}_U \mathcal{O}_U[T] = U.$$

For (ii), we note that for any affine open set V in X, the sheaf $\mathcal{O}_{X'}(1)$ on $\mathrm{Proj}(\mathcal{R}(V))$ is the sheaf associated to the graded $\mathcal{R}(V)$-module

$$\mathcal{R}(V)(1) = \bigoplus_{n \geq 0} \mathcal{I}^{n+1}(V).$$

But this is equal to the ideal $\mathcal{I}\mathcal{R}(V)$ generated by $\mathcal{I}(V)$ in $\mathcal{R}(V)$. This proves (ii), and the concludes the proof of the theorem. $\square$

THEOREM 3.2 (Universality of Blow-ups). *Let*

$$\pi\colon \mathrm{Bl}_Y(X) \longrightarrow X$$

be the blow up of a sheaf of ideals $\mathcal{I}$ in X. Let

$$f\colon Z \longrightarrow X$$

be a morphism such that $\mathcal{I}\mathcal{O}_Z$ is an invertible sheaf of ideals on Z. Then there exists a unique morphism $f_1\colon Z \to \mathrm{Bl}_Y(X)$ such that the following diagram is commutative.

$$\begin{tikzcd} Z \arrow[rr, "f_1"] \arrow[dr, "f"'] & & \mathrm{Bl}_Y(X) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

PROOF. To construct f_1 we use Theorem 2.8, taking $\mathcal{L} = \mathcal{I}\mathcal{O}_Z$ and h to be the natural map

$$h\colon f^*\mathcal{R}_1 = f^*\mathcal{I} \longrightarrow \mathcal{I}\mathcal{O}_Z = \mathcal{L}.$$

Note that $\mathrm{Symm}(h)$ factors through $\bigoplus \mathcal{L}^n$.

To see that f_1 is unique, take a sufficiently small affine open piece $\mathrm{Spec}(R)$ of Z in which $\mathcal{I}\mathcal{O}_Z$ is $\widetilde{(aR)}$, $a \in \mathcal{I}$. Then a is a non-zero divisor in R by hypothesis. Now $\mathrm{Spec}(R_a)$ lies over $X \setminus Y$, over which π is an isomorphism:

$$\begin{tikzcd} \mathrm{Spec}(R_a) \arrow[dr] & & \mathrm{Bl}_Y(X) \setminus \pi^{-1}(Y) \arrow[dl, "\approx"', "\pi"] \\ & X \setminus Y & \end{tikzcd}$$

Therefore f_1 is unique on $\mathrm{Spec}(R_a)$. But since a is not a zero-divisor, any morphism on $\mathrm{Spec}(R_a)$ has at most one extension to $\mathrm{Spec}(R)$. This is because $R \to R_a$ is injective and hence a homomorphism $S \to R$ is determined by the composition $S \to R_a$. This concludes the proof. $\square$

THEOREM 3.3. *Let Y' be the restriction of $\mathrm{Bl}_Y(X)$ to Y, or in other words*

$$Y' = Y \times_X \mathrm{Bl}_Y(X).$$

Then $Y' = \mathrm{Proj}_Y \mathrm{gr}_{\mathcal{I}}(\mathcal{O}_X)$ where $\mathrm{gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}$. In other words we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Proj}_Y \mathrm{gr}_{\mathcal{I}}(\mathcal{O}_X) = Y' \arrow[r] \arrow[d] & \mathrm{Bl}_Y(X) = \mathrm{Proj}_X(\bigoplus \mathcal{I}^n) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

PROOF. Let $\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^n$ as before. Then $\mathcal{I}\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^{n+1}$, where $\mathcal{I}^{n+1}$ is the n-th graded component, and is a homogeneous ideal sheaf of $\mathcal{R}$. The restriction to Y is given by the graded ring homomorphism

$$\mathcal{R} \longrightarrow \mathcal{R}/\mathcal{I}\mathcal{R},$$

which induces the restriction of $\mathrm{Proj}_X(\mathcal{R})$ to Y. Hence this restriction is equal to $\mathrm{Proj}_Y(\mathcal{R}/\mathcal{I}\mathcal{R})$, viewing $\mathcal{R}/\mathcal{I}\mathcal{R}$ as an $\mathcal{O}_X/\mathcal{I} = \mathcal{O}_Y$-sheaf of graded algebras. But

$$\mathcal{R}/\mathcal{I}\mathcal{R} = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

This proves the theorem. $\square$

In general, nothing much more can be said about the sheaf

$$\mathrm{gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

However, under some conditions, this sheaf is the symmetric algebra of $\mathcal{I}/\mathcal{I}^2$. Assume that A is a noetherian ring and I an ideal of A. We say that a sequence of elements $(a_1, \ldots, a_r)$ is a *regular sequence* in I if a_1 is not a divisor of 0, and if a_{i+1} is not a divizor of 0 in $I/(a_1, \ldots, a_i)$ for all $i \geq 1$.

LEMMA 3.4. *Assume that I is generated by a regular sequence of length r. Then there is a natural isomorphism*

$$\mathrm{Symm}_{A/I}(I/I^2) \approx \bigoplus_{n\geq 0} I^n/I^{n+1}$$

and I/I^2 is free of dimension r over A/I.

PROOF. See Matsumura [**78**, Chapter 6]. □

Now suppose X is a noetherian scheme and $\mathcal{I}$ is a sheaf of ideals as before, defining the subscheme Y. We say that Y is a *local complete intersection in X of codimension* r if each point $y \in Y$ has an affine open neighborhood $\mathrm{Spec}(A)$ in X, such that if I is the ideal corresponding to $\mathcal{I}$ over $\mathrm{Spec}(A)$, then I is generated by a regular sequence of length r. The elementary commutative algebra of regular sequences shows that if this condition is true over $\mathrm{Spec}(A)$, then it is true over $\mathrm{Spec}(A_f)$ for any element $f \in A$. Lemma 3.4 then globalizes to an isomorphism

$$\mathrm{Symm}_Y(\mathcal{I}/\mathcal{I}^2) \approx \mathrm{gr}_{\mathcal{I}}(\mathcal{O}_X) = \bigoplus_{n\geq 0} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Furthermore $\mathcal{I}/\mathcal{I}^2$ is locally free of rank r over $\mathcal{O}_Y$. Therefore we may rephrase Theorem 3.3 as follows:

THEOREM 3.5. *Suppose that Y is a local complete intersection of codimension r in X, and is defined by the sheaf of ideals $\mathcal{I}$. Let Y' be the restriction of $\mathrm{Bl}_Y(X)$ to Y. Then we have a commutative diagram:*

$$\begin{tikzcd} Y' = \mathbb{P}_Y(\mathcal{I}/\mathcal{I}^2) \arrow[r] \arrow[d] & \mathrm{Bl}_Y(X) \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

In particular, if y is a closed local complete intersection point, then

$$\mathbb{P}_y(\mathcal{I}/\mathcal{I}^2) = \mathbb{P}^r_k$$

where k is the residue class field of the point. Thus the fibre of the blow up of such a point is a projective space.

We shall now apply blow ups to resolve indeterminacies of rational maps.

Let X be a noetherian scheme and let $\mathcal{L}$ be an invertible sheaf on X. Let $s_0, \ldots, s_r$ be global sections of $\mathcal{L}$. By Lemma 2.3, the set of points $x \in X$ such that $(s_0)_x, \ldots, (s_r)_x$ generate $\mathcal{L}_x$ is an open set U_s, and these sections generate $\mathcal{L}$ over U_s. Here s denotes the r-tuple

$$s = (s_0, \ldots, s_r).$$

Then s defines a morphism

$$f_s \colon U_s \longrightarrow \mathbb{P}^r_X$$

of U_s into projective r-space over X, in line with Theorem 2.2 and the remarks following it. We shall now define a closed subscheme of X whose support is the complement of U_s, and we shall define a canonical blow up (depending on the given sections) so that the morphism f_s extends to a morphism of this blow up.

Let $s_0, \ldots, s_r$ be sections of $\mathcal{L}$. We shall define an associated sheaf of ideals $\mathcal{I}_s$ as follows. Let U be an open affine set where $\mathcal{L}$ is free, and so

$$\mathcal{L}|_U \approx \mathcal{O}_X|_U.$$

Under this isomorphism, the sections become sections of $\mathcal{O}_X$ over U. We let $\mathcal{I}_U$ be the sheaf of ideals generated by these sections over U. If $U = \operatorname{Spec}(A)$, then the sections can be identified with elements of A, and the ideal corresponding to this sheaf is the ideal $(s_0, \ldots, s_r)$ generated by these elements. It is immediately verified that this ideal is independent of the trivialization of $\mathcal{L}|_U$, and that the sheaf $\mathcal{I}_U$ agrees with the similarly defined sheaf $\mathcal{L}|_V$ on the intersection $U \cap V$ of two affine open sets U and V. This is the *sheaf of ideals* which we call $\mathcal{I}_s$, *determined by* or *associated with the family of sections* s.

Since X is assumed noetherian, $\mathcal{I}_s$ is a coherent sheaf of ideals, or in other words, it is locally finitely generated.

U_s is the open subset of X which is the complement of the support of $\mathcal{O}_X/\mathcal{I}_s$. Thus $\mathcal{I}_s$ defines a closed subscheme Y, and U_s is the complement of Y. We view U_s as a scheme, whose structure sheaf is $\mathcal{O}_X|_{U_s}$.

PROPOSITION 3.6. *Let $s = (s_0, \ldots, s_r)$ be sections of an invertible sheaf $\mathcal{L}$ over X as above. Let $\mathcal{I} = \mathcal{I}_s$ be the associated sheaf of ideals, defining the subscheme Y, and let $\pi\colon X' \to X$ be the blow up of X along Y. Then the sections $\pi^*s_0, \ldots, \pi^*s_r$ generate an invertible subsheaf of $\pi^*\mathcal{L}$, and thus define a morphism*

$$f_{\pi^*s}\colon X' \longrightarrow \mathbb{P}^r_X,$$

such that the following diagram is commutative:

$$\begin{tikzcd} \pi^{-1}(U_s) \arrow[r, "f_{\pi^*s}"] \arrow[d, leftrightarrow, "\text{isomorphism}"'] & \mathbb{P}^r_X \\ U_s \arrow[r, "f_s"'] & \mathbb{P}^r_{U_s} \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

PROOF. By Theorem 3.1 we know that $\mathcal{I}\mathcal{O}_X$ is invertible, and the sections $\pi^*s_0, \ldots, \pi^*s_r$ generate this subsheaf of $\pi^*\mathcal{L}$.

Thus the assertion of the proposition is immediate. $\square$

In this manner, we have a globally defined morphism on the blow up X' which "coincides" with f_s on the open set U_s.

4. Quasi-coherent sheaves on $\operatorname{Proj} R$

Throughout this section we let R be a graded ring, generated by R_1 over R_0. We let $\mathbb{P} = \operatorname{Proj} R$. We assume moreover that R_1 is a finitely generated R_0-module, hence $\mathbb{P}$ is quasi-compact.

The purpose of this section is to classify quasi-coherent sheaves in terms of graded modules on projective schemes in a manner analogous to the classification of quasi-coherent sheaves in terms of ordinary modules over affine schemes. We start with a lemma.

Let $\mathcal{L}$ be an invertible sheaf on a scheme X. Let $f \in \Gamma(X, \mathcal{L})$ be a section. We let:

$$X_f = \text{set of points } x \text{ such that } f(x) \neq 0.$$

We recall that $f(x)$ is the value of f in $\mathcal{L}_x/\mathfrak{m}_x\mathcal{L}_x$, as distinguished from $f_x \in \mathcal{L}_x$.

LEMMA 4.1. *Let $\mathcal{L}$ be an invertible sheaf on the scheme X. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Assume X is quasi-compact.*

i) *Let $s \in \Gamma(X, \mathcal{F})$ be a section whose restriction to X_f is 0. Then for some $n > 0$ we have $f^n s = 0$, where $f^n s \in \Gamma(\mathcal{L}^n \otimes \mathcal{F}) \approx \Gamma(\mathcal{F} \otimes \mathcal{L}^n)$.*

ii) *Suppose X has a finite covering by open affine subsets U_j such that $\mathcal{L}|_{U_j}$ is free for each j. Let $t \in \Gamma(X_f, \mathcal{F})$ be a section over X_f. Then there exists $n > 0$ such that the section $f^n t \in \Gamma(X_f, \mathcal{F} \otimes \mathcal{L}^n)$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^n$ over X.*

PROOF. There is a covering of X by affine open sets on which $\mathcal{L}$ is free, and since X is assumed quasi-compact, we can take this covering to be finite. Hence it suffices to prove that if $U = \operatorname{Spec}(A)$ is affine open such that $\mathcal{L}|_U$ is free, then there is some $n > 0$ such that $f^n s = 0$ on U. But $\mathcal{F}|_U = \widetilde{M}$ with some A-module M by Proposition-Definition I.5.1. Then we can view s as an element of M, and f as an element of A under an isomorphism $\mathcal{L}|_U \approx \mathcal{O}_X|_U$. By definition of the localization, the fact that the restriction of s to X_f is 0 means that s is 0 in M_f, and so there is some n such that $f^n s = 0$. This has an intrinsic meaning in $\mathcal{L}^n \otimes \mathcal{F}$, independently of the choice of trivialization of $\mathcal{L}$ over U, whence (i) follows.

For (ii), let $t \in \Gamma(X_f, \mathcal{F})$. We can cover X by a finite number of affine open $U_i = \operatorname{Spec}(A_i)$ such that $\mathcal{L}|_{U_i}$ is free. On each U_i there is an A_i-module M_i such that $\mathcal{F}|_{U_i} = \widetilde{M_i}$. The restriction of t to $X_f \cap U_i = (U_i)_f$ is in $(M_i)_{f_i}$, where $f_i = f|_{U_i}$ can be viewed as an element of A_i since $\mathcal{L}|_{U_i}$ is free of rank one. By definition of the localization, for each i there is an integer n and a section $t_i \in \Gamma(U_i, \mathcal{F})$ such that the restriction of t_i to $(U_i)_{f_i}$ is equal to $f^n t$ (that is $f^n \otimes t$) over $(U_i)_{f_i}$. Since we are dealing with a finite number of such open sets, we can select n large to work for all i. On $U_i \cap U_j$ the two sections t_i and t_j are defined, and are equal to $f^n t$ when restricted to $X_f \cap U_i \cap U_j$. By the first part of the lemma, there is an integer m such that $f^m(t_i - t_j) = 0$ on $U_i \cap U_j$ for all i, j, again using the fact that there is only a finite number of pairs (i, j). Then the section $f^m t_i \in \Gamma(U_i, \mathcal{L}^m \otimes \mathcal{F})$ define a global section of $\mathcal{L}^m \otimes \mathcal{F}$, whose restriction to X_f is $f^{n+m}t$. This concludes the proof of the lemma. $\square$

We turn to the application in the case of sheaves over $\mathbb{P} = \operatorname{Proj}(R)$. The sheaf $\mathcal{L}$ of Lemma 4.1 will be $\mathcal{O}_{\mathbb{P}}(1)$.

Let M be a graded module over R. Then $\widetilde{M}$ is a sheaf on $\mathbb{P}$. Suppose that N is a graded module such that $N_d = M_d$ for all $d \geq d_0$. Then

$$\widetilde{M} = \widetilde{N}.$$

This is easily seen, because for $f \in R_1$, we know that $\mathbb{P}$ is covered by the affine open sets $\mathbb{P}_f$. Then any section of $\widetilde{M}$ over $\mathbb{P}_f$ can be written in the form x/f^n for some $x \in M_n$, but we can also write such an element in the form

$$\frac{x}{f^n} = \frac{f^m x}{f^{m+n}}$$

so we can use only homogeneous elements of arbitrarily high degree. Hence changing a finite number of graded components in M does not affect M_f, nor $\widetilde{M}$.

If M is finitely generated, it is therefore natural to say that M is *quasi-equal* to N if $M_d = N_d$ for all d sufficiently large. Quasi-equality is an equivalence relation. Two graded homomorphisms

$$f,\, g \colon M \longrightarrow N$$

are called *quasi-equal* if $f_d = g_d$ for all d sufficiently large. ($f_d, g_d\colon M_d \to N_d$ are the restrictions of f, g.) More generally, we define:

$$\operatorname{Hom}_{\mathrm{qe}}(M,N) = \varinjlim_n \operatorname{Hom}(M_{\geq n}, N_{\geq n}),$$

where $M_{\geq n}$ denotes the submodule of M of components of degree $\geq n$. This defines a category which we call the category of *graded modules modulo quasi-equality*, and denote by $\operatorname{GrMod}_{\mathrm{qe}}(R)$.

The association

$$M \longmapsto \widetilde{M} \quad \text{(projective tilde)}$$

is a functor from this category to the category of quasi-coherent sheaves on $\mathbb{P}$.

Our object is now to drive toward Theorem 4.8, which states that under suitable finiteness assumptions, this functor establishes an equivalence of categories. Some of the arguments do not use all the assumptions, so we proceed stepwise. The first thing to show is that every quasi-coherent sheaf is some $\widetilde{M}$. Let $\mathcal{F}$ be quasi-coherent over $\mathbb{P}$. Then in §1 we had defined

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(\mathbb{P}, \mathcal{F}(n)).$$

PROPOSITION 4.2. *Let $\mathcal{F}$ be a quasi-coherent sheaf over $\mathbb{P}$. Let $M = \Gamma_*(\mathcal{F})$. Then $\mathcal{F} \approx \widetilde{M}$.*

PROOF. Let $f \in R_1$. We want to establish an isomorphism

$$(M_f)_0 \xrightarrow{\approx} \mathcal{F}(\mathbb{P}_f).$$

The left hand side is the module of sections of $\widetilde{M}$ over $\mathbb{P}_f$. The compatibility as f varies will be obvious from the definition, and this isomorphism will give the desired isomorphism of $\widetilde{M}$ with $\mathcal{F}$. Multiplication by f gives a homomorphism

$$\mathcal{F}(n) \xrightarrow{f} \mathcal{F}(n+1)$$

whence a corresponding homomorphism on global sections. There is a natural isomorphism

$$(M_f)_0 \approx \varinjlim_n (M_n, f) \approx \varinjlim_n (\Gamma\mathcal{F}(n), f)$$

where the right hand side is the direct limit of the system:

$$M_0 \xrightarrow{f} M_1 \xrightarrow{f} M_2 \xrightarrow{f} \cdots \xrightarrow{f} M_n \xrightarrow{f} \cdots$$

Indeed, an element of $(M_f)_0$ can be represented as a quotient x/f^n with $x \in \Gamma\mathcal{F}(n)$. There is an equality

$$\frac{x}{f^n} = \frac{y}{f^m}$$

with $y \in \Gamma\mathcal{F}(m)$ if and only if there is some power f^d such that

$$f^{d+m}x = f^{d+n}y.$$

This means precisely that an element of $(M_f)_0$ corresponds to an element of the direct limit as stated.

On the other hand, let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$. We have an isomorphism

$$\mathcal{O}|_{\mathbb{P}_f} \underset{\approx}{\xrightarrow{f^n}} \mathcal{O}(n)|_{\mathbb{P}_f}$$

and since $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}(n)$ by definition, we get an isomorphism

$$\mathcal{F}|_{\mathbb{P}_f} \underset{\approx}{\xrightarrow{f^n}} \mathcal{F}(n)|_{\mathbb{P}_f}.$$

Now we look at the directed system and commutative diagrams:

$$\begin{tikzcd}
\Gamma\mathcal{F}(n) \arrow[r, "\mathrm{res}"] & \mathcal{F}(n)(\mathbb{P}_f) & \mathcal{F}(\mathbb{P}_f) \arrow[l, "f^n", "\approx"'] \arrow[dl, "\approx"', "f^{n-1}"] \\
\Gamma\mathcal{F}(n-1) \arrow[u, "f"] \arrow[r, "\mathrm{res}"'] & \mathcal{F}(n-1)(\mathbb{P}_f) \arrow[u, "f"] &
\end{tikzcd}$$

The top row gives a homomorphism

$$f^{-n} \circ \mathrm{res}\colon \Gamma\mathcal{F}(n) \longrightarrow \mathcal{F}(\mathbb{P}_f).$$

The commutativity of the square and triangle induces a homomorphism on the direct limit

$$(M_f)_0 \approx \varinjlim_n (\Gamma\mathcal{F}(n), f) \longrightarrow \mathcal{F}(\mathbb{P}_f).$$

The first part of Lemma 4.1 shows that this map is injective. Using the quasi-compactness of $\mathbb{P}$, the second part shows that this map is surjective, whence the desired isomorphism. We leave to the reader the verification of the compatibility condition as f varies in R_1, to conclude the proof. $\square$

THEOREM 4.3 (Serre). *Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on $\mathbb{P}$. Then there is some n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ is generated by a finite number of global sections.*

PROOF. Let $f_0, \ldots, f_r$ generate R_1 over R_0, and let $\mathbb{P}_i = \mathbb{P}_{f_i}$. For each i there is a finitely generated module M_i over $\mathcal{O}(\mathbb{P}_i)$ such that $\mathcal{F}|_{\mathbb{P}_i} = \widetilde{M_i}$. For each i, let s_{ij} be a finite number of sections in M_i generating M_i over $\mathcal{O}(\mathbb{P}_i)$. By Lemma 4.1 there is an integer n such that for all i, j the sections $f_i^n s_{ij}$ extend to global sections of $\mathcal{F}(n)$. But for fixed i, the global sections $f_i^n s_{ij}$ (j variable) generate M_i over $\mathcal{O}(\mathbb{P}_i)$ since f_i^n is invertible over $\mathcal{O}(\mathbb{P}_i)$. Since the open sets $\mathbb{P}_i$ ($i = 0, \ldots, r$) cover $\mathbb{P}$, this concludes the proof. $\square$

PROPOSITION 4.4. *Let $\mathcal{F}$ be a finitely generated quasi-coherent sheaf on $\mathbb{P}$. Then there is a finitely generated R-submodule N of $\Gamma_*\mathcal{F}$ such that $\mathcal{F} = \widetilde{N}$.*

PROOF. As in Proposition 4.2, let $M = \Gamma_*\mathcal{F}$, so $\widetilde{M} = \mathcal{F}$. By Theorem 4.3, there exists n such that $\mathcal{F}(n)$ is generated by global sections in $\Gamma(\mathbb{P}, \mathcal{F}(n))$. Let N be the R-submodule of M generated by this finite number of global sections. The inclusion $N \hookrightarrow M$ induces an injective homomorphism of sheaves

$$0 \longrightarrow \widetilde{N} \longrightarrow \widetilde{M} = \mathcal{F}$$

whence an injective homomorphism obtained by twisting n times

$$0 \longrightarrow \widetilde{N}(n) \longrightarrow \widetilde{M}(n) = \mathcal{F}(n).$$

This homomorphism is an isomorphism because $\mathcal{F}(n)$ is generated by the global sections in N. Twisting back by $-n$ we get the isomorphism $\widetilde{N} \approx \mathcal{F}$, thereby concluding the proof. $\square$

We have now achieved part of our objective to relate quasi-equal graded modules with coherent sheaves. We proceed to the inverse construction, and we consider the morphisms.

PROPOSITION 4.5. *Assume that M is a finitely presented graded module over R. Let N be a graded module. Then we have an isomorphism*

$$\varinjlim_n \mathrm{Hom}(M_{\geq n}, N_{\geq n}) \xrightarrow{\approx} \mathrm{Hom}(\widetilde{M}, \widetilde{N}).$$

PROOF. Consider a finite presentation

$$R^p \longrightarrow R^q \longrightarrow M \longrightarrow 0.$$

In such a presentation, the homomorphism are not of degree 0, and we rewrite it in the form

$$F \longrightarrow E \longrightarrow M \longrightarrow 0$$

where each of F, E is a direct sum of free graded module of type $R(d)$ with $d \in \mathbb{Z}$. We then obtain an exact and commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(\widetilde{M}, \widetilde{N}) \arrow[r] & \operatorname{Hom}(\widetilde{E}, \widetilde{N}) \arrow[r] & \operatorname{Hom}(\widetilde{F}, \widetilde{N}) \\ 0 \arrow[r] & \varinjlim_n \operatorname{Hom}(M_{\geq n}, N_{\geq n}) \arrow[r] \arrow[u] & \varinjlim_n \operatorname{Hom}(E_{\geq n}, N_{\geq n}) \arrow[r] \arrow[u] & \varinjlim_n \operatorname{Hom}(F_{\geq n}, N_{\geq n}). \arrow[u] \end{tikzcd}$$

It will suffice to prove that the two vertical arrows on the right are isomorphisms. In light of the direct sum structure of E and F, it suffices to prove that

$$\varinjlim_n \operatorname{Hom}(R(d)_{\geq n}, N_{\geq n}) \longrightarrow \operatorname{Hom}(\widetilde{R(d)}, \widetilde{N})$$

is an isomorphism, and twisting by $-d$, it suffices to prove that

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N_{\geq n}) \longrightarrow \operatorname{Hom}(\widetilde{R}, \widetilde{N})$$

is an isomorphism for any graded module N. But $\widetilde{R} = \mathcal{O}_{\mathbb{P}}$ and thus

$$\operatorname{Hom}(\widetilde{R}, \widetilde{N}) = \operatorname{Hom}(\mathcal{O}_{\mathbb{P}}, \widetilde{N}) = \Gamma \widetilde{N}.$$

Thus it suffices to prove the following lemma.

LEMMA 4.6. *Let N be a graded R-module. Then we have an isomorphism*

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N_{\geq n}) \xrightarrow{\sim} \Gamma \widetilde{N}.$$

PROOF OF LEMMA 4.6. Corresponding to a finite set of generators of R_1 over R_0, we have a graded surjective homomorphism

$$R_0[T_0, \dots, T_r] \longrightarrow R_0[R_1] = R \longrightarrow 0,$$

which makes $\mathbb{P} = \operatorname{Proj} R$ into a closed subscheme of $\mathbb{P}^r_A$ where $A = R_0$. We can view the module N as graded module over $\mathbb{P}^r_A$, and the sheaves are sheaves over $\mathbb{P}^r_A$. We also view R as graded module over the polynomial ring $A[T_0, \dots, T_r]$. The relation to be proved is then concerned with objects on $\mathbb{P}^r_A$.

In this notation, the arrow in the lemma is given as follows: For a homomorphism

$$\alpha \colon A[T_0, \dots, T_r]_{\geq n} \to N_{\geq n}$$

of graded $A[T_0, \dots, T_r]$-modules, the global section of $\widetilde{N}$ corresponding to α is given by $\alpha(T^I)/(T^I)$, where $I = (i_0, \dots, i_r)$ is an $(r+1)$-tuple of nonnegative integers with $|I| := i_0 + \dots + i_r \geq n$ and $T^I := T_0^{i_0} \cdots T_r^{i_r}$. In other words, the restriction of this section to the affine open subset $(\mathbb{P}^r_A)_{T^I}$ is $\alpha(T^I)/T^I$, an element of degree 0 in the localization N_{T^I}.

We have to prove the surjectivity and injectivity of the arrow. For surjectivity, let $x \in \Gamma \widetilde{N}$. Let $\mathbb{P}_i$ be the complement of the hyperplane $T_i = 0$ as usual. Then

$$\operatorname{res}_{\mathbb{P}_i}(x) = \frac{x_i}{T_i^n}$$

with n sufficiently large and some $x_i \in N_n$. Increasing n further, we may assume that

$$T_i^n x_j = T_j^n x_i$$

because $x_i/T_i^n = x_j/T_j^n$ in $N_{T_iT_j}$ for all i, j. Therefore there exists a homomorphism of the ideal $(T_0^n, \ldots, T_r^n)$ into N,

$$\varphi\colon (T_0^n, \ldots, T_r^n) \longrightarrow N$$

sending $T_i^n \mapsto x_i$ for each i, and this homomorphism maps on x by the arrow

$$\operatorname{Hom}(R_{\geq m}, N_{\geq m}) \longrightarrow \Gamma\widetilde{N},$$

for m sufficiently large, because $R_{\geq m} \subset (T_0^n, \ldots, T_r^n)$ for m large compared to n. In fact, the ideals $(T_0^n, \ldots, T_r^n)$ are cofinal with the modules $R_{\geq m}$ as m, n tend to infinity. This shows that the map

$$\varinjlim_n \operatorname{Hom}(R_{\geq n}, N_{\geq n}) \longrightarrow \Gamma\widetilde{N}$$

is surjective. The injectivity is proved in the same way. This concludes the proof of the lemma, and also the proof of Proposition 4.5. $\square$

$\square$

The proof of the next proposition relies on the following:

FACT. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P} = \operatorname{Proj} R$ with R_0 noetherian. Then $\Gamma_*\mathcal{F}$ is a finitely presented R-module.*

The proof of this fact will be given as a consequence of theorems in cohomology, by descending induction, and is therefore postponed to Chapter VII (cf. Theorem VII.6.1, which is the fundamental theorem of Serre [**99**], and its proof.)

PROPOSITION 4.7. *Let M be a finitely presented graded module over R with R_0 noetherian. Then the natural map*

$$M \longrightarrow \Gamma_*\widetilde{M}$$

is an isomorphism modulo quasi-equality.

PROOF. By Proposition 4.2 we have an isomorphism

$$\varphi\colon \widetilde{(\Gamma_*\widetilde{M})} \xrightarrow{\approx} \widetilde{M},$$

so by Proposition 4.5, and the "Fact" above:

$$\varphi \in \operatorname{Hom}(\widetilde{(\Gamma_*\widetilde{M})}, \widetilde{M}) \approx \varinjlim_n \operatorname{Hom}((\Gamma_*\widetilde{M})_{\geq n}, M_{\geq n}).$$

Therefore φ comes from a homomorphism

$$h_n\colon (\Gamma_*\widetilde{M})_{\geq n} \longrightarrow M_{\geq n}$$

for n sufficiently large since M is finitely presented over R, that is $\varphi = \widetilde{h_n}$. But since φ is an isomorphism, it follows from applying Proposition 4.5 to φ^{-1} that h_n has to be an isomorphism for n large. This concludes the proof. $\square$

We can now put together Propositions 4.2 and 4.7 to obtain the goal of this section.

THEOREM 4.8. *If R_0 is noetherian, then the association*

$$M \longmapsto \widetilde{M}$$

is an equivalence of categories between finitely presented graded modules over R modulo quasi-equality and coherent sheaves on $\mathbb{P}$. The inverse functor is given by

$$\mathcal{F} \longmapsto \Gamma_* \mathcal{F}.$$

This theorem now allows us to handle sheaves like graded modules over R. For example we have the immediate application:

COROLLARY 4.9. *Let $\mathcal{F}$ be a coherent sheaf on* Proj R *with R_0 noetherian. Then there exists a presentation*

$$\mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}$ is a finite direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$ with $d \in \mathbb{Z}$.

PROOF. The corresponding assertion is true for graded modules, represented as quotients of finite direct sums of modules $R(d)$ with $d \in \mathbb{Z}$. Taking the tilde gives the result for coherent sheaves. □

5. Ample invertible sheaves

There will be two notions of ampleness, one absolute and the other relative. We start with the absolute notion. For simplicity, we develop the theory only in the notherian case.

DEFINITION 5.1. Let X be a noetherian scheme. An invertible sheaf $\mathcal{L}$ on X is called *ample* if for all coherent sheaves $\mathcal{F}$ on X there exists n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections if $n \geq n_0$.

EXAMPLE. Serre's Theorem 4.3 gives the fundamental example of an ample $\mathcal{L}$, namely $\mathcal{O}_{\mathbb{P}}(1)$ where $\mathbb{P} = \operatorname{Proj} R$ with R noetherian.

It is obvious that if $\mathcal{L}$ is ample, then $\mathcal{L}^m$ is ample for any positive integer m. It is convenient to have a converse version of this fact.

LEMMA 5.2. *If $\mathcal{L}^m$ is ample for some positive integer m, then $\mathcal{L}$ is ample.*

PROOF. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F} \otimes \mathcal{L}^{mn}$ is generated by global sections for all $n \geq n_0$. Furthermore, for each $i = 0, \dots, m-1$ the sheaf

$$\mathcal{F} \otimes \mathcal{L}^i \otimes \mathcal{L}^{mn}$$

is generated by global sections for $n \geq n_i$. We let N be the maximum of $n_0, \dots, n_{m-1}$. Then $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for $n \geq N$, thus proving the lemma. □

DEFINITION 5.3. Let $\varphi \colon X \to Y$ be a morphism of finite type over a noetherian base Y. Let $\mathcal{L}$ be an invertible sheaf on X. We say that $\mathcal{L}$ is *relatively very ample with respect to* φ, or *φ-relatively very ample*, if there exists a coherent sheaf $\mathcal{F}$ on Y and an immersion (not necessarily closed)

$$\iota \colon X \longrightarrow \mathbb{P}_Y(\mathcal{F})$$

over Y, i.e., making the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & \mathbb{P}_Y(\mathcal{F}) \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

such that $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$. We say that $\mathcal{L}$ is *relatively ample* if for some $n \geq 1$, $\mathcal{L}^{\otimes n}$ is relatively very ample.

The definition is adjusted to be able to deal with a wide assortment of base scheme Y. However, when $Y = \mathrm{Spec}(A)$ is affine, then it turns out that one can replace $\mathbb{P}_Y(\mathcal{F})$ by $\mathbb{P}^r_A$ for some r, as in the following theorem. Observe that in the affine case, we have

$$\mathbb{P}^r_A = \mathbb{P}_Y(\mathcal{F}) \quad \text{with} \quad \mathcal{F} = \mathcal{O}_Y^{\oplus(r+1)}.$$

THEOREM 5.4. *Let X be a scheme of finite type over a noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}$ is relatively ample over* $\mathrm{Spec}(A)$. *Moreover, when this holds the immersion $\iota\colon X \to \mathbb{P}_A(\mathcal{F})$ such that $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ can be taken into projective space $\mathbb{P}^r_A$.*

REMARK. Serre's cohomological criterion for ampleness will be given in Theorem VII.8.2.

PROOF. Suppose that there is an immersion $\iota\colon X \to \mathbb{P}^r_A$. The only problem to show that $\mathcal{L}$ is ample is that X need not be closed in $\mathbb{P}^r_A$, because if X is closed then we can apply Theorem 4.3. The next result is designed to take care of this problem.

PROPOSITION 5.5. *Let $\mathcal{F}$ be a quasi-coherent sheaf on a noetherian scheme X. Let U be an open subscheme of X, and let $\mathcal{G}_U$ be a coherent subsheaf of $\mathcal{F}|_U$. Then there exists a coherent subsheaf $\mathcal{G}$ of $\mathcal{F}$ on X such that*

$$\mathcal{G}|_U = \mathcal{G}_U.$$

PROOF. Consider all pairs $(\mathcal{G}, W)$ consisting of an open subscheme W of X and a coherent subsheaf $\mathcal{G}$ of $\mathcal{F}|_W$ extending $(\mathcal{G}_U, U)$. Such pairs are partially ordered by inclusion of W's and are in fact inductively ordered because the notion of a coherent sheaf is local, so the usual union over a totally ordered subfamily gives a pair dominating every element of the family. By Zorn's lemma, there exists a maximal element, say $(\mathcal{G}, W)$. We reduce the proposition to the affine case as follows. If $W \neq X$, then there is an affine open subscheme $V = \mathrm{Spec}(A)$ in X such that $V \not\subset W$. Then $W \cap V$ is an open subscheme of V, and if we have the proposition in the affine case, then we extend $\mathcal{G}$ from $W \cap V$ to V, thus extending $\mathcal{G}$ to a larger subscheme than W, contradicting the maximality.

We now prove the proposition when X is affine. In that case, we note that the coherent subsheaves of $\mathcal{G}_U$ satisfy the ascending chain condition. We let $\mathcal{G}_1$ be a maximal coherent subsheaf which admits a coherent extension $\mathcal{G}$ which is a subsheaf of $\mathcal{F}$. We want to prove that $\mathcal{G}_1 = \mathcal{G}_U$. If $\mathcal{G}_1 \neq \mathcal{G}_U$ then there exists an affine open $X_f \subset U$ and a section $s \in \mathcal{G}_U(X_f)$ such that $s \notin \mathcal{G}_1(X_f)$. By Lemma 4.1 (ii), there exists n such that $f^n s$ extends to a section $s' \in \mathcal{F}(X)$ and the restriction of s' to U is in $\mathcal{F}(U)$. By Lemma 4.1 (i) there exists a still higher power f^m such that

$$f^m(s'|_U) = 0 \quad \text{in} \quad (\mathcal{F}/\mathcal{G})(U).$$

Then $\mathcal{G}_1 + f^m s'\mathcal{O}_X$ is a coherent subsheaf of $\mathcal{F}$ which is bigger than $\mathcal{G}_1$, contradiction. This concludes the proof of the proposition. □

COROLLARY 5.6. *Let X be a noetherian scheme. Let U be an open subscheme, and let $\mathcal{G}$ be a coherent sheaf on U. Then $\mathcal{G}$ has a coherent extension to X, and this coherent extension may be taken as a subsheaf of $\iota_*\mathcal{G}$, where $\iota\colon U \to X$ is the open immersion.*

PROOF. By Proposition II.4.10 we know that $\iota_*\mathcal{G}$ is quasi-coherent, and so we can apply Proposition 5.5 to finish the proof. □

We can now finish one implication in Theorem 5.4. Assuming that we have the projective immersion $\iota\colon X \to \mathbb{P}^r_A$, we consider the closure $\overline{X}$ and apply Theorem 4.3 to an extension $\overline{\mathcal{F}}$ of a coherent sheaf $\mathcal{F}$ on X. Then $\overline{\mathcal{F}} \otimes \mathcal{O}_{\overline{X}}(n)$ is generated by global sections for $n \geq n_0$, and the restrictions of these sections to $\mathcal{F}$ generate $\mathcal{F}$, thus concluding the proof of one half of the theorem.

To prove the converse, we need a lemma.

LEMMA 5.7. *Let $\mathcal{L}$ be an ample sheaf on a noetherian scheme X. Then there exists an open affine covering of X by subschemes defined by the property $s(x) \neq 0$, for some global section s of $\mathcal{L}^n$, some n.*

PROOF. Given a point $x \in X$, there is an open affine neighborhood U of x such that $\mathcal{L}|_U$ is free. Let $Y = X \setminus U$ be the complement of U, with the reduced scheme structure, so that Y is a closed subscheme, defined by a sheaf of ideals $\mathcal{I}_Y$, which is coherent on X. There exists n such that $\mathcal{I}_Y \otimes \mathcal{L}^n$ is generated by global sections, and in particular, there is a section s of $\mathcal{I}_Y \otimes \mathcal{L}^n$ such that $s(x) \neq 0$, or equivalently, $s_x \notin \mathfrak{m}_x(\mathcal{I}_Y \otimes \mathcal{L}^n)_x$. Since $\mathcal{L}^n$ is free, we can view $\mathcal{I}_Y \otimes \mathcal{L}^n$ as a subsheaf of $\mathcal{L}^n$. Then by Lemma 2.3 the set X_s of points z such that $s(z) \neq 0$ is open and is contained in U because $s(y) \in \mathfrak{m}_y\mathcal{L}^n_y$ for $y \in Y$. The section s restricted to U can be viewed as an element of $\mathcal{L}^n(U)$, and since $\mathcal{L}$, so $\mathcal{L}^n$, are free over U, it follows that s corresponds to a section f of $\mathcal{O}_U$ and that $X_s = U_f$ so X_s is affine. $\square$

Thus we have proved that for each point $x \in X$ there is an affine open neighborhood X_s defined by a global section s of $\mathcal{L}^{n(x)}$ such that $s(x) \neq 0$. Since X is quasi-compact, we can cover X by a finite number of such affine open sets, and we let m to be the least common multiple of the finite number of exponents $n(x)$.

Since we wish to prove that $\mathcal{L}^n$ is very ample for sufficiently large n, we may now replace $\mathcal{L}$ by $\mathcal{L}^m$ without loss of generality. We are then in the situation when we have a finite number of global sections $s_1, \dots, s_r$ of $\mathcal{L}$ which generate $\mathcal{L}$, such that X_{s_i} is affine for all i, and such that the open sets X_{s_i} cover X. We abbreviate X_{s_i} by X_i.

Let B_i be the affine algebra of X_i over A. By assumption X is of finite type over A, so B_i is finitely generated as A-algebra, say by elements b_{ij}. By Lemma 4.1 there exists an integer N such that for all i, j the section $s_i^N b_{ij}$ extends to a global section t_{ij} of $\mathcal{L}^N$. The family of sections s_i^N, t_{ij} for all i, j generates $\mathcal{L}^N$ since already the sections $s_1^N, \dots, s_r^N$ generate $\mathcal{L}^N$, and hence they define a morphism

$$\psi\colon X \longrightarrow \mathbb{P}^M_A$$

for some integer M. It will now suffice to prove that ψ is a closed immersion. Let T_i, T_{ij} be the homogeneous coordinates of $\mathbb{P}^M_A$, and put $\mathbb{P} = \mathbb{P}^M_A$ for simplicity. If $\mathbb{P}_i$ is the complement of the hyperplane $T_i = 0$ then $X_i = \psi^{-1}(\mathbb{P}_i)$. The morphism induces a morphism

$$\psi_i\colon X_i \longrightarrow \mathbb{P}_i$$

which corresponds to a homomorphism of the corresponding affine algebras

$$A[z_k, z_{kj}] \longrightarrow B_i,$$

where z_k, z_{kj} are the affine coordinates: $z_k = T_k/T_i$ and $z_{kj} = T_{kj}/T_i$. We see that z_{ij} maps on $t_{ij}/s_i^N = b_{ij}$ so the affine algebra homomorphism is surjective. This means that ψ_i is a closed immersion of X_i in $\mathbb{P}_i$. Since X is covered by the finite number of affine open sets $X_1, \dots, X_r$ it follows by Corollary II.3.5 that ψ itself is a closed immersion. This concludes the proof of Theorem 5.4. $\square$

Next we want to investigate the analogous situation when the base Y is not affine.

PROPOSITION 5.8. *Let U be open in X and $\mathcal{L}$ ample on X. Then $\mathcal{L}|_U$ is ample on U.*

PROOF. By Corollary 5.6, a coherent sheaf $\mathcal{F}$ on U has an extension to a coherent sheaf on X. Global sections which generate this extension restrict to sections of $\mathcal{F}$ on U which generate $\mathcal{F}$ on U, so the proposition is immediate. □

Now comes the globalized version of Theorem 5.4.

THEOREM 5.9. *Let $\varphi\colon X \to Y$ be of finite type with X, Y noetherian. The following conditions are equivalent.*

i) *There exists a positive integer n such that $\mathcal{L}^n$ is relatively very ample for φ.*
ii) *There exists an open affine covering $\{V_i\}$ of Y such that $\mathcal{L}|_{\varphi^{-1}V_i}$ is ample for all i.*
iii) *For all affine open subsets V of Y the restriction $\mathcal{L}|_{\varphi^{-1}V}$ is ample.*

PROOF. The implication (iii) $\implies$ (ii) is trivial and (i) implies (iii) follows immediately from Theorem 5.4.

We must show that (ii) implies (i). We have done this when the base Y is affine in Theorem 5.4, and we must globalize the construction. When Y is affine, we could take the immersion of X into a projective space, but now we must use $\mathbb{P}_Y(\mathcal{F})$ with some sheaf $\mathcal{F}$ which need not be locally free.

Applying Theorem 5.4 to $\mathcal{L}|_{\varphi^{-1}V_i}$, we get coherent sheaves $\mathcal{F}_i$ on V_i and immersions ψ_i

$$\begin{tikzcd} \varphi^{-1}(V_i) \arrow[rr, "\psi_i"] \arrow[dr, "\operatorname{res}\varphi"'] & & \mathbb{P}_{V_i}(\mathcal{F}_i) \arrow[dl] \\ & V_i & \end{tikzcd}$$

satisfying $\psi_i^*(\mathcal{O}(1)) \approx \mathcal{L}^{n_i}|_{\varphi^{-1}V_i}$. We first make two reductions. First of all, we may assume the n_i are equal because if $n = \mathrm{l.c.m}(n_i)$ and $m_i = n/n_i$ then

$$\begin{aligned} \mathbb{P}_{V_i(\mathcal{F}_i)} &= \operatorname{Proj}_{V_i}(\operatorname{Symm}(\mathcal{F}_i)) \\ &\approx \operatorname{Proj}_{V_i}\left(\bigoplus_k \operatorname{Symm}^{m_i k}(\mathcal{F}_i)\right) \\ &= \operatorname{Proj}_{V_i}(\operatorname{Symm}(\operatorname{Symm}^{m_i}(\mathcal{F}_i))/I_i) \quad \text{for some ideal } I_i \\ &\subset \mathbb{P}_{V_i}(\operatorname{Symm}^{m_i}(\mathcal{F}_i)). \end{aligned}$$

Replacing $\mathcal{F}_i$ by $\operatorname{Symm}^{m_i}(\mathcal{F}_i)$, we find $\psi_i^*(\mathcal{O}(1)) \approx \mathcal{L}^n|_{\varphi^{-1}V_i}$ for the new ψ_i.

Secondly, ψ_i gives us the canonical surjective homomorphisms

$$\alpha_i\colon (\operatorname{res}\varphi)^*(\mathcal{F}_i) \twoheadrightarrow \mathcal{L}^n|_{\varphi^{-1}V_i}$$

hence

$$\beta_i\colon \mathcal{F}_i \longrightarrow (\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i}) \quad \text{(cf. (I.5.11))}.$$

We may assume that β_i is injective. In fact, let $\mathcal{F}_i'$ be the image of $\mathcal{F}_i$ in $(\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i})$. Then $\mathcal{F}_i'$ is still coherent because $(\operatorname{res}\varphi)_*(\mathcal{L}^n|_{\varphi^{-1}V_i})$ is quasi-coherent (cf. Proposition II.4.10), and the morphism ψ_i factors

$$\varphi^{-1}(V_i) \to \mathbb{P}_{V_i}(\mathcal{F}_i') \hookrightarrow \mathbb{P}_{V_i}(\mathcal{F}_i).$$

We now apply Corollary 5.6 to choose a coherent subsheaf $\mathcal{G}_i \subset \varphi_*\mathcal{L}^n$ such that $\mathcal{G}_i|_{V_i} \approx \mathcal{F}_i$. Now the homomorphism

$$\beta\colon \bigoplus \mathcal{G}_i \longrightarrow \varphi_*\mathcal{L}^n$$

defines

$$\alpha\colon \varphi^*\left(\bigoplus \mathcal{G}_i\right) \longrightarrow \mathcal{L}^n$$

(cf. (I.5.11)) and α is surjective because on each V_i,

$$\varphi^*\mathcal{G}_i|_{V_i} \longrightarrow \mathcal{L}^n|_{V_i}$$

is surjective. By the universal mapping property of $\mathbb{P}_Y$, $\mathcal{L}^n$ and α define a morphism:

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & \mathbb{P}_Y\left(\bigoplus \mathcal{G}_i\right) \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

I claim this is an immersion. In fact, restrict the morphisms to $\varphi^{-1}(V_i)$. The functoriality of Proj (cf. §II.5, Remark h)) plus the homomorphism

$$\mathrm{Symm}(\mathcal{G}_i|_{V_i}) \hookrightarrow \mathrm{Symm}(\bigoplus \mathcal{G}_j|_{V_i})$$

gives us an open set $W_i \subset \mathbb{P}_Y(\bigoplus \mathcal{G}_j)$ and a "projection" morphism:

$$\begin{tikzcd} W_i \arrow[d] & \subset \; \mathbb{P}_Y(\bigoplus \mathcal{G}_j) \\ \mathbb{P}_Y(\mathcal{G}_i) & \end{tikzcd}$$

It is not hard to verify that $\psi(\varphi^{-1}(V_i)) \subset W_i$, and that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\psi"] & \mathbb{P}_Y(\bigoplus \mathcal{G}_j) \\ \varphi^{-1}(V_i) \arrow[u, phantom, "\cup"] \arrow[r, "\mathrm{res}\,\varphi"] \arrow[ddr, "\psi_i"'] & W_i \cap \pi^{-1}(V_i) \arrow[u, phantom, "\cup"] \arrow[d] \\ & \mathbb{P}_{V_i}(\mathcal{G}_i|_{V_i}) \arrow[d, "\approx"] \\ & \mathbb{P}_{V_i}(\mathcal{F}_i) \end{tikzcd}$$

Since ψ_i is an immersion, so is $\mathrm{res}\,\psi$ (cf. Proposition II.3.14), and since this holds for all i, it follows that ψ is an immersion. $\square$

A final result explains further why relatively ample is the relative version of the concept ample.

THEOREM 5.10. *Let $f\colon X \to Y$ be of finite type with X, Y noetherian. Let $\mathcal{L}$ be relatively ample on X with respect to f, and $\mathcal{M}$ ample on Y. Then $\mathcal{L} \otimes f^*\mathcal{M}^k$ is ample on X for all k sufficiently large.*

PROOF. The first step is to fix a coherent sheaf $\mathcal{F}$ on X and to show that for all n_1 sufficiently large, there exists n_2 such that

$$\mathcal{F} \otimes \mathcal{L}^{n_1} \otimes f^*\mathcal{M}^{n_2}$$

is generated by global sections. This goes as follows: because $\mathcal{M}$ is ample, Y can be covered by affine open sets Y_{s_i}, with $s_i \in \Gamma(Y, \mathcal{M}^{m_1})$ for suitable m_1 by Lemma 5.7. Then $\mathcal{L}|_{f^{-1}(Y_{s_i})}$ is ample by Theorem 5.4. Thus $\mathcal{F} \otimes \mathcal{L}^{n_1}|_{f^{-1}(Y_{s_i})}$ is generated by sections $t_{i1}, \ldots, t_{iN}$ if n_1 is sufficiently large. But by Lemma 4.1, for large m_2 all the sections

$$s_i^{m_2} t_{ij}$$

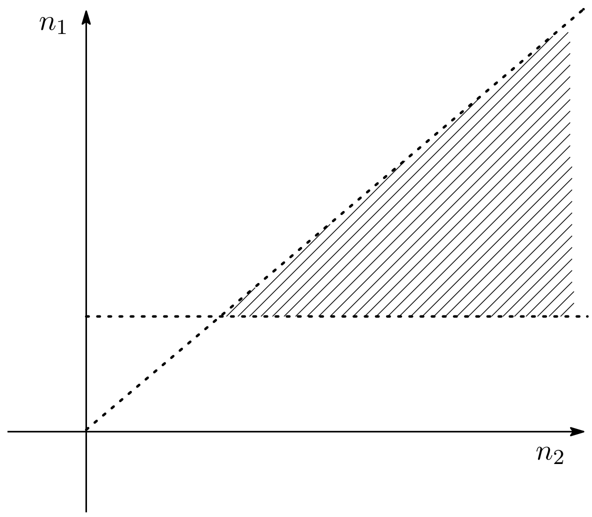

FIGURE III.1

extend from X_{s_i} to X as sections of $\mathcal{F}\otimes\mathcal{L}^{n_1}\otimes f^*(\mathcal{M}^{m_1m_2})$. Let $n_2 = m_1m_2$. Then this collection of global sections generates

$$\mathcal{F}\otimes\mathcal{L}^{n_1}\otimes f^*(\mathcal{M}^{n_2}).$$

There remains to "rearrange the order of the quantifiers", i.e., to pick an upper bound of n_2/n_1 independent of $\mathcal{F}$. The simplest way to do this is to consider the set:

$$S = \{(n_1, n_2) \mid \mathcal{L}^{n_1}\otimes f^*(\mathcal{M}^{n_2}) \text{ is generated by global sections}\}.$$

Note that:

(a) S is a semi-group;
(b) $S \supset (0)\times(n_0+\mathbb{N})$ for some n_0 because $\mathcal{M}$ is ample on Y ($\mathbb{N}$ is the set of positive integers);
(c) there exists n_0' such that if $n_1 \geq n_0'$ then

$$(n_1, n_2) \in S \quad \text{for some } n_2.$$

For this last part, apply Step I with $\mathcal{F} = \mathcal{O}_X$.

A little juggling will convince you that such an S must satisfy

$$S \supset \{(n_1, n_2) \mid n_2 \geq k_0 n_1 \geq n_0\}$$

for suitable k_0, n_0 (see Figure III.1). Now take any $k > k_0$ (strictly greater). Then I claim $\mathcal{L}\otimes f^*\mathcal{M}^k$ is ample. In fact, for any $\mathcal{F}$,

$$\mathcal{F}\otimes\mathcal{L}^{n_1}\otimes f^*\mathcal{M}^{n_2}$$

is generated by its sections for some n_1, n_2. Then so is

$$\mathcal{F}\otimes\mathcal{L}^{n_1+n_1'}\otimes f^*\mathcal{M}^{n_2+n_2'} \quad \text{if } (n_1', n_2') \in S.$$

But $(n, nk) - (n_1, n_2) \in S$ if $n \gg 0$, so we are OK. This concludes the proof of the theorem. $\square$

6. Invertible sheaves via cocycles, divisors, line bundles

There is a natural correspondence between the four objects occurring in the title of this section. We have already met the invertible sheaves. We shall define the other three and establish this correspondence. We then relate these to Weil divisors.

Basic to all the constructions is the following definition. Let X be a scheme. We define the subsheaf of *units* $\mathcal{O}_X^*$ of $\mathcal{O}_X$ to be the sheaf such that for any open U we have

$$\begin{aligned}\mathcal{O}_X^*(U) = \mathcal{O}_X(U)^* &= \text{units in } \mathcal{O}_X(U)\\ &= \{f \in \mathcal{O}_X(U) \text{ such that } f(x) \neq 0 \text{ for all } x \in U\}.\end{aligned}$$

1-cocycles of units. Let X be a scheme and let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_X$-modules or as we also say, an invertible sheaf over X. Let $\{U_i\} = \mathcal{U}$ be an open covering such that the restriction $\mathcal{L}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$ for each i. Thus we have isomorphisms

$$\varphi_i \colon \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}.$$

It follows that

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \colon \mathcal{O}_X|_{(U_i \cap U_j)} \longrightarrow \mathcal{O}_X|_{(U_i \cap U_j)}$$

is an automorphism, which is $\mathcal{O}_X$-linear, and so is given by multiplication with a unit in $\mathcal{O}_X(U_i \cap U_j)^*$. We may therefore identify φ_{ij} with such a unit. The family of such units $\{\varphi_{ij}\}$ satisfies the condition

$$\varphi_{ij}\varphi_{jk} = \varphi_{ik}.$$

A family of units satisfying this condition is called a 1-*cocycle*. The group of these is denoted $Z^1(\mathcal{U}, \mathcal{O}_X^*)$. By a *coboundary* we mean a cocycle which can be written in the form $f_i f_j^{-1}$, where $f_i \in \mathcal{O}_X(U_i)^*$. These form a subgroup of $Z^1(\mathcal{U}, \mathcal{O}_X^*)$ written $B^1(\mathcal{U}, \mathcal{O}_X^*)$. The factor group $Z^1(\mathcal{U}, \mathcal{O}_X^*)/B^1(\mathcal{U}, \mathcal{O}_X^*)$ is called $H^1(\mathcal{U}, \mathcal{O}_X^*)$. If $\mathcal{U}'$ is a refinement of $\mathcal{U}$, i.e., for each $U_i' \in \mathcal{U}'$, there is a $U_j \in \mathcal{U}$ such that $U_i' \subset U_j$, then there is a natural homomorphism

$$H^1(\mathcal{U}, \mathcal{O}_X^*) \longrightarrow H^1(\mathcal{U}', \mathcal{O}_X^*),$$

(for details, see §VII.1). The direct limit taken over all open coverings $\mathcal{U}$ is called the *first Čech cohomology group* $H^1(X, \mathcal{O}_X^*)$.

Suppose

$$f \colon \mathcal{L} \longrightarrow \mathcal{M}$$

is an isomorphism of invertible sheaves. We can find a covering $\mathcal{U}$ by open sets such that on each U_i of $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{M}$ are free. Then f is represented by an isomorphism

$$f_i \colon \mathcal{O}_X|_{U_i} \longrightarrow \mathcal{O}_X|_{U_i}$$

which can be identified with an element of $\mathcal{O}_X(U_i)^*$. We then see that the cocycles φ_{ij} and φ'_{ij} associated to $\mathcal{L}$ and $\mathcal{M}$ with respect to this covering differ by multiplication by $f_i f_j^{-1}$. This yields a homomorphism (cf. Definition 1.2)

$$\operatorname{Pic}(X) \longrightarrow H^1(X, \mathcal{O}_X^*).$$

Proposition 6.1. *This map* $\operatorname{Pic}(X) \to H^1(X, \mathcal{O}_X^*)$ *is an isomorphism.*

Proof. The map is injective, for if two cocycles associated with $\mathcal{L}$, $\mathcal{M}$ give the same element in $H^1(X, \mathcal{O}_X^*)$, then the quotient of these cocycles is a coboundary which can be used to define an isomorphism between the invertible sheaves. Conversely, given a cocycle $\varphi_{ij} \in Z^1(\mathcal{U}, \mathcal{O}_X^*)$ it constitutes glueing data in the sense of §I.5 and there exists a unique sheaf $\mathcal{L}$ which corresponds to this glueing data. $\square$

Cartier divisors. Let X be a scheme. Let $U = \operatorname{Spec}(A)$ be an open affine subset of X. Let S be the multiplicative subset of elements of A which are not zero-divisors, and let $K(U) = S^{-1}A$ be the localization of A with this subset. We call $K(U)$, also denoted by $K(A)$, the *total quotient ring* of A. If A has no divisors of 0, then $K(A)$ is the usual quotient field.

The association $U \mapsto K(U)$ defines a presheaf, whose associated sheaf is the sheaf of total quotient rings of $\mathcal{O}_X$, and is denoted by $\mathcal{K}_X$. If X is integral, then all the rings $\mathcal{O}_X(U)$ for affine open U can be identified as subrings of the same quotient field K and $\mathcal{K}_X$ is the constant sheaf with global sections K. ($K = \mathbf{R}(X)$, the function field of X, in the notation of Proposition II.2.5.)

We now consider pairs (U, f) consisting of an open set U and an element $f \in \mathcal{K}^*(U)$, where $\mathcal{K}^*(U)$ is the group of invertible elements of $\mathcal{K}(U)$. We say that two such pairs (U, f) and (V, g) are *compatible* if $fg^{-1} \in \mathcal{O}(U \cap V)^*$, that is, fg^{-1} is a unit in the sheaf of rings over $U \cap V$. Let $\{(U_i, f_i)\}$ be a family of compatible pairs such that the open sets U_i cover X. Two such families are called *compatible* if each pair from one is compatible with all the pairs from the other. A compatibility class of such covering families is a *Cartier divisor* D. As usual, we can say that a Cartier divisor is a maximal family of compatible pairs, covering X. If $f \in \mathcal{K}^*(U)$ and (U, f) belongs to the compatibility class, then we say that the divisor is *represented* by f over U, and we write $D|_U = (f)$. We also say that $f = 0$ is a *local equation* for D over U.

This amounts to saying that a Cartier divisor is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. We can define the *support* of a Cartier divisor D, and denote by $\operatorname{Supp}(D)$, the set of points x such that if D is represented by (U, f) on an open neighborhood of x, then $f \notin \mathcal{O}_x^*$. It is easy to see that the support of D is closed.

A Cartier divisor is called *principal* if there exists an element $f \in \Gamma(X, \mathcal{K}^*)$ such that for every open set U, the pair (U, f) represents the divisor. We write (f) for this principal divisor.

Let D, E be Cartier divisors. Then there exists a unique Cartier divisor $D + E$ having the following property. If (U, f) represents D and (U, g) represents E, then (U, fg) represents $D + E$. This is immediate, and one then sees that Cartier divisors form a group $\operatorname{Div}(X)$ having the principal divisors as subgroup. The group is written additively, so $-D$ is represented by (U, f^{-1}). We can take f^{-1} since $f \in \mathcal{K}^*(U)$ by definition.

We introduce a partial ordering in the group of divisors. We say that a divisor D is *effective* if for every representative (U, f) of the divisor, the function f is a morphism on U, that is, $f \in \mathcal{O}_X(U)$. The set of effective divisors is closed under addition. We write $D \geq 0$ if D is effective, and $D \geq E$ if $D - E$ is effective. *Note*: although sometimes one also calls D positive, there are other positive cones which can be introduced in the group of divisors, such as the ample cone. The word "positive" is usually reserved for these other cones.

REMARK. It may be that the function f is not on $\mathcal{O}_X(U)$ but is integral over $\mathcal{O}_X(U)$. Thus the function f may be finite over a point, without being a morphism. If X is integral, and all the local rings $\mathcal{O}_x$ for $x \in X$ are integrally closed, then this cannot happen. See below, where we discuss divisors in this context. In this case, the support of D turns out to be the union of the codimension one subschemes where the representative function f has a zero or a pole. This difference in behavior is one of the main differences between Cartier divisors and the other divisors discussed below.

Let D be an effective Cartier divisor. If (U, f) is a representative of D, then f generates a principal ideal in $\mathcal{O}_X(U)$, and this ideal does not depend on the choice of f. In this way we can define a sheaf of ideals, denoted by $\mathcal{I}_D$. It defines a closed subscheme, which is often identified with D.

Two Cartier divisors D, E are called *linearly equivalent*, and we write $D \sim E$, if there exists $f \in \Gamma(X, \mathcal{K}_X^*)$ such that

$$D = E + (f).$$

In other words, $D - E$ is principal. We define the group of *divisor classes*

$$\mathrm{DivCl}(X) = \mathrm{Div}(X)/\mathcal{K}_X^*(X)$$

to be the factor group of Cartier divisors mod principal divisors.

To each Cartier divisor D we shall now associate an invertible sheaf $\mathcal{O}_X(D) = \mathcal{O}(D)$ as follows. If $\{(U_i, f_i)\}$ is a covering family of pairs representing D, then there is a unique subsheaf $\mathcal{L}$ of $\mathcal{K}_X$ such that

$$\mathcal{L}(U_i) = \mathcal{O}(U_i) f_i^{-1}.$$

This subsheaf is denoted by $\mathcal{O}(D)$. Since f_i is a unit in $\mathcal{K}_X(U_i)$, it follows that $\mathcal{L}(U_i)$ is free of rank one over $\mathcal{O}(U_i)$, so $\mathcal{O}(D)$ is invertible. $\mathcal{I}_D = \mathcal{O}_X(-D)$ if D is effective.

PROPOSITION 6.2. *The association*

$$D \longmapsto \mathcal{O}(D)$$

is an isomorphism between Cartier divisors and invertible subsheaves of $\mathcal{K}_X$ *(under the tensor product).*

It induces an injective homomorphism on the classes

$$0 \longrightarrow \mathrm{DivCl}(X) \longrightarrow \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ *is the group of isomorphism classes of invertible sheaves. In other words,* $D \sim E$ *if and only if* $\mathcal{O}(D) \approx \mathcal{O}(E)$*. If* X *is an integral scheme, then this homomorphism is surjective, so we have a natural isomorphism*

$$\mathrm{DivCl}(X) \approx \mathrm{Pic}(X).$$

PROOF. The fact that the map $D \mapsto \mathcal{O}(D)$ is homomorphic is immediate from the definitions. From an invertible subsheaf of $\mathcal{K}_X$ we can define a Cartier divisor by the inverse construction that we used to get $\mathcal{O}(D)$ from D. That is, D is represented by f on U if and only if $\mathcal{O}(D)$ is free with basis f^{-1} over U. If $D \sim E$, say $D = E + (f)$, then multiplication by f induces an isomorphism from $\mathcal{O}(D)$ to $\mathcal{O}(E)$. Conversely suppose $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}(E)$. Then $\mathcal{O}(D - E)$ is isomorphic to $\mathcal{O} = \mathcal{O}_X$, so we must prove that if $\mathcal{O}(D) \approx \mathcal{O}$ then $D = 0$. But the image of the global section $1 \in \mathcal{K}^*(X)$ then represents D as a principal divisor.

Finally, suppose X integral. We must show that every invertible sheaf is isomorphic to $\mathcal{O}(D)$ for some divisor D. Let

$$\varphi_i \colon \mathcal{L}|_{U_i} \longrightarrow \mathcal{O}|_{U_i}$$

be an isomorphism and let $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \in \mathcal{O}(U_i \cap U_j)^*$ be the associated cocycle. We have seen already that this constitutes glueing data to define an invertible sheaf. But now we may view all rings $\mathcal{O}(U_i)$ or $\mathcal{O}(U_i \cap U_j)$ as contained in the quotient field K of X since X is integral. We fix an index j, and define the divisor D by the covering $\{U_i\}$, and the local equation φ_{ij}. In other words, the family of pairs (U_i, φ_{ij}) (with j fixed) is a compatible family, defining a Cartier divisor D. Then it is immediately verified that $\mathcal{O}(D)$ is isomorphic to $\mathcal{L}$. This concludes the proof. $\square$

Line bundles. Let $L \to X$ be a scheme over X. Let $\mathbb{A}^1$ be the affine line. We shall say that L is a *line bundle* over X if one is given an open affine covering $\{U_i\}$ of X and over each U_i an isomorphism of schemes

$$f_i \colon L|_{U_i} \longrightarrow U_i \times \mathbb{A}^1$$

over U_i such that the automorphism

$$f_i \circ f_j^{-1} \colon (U_i \cap U_j) \times \mathbb{A}^1 \longrightarrow (U_i \cap U_j) \times \mathbb{A}^1$$

over $U_i \cap U_j$ is given by an $\mathcal{O}(U_i \cap U_j)$-linear map. Such a map is then represented by a unit $\varphi_{ij} \in \mathcal{O}(U_i \cap U_j)^*$, and such units satisfy the cocycle condition. Consequently, there is an invertible sheaf $\mathcal{L}$ corresponding to this cocycle.

One defines an isomorphism of line bundles over X in the obvious way, so that they are linear on the affine line when given local representations as above.

PROPOSITION 6.3. *The above association of a cocycle to a line bundle over X induces a bijection between isomorphism classes of line bundles over X and $H^1(X, \mathcal{O}_X^*)$. If $\mathcal{L}$ is an invertible sheaf corresponding to the cocycle, then we have an isomorphism*

$$L \approx \mathrm{Spec}_X(\mathrm{Symm}^*(\mathcal{L})).$$

PROOF. Left to the reader. □

Weil divisors. The objects that we have called Cartier divisors are rather different from the divisors that we defined in Part I [**87**, §1C]. In good cases we can bring these closer together. The problem is: for which integral domain R can we describe the structure of K^*/R^* more simply?

DEFINITION 6.4. A (not necessarily integral) scheme X is called *normal* if all its local rings $\mathcal{O}_{x,X}$ are integral domains, integrally closed in their quotient field (integrally closed, for short); *factorial* if all its local rings $\mathcal{O}_{x,X}$ are unique factorization domains (UFD).

In particular, note that:

$$\begin{aligned} &X \text{ factorial} \implies X \text{ normal} \\ &\qquad\qquad\qquad\text{(all UFD's are integrally closed,} \\ &\qquad\qquad\qquad\text{see Zariski-Samuel [}\mathbf{119}\text{, vol. I, Chapter V, §3, p. 261])} \\ &X \text{ normal} \implies X \text{ reduced.} \end{aligned}$$

Now the fundamental structure theorem for integrally closed ring states:

THEOREM 6.5 (Krull's Structure Theorem). *Let R be a noetherian integral domain. Then*

$$R \textit{ integrally closed} \iff \begin{cases} \text{a)} & \forall \textit{(non-zero) minimal prime ideal } \mathfrak{p} \subset R, \\ & R_{\mathfrak{p}} \textit{ is a discrete valuation ring,} \\ \text{b)} & R = \bigcap_{\mathfrak{p} \textit{ (non-zero) minimal}} R_{\mathfrak{p}} \end{cases}$$

(cf. Zariski-Samuel [**119**, vol. I, Chapter V, §6]; Bourbaki [**27**, Chapter 7]).

COROLLARY 6.6. *Assume a noetherian domain R to be integrally closed. Let*

$$\mathcal{S} = \textit{set of (non-zero) minimal prime ideals of } R$$

$$Z^1(R) = \textit{free abelian group generated by } \mathcal{S}.$$

If $\mathfrak{p} \in \mathcal{S}$

$$\mathrm{ord}_{\mathfrak{p}} = \left\{ \begin{array}{l} \textit{valuation on } K^* \textit{ defined by the valuation ring } R_{\mathfrak{p}} \\ \textit{i.e., if } \pi \cdot R_{\mathfrak{p}} = \textit{ maximal ideal, } f = \pi^{\mathrm{ord}_{\mathfrak{p}} f} \cdot u, \; u \in R_{\mathfrak{p}}^* \end{array} \right\}.$$

Then the homomorphism:

$$\text{ord}\colon K^*/R^* \longrightarrow Z^1(R)$$

given by $\text{ord}(f) = \sum_{\mathcal{S}}(\text{ord}_{\mathfrak{p}} f) \cdot \mathfrak{p}$ *is injective.* ord *is surjective if and only if R is a UFD.*

Proof. Everything is a straightforward consequence of Theorem 6.5 except for the last assertion. This follows from the well known characterization of UFD's among all noetherian domains—that the (non-zero) minimal prime ideals should be principal, i.e., that Image(ord) $\ni$ the cycle $\mathfrak{p}$ (cf. Zariski-Samuel [**119**, vol. I, Chapter IV, §14, p. 238]). □

Corollary 6.7. *Assume X is a normal irreducible noetherian scheme. Let*

$$\mathcal{S} = \textit{set of maximal closed irreducible subsets } Z \subsetneqq X$$

$$Z^1(X) = \textit{free abelian group generated by } \mathcal{S}.$$

$Z^1(X)$ is called the group of Weil divisors on X. If $Z \in \mathcal{S}$, let

$$\text{ord}_Z = \left\{ \begin{array}{l} \textit{valuation on } \mathbf{R}(X) \textit{ defined by the valuation ring} \\ \mathcal{O}_{z,X},\ z = \textit{generic point of } Z \end{array} \right\}.$$

Then there is a well-defined homomorphism:

$$\text{ord}\colon \text{Div}(X) \longrightarrow Z^1(X)$$

given by $\text{ord}(D) = \sum_{\mathcal{S}}(\text{ord}_Z(f_z))\cdot Z$ *(where f_z = local equation of D near the generic point $z \in Z$), and it is injective.* ord *is surjective if and only if X is factorial.*

Proof. Straightforward. □

Remark. Let X be a normal irreducible noetherian scheme with the function field $\mathbf{R}(X)$, and let D be a Cartier divisor on X. Then for $f \in \mathbf{R}(X)^*$, one has $(f) + D \geq 0$ if and only if $f \in \Gamma(X, \mathcal{O}_X(D))$. Thus the set of effective Cartier divisors linearly equivalent to D is controlled by the space $\Gamma(X, \mathcal{O}_X(D))$ of global sections of the invertible sheaf $\mathcal{O}_X(D)$.

Exercise

For some of the notions and terminology in the following, the reader is referred to Part I [**87**].

(1) A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be locally free of rank r if each point $x \in X$ has a neighborhood U such that there is an isomorphism

$$(\mathcal{O}_X|_U)^{\oplus r} \xrightarrow{\approx} \mathcal{F}|_U$$

(cf, Definition I.5.3). As a generalization of Proposition 6.1, show that such an $\mathcal{F}$ may be explicitly described in terms of $H^1(X, \text{GL}_r(\mathcal{O}_X))$. As a generalization of Proposition 6.3, show that the isomorphism classes of vector bundles over X and those of locally free $\mathcal{O}_X$-modules are in one-to-one correspondence: Given a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank r, let $\check{\mathcal{F}} = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ be the dual $\mathcal{O}_X$-module. Let

$$\mathbb{V}(\mathcal{F}) = \text{Spec}_X(\bigoplus_{n=0}^{\infty} \text{Symm}^n(\check{\mathcal{F}})),$$

and let $\pi\colon \mathbb{V}(\mathcal{F}) \to X$ be the projection. $\pi\colon \mathbb{V}(\mathcal{F}) \to X$ is the vector bundle of rank r over X, and $\mathcal{F}$ is the sheaf of germs of sections of π.

(2) Prove that the Segre embedding (cf. Example I.8.11 and Proposition II.1.2)

$$i\colon \mathbb{P}^{n_1}_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^{n_2}_{\mathbb{Z}} \hookrightarrow \mathbb{P}^{n_1n_2+n_1+n_2}_{\mathbb{Z}}$$

corresponds in Theorem 2.2 to the invertible sheaf $\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1)$ and the surjective homomorphism

$$(\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}})^{\oplus(n_1+1)} \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}})^{\oplus(n_2+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1) \otimes_{\mathbb{Z}} \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1)$$

obtained as the tensor product over Z of the canonical surjective homomorphisms

$$(\mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}})^{\oplus(n_1+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_1}_{\mathbb{Z}}}(1)$$
$$(\mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}})^{\oplus(n_2+1)} \longrightarrow \mathcal{O}_{\mathbb{P}^{n_2}_{\mathbb{Z}}}(1).$$

(3) Let X be of finite type over R. Prove that if $\mathcal{L}_1$, $\mathcal{L}_2$ are very ample (resp. ample) invertible sheaves on X, then $\mathcal{L}_1 \otimes \mathcal{L}_2$ is very ample (resp. ample). Referred to in the proof of Theorem VIII.5.5.

(4) Let k be a field and consider $\mathbb{P}^n_k$.
  a) All maximal irreducible subsets of $\mathbb{P}^n_k$ are of the form $V(f)$, $f \in k[X_0, \dots, X_n]$ homogeneous and irreducible.
  b) All effective Cartier divisors D on $\mathbb{P}^n_k$, considered via (a) above as subschemes of $\mathbb{P}^n_k$, are equal to $V(f)$, some homogeneous $f \in k[X_0, \dots, X_n]$.
  c) Two effective divisors $D_1 = V(f_1)$ and $D_2 = V(f_2)$ are linearly equivalent if and only if $\deg f_1 = \deg f_2$; hence the set of all effective divisors D given by subschemes $V(f)$, $\deg f = d$, is a complete linear system; the canonical map

$$k[X_0, \dots, X_n]_d \longrightarrow \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$$

  is an isomorphism and $\operatorname{Pic}(\mathbb{P}^n_k) \cong \mathbb{Z}$, with $\mathcal{O}_{\mathbb{P}^n_k}(1)$ being a generator.
  d) If $\sigma\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ is an automorphism over k, then $\sigma^*(\mathcal{O}_{\mathbb{P}^n_k}(1)) \cong \mathcal{O}_{\mathbb{P}^n_k}(1)$. Using the induced action on $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$, show that σ is induced by the linear change of homogeneous coordinates $A \in \mathrm{GL}_{n+1}(k)$.

(5) Work over a field k. Let $T \subset \mathbb{P}^2$ be the "triangle" defined by $x_0x_1x_2 = 0$, a closed subscheme. Let $f\colon \mathbb{P}^2 \backslash T \to \mathbb{P}^2 \backslash T$ be the isomorphism defined in projective coordinates by

$$(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right).$$

Let Z be the Zariski closure of the graph of f in $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$, a closed subscheme of $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$. Let $p_1\colon Z \to \mathbb{P}^2$ be the projection to the first factor of $\mathbb{P}^2 \times_{\mathrm{Spec}(k)} \mathbb{P}^2$, thought of as the source of the birational map f. Relate $p_1\colon Z \to \mathbb{P}^2$ to a suitable blow up of $\mathbb{P}^2$.

(6) Work over a base field k. Let y be a k-point of $\mathbb{P}^2$, and let $f\colon Y \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ with center y. Let E be the exceptional divisor for $Y \to \mathbb{P}^2$. Let L be a line on $\mathbb{P}^2$ passing through y, and let $\widetilde{L}$ be the strict transform of L in Y. Let h and e be the class of $f^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{O}_Y(E)$ in $\operatorname{Pic}(Y)$, respectively.
  (i) Show h, e form a $\mathbb{Z}$-basis of the Picard group of Y, with $(h \cdot h) = 1$, $(h \cdot e) = 0$, $(e \cdot e) = -1$.
  (ii) Prove that an element $ah - be$ in $\operatorname{Pic}(Y)$ with $a, b \in \mathbb{Z}$ is the class of an effective divisor if and only if $a \geq b \geq 0$.
  (iii) Prove that an element $ah - be$ in $\operatorname{Pic}(Y)$ is the class of an ample invertible $\mathcal{O}_Y$-module if and only if $a > b > 0$.

(iv) Let $\mathcal{F} := f^*\mathcal{O}_{\mathbb{P}^2}(2)\otimes\mathcal{O}_Y(-E)$. Show that the linear system $|\Gamma(Y,\mathcal{F})|$ is canonically isomorphic to the linear system of quadrics on $\mathbb{P}^2$ passing through y, and defines an embedding $\iota_{\mathcal{F}}\colon Y \hookrightarrow \mathbb{P}^4$ of Y as a surface of degree 3 in $\mathbb{P}^4$.

(v) Show that the linear pencil $|\Gamma(Y, f^*\mathcal{O}_{\mathbb{P}^2}(1)\otimes\mathcal{O}_Y(-E)|$ is base point free, and defines a fibration $g\colon Y \to \mathbb{P}^1$.

(vi) Let $Y_1 := \mathrm{Proj}_{\mathbb{P}^1}(\mathrm{Symm}^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)))$, and let $g\colon Y_1 \to \mathbb{P}^1$ be the structure morphism for $\mathrm{Proj}_{\mathbb{P}^1}(\mathrm{Symm}^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1)))$. Show that $(g\colon Y\to\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^2}(1))$ is isomorphic to $(g_1\colon Y_1\to\mathbb{P}^1,\mathcal{O}(1))$, where the last term $\mathcal{O}(1)$ is the universal invertible quotient $\mathcal{O}_Y$-module of $g_1^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1))$ on Y_1.

(vii) Which ones among ample invertible $\mathcal{O}_Y$-modules are very ample?

(7) Work over a base field k. Let X be a smooth quadric in $\mathbb{P}^3$, x_0 a k-rational point of X, and $g\colon\ X \dashrightarrow \mathbb{P}^2$ be the projection from x_0 to a plane disjoint from x_0, a rational map which is regular on $X\setminus\{x_0\}$.

(i) Show that g does not extend to a morphism on X.

(ii) Show that g is a birational map.

(iii) Determine all $\mathbb{P}^1$'s contracted by g.

(iv) Let $\alpha\colon B\to X$ be the blow up of X at x_0. Show that the birational map g induces a morphism $\beta\colon B\to\mathbb{P}^2$.

(v) Let y_1 and y_2 be the images in $\mathbb{P}^2$ of the two lines in X contracted under g. Show that B is isomorphic to the blow up of $\mathbb{P}^2$ at y_1 and y_2.

(vi) Show that the birational map $g^{-1}\colon \mathbb{P}^2 \dashrightarrow X$ is given by the linear system of conics on $\mathbb{P}^2$ passing through y_1 and y_2.

(vii) Show that X is *not* isomorphic to the blow up of $\mathbb{P}^2$ centered at a closed point.

(8) (Continuation of the previous exercise) Let E be the exception divisor for α. Let $l_1\cup l_2$ be the intersection of X with its tangent plane $T_{x_0}X$ at x_0, and let E_1, E_2 be the strict transforms of l_1, l_2, respectively. Then the total transform on B of l_i is E_i+E (as a divisor), $i=1,2$. We saw that E_1 and E_2 are the two exceptional divisors for the morphism β with $\beta(E_i)=y_i$ for $i=1,2$. Let h, h_1, h_2 be the classes of $\beta^*\mathcal{O}_{\mathbb{P}^2}(1)$, $\alpha^*\mathcal{O}(l_1)$, $\alpha^*\mathcal{O}(l_2)$ in $\mathrm{Pic}(B)$, respectively. Similarly, denote by e, e_1, e_2 the classes of $\mathcal{O}_B(E)$, $\mathcal{O}_B(E_1)$ and $\mathcal{O}_B(E_2)$, respectively. So we have 6 elements h, e_1, e_2, h_1, h_2, e in $\mathrm{Pic}(B)$.

(i) Show that E is the strict transform on B of the line $\overline{y_1\,y_2}$ on $\mathbb{P}^2$.

(ii) Show that h, e_1, e_2 form a $\mathbb{Z}$-basis of $\mathrm{Pic}(B)$, and so do h_1, h_2, e. These two bases are related by

$$\begin{cases} e_1 &= h_1 - e\\ e_2 &= h_2 - e\\ h &= h_1+h_2-e\end{cases}\qquad\qquad \begin{cases} h_1 &= h-e_2\\ h_2 &= h-e_1\\ e &= h-e_1-e_2\end{cases}$$

A third $\mathbb{Z}$-basis is $\{e,e_1,e_2\}$, and we have

$$h_1=e_1+e,\quad h_2=e_1+e,\quad h=e_1+e_2+e.$$

The classes e, e_1, e_2, h, h_1, h_2 are all effective.

(iii) Verify that the intersection numbers for the elements h_1, h_2, h, e, e_1, e_2 are given by

$$e\cdot e=e_1\cdot e_1=e_2\cdot e_2=-1,$$

$$h\cdot e_1=h\cdot e_2=h_1\cdot e_1=h_2\cdot e_2=h_1\cdot e=h_2\cdot e=h_1\cdot h_1=h_2\cdot h_2=e_1\cdot e_2=0,$$

$$e_1\cdot e=e_1\cdot e=h\cdot h=h\cdot h_1=h\cdot h_2=h_1\cdot h_2=h_1\cdot e_2=h_2\cdot e_1=h\cdot e=1.$$

(iii) Show that $ae + b_1e_1 + b_2e_2$ is the class of an effective divisor if and only if a, b_1, b_2 are non-negative integers.
Hint: If $ae+b_1e_1+b_2e_2$ is the class of an effective divisor D which does not contain E, E_1, E_2, then $(D \cdot E) \geq 0$, $(D \cdot E_i) \geq 0$.

(iv) Deduce that an invertible $\mathcal{O}_B$-module $\beta^*\mathcal{O}_{\mathbb{P}^2}(a)\otimes\mathcal{O}_B(b_1E_1+b_2E_2)$ with $a, b_1, b_2 \in \mathbb{Z}$ is ample if and only if $b_1 > 0$, $b_2 > 0$ and $a - b_1 - b_2 > 0$.

(v) Which among these ample divisors on $\mathcal{O}_B$ are very ample?

(9) Let k be a field. Let $X = \mathrm{Bl}_y(\mathbb{P}^n)$, the blow up of $\mathbb{P}^n$ at a k-rational point $y \in \mathbb{P}^n$, $n \geq 2$. Let $N = (n+1)(n+2)/2-2$. Let $g' \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^N$ be the rational map defined by the linear system of quadrics on $\mathbb{P}^n$ passing through y.

(i) Show that the rational map g' extends to a morphism $g \colon X \to \mathbb{P}^N$, which is a closed immersion.

(ii) Show that $g^*\mathcal{O}_{\mathbb{P}^N}(1)$ is isomorphic to $f^*(\mathcal{O}_{\mathbb{P}^n}(2))(-E)$, where $f \colon X \to \mathbb{P}^n$ is the blow up, and E is the exceptional divisor above y.

(iii) Determine $h^0(X, \mathcal{L}^{\otimes m})$, where $\mathcal{L}$ is the ample invertible sheaf $f^*(\mathcal{O}_{\mathbb{P}^n}(2))(-E)$ on X.

(iv) Conclude from (iii) that $\deg(g(X)) = 2^n - 1$, i.e., $g(X)$ is a subvariety of $\mathbb{P}^N$ of degree $2^n - 1$.

(10) Let $f \colon Y \to \mathbb{P}^n$ be the blow up of a linear subspace $L \cong \mathbb{P}^{n-2}$ in $\mathbb{P}^n$, $n \geq 2$. Let $E \subset Y$ be the exceptional divisor for f. Let $g' \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ be the linear projection with center L, a rational map from $\mathbb{P}^n$ to $\mathbb{P}^1$.

(i) Show that the rational map g' extends to a morphism $g \colon Y \to \mathbb{P}^1$.

(ii) Let $Y_1 := \mathrm{Proj}_{\mathbb{P}^1}(\mathrm{Symm}^*(\mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)))$, let $g_1 \colon Y_1 \to \mathbb{P}^1$ be the structure morphism, denote by $\mathcal{L}$ the universal invertible quotient $\mathcal{O}_{Y_1}$-module on Y. Show that the pairs

$$(Y \xrightarrow{g} \mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^n}(1)) \quad \text{and} \quad (Y_1 \xrightarrow{g_1} \mathbb{P}^1, \mathcal{L})$$

are isomorphic.

(iii) Show that E is identified with the closed subscheme $\mathrm{Proj}_{\mathbb{P}^1}(\mathrm{Symm}^*(\mathcal{O}_{\mathbb{P}^1}^{\oplus(n-1)}))$ under the isomorphism in (ii). In particular $E \cong L \times \mathbb{P}^1$.

(iv) Show that $\mathcal{L} \otimes \mathcal{O}_E$ is isomorphic to $p_L^*\mathcal{O}_L(1)$, where $p_L \colon E \to L$ is the natural projection.

(v) Show that $\mathcal{N}_{E/Y} \cong p_L^*\mathcal{O}_L(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-1)$, where $\mathcal{N}_{E/Y}$ denotes the normal bundle for $E \hookrightarrow Y$, and $p_2 \colon Y \cong L \times \mathbb{P}^1$ is the projection to $\mathbb{P}^1$.

(vi) Show that $\mathcal{F} := \mathcal{L}^{\otimes 2} \otimes \mathcal{O}_Y(-E)$ is a very ample invertible sheaf on Y.

(vii) Determine the degrees of Y and E with respect to the very ample invertible sheaf $\mathcal{F}$ on Y.
Hint: Use (vi) to show that $\deg_{\mathcal{F}}(E) = n - 1$.

(11) Work over a field k. Let H be a hyperplane in $\mathbb{P}^n$, $n \geq 2$. Let $Z \subset H$ be a smooth hypersurface in H of degree d, $d \geq 2$. Let $f \colon X \to \mathbb{P}^n$ be the blow up of $\mathbb{P}^n$ with center Z, and let Y be the strict transform of H. By the universal property of blow ups, the $\mathcal{O}_X$-module $\mathcal{J} := f^{-1}\mathcal{I}_Z \cdot \mathcal{O}_X$, i.e., the ideal in $\mathcal{O}_X$ generated by the image of the sheaf of ideals $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^n}$ for $Z \subset \mathbb{P}^n$, is an invertible $\mathcal{O}_X$-module isomorphic to the sheaf "$\mathcal{O}_X(1)$" on $X = \mathrm{Proj}_{\mathbb{P}^n}(\oplus_{n\geq 0}\mathcal{I}_Z^n)$. Show that Y is isomorphic to H under the morphism f, and $\mathcal{J} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is isomorphic to $f^*\mathcal{O}_{\mathbb{P}^n}(-d) \otimes_{\mathcal{O}_X} \mathcal{O}_Y$.

(12) Work over a field k. Let X be a smooth quadric in $\mathbb{P}^{n+1}$, $n \geq 2$. Let x be a k-point of X, and let $p\colon X - - \to \mathbb{P}^n$ be the linear projection from x. Let $\pi\colon B \to X$ be the blow up of X with center x.
  (i) Show that the birational map $p\colon X - - \to \mathbb{P}^n$ induces a morphism $f\colon B \to \mathbb{P}^n$.
  (ii) Let $C(x)$ be the intersection of X with the hyperplane $T(x) \subset \mathbb{P}^n$ tangent to X at x. Show that $C(x)$ is a cone over a smooth conic Q' in $T(x)$. Moreover, Q' is mapped isomorphically under the projection p to a smooth quadric Q in a hyperplane of $\mathbb{P}^n$.
  (iii) Show that $f\colon B \to \mathbb{P}^n$ factors through the blow up of $\mathrm{Bl}_Q(\mathbb{P}^n)$ of $\mathbb{P}^n$ with center Q; the resulting morphism $f_1\colon B \to \mathrm{Bl}_Q(\mathbb{P}^n)$ is an isomorphism.
  (iv) Prove the birational map $p^{-1}\colon \mathbb{P}^n - - \to X$ corresponds to the linear system on $\mathbb{P}^n$ consisting of all quadrics on $\mathbb{P}^n$ passing through Q.

(13) (Converse to the previous problem) Let Q be a smooth conic in a hyperplane $H \subset \mathbb{P}^n$. Let L be the linear system on $\mathbb{P}^n$ consisting of all quadrics passing through Q. Let $\pi\colon \mathrm{Bl}_Q(\mathbb{P}^n) \to \mathbb{P}^n$ be the blow up with center Q, and let $E = \pi^{-1}(Q)$ be the exceptional divisor in $\mathrm{Bl}_Q(\mathbb{P}^n)$.
  (i) Show that the rational map corresponding to the linear system L is represented by a morphism $\alpha\colon \mathbb{P}^n \setminus Q \to \mathbb{P}^{n+1}$.
  (ii) Show that $\alpha(\mathbb{P}^n \setminus Q)$ is contained in a quadric $X \subset \mathbb{P}^{n+1}$.
  (iii) Show that α extends to a morphism $\beta\colon \mathrm{Bl}_Q(\mathbb{P}^n) \to \mathbb{P}^{n+1}$, and $\beta^*\mathcal{O}_{\mathbb{P}^n}(1)$ is isomorphic to $\pi^*\mathcal{O}_{\mathbb{P}^2}(2)(-E)$.
  (iv) Let D be the strict transform of the hyperplane H in $\mathrm{Bl}_Q(\mathbb{P}^n)$. Show that $\alpha(D)$ is a point $x \in X$.
  (v) Prove that X is smooth, and the morphism $\beta\colon \mathrm{Bl}_Q(\mathbb{P}^n) \to X$ identifies $\mathrm{Bl}_Q(\mathbb{P}^n)$ as the blow up of X with center x.

(14) Let $X = F(a_1, \ldots, a_n) := \mathrm{Proj}_{\mathbb{P}^1} \mathrm{Symm}^* \left(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1(a_n)}\right)$. Assume for simplicity that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $\pi\colon X \to \mathbb{P}^1$ be the structure morphism, so that X is a family of $\mathbb{P}^{n-1}$'s parametrized by $\mathbb{P}^1$. Denote by $\mathcal{O}_X(1)$ the universal invertible quotient $\mathcal{O}_X$-module of

$$\pi^* \left(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1(a_n)}\right).$$

  (i) For every local ring $(R, \mathfrak{m})$, let S_R be the set

$$\left\{ (t_0, t_1; x_1 : x_2 : \ldots : x_n) \in R^{n+2} \mid t_0 R + t_1 R = R,\ x_1 R + \cdots + x_n R = R \right\}$$

  modulo the equivalence relation generated by

$$\begin{array}{lcll} (t_0, t_1; x_1 : x_2 : \ldots : x_n) & \sim & (t_0, t_1; \mu x_1 : \mu x_2 : \ldots : \mu x_n) & \mu \in R^* \\ (t_0, t_1; x_1 : x_2 : \ldots : x_n) & \sim & (\lambda t_0, \lambda t_1; \lambda^{-a_1} x_1 : \lambda^{-a_2} x_2 : \ldots : \lambda^{-a_n} x_n) & \lambda \in R^*. \end{array}$$

  Show that there is a functorial bijection between $X(R)$ and the set S_R for every local ring $(R, \mathfrak{m})$.
  (ii) Show that the complete linear system $|\Gamma(X, \mathcal{O}_X(1))|$ is base point free if $a_i \geq 0$ for all $i = 1, \ldots, n$.
  (iii) Suppose that $a_i > 0$ for all i. Show that the complete linear system $|\Gamma(X, \mathcal{O}_X(1))|$ defines an closed immersion $\phi_{\mathcal{O}_X(1)}\colon X \hookrightarrow \mathbb{P}^N$, where $N = a_1 + \cdots + a_n + n - 1$. Moreover, under the morphism $\phi_{\mathcal{O}_X(1)}$, every fibre of π is embedded into a linear $\mathbb{P}^{n-1}$ in $\mathbb{P}^N$, and $\phi_{\mathcal{O}_X(1)}(X)$ is a subvariety of $\mathbb{P}^N$ of degree $a_1 + \ldots + a_n$. (The subvariety $\phi_{\mathcal{O}_X(1)}(X)$ is called a *rational scroll* in $\mathbb{P}^N$.)

(iv) Show that $\phi(F(1,1))$ is a smooth quadric surface in $\mathbb{P}^3$, $\phi(F(2,1)$ is a smooth surface of degree 3 in $\mathbb{P}^4$, $\phi\colon F(0,1) \to \mathbb{P}^2$ is the blow up of a point, and $\phi\colon F(0,2) \to \phi(F(0,2)) \subset \mathbb{P}^3$ is the resolution of singularities of the cone over a plane conic curve in $\mathbb{P}^3$.

(v) Suppose that $a_1 \le \cdots \le a_m < b \le a_{m+1} \le \cdots \le a_n$, $b \in \mathbb{Z}$. Show that the base locus of the complete linear system $|\Gamma(X, \mathcal{O}_X(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-b)|$ is the closed subscheme

$$\operatorname{Proj}_{\mathbb{P}^1} \operatorname{Symm}^* \left(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1(a_m)}\right) \subset X$$

corresponding to the obvious surjection

$$\operatorname{Symm}^* \left(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1(a_n)}\right) \twoheadrightarrow \operatorname{Symm}^* \left(\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1(a_m)}\right).$$

CHAPTER IV

Ground fields and base rings

1. Kronecker's big picture

For all schemes X, there is a unique morphism:

$$\pi\colon X \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

This follows from Theorem I.3.7, since there is a unique homomorphism

$$\pi^*\colon \mathbb{Z} \longrightarrow \Gamma(\mathcal{O}_X).$$

Categorically speaking, $\operatorname{Spec}\mathbb{Z}$ is the final object in the category of schemes. $\operatorname{Spec}\mathbb{Z}$ itself is something like a line, but in which the variable runs not over constants in a fixed field but over primes p. In fact $\mathbb{Z}$ is a principal ideal domain like $k[X]$ and its prime ideals are (p) or $p\cdot\mathbb{Z}$, p a prime number, and (0). (cf. Figure IV.1) The stalk of the structure sheaf at $[(p)]$ is the discrete valuation ring $\mathbb{Z}_{(p)} = \{m/n \mid p \nmid n\}$ and at $[(0)]$ is the field $\mathbb{Q}$. $\operatorname{Spec}\mathbb{Z}$ is reduced and irreducible with "function field" $\mathbf{R}(\operatorname{Spec}\mathbb{Z}) = \mathbb{Q}$. The non-empty open sets of $\operatorname{Spec}\mathbb{Z}$ are gotten by throwing away finitely many primes $p_1,\dots,p_n$. If $m = \prod p_i$, then this is a distinguished open set:

$$\operatorname{Spec}(\mathbb{Z})_m, \qquad \text{with ring} \quad \mathbb{Z}_m = \left\{\frac{a}{m^n} \mid a,n\in\mathbb{Z}\right\}.$$

The residue fields are:

$$\Bbbk([(p)]) = \mathbb{Z}/p\mathbb{Z}$$
$$\Bbbk([(0)]) = \mathbb{Q},$$

i.e., each prime field occurs exactly once.

If X is an arbitrary scheme, then set-theoretically the morphism

$$\pi\colon X \longrightarrow \operatorname{Spec}\mathbb{Z}$$

is just the map

$$x \longmapsto [(\operatorname{char}\Bbbk(x))],$$

because if $\pi(x) = y$, then we get

$$\Bbbk(x) \xhookleftarrow[\pi_x^*]{} \Bbbk(y) = \begin{cases} \mathbb{Z}/p\mathbb{Z} \\ \text{or} \\ \mathbb{Q} \end{cases}$$

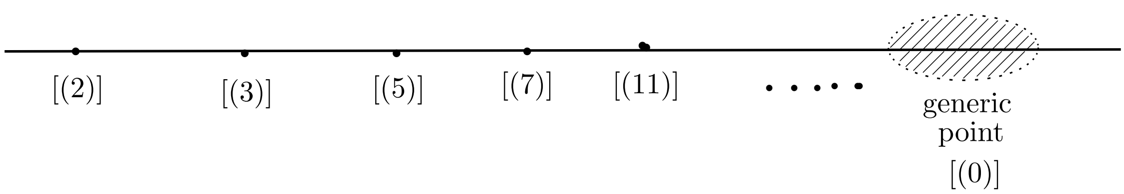

FIGURE IV.1. $\operatorname{Spec}\mathbb{Z}$

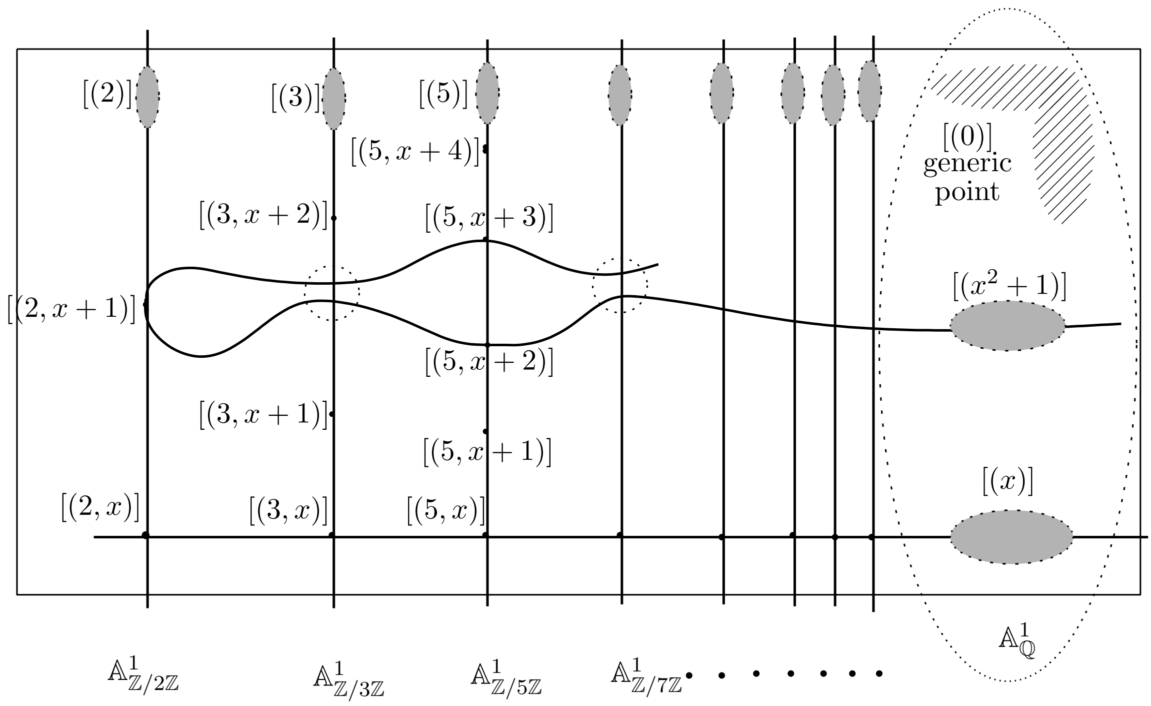

FIGURE IV.2. $\mathbb{A}^1_{\mathbb{Z}}$

hence

$$\begin{aligned} \operatorname{char} \Bbbk(x) = p > 0 &\Longrightarrow \pi(x) = [(p)] \\ \operatorname{char} \Bbbk(x) = 0 &\Longrightarrow \pi(x) = [(0)]. \end{aligned}$$

Thus every scheme X is a kind of fibred object, made up out of separate schemes (possibly empty),

$$X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \begin{cases} \mathbb{Z}/p\mathbb{Z} \\ \text{or} \\ \mathbb{Q}, \end{cases}$$

of each characteristic! For instance, we can "draw" a sort of picture of the scheme $\mathbb{A}^1_{\mathbb{Z}}$, showing how it is the union of the affine lines $\mathbb{A}^1_{\mathbb{Z}/p\mathbb{Z}}$ and $\mathbb{A}^1_{\mathbb{Q}}$. The prime ideals in $\mathbb{Z}[X]$ are:

i) (0),
ii) principal prime ideals (f), where f is either a prime number p, or a $\mathbb{Q}$-irreducible integral polynomial written so that its coefficients have greatest common divisor 1,
iii) maximal ideals (p, f), p a prime and f a monic integral polynomial irreducible modulo p.

The whole should be pictured as in Figure IV.2. (The picture is misleading in that $\mathbb{A}^1_{\mathbb{Z}/p\mathbb{Z}}$ for any p has actually an infinite number of closed points: i.e., in addition to the maximal ideals $(p, X-a)$, $0 \le a \le p-1$, with residue field $\mathbb{Z}/p\mathbb{Z}$, there will be lots of others $(p, f(x))$, $\deg f > 1$, with residue fields $\mathbb{F}_{p^n} =$ finite field with p^n elements, $n > 1$.)

An important property of schemes of finite type over $\mathbb{Z}$ is:

PROPOSITION 1.1. *Let X be of finite type over $\mathbb{Z}$ and let $x \in X$. Then*

$$[x \textit{ is closed}] \Longleftrightarrow [\Bbbk(x) \textit{ is finite}].$$

PROOF. Let $\pi\colon X \to \operatorname{Spec}\mathbb{Z}$ be the morphism. By Theorem II.2.9 (Chevalley's Nullstellensatz),

$$x \text{ closed} \Longrightarrow \{\pi(x)\} \text{ constructible} \Longrightarrow \pi(x) \text{ closed}.$$

If $\pi(x) = [(p)]$, then $x \in X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}/p\mathbb{Z}$ — call this scheme X_p. Then x is a closed point of X_p, so by Corollary II.2.11, x is an algebraic point, i.e., $\Bbbk(x)$ is algebraic over $\mathbb{Z}/p\mathbb{Z}$, so $\Bbbk(x)$ is finite. Conversely, if $\Bbbk(x)$ is finite, let p be its characteristic. Then $x \in X_p$ and by Corollary II.2.11, x is closed in X_p and since X_p is closed in X, x is closed in X. □

From the point of view of arithmetic, schemes of finite type over $\mathbb{Z}$ are the basic objects. The classical problem in Diophantine equations is always to find all $\mathbb{Z}$- or $\mathbb{Q}$-valued points of various schemes X (recall Definition I.6.2). For instance, if $f \in \mathbb{Z}[X_1, \ldots, X_n]$, the solutions

$$f(a_1, \ldots, a_n) = 0$$

with a_i in any ring R are just the R-valued points of the affine scheme

$$\operatorname{Spec}\mathbb{Z}[X_1, \ldots, X_n]/(f)$$

(see Theorem I.3.7). Because of its homogeneity, however, Fermat's last theorem may also be interpreted via the "plane curve"

$$V(X_1^n + X_2^n - X_0^n) \subset \mathbb{P}^2_{\mathbb{Z}}$$

and the conjecture[1] asserts that if $n \geq 3$, its only $\mathbb{Q}$-valued points are the trivial ones, where either X_0, X_1, or X_2 is 0. Moreover, it is for such schemes that a zeta-function can be introduced formally:

$$\zeta_X(s) = \prod_{\substack{\text{closed}\\ \text{points}\\ x \in X}} \left(1 - \frac{1}{(\#\Bbbk(x))^s}\right)^{-1}, \qquad \# = \text{cardinality} \tag{1.2}$$

which one expands formally to the Dirichlet series

$$\begin{aligned} \zeta_X(s) &= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \\ a_n &= \left\{ \begin{array}{l} \text{number of 0-cycles } \mathfrak{a} = \sum n_i x_i \text{ on } X, \\ \text{where } n_i > 0,\ x_i \in X \text{ closed and } \deg\mathfrak{a} \overset{\text{def}}{=} \sum n_i \#\Bbbk(x_i) \text{ is } n \end{array} \right\}. \end{aligned} \tag{1.3}$$

This is known to converge if $\operatorname{Re} s \gg 0$ and is conjectured to be meromorphic in the whole s-plane—cf. Serre's talk [**106**] for a general introduction.

But these schemes also play a fundamental role for many geometric questions because of the following simple but very significant observation:

> Suppose $X \subset \mathbb{A}^n_{\mathbb{C}}$ (resp. $X \subset \mathbb{P}^n_{\mathbb{C}}$) is a complex affine (resp. projective) variety. Let its ideal be generated by polynomials (resp. homogeneous polynomials) $f_1, \ldots, f_k$. Let $R \subset \mathbb{C}$ be a subring finitely generated over $\mathbb{Z}$ containing the coefficients of the f_i: Then $f_1, \ldots, f_k$ define $X_0 \subset \mathbb{A}^n_R$ (resp. $X_0 \subset \mathbb{P}^n_R$) such that
>
> a) $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec}\mathbb{C}$
>
> b) X_0 is of finite type over R, hence is of finite type over $\mathbb{Z}$.

More generally, we have:

[1] (Added in publication) The conjecture has since been settled affirmatively by Wiles [**117**].

PROPOSITION 1.4. *Let X be a scheme of finite type over $\mathbb{C}$. Then there is a subring $R \subset \mathbb{C}$, finitely generated over $\mathbb{Z}$ and a scheme X_0 of finite type over R such that*

$$X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}.$$

PROOF. Let $\{U_i\}$ be a finite affine open covering of X and write

$$U_i = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_{n_i}]/(f_{i,1}, \ldots, f_{i,k_i}) = \operatorname{Spec} R_i.$$

For each i, j, cover $U_i \cap U_j$ by open subsets which are distinguished affines in U_i and U_j and let each of these subsets define an isomorphism

$$\phi_{ij,l} \colon (R_i)_{g_{ij,l}} \xrightarrow{\approx} (R_j)_{g_{ji,l}}.$$

The fact that

$$(U_i)_{g_{ij,l} \cdot \phi_{ij,l}^{-1}(g_{jk,l'})} \subset U_i \cap U_j \cap U_k \subset \bigcup_{l''} (U_i)_{g_{ik,l''}}$$

means that

$$(*) \qquad \left[g_{ij,l} \cdot \phi_{ij,l}^{-1}(g_{jk,l'}) \right]^N = \sum_{l''} a_{ijkll'l''} g_{ikl''}, \quad \text{suitable } a\text{'s in } R_i.$$

Let R be generated by the coefficients of the f_{ij}'s, the g's and a's (lifted to $\mathbb{C}[X]$) and of the polynomials defining the $\phi_{ij,l}$'s. Define

$$U_{i,0} = \operatorname{Spec} R[X_1, \ldots, X_{n_i}]/I_i = \operatorname{Spec} R_{i,0}$$

where $I_i = \operatorname{Ker}[R[X] \to \mathbb{C}[X]/(f_{i,1}, \ldots, f_{i,k_i})]$, i.e., I_i consists of the f_{ij}'s plus enough other polynomials to make $R_{i,0}$ into a subring of R_i. Clearly $R_i \cong R_{i,0} \otimes_R \mathbb{C}$. Then $g_{ij,l}$ is in the subring $R_{i,0}$ and $\phi_{ij,l}$ restricts to an isomorphism $(R_{i,0})_{g_{ij,l}} \xrightarrow{\approx} (R_{j,0})_{g_{ji,l}}$, hence ϕ defines:

$$(U_{i,0})_{g_{ij,l}} \xrightarrow{\approx} (U_{j,0})_{g_{ji,l}}.$$

Let $U_{i,0}^{(j)} = \bigcup_l (U_{i,0})_{g_{ij,l}}$ and glue $U_{i,0}^{(j)}$ to $U_{j,0}^{(i)}$ by these ϕ's: the fact that $\phi_{ij,l} = \phi_{ij,l'}$ on overlaps is guaranteed by the fact that $R_{i,0} \subset R_i$. Moreover the identity $(*)$ still holds because we smartly put the coefficients of the a's in R, hence points of $U_{i,0}$ which are being glued to points of $U_{j,0}$ which in turn are being glued to points of $U_{k,0}$ are being *directly* glued to points of $U_{k,0}$; Moreover the direct and indirect glueing maps again agree because $R_{i,0} \subset R_i$. Thus an X_0 can be constructed by glueing all the $U_{i,0}$'s and clearly $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}$. □

The idea of Kroneckerian geometry is that when you have $X \cong X_0 \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}$, then (a) classical geometric properties of X over $\mathbb{C}$ may influence Diophantine problems on X_0, and (b) Diophantine properties of X_0, even for instance the characteristic p fibres of X_0, may influence the geometry on X. In order to go back and forth in this way between schemes over $\mathbb{C}$, $\mathbb{Z}$ and finite fields, one must make use of all possible homomorphisms and intermediate rings that nature gives us. These "God-given" natural rings form a diagram as in Figure IV.3 (with various Galois groups acting too), where the completion $\widehat{\overline{\mathbb{Q}_p}}$ of the algebraic closure $\overline{\mathbb{Q}_p}$ of the p-adic number field $\mathbb{Q}_p$ is known to be algebraically closed, $\widehat{\widetilde{\mathbb{Z}_p}}$ is the completion of the integral closure $\widetilde{\mathbb{Z}_p}$ in $\overline{\mathbb{Q}_p}$ of the ring of p-adic integers $\mathbb{Z}_p$, the field of algebraic numbers $\overline{\mathbb{Q}}$ is the algebraic closure of the rational number field $\mathbb{Q}$, and $\overline{\mathbb{Z}/p\mathbb{Z}}$ is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$: Thus given any $X \to \operatorname{Spec} \mathbb{Z}$, say of finite type, one gets a big diagram of schemes as in Figure IV.4 (where we have written X_R for $X \times \operatorname{Spec} R$, and $\overline{R}$ for the algebraic closure or integral closure of R, or completions thereof.)

In order to use the diagram (1.6) effectively, there are two component situations that must first be studied in detail:

(1.5)

$$\begin{tikzcd}
\mathbb{C} \arrow[rrr, dashed, dash, "\text{non-canonical isomorphism}"] & & & \widehat{\overline{\mathbb{Q}_p}} & \\
\cup & \overline{\mathbb{Q}} \arrow[ul, hook] \arrow[urr, hook] & & & \\
\mathbb{R} & \cup & \mathbb{Q}_p \arrow[uur, hook] & \cup & \\
& \mathbb{Q} \arrow[ul, hook] \arrow[ur, hook] & \cup & \widehat{\overline{\mathbb{Z}_p}} \arrow[dr, two heads] & \\
& \cup & \mathbb{Z}_p \arrow[ur, hook] \arrow[dr, two heads] & & \overline{\mathbb{Z}/p\mathbb{Z}} \\
& \mathbb{Z} \arrow[ur, hook] & & \mathbb{Z}/p\mathbb{Z} \arrow[ur, hook] &
\end{tikzcd}$$

FIGURE IV.3. The diagram formed by "God-given" natural rings

(1.6)

$$\begin{tikzcd}
X_{\mathbb{C}} \arrow[rrr, dashed, dash, "\text{non-canonical isomorphism}"] \arrow[dd] \arrow[dr] & & & X_{\overline{\mathbb{Q}_p}} \arrow[dl, shorten <=2em] \arrow[dll] \arrow[ddd] & \\
& X_{\overline{\mathbb{Q}}} \arrow[dd] & & & \\
X_{\mathbb{R}} \arrow[dr] & & X_{\mathbb{Q}_p} \arrow[dl] \arrow[dd] & & \\
& X_{\mathbb{Q}} \arrow[dd] & & X_{\overline{\mathbb{Z}_p}} \arrow[dl] & \\
& & X_{\mathbb{Z}_p} \arrow[dl] & & X_{\overline{\mathbb{Z}/p\mathbb{Z}}} \arrow[ul] \arrow[dl] \\
& X_{\mathbb{Z}} & & X_{\mathbb{Z}/p\mathbb{Z}} \arrow[ul] &
\end{tikzcd}$$

FIGURE IV.4. The big diagram of schemes

1.7. *Given*

$$\begin{cases} k \text{ a field} \\ \overline{k} = \text{algebraic closure of } k \\ X \text{ of finite type over } k \end{cases}$$

consider:

$$\begin{tikzcd}
\overline{X} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} \overline{k} \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

where $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. *Compare* X *and* $\overline{X}$.

1.8. *Given*

$$\begin{cases} R \text{ a valuation ring} \\ K \text{ its quotient field} \\ k \text{ its residue field} \\ X \text{ of finite type over } R \end{cases}$$

consider:

$$\begin{tikzcd}
X_\eta \arrow[r] \arrow[d] & X \arrow[d] & X_0 \arrow[l] \arrow[d] \\
\operatorname{Spec} K \arrow[r] & \operatorname{Spec} R & \operatorname{Spec} k \arrow[l]
\end{tikzcd}$$

where $X_\eta = X \times_{\operatorname{Spec} R} \operatorname{Spec} K$, $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$. *Compare* X_0 *and* X_η.

We take these situations up in §§2–3 and §§4–6 separately. In §VIII.5 we will give an illustration of how the big picture is used. The idea will also be used in the proof of Belyi's three point theorem (Theorem IX.2.1).

Classical geometry was the study of varieties over $\mathbb{C}$. But it did not exploit the fact that the defining equations of a variety can have coefficients in a *subfield* of $\mathbb{C}$. This possibility leads us directly to the analysis of schemes over non-algebraically closed fields (1.7), and to the relation between schemes over two different fields given by (1.8).

2. Galois theory and schemes

For this whole part, fix a field k and an algebraic closure $\overline{k}$. We write $\mathrm{Gal}(\overline{k}/k)$ for the Galois group, and for each scheme X over k, we write $\overline{X}$ for $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. First consider the action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{k}^n$ by conjugation:

1. For $\sigma \in \mathrm{Gal}(\overline{k}/k)$

$$(a_1, \dots, a_n) \longmapsto (\sigma a_1, \dots, \sigma a_n), \qquad \overline{k}^n \longrightarrow \overline{k}^n.$$

   If we identify $\overline{k}^n$ with the set of closed points of $\mathbb{A}^n_{\overline{k}}$, then this map *extends in fact to an automorphism of* $\mathbb{A}^n_{\overline{k}}$:

2. Define $\sigma_{\mathbb{A}^n} \colon \mathbb{A}^n_{\overline{k}} \to \mathbb{A}^n_{\overline{k}}$ by

$$(\sigma_{\mathbb{A}^n})^* \colon \overline{k}[X_1, \dots, X_n] \longrightarrow \overline{k}[X_1, \dots, X_n]$$

   where

$$\sigma^*_{\mathbb{A}^n}(X_i) = X_i, \qquad \sigma^*_{\mathbb{A}^n}(a) = \sigma^{-1} a, \qquad a \in \overline{k}.$$

   In fact, for all prime ideals $\mathfrak{p} \subset \overline{k}[X_1, \dots, X_n]$,

$$\sigma_{\mathbb{A}^n}([\mathfrak{p}]) = [(\sigma^*_{\mathbb{A}^n})^{-1} \mathfrak{p}]$$

   and if $\mathfrak{p} = (X_1 - a_1, \dots, X_n - a_n)$, then since $\sigma^*_{\mathbb{A}^n}(X_i - \sigma a_i) = X_i - a_i$, we find $(\sigma^*_{\mathbb{A}^n})^{-1}\mathfrak{p} \supset (X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$; since $(X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$ is maximal, $(\sigma^*_{\mathbb{A}^n})^{-1}\mathfrak{p} = (X_1 - \sigma a_1, \dots, X_n - \sigma a_n)$.

   Note that $\sigma_{\mathbb{A}^n}$ is a k-morphism but *not a $\overline{k}$-morphism.* For this reason, $\sigma_{\mathbb{A}^n}$ will have, for instance, a graph in

$$\mathbb{A}^n_{\overline{k}} \times_{\operatorname{Spec} k} \mathbb{A}^n_{\overline{k}} = \operatorname{Spec}\left((\overline{k} \otimes_k \overline{k})[X_1, \dots, X_n, Y_1, \dots, Y_n]\right),$$

   but not in $\mathbb{A}^n_{\overline{k}} \times_{\operatorname{Spec} \overline{k}} \mathbb{A}^n_{\overline{k}} = \mathbb{A}^{2n}_{\overline{k}}$. Thus when $\overline{k} = \mathbb{C}$, $\sigma_{\mathbb{A}^n}$ will not be a correspondence nor will it act at all continuously in the classical topology (with the one exception $\sigma =$ complex conjugation).

3. Now we may also define $\sigma_{\mathbb{A}^n}$ as:

$$\sigma_{\mathbb{A}^n} = 1_{\mathbb{A}^n_k} \times \sigma_k \colon \mathbb{A}^n_k \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \longrightarrow \mathbb{A}^n_k \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$$

   where $\sigma_k \colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$ is defined by $(\sigma_k)^* a = \sigma^{-1} a$.

The third form clearly generalizes to all schemes of the form X:

Definition 2.1. For every k-scheme X, define the conjugation action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{X}$ to be:

$$\sigma_X = 1_X \times \sigma_k \colon \overline{X} \to \overline{X}, \quad \text{all } \sigma \in \mathrm{Gal}(\overline{k}/k).$$

$$\begin{tikzcd}
& & & \Omega \\
& (R\otimes_k \overline{k})/\mathfrak{p}_1 \arrow[urr, dashed, "\alpha_1"] & (R\otimes_k \overline{k})/\mathfrak{p}_2 \arrow[ur, dashed, "\alpha_2"'] & \\
\overline{k} \arrow[ur, hook] \arrow[urr, hook] & & & R/\mathfrak{p} \arrow[ull, hook] \arrow[ul, hook] \arrow[uu, hook] \\
& & k \arrow[ull, hook] \arrow[ur, hook] &
\end{tikzcd}$$

FIGURE IV.5

Then σ_X is *not* a $\overline{k}$-morphism, but rather fits into a diagram:

$$\begin{array}{ccc}
\overline{X} & \xrightarrow{\sigma_X} & \overline{X} \\
\downarrow & & \downarrow \\
\operatorname{Spec}\overline{k} & \xrightarrow{\sigma_k} & \operatorname{Spec}\overline{k}.
\end{array}$$

What this means is that if $f \in \mathcal{O}_{\overline{X}}(U)$ then $\sigma_X^* f \in \mathcal{O}_{\overline{X}}(\sigma_X^{-1}U)$ has value at a point $x \in \sigma_X^{-1}U$ given by:

$$(\sigma_X^* f)(x) = \sigma^{-1} \cdot f(\sigma_X \cdot x), \tag{2.2}$$

i.e., set-theoretically, σ_X^* is *not* "pull-back" on functions. This can be proven as follows:

$$\begin{aligned}
f - f(\sigma_X \cdot x) \in \mathfrak{m}_{\sigma_X x, \overline{X}} &\Longrightarrow \sigma_X^*(f - f(\sigma_X \cdot x)) \in \mathfrak{m}_{x,\overline{X}} \\
&\Longrightarrow \sigma_X^* f - \sigma^{-1} \cdot f(\sigma_X \cdot x) \in \mathfrak{m}_{x,\overline{X}} \\
&\Longrightarrow \sigma_X^* f(x) = \sigma^{-1} \cdot f(\sigma_X \cdot x).
\end{aligned}$$

I want next to analyze the relationship between X and $\overline{X}$. The first point is that topologically X is the quotient of $\overline{X}$ by the action of $\operatorname{Gal}(\overline{k}/k)$.

THEOREM 2.3. *Let X be a scheme of finite type over k, let*

$$\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec}\overline{k}$$

and let $p\colon \overline{X} \to X$ be the projection. Then

1) *p is surjective and both open and closed (i.e., maps open (resp. closed) sets to open (resp. closed) sets);*
2) *$\forall x, y \in \overline{X}$, $p(x) = p(y)$ iff $x = \sigma_X(y)$ for some $\sigma \in \operatorname{Gal}(\overline{k}/k)$;*
3) *$\forall x \in X$, let $Z =$ closure of $\{x\}$. Then $p^{-1}(x) =$ the set of generic points of the components of $p^{-1}(Z)$. In particular, $p^{-1}(x)$ is finite.*

PROOF. Since all these results are local on X, we may as well replace X by an open affine subset U, and replace $\overline{X}$ by $p^{-1}U$. Therefore assume $X = \operatorname{Spec} R$, $\overline{X} = \operatorname{Spec} R \otimes_k \overline{k}$. First of all, p is surjective by Corollary I.4.4. Secondly, p is closed because $R \otimes_k \overline{k}$ is integrally dependent on R (cf. Proposition II.6.5; this is an easy consequence of the Going-up theorem). Thirdly, let's prove (2). If $\mathfrak{p}_1, \mathfrak{p}_2 \subset R \otimes_k \overline{k}$ are two prime ideals, we must show:

$$\mathfrak{p}_1 \cap R = \mathfrak{p}_2 \cap R \Longleftrightarrow \exists \sigma \in \operatorname{Gal}(\overline{k}/k), \quad \mathfrak{p}_1 = (1_R \otimes \sigma)\mathfrak{p}_2.$$

$\Longleftarrow$ is obvious, so assume $\mathfrak{p}_1 \cap R = \mathfrak{p}_2 \cap R$. Call this prime $\mathfrak{p}$. Let Ω be an algebraically closed field containing $R/\mathfrak{p}$. Consider the *solid* arrows in Figure IV.5. It follows that there exist injective

k-homomorphisms α_1, α_2 as indicated. Then $\alpha_1(\overline{k})$ and $\alpha_2(\overline{k})$ both equal the algebraic closure of k in Ω, so for some $\sigma \in \mathrm{Gal}(\overline{k}/k)$, $\alpha_2 = \alpha_1 \circ \sigma$ on $\overline{k}$. But then if $x_i \in R$, $y_i \in \overline{k}$:

$$\begin{aligned}\sum x_i \otimes y_i \in \mathfrak{p}_2 &\iff \sum x_i \cdot \alpha_2(y_i) = 0 \text{ in } \Omega \\ &\iff \sum x_i \cdot \alpha_1(\sigma(y_i)) = 0 \text{ in } \Omega \\ &\iff \sum x_i \otimes \sigma(y_i) \in \mathfrak{p}_1,\end{aligned}$$

so $(1_R \otimes \sigma)\mathfrak{p}_2 = \mathfrak{p}_1$. Fourthly, p is an open map. In fact, let $U \subset \overline{X}$ be open. Then

$$U' = \bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(U)$$

is also open, and by (2), $p(U) = p(U')$ and $U' = p^{-1}(p(U'))$. Therefore $X \backslash p(U) = p(\overline{X} \backslash U')$ which is closed since p is a closed map. Therefore $p(U)$ is open. Finally, let $x \in X$, $Z =$ closure of $\{x\}$. Choose $w \in p^{-1}(x)$ and let $W =$ closure of $\{w\}$. Since p is closed, $p(W)$ is a closed subset of Z containing x, so $p(W) = Z$. Therefore $\bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(W)$ is Gal-invariant and maps onto Z, so by (2):

$$\bigcup_{\sigma \in \mathrm{Gal}} \sigma_X(W) = p^{-1}Z.$$

Therefore every component of $p^{-1}Z$ equals $\sigma_X(W)$ for some σ, and since they are all conjugate, the $\sigma_X(W)$'s are precisely the components of $p^{-1}Z$. (3) now follows easily. □

Suppose now X is a k-variety. Is $\overline{X}$ necessarily a $\overline{k}$-variety?

THEOREM 2.4. *Let X be a k-variety and let $\overline{X} = X \times_k \overline{k}$.*

i) *Let*

$$L = \{x \in \mathbf{R}(X) \mid x \textit{ separable algebraic over } k\}.$$

Then L is a finite algebraic extension of k. Let $U \subset X$ be an open set such that the elements of L extend to sections of $\mathcal{O}_X$ over U. Then the basic morphism from X to Spec k *factors:*

$$\begin{tikzcd} U \arrow[r, phantom, "\subset"] \arrow[d, "f"'] & X \arrow[dd] \\ \operatorname{Spec} L \arrow[dr] & \\ & \operatorname{Spec} k \end{tikzcd}$$

and taking fibre products with Spec $\overline{k}$, *we get:*

$$\begin{tikzcd} \overline{U} \arrow[r, phantom, "\subset"] \arrow[d, "\overline{f}"'] & \overline{X} \arrow[dd] \\ \operatorname{Spec} L \otimes_k \overline{k} \arrow[dr] & \\ & \operatorname{Spec} \overline{k}. \end{tikzcd}$$

Then

$$
\begin{aligned}
L \otimes_k \overline{k} &\cong \prod_{i=1}^{t} \overline{k} \\
\operatorname{Spec} L \otimes_k \overline{k} &= \textit{disjoint union of } t \textit{ reduced closed points } P_1, \dots, P_t \\
\overline{U} &= \textit{disjoint union of } t \textit{ irreducible pieces } \overline{U}_i = \overline{f}^{-1}(P_i) \\
\overline{X} &= \textit{union of } t \textit{ irreducible components } \overline{X}_i, \textit{ with } \overline{X}_i = \textit{closure}(\overline{U}_i).
\end{aligned}
$$

This induces an isomorphism of sets:

$$(\textit{Components of } \overline{X}) \cong \operatorname{Hom}_k(L, \overline{k}),$$

commuting with the action of the Galois group $\operatorname{Gal}(\overline{k}/k)$.

ii) *If* $y_i =$ *generic point of* $\overline{X}_i$, *then* y_i *maps to the generic point of* X *and*

$$\prod_{i=1}^{t} \mathcal{O}_{y_i, \overline{X}} \cong \mathbf{R}(X) \otimes_k \overline{k}$$

hence $\dim \overline{X}_i = \dim X$ *for all* i, *and:*

$$
\begin{aligned}
\overline{X} \textit{ is reduced} &\iff \mathcal{O}_{y_i, \overline{X}} \textit{ has no nilpotents, for all } i \\
&\iff \mathbf{R}(X) \textit{ is separable over } k.
\end{aligned}
$$

PROOF OF THEOREM 2.4, (i). Let $L_1 \subset L$ be a subfield which is finite algebraic over k. Then $L_1 \otimes_k \overline{k}$ is a finite-dimensional separable $\overline{k}$-algebra, hence by the usual Wedderburn theorems,

$$L_1 \otimes_k \overline{k} \cong \prod_{i=1}^{t} \overline{k}, \quad \text{where } t = [L_1 : k]$$

and $\operatorname{Spec} L_1 \otimes_k \overline{k} = \{P_1, \dots, P_t\}$ as asserted. Elements of a basis of L_1 extend to sections of $\mathcal{O}_X$ over some open set U_1, and we get a diagram

$$
\begin{tikzcd}
\overline{U}_1 \arrow[d, "\overline{f}_1"'] & \subset & \overline{X} \arrow[dd] \\
\{P_1, \dots, P_t\} \arrow[drr] & & \\
& & \operatorname{Spec} \overline{k}.
\end{tikzcd}
$$

Therefore $\overline{U}_1$ is the disjoint union of open sets $\overline{f}_1^{-1}(P_i)$. Therefore $\overline{X}$ has *at least* t components, i.e., components of the closure of $\overline{f}_1^{-1}(P_i)$ in $\overline{X}$. But $\overline{X}$ has only a finite number of components, hence t is bounded above. Therefore L itself is finite over k. Now take $L_1 = L$. The main step consists in showing that $\overline{f}^{-1}(P_i)$ is irreducible. In fact

$$
\begin{aligned}
\overline{f}^{-1}(P_i) &\cong \overline{U} \times_{\operatorname{Spec} L \otimes_k \overline{k}} \operatorname{Spec} \overline{k} \\
&\cong U \times_{\operatorname{Spec} L} \operatorname{Spec} \overline{k}, \quad \text{via } L \to L \otimes_k \overline{k} \xrightarrow[i\text{-th factor}]{\text{projection on}} \overline{k}
\end{aligned}
$$

so in effect this step amounts to checking the special case:

$$k \text{ separable algebraically closed in } \mathbf{R}(X) \Longrightarrow \overline{X} \text{ is irreducible.}$$

The rest of part (i) follows from two remarks: first, by Theorem 2.3, (3), each component of $\overline{X}$ is the closure of a component of $\overline{U}$; secondly, there is an isomorphism of sets commuting with Gal:

$$\begin{aligned}\{\text{Maximal ideals of } L \otimes_k \overline{k}\} &\cong \{\text{Kernels of the various projections } L \otimes_k \overline{k} \to \overline{k}\}\\ &\cong \mathrm{Hom}_k(L, \overline{k}).\end{aligned}$$

Now consider the special case. If $\overline{X} = \bigcup_{i=1}^t \overline{X}_i$ is reducible, we can find an affine open $U \subset X$ such that the sets $p^{-1}(U) \cap \overline{X}_i = \overline{U}_i$ are disjoint. Let $U = \operatorname{Spec} R$, so that

$$\coprod_{i=1}^t \overline{U}_i = \operatorname{Spec} R \otimes_k \overline{k}.$$

Let ϵ_i be the function which equals 1 on $\overline{U}_i$ and 0 on the other $\overline{U}_j$. Then $\epsilon_i^n = \epsilon_i$ for all n and $\epsilon_i \in R \otimes_k \overline{k}$. Write

$$\epsilon_i = \sum_j \beta_{ij} \otimes \gamma_{ij}, \quad \beta_{ij} \in R, \gamma_{ij} \in \overline{k}.$$

Then if the characteristic is $p > 0$,

$$\epsilon_i = \epsilon_i^{p^n} = \sum \beta_{ij}^{p^n} \otimes \gamma_{ij}^{p^n}$$

and if $n \gg 0$, $\gamma_{ij}^{p^n} \in k_s =$ separable closure of k. Thus if $p > 0$, we find $\epsilon_i \in R \otimes_k k_s$ too. Let L_s be the k_s-subalgebra of $R \otimes_k \overline{k}$ that the ϵ_i generate. The Galois group, acting on $\overline{X}$, permutes the $\overline{X}_i$; hence acting on $R \otimes_k \overline{k} = \Gamma(\coprod \overline{U}_i, \mathcal{O}_{\overline{X}})$, it permutes the ϵ_i. Therefore L_s is a Gal-invariant subspace of $R \otimes_k k_s$. Now apply:

Lemma 2.5. *Let V be a k-vector space and let $W' \subset V \otimes_k k_s$ be a k_s-subspace. Then*

$$\left[\begin{array}{l} W' = W \otimes_k k_s \textit{ for} \\ \textit{some } k\textit{-subspace } W \subset V \end{array}\right] \iff \left[\begin{array}{l} W' \textit{ is invariant} \\ \textit{under } \mathrm{Gal}(k_s/k) \end{array}\right].$$

Proof of Lemma 2.5. "$\Longrightarrow$" is obvious. To prove "$\Longleftarrow$", first note that any $w \in W'$ has only a finite number of conjugates w^σ, $\sigma \in \mathrm{Gal}(k_s/k)$, hence $\sum_\sigma k_s \cdot w^\sigma$ is a *finite-dimensional* Gal-invariant subspace of W' containing w. Thus it suffices to prove "$\Longleftarrow$" when $\dim W' < \infty$. Let $\{e_\alpha\}_{\alpha \in S}$ be a basis of V and let $f_1, \ldots, f_t$ be a basis of W'. Write $f_i = \sum c_{i\alpha} e_\alpha$, $c_{i\alpha} \in k_s$. Since the f's are independent, some $t \times t$-minor of the matrix $(c_{i\alpha})$ is non-zero: say $(c_{i,\alpha_j})_{1 \le i,j \le t}$. Then W' has a *unique* basis f_i' of the form

$$f_i' = e_{\alpha_i} + \sum_{\beta \notin \{\alpha_1, \ldots, \alpha_t\}} c_{i\beta}' e_\beta.$$

Since $\forall \sigma \in \mathrm{Gal}$, $W'^\sigma = W'$, it follows that $(f_i')^\sigma = f_i'$, hence $(c_{i\beta}')^\sigma = c_{i\beta}'$, hence $c_{i\beta}' \in k$, hence $f_i' \in V$. If $W = \sum k f_i'$, then $W' = W \otimes_k k_s$. □

By the lemma, $L_s = L' \otimes_k k_s$ for some subspace $L' \subset R$. But L' is clearly unique and since for all $a \in L'$, $a \cdot L_s \subset L_s$, therefore $a \cdot L' \subset L'$. So L' is a subalgebra of R and hence of $\mathbf{R}(X)$ of dimension t, separable over k because L_s is separable over k_s. Therefore $L' = k$ and $t = 1$. This proves Theorem 2.4, (i). □

Proof of Theorem 2.4, (ii). Let $U = \operatorname{Spec} R$ be any open affine in X so that $p^{-1}(U) = \operatorname{Spec} R \otimes_k \overline{k}$. Since $R \subset \mathbf{R}(X)$, $R \otimes_k \overline{k} \subset \mathbf{R}(X) \otimes_k \overline{k}$. Thus if $\overline{X}$ is not reduced, some ring $R \otimes_k \overline{k}$ has nilpotents, hence $\mathbf{R}(X) \otimes_k \overline{k}$ must have nilpotent elements in it. On the other hand, if U

is small enough as we saw above, $p^{-1}(U) = \coprod \overline{U}_i$, where $\overline{U}_i$ is irreducible, open and $y_i \in \overline{U}_i$. Therefore

$$R \otimes_k \overline{k} = \prod_{i=1}^{t} \mathcal{O}_{\overline{X}}(\overline{U}_i).$$

Replacing R by R_f, U by U_f and $\overline{U}_i$ by $(\overline{U}_i)_f$, and passing to the limit over f's, this shows that

$$\mathbf{R}(X) \otimes_k \overline{k} \cong \prod_{i=1}^{t} \varinjlim_{\substack{\text{distinguished} \\ \text{open sets } (\overline{U}_i)_f}} \mathcal{O}_{\overline{X}}((\overline{U}_i)_f) = \prod_{i=1}^{t} \mathcal{O}_{y_i,\overline{X}}.$$

In particular, if $\mathbf{R}(X) \otimes_k \overline{k}$ has nilpotents, so does one of the rings $\mathcal{O}_{y_i,\overline{X}}$ and hence $\overline{X}$ is not reduced. Now recall that the separability of $\mathbf{R}(X)$ over k means by definition that one of the equivalent properties holds:

Let $k^{p^{-\infty}}$ = perfect closure of k.

a) $\mathbf{R}(X)$ and $k^{p^{-\infty}}$ are linearly disjoint over k.
b) $\mathbf{R}(X) \otimes_k k^{p^{-\infty}} \longrightarrow \mathbf{R}(X)^{p^{-\infty}}$ is injective.
c) $\mathbf{R}(X)$ and $k^{p^{-1}}$ are linearly disjoint over k.
d) $\mathbf{R}(X) \otimes_k k^{p^{-1}} \longrightarrow \mathbf{R}(X)^{p^{-1}}$ is injective.

(cf. Zariski-Samuel [**119**, vol. I, pp. 102–113]; or Lang [**75**, pp. 264–265]. A well-known theorem of MacLane states that these are also equivalent to $\mathbf{R}(X)$ being separable *algebraic* over a purely transcendental extension of k.)

Note that the kernel of $\mathbf{R}(X) \otimes_k k^{p^{-\infty}} \longrightarrow \mathbf{R}(X)^{p^{-\infty}}$ is precisely the ideal $\sqrt{(0)}$ of nilpotent elements in $\mathbf{R}(X) \otimes_k k^{p^{-\infty}}$: because if $a_i \in \mathbf{R}(X)$, $b_i \in k^{p^{-n}}$, then

$$\begin{aligned} \sum a_i b_i = 0 \text{ in } \mathbf{R}(X)^{p^{-\infty}} &\Longrightarrow \sum a_i^{p^n} b_i^{p^n} = 0 \text{ in } \mathbf{R}(X) \\ &\Longrightarrow (\sum a_i \otimes b_i)^{p^n} = \sum a_i^{p^n} b_i^{p^n} \otimes 1 = 0. \end{aligned}$$

Now if N = ideal of nilpotents in $\mathbf{R}(X) \otimes_k \overline{k}$, then N is Gal-invariant, so by Lemma 2.5 applied to $\overline{k}$ over $k^{p^{-\infty}}$, $N = N_0 \otimes_{(k^{p^{-\infty}})} \overline{k}$ for some $N_0 \subset \mathbf{R}(X) \otimes_k k^{p^{-\infty}}$. Hence

$$N \neq (0) \Longleftrightarrow N_0 \neq (0) \Longleftrightarrow \operatorname{Ker}\left(\mathbf{R}(X) \otimes_k k^{p^{-\infty}} \to \mathbf{R}(X)^{p^{-\infty}}\right) \neq (0).$$

$\square$

COROLLARY 2.6 (Zariski). *If X is a k-variety, then $\overline{X}$ is a $\overline{k}$-variety if and only if $\mathbf{R}(X)$ is separable over k and k is algebraically closed in $\mathbf{R}(X)$.*

COROLLARY 2.7. *Let X be any scheme of finite type over k and let $p\colon \overline{X} \to X$ be as before. Then for any $x \in X$, if $L = \{a \in \Bbbk(x) \mid a$ separable algebraic over $k\}$, $\exists$ an isomorphism of sets:*

$$p^{-1}x \cong \operatorname{Hom}_k(L, \overline{k})$$

commuting with $\operatorname{Gal}(\overline{k}/k)$, and the scheme-theoretic fibre is given by:

$$p^{-1}x \cong \operatorname{Spec} \Bbbk(x) \otimes_k \overline{k},$$

hence is reduced if and only if $\Bbbk(x)$ is separable over k.

PROOF. If we let $Z = \overline{\{x\}}$ with reduced structure, then we can replace X by Z and so reduce to the case X a k-variety, x = generic point. Corollary 2.7 then follows from Theorem 2.4. $\square$

COROLLARY 2.8. *Let X be any scheme of finite type over k and let $p\colon \overline{X} \to X$ be as before. Let $x \in \overline{X}$ be a closed or $\overline{k}$-rational point. By k-coordinates near x, we mean: take an affine neighborhood U of $p(x)$, generators $f_1, \dots, f_n$ of $\mathcal{O}_X(U)$, and then define a closed immersion:*

$$\begin{array}{ccc} p^{-1}U & \longrightarrow & \mathbb{A}^n_{\overline{k}} \\ \cap & & \\ \overline{X} & & \end{array}$$

by the functions $f_1, \dots, f_n$. Then

i) $\#($*Galois orbits of* $x) = [\Bbbk(p(x)) : k]_s$

ii) *The following are equivalent:*

   a) $p^{-1}(\{p(x)\}) =$ *the reduced closed subscheme* $\{x\}$,

   b) $p(x)$ *is a k-rational point of X,*

   c) *In k-coordinates, x goes to a point in $k^n \subset \mathbb{A}^n_{\overline{k}}$.*

   If these hold, we say that x is defined over *k.*

iii) *If k is perfect, these are equivalent to*

   d) *x is a fixed point of the Galois action on $\overline{X}$.*

PROOF. (i) and the equivalence of (a) and (b) are restatements of Corollary 2.7 for closed points; as for (c), note that the values of the "proper coordinates" at x are $f_1(x), \dots, f_n(x)$ and that $\Bbbk(p(x)) = k(f_1(x), \dots, f_n(x))$, hence (b) $\Longleftrightarrow$ (c). (iii) is clear. $\square$

In case k is perfect, Corollary 2.8 suggests that there are further ties between X and $\overline{X}$:

THEOREM 2.9. *Let k be a perfect field and $p\colon \overline{X} \to X$ as before. Then*

i) *$\forall U \subset X$ open,*

$$\mathcal{O}_X(U) = \left\{ f \in \mathcal{O}_{\overline{X}}(p^{-1}U) \mid \sigma_X^* f = f,\ \forall \sigma \in \mathrm{Gal}(\overline{k}/k) \right\}.$$

ii) *$\forall$ closed subschemes $\overline{Y} \subset \overline{X}$*

$$\overline{Y} \text{ is Gal-invariant} \Longleftrightarrow \exists \text{ closed subschemes } Y \subset X \text{ with } \overline{Y} = Y \otimes_k \overline{k}$$

   and if this holds, Y is unique, and one says that $\overline{Y}$ is defined over *k.*

iii) *If $x \in \overline{X}$ and $H = \{\sigma \in \mathrm{Gal} \mid \sigma_X(x) = x\}$, then $\Bbbk(p(x)) = \Bbbk(x)^H$.*

iv) *If Y is another scheme of finite type over k and $\overline{Y} = Y \times_k \overline{k}$, then every $\overline{k}$-morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ that commutes with the Galois action (i.e., $\sigma_Y \circ \overline{f} = \overline{f} \circ \sigma_X$, for all $\sigma \in$ Gal) is of the form $f \times 1_{\overline{k}}$ for a unique k-morphism $f\colon X \to Y$, and one says that $\overline{f}$ is* defined over *k.*

PROOF OF (i). Let

$$\mathcal{F}(U) = \left\{ f \in \mathcal{O}_{\overline{X}}(p^{-1}U) \mid \sigma_X^* f = f,\ \text{for all } \sigma \right\}.$$

Then $\mathcal{F}$ is easily seen to be a sheaf and whenever U is affine, say $U = \operatorname{Spec} R$, then

$$\begin{aligned} \mathcal{F}(U) &= \left\{ f \in R \otimes_k \overline{k} \mid (1_R \otimes \sigma) f = f,\ \text{for all } \sigma \right\} \\ &= R, \quad \text{since } k \text{ is perfect} \\ &= \mathcal{O}_X(U). \end{aligned}$$

Thus $\mathcal{F} \cong \mathcal{O}_X$. $\square$

PROOF OF (ii). Suppose $\overline{Y} \subset \overline{X}$ is Gal-invariant. Then for all open affine $U = \operatorname{Spec} R$ in X, $\overline{Y} \cap p^{-1}U$ is defined by an ideal $\overline{\mathfrak{a}} \subset R \otimes_k \overline{k}$. Then $\overline{\mathfrak{a}}$ is Gal-invariant so by Lemma 2.5, $\overline{\mathfrak{a}} = \mathfrak{a} \otimes_k \overline{k}$ for some k-subspace $\mathfrak{a} \subset R$. Since $a\overline{\mathfrak{a}} \subset \overline{\mathfrak{a}}$ for all $a \in R$, it follows that $a\mathfrak{a} \subset \mathfrak{a}$ and so $\mathfrak{a}$ is an ideal. It is easy to see that these ideals $\mathfrak{a}$ define the unique $Y \subset X$ such that $\overline{Y} = Y \times_k \overline{k}$. $\square$

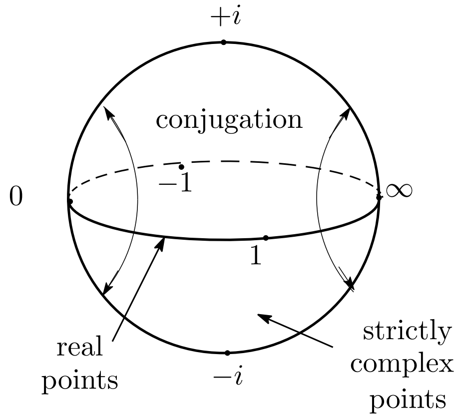

FIGURE IV.6. $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$

PROOF OF (iii). As in Corollary 2.7 above, we can replace X by the closure of $p(x)$ and so reduce to the case where X is a variety with generic point $p(x)$ and $x = x_1$ is one of the generic points $x_1, \dots, x_t$ of $\overline{X}$. By Theorem 2.4, X is reduced and we have

$$\prod_{i=1}^{t} \Bbbk(x_i) = \prod_{i=1}^{t} \mathbf{R}(\overline{X}_i) \cong \mathbf{R}(X) \otimes_k \overline{k} = \Bbbk(p(x)) \otimes_k \overline{k}.$$

Thus

$$\begin{aligned} \Bbbk(p(x)) &\cong \left\{ (x'_1, \dots, x'_t) \in \prod \Bbbk(x_i) \mid (x'_1, \dots, x'_t) \text{ Gal-invariant} \right\} \\ &\cong \left\{ x'_1 \in \Bbbk(x_1) \mid x'_1 \text{ is } H\text{-invariant} \right\}. \end{aligned}$$

□

PROOF OF (iv). Left to the reader. □

Note that when $\overline{Y}$ = one point x, then $\{x\}$ is defined over k as in Theorem 2.9 above if and only if it is defined over k as in Corollary 2.8.

When k is not perfect, the theorem is false. One still says "$\overline{Y}$ is *defined over* k" if $\overline{Y} = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ for some closed subscheme $Y \subset X$, and Y is still unique if it exists. But being Gal-invariant is not strong enough to guarantee being defined over k. For instance, if $\overline{Y}$ is a reduced Gal-invariant subscheme, one can try by setting $Y' = p(\overline{Y})$ with reduced structure. Then $\overline{Y}' = Y' \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ will be a subscheme of $\overline{X}$ defined over k, with the same point set as $\overline{Y}$ and $\overline{Y} \subset \overline{Y}'$ but in general $\overline{Y}'$ need not be reduced: i.e., the *subset* $\overline{Y}$ is defined over k but the *subscheme* $\overline{Y}$ is not (cf. Example 4 below).

The theory can be illustrated with very pretty examples in the case:

$$\begin{aligned} k &= \mathbb{R} \\ \overline{k} &= \mathbb{C} \\ \operatorname{Gal}(\overline{k}/k) &= \{\mathrm{id}, *\}, \quad * = \text{complex conjugation}. \end{aligned}$$

In this case, $*_X : \overline{X} \to \overline{X}$ is continuous in the classical topology and can be readily visualized.

EXAMPLE. 1. Let $X = \mathbb{P}^1_{\mathbb{R}}$, $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$. Ignoring the generic point, $\mathbb{P}^1_{\mathbb{C}}$ looks like Figure IV.6. Identifying conjugate points, $\mathbb{P}^1_{\mathbb{R}}$ looks like Figure IV.7.

EXAMPLE. 2. Let $\overline{X} = \mathbb{P}^1_{\mathbb{C}}$ again. Then in fact there are exactly two real forms of $\mathbb{P}^1_{\mathbb{C}}$: schemes X over $\mathbb{R}$ such that $X \times_{\mathbb{R}} \mathbb{C} \cong \overline{X}$. One is $\mathbb{P}^1_{\mathbb{R}}$ which was drawn in Example 1. The other

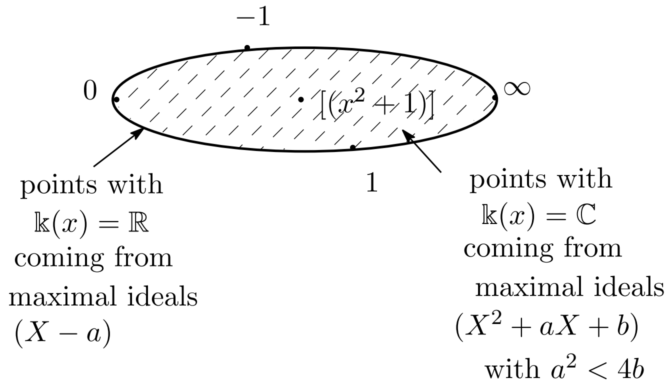

FIGURE IV.7. $X = \mathbb{P}^1_{\mathbb{R}}$

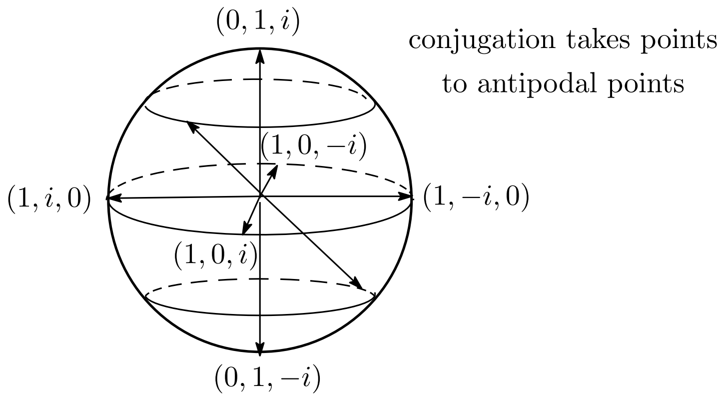

FIGURE IV.8. $X = V(X_0^2 + X_1^2 + X_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$

is represented by the conic:

$$X = V(X_0^2 + X_1^2 + X_2^2) \subset \mathbb{P}^2_{\mathbb{R}}.$$

Then $\overline{X}$ is the same conic over $\mathbb{C}$ and, projecting from any closed point $x \in \overline{X}$, we find as in Part I [**87**] an isomorphism between $\overline{X}$ and $\mathbb{P}^1_{\mathbb{C}}$. Since X has in fact *no* $\mathbb{R}$-rational points at all ($\forall(a_0, a_1, a_2)$, $a_0^2 + a_1^2 + a_2^2 > 0!$) we cannot find a projection $\overline{X} \to \mathbb{P}^1_{\mathbb{C}}$ defined over $\mathbb{R}$. The picture is as in Figure IV.8, so X is homeomorphic in the classical topology to the real projective plane $S^2/(\text{antipodal map})$ and for all its closed points $x \in X$, $\Bbbk(x) = \mathbb{C}$.

EXAMPLE. 3. Let X be the curve $X_1^2 = X_0(X_0^2 - 1)$ in $\mathbb{P}^2_{\mathbb{R}}$. One can work out the picture by thinking of X as a double covering of the X_0-line gotten by considering the two values $\pm\sqrt{X_0(X_0^2 - 1)}$. We leave the details to the reader. One finds the picture in Figure IV.9.

EXAMPLE. 4. To see how $\overline{X}$ may be reducible when X is irreducible, look at the affine curve

$$X_0^2 + X_1^2 = 0$$

in $\mathbb{A}^2_{\mathbb{R}}$. Then $\overline{X}$ is given by:

$$(X_0 + iX_1)(X_0 - iX_1) = 0$$

and the picture is as in Figure IV.10. If $U = X \setminus \{(0, 0)\}$, then U is actually already a variety over $\mathbb{C}$ via

$$p\colon U \longrightarrow \operatorname{Spec} \mathbb{C}, \quad p^*(a + ib) = a + \frac{X_1}{X_0} \cdot b$$

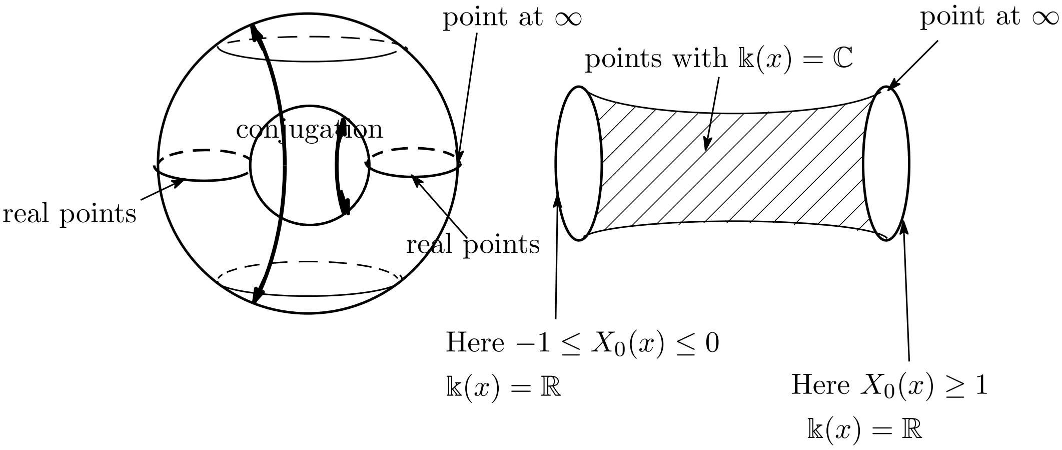

FIGURE IV.9. $X = V(X_1^2 X_2 - X_0(X_0^2 - X_2^2)) \subset \mathbb{P}^2_{\mathbb{R}}$

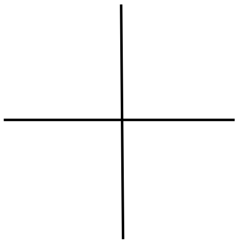

FIGURE IV.10. $X = V(X_0^2 + X_1^2) \subset \mathbb{A}^2_{\mathbb{R}}$

and in fact,

$$\big(\mathbb{R}[X_0, X_1]/(X_0^2 + X_1^2)\big)\left[\frac{1}{X_0}\right] \cong \mathbb{R}\left[\frac{X_1}{X_0}, X_0, X_0^{-1}\right] \Big/ \left(\left(\frac{X_1}{X_0}\right)^2 + 1\right)$$
$$\cong \mathbb{C}[X_0, X_0^{-1}]$$

so $U \cong \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$:

To go deeper into the theory of one-dimensional varieties over $\mathbb{R}$, see Alling-Greenleaf [**12**].

To illustrate how X may be reduced and yet have hidden nilpotents, we must look in characteristic p.

EXAMPLE. 5. Let k be an imperfect field, and consider the hypersurface $X \subset \mathbb{P}^n_k$ defined by

$$a_0 X_0^p + \cdots + a_n X_n^p = 0, \quad a_i \in k.$$

In $\overline{k}$, each a_i will be a p-th power, say $a_i = b_i^p$, so $\overline{X} \subset \mathbb{P}^n_{\overline{k}}$ is defined by

$$(b_0 X_0 + \cdots + b_n X_n)^p = 0.$$

Thus $\overline{X}$ is a "p-fold hyperplane" and the function $\sum b_i X_i / X_0$ is nilpotent and non-zero. However, provided that at least one ratio $a_i/a_j \notin k^p$, then $\sum a_i X_i^p$ is *irreducible* over k, hence X is a k-variety: Put another way, the hyperplane $L : \sum b_i X_i = 0$ in $\mathbb{P}^n_{\overline{k}}$ is "defined over k" as a set in the sense that it is Gal-invariant, hence is set-theoretically $p^{-1}(p(L))$ using $p \colon \mathbb{P}^n_{\overline{k}} \to \mathbb{P}^n_k$; but it is not "defined over k" as a subscheme of $\mathbb{P}^n_k$ unless $b_i/b_j = (a_i/a_j)^{1/p} \in k$ all i, j.

Before leaving this subject, I would like to indicate briefly the main ideas of *Descent theory* which arise when you pursue deeply the relations between X and $\overline{X}$.

(I.) If you look at Theorem 2.9, (ii) as expressing when a quasi-coherent sheaf of ideals $\overline{\mathcal{I}} \subset \mathcal{O}_{\overline{X}}$ is defined over k, it is natural to generalize it to arbitrary quasi-coherent sheaves of *modules*. The result is (assuming k perfect): given a quasi-coherent sheaf $\overline{\mathcal{F}}$ of $\mathcal{O}_{\overline{X}}$-modules, plus an action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{\mathcal{F}}$ compatible with its action on $\mathcal{O}_{\overline{X}}$, i.e., $\forall\sigma \in \mathrm{Gal}$, $U \subset \overline{X}$ isomorphisms

$$\sigma^U_{\mathcal{F}} \colon \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{F}}(\sigma_X^{-1}(U))$$

such that

$$\sigma_X^*(a) \cdot \sigma_{\mathcal{F}}^U(b) = \sigma_{\mathcal{F}}^U(a \cdot b), \quad a \in \mathcal{O}_{\overline{X}}(U), b \in \overline{\mathcal{F}}(U)$$

$$(\sigma\tau)^U_{\mathcal{F}} = \sigma_{\mathcal{F}}^{\tau_X^{-1}(U)} \circ \tau^U_{\mathcal{F}}, \quad \sigma, \tau \in \mathrm{Gal}$$

and commuting with restrictions, then there is one, and up to canonical isomorphism, only one quasi-coherent $\mathcal{F}$ on X such that (i) $\overline{\mathcal{F}} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$ and (ii) the Gal-action on $\overline{\mathcal{F}}$ goes over via this isomorphism to the Gal-action $\sigma_{\overline{\mathcal{F}}}(b \otimes a) = b \otimes \sigma_X^* a$ on $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\overline{X}}$. More precisely, there is an *equivalence of categories* between the category of pairs $(\overline{\mathcal{F}}, \sigma^U_{\mathcal{F}})$ of quasi-coherent sheaves on $\overline{X}$ plus Gal-action and the category of quasi-coherent $\mathcal{F}$ on X.

(II.) The whole set-up in fact generalizes to a much bigger class of morphisms than $p \colon \overline{X} \to X$:

DEFINITION 2.10. Given a morphism $f \colon X \to Y$, a quasi-coherent sheaf $\mathcal{F}$ on X is flat[2] over Y if for every $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{f(x),Y}$-module. f itself is flat if $\mathcal{O}_X$ is flat. f is faithfully flat if f is flat and surjective.

Grothendieck has then proven that for any faithfully flat quasi-compact $f \colon X \to Y$, there is an equivalence of categories between:

a) the category of quasi-coherent sheaves $\mathcal{G}$ on Y,
b) the category of pairs $(\mathcal{F}, \alpha)$, $\mathcal{F}$ a quasi-coherent sheaf on X and α being "descent data", i.e., an isomorphism on $X \times_Y X$:

$$\alpha \colon p_1^* \mathcal{F} \xrightarrow{\approx} p_2^* \mathcal{F}$$

satisfying a suitable associativity condition on $X \times_Y X \times_Y X$ and restricting to the identity on the diagonal $\Delta \colon X \to X \times_Y X$.

In the special case $f = p$, k perfect, it turns out that descent data α is just another way of describing Galois actions. A good reference is Grothendieck's SGA1 [**4**, Exposé VIII][3].

(III.) The final and most interesting step of all is the problem: given $\overline{X}$ over $\overline{k}$, classify the set of all possible X's over k plus $\overline{k}$-isomorphisms $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \cong \overline{X}$, up to isomorphism over k. Such an X is called a *form of* $\overline{X}$ *over* k and to find an X is called *descending* $\overline{X}$ *to* k. If k is perfect, then (cf. Exercise below) it is easy to see that each form of $\overline{X}$ over k is determined up to k-isomorphism by the Galois action $\{\sigma_X \mid \sigma \in \mathrm{Gal}(\overline{k}/k)\}$ on $\overline{X}$ that it induces. What is much harder, and is only true under restrictive hypotheses (such as $\overline{X}$ affine or $\overline{X}$ quasi-projective with Gal also acting on its ample sheaf $\overline{L}$, cf. Chapter III) is that every action of $\mathrm{Gal}(\overline{k}/k)$ is an *effective descent data*, i.e., comes from a descended form X of $\overline{X}$ over k. For a discussion of these matters, cf. Serre [**103**, Chapter V, §4, No. 20, pp. 102–104]. All sorts of beautiful results are known about k-forms: for instance, there is a canonical bijection between the set of k-forms of $\mathbb{P}^n_{\overline{k}}$

[2] We will discuss the meaning of this concept shortly: see §4.

[3] (Added in publication) See also FAG [**3**].

and the set of central simple k-algebras of rank n^2 (cf. Serre [**105**, Chapter X, §6, p. 160]).

3. The frobenius morphism

The most remarkable example of the theory of Galois actions is the case:

$$k = \mathbb{F}_q, \quad \text{the finite field with } q \text{ elements, } q = p^\nu$$

$$\overline{k} = \bigcup_{n=1}^{\infty} \mathbb{F}_{q^n}$$

$$\mathrm{Gal}(\overline{k}/k) = \text{pro-finite cyclic group generated by the frobenius}$$

$$\mathbf{f} \colon \overline{k} \to \overline{k}, \quad \mathbf{f}(x) = x^q.$$

$\mathbf{f}$ is the only automorphism of a field that is given by a polynomial! This has amazing consequences:

Definition 3.1. If X is any scheme in prime characteristic p, i.e., $p = 0$ in $\mathcal{O}_X$, define a morphism

$$\phi_X \colon X \longrightarrow X$$

by

a) set-theoretically, $\phi_X =$ identity,
b) $\forall U$ and $\forall a \in \mathcal{O}_X(U)$, define $\phi_X^* a = a^p$.

Definition 3.2. If X is a scheme of finite type over $k = \mathbb{F}_q$, $\overline{X} = X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$, then:

i) Note that $\mathbf{f}_k \colon \mathrm{Spec}\, \overline{k} \to \mathrm{Spec}\, \overline{k}$ (in the notation at the beginning of §2) is the automorphism $(\phi_{\mathrm{Spec}\, \overline{k}})^{-\nu}$, hence the conjugation $\mathbf{f}_X \colon \overline{X} \to \overline{X}$ defined in Definition 2.1 above is
$$1_X \times (\phi_{\mathrm{Spec}\, \overline{k}})^{-\nu}.$$
We write this now $\mathbf{f}_X^{\mathrm{arith}} \colon \overline{X} \to \overline{X}$.
ii) Set-theoretically identical with this morphism will be a $\overline{k}$-morphism called the geometric frobenius
$$\begin{aligned} \mathbf{f}_X^{\mathrm{geom}} &= \phi_{\overline{X}}^{\nu} \circ (1_X \times \phi_{\mathrm{Spec}\, \overline{k}}^{-\nu}) \\ &= \phi_X^{\nu} \times 1_{\mathrm{Spec}\, \overline{k}} \colon \overline{X} \to \overline{X}. \end{aligned}$$

In other words, there are two morphisms both giving the right conjugation map: an automorphism $\mathbf{f}_X^{\mathrm{arith}}$ which does not preserve scalars, and a $\overline{k}$-morphism $\mathbf{f}_X^{\mathrm{geom}}$ which however is not an automorphism. For instance, look at the case $X = \mathbb{A}_k^n$. All morphisms $\mathbb{A}_{\overline{k}}^n \to \mathbb{A}_{\overline{k}}^n$ are described by their actions on $\overline{k}[X_1, \ldots, X_n]$ and we find:

$$\phi_{\mathbb{A}^n}^* \begin{cases} X_i & \longmapsto & X_i^p \\ a & \longmapsto & a^p \end{cases}$$

$$\left(\mathbf{f}_{\mathbb{A}^n}^{\mathrm{arith}}\right)^* \begin{cases} X_i & \longmapsto & X_i \\ a & \longmapsto & a^{q^{-1}} \end{cases}, \quad \text{an automorphism of } \overline{k}[X_1, \ldots, X_n]$$

$$\left(\mathbf{f}_{\mathbb{A}^n}^{\mathrm{geom}}\right)^* \begin{cases} X_i & \longmapsto & X_i^q \\ a & \longmapsto & a \end{cases}, \quad \text{a } \overline{k}\text{-homomorphism of } \overline{k}[X_1, \ldots, X_n] \text{ into itself,}$$

where $a \in \overline{k}$. This means that completely unlike other conjugations, the graph of $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}}$ is closed in $\overline{X} \times_{\mathrm{Spec}\, \overline{k}} \overline{X}$. Corollary 2.8 gives us a very interesting expansion of the zeta-function of X in terms of the number of certain points on $\overline{X}$:

For every $\nu \geq 1$, we say that a closed point $x \in \overline{X}$ is *defined over* $\mathbb{F}_{q^\nu}$ if any of the equivalent conditions hold:

i) In $\mathbb{F}_q$-coordinates, x corresponds to a point of $(\mathbb{F}_{q^\nu})^n \subset \mathbb{A}^n_{\overline{k}}$,
ii) $\Bbbk(p(x)) \subset \mathbb{F}_{q^\nu}$ ($p\colon \overline{X} \to X$ is the projection in Theorem 2.3),
iii) $\mathbf{f}_X^\nu(x) = x$, i.e., x a fixed point of the morphism $\mathbf{f}_X^\nu$.

(Apply Corollary 2.8 to $\overline{k} \supset \mathbb{F}_{q^\nu}$ and to $\overline{X} \to (X \times_{\mathbb{F}_q} \mathbb{F}_{q^\nu})$.) The set of all these points we call $\overline{X}(\mathbb{F}_{q^\nu})$. Then if

$$N_\nu = \#\overline{X}(\mathbb{F}_{q^\nu}),$$

I claim that formally:

3.3.

$$\zeta_X(s) = Z_X(q^{-s}),$$

where $Z_X(t)$ is given by

$$\begin{aligned}\frac{dZ_X}{Z_X} &= \left(\sum_{\nu=1}^{\infty} N_\nu \cdot t^{\nu-1}\right) dt\\ Z_X(0) &= 1.\end{aligned}$$

Proof. If $M_\nu =$ number of $x \in X$ with $\Bbbk(x) \cong \mathbb{F}_{q^\nu}$, then each such point splits in $\overline{X}$ into ν distinct closed points which are in $\overline{X}(\mathbb{F}_{q^\mu})$ if $\nu \mid \mu$. Thus

$$N_\mu = \sum_{\nu \mid \mu} \nu \cdot M_\nu.$$

By definition:

$$\zeta_X(s) = \prod_{\nu=1}^{\infty} \left(1 - \frac{1}{q^{\nu s}}\right)^{-M_\nu}$$

so if we set

$$Z_X(t) = \prod_{\nu=1}^{\infty} (1 - t^\nu)^{-M_\nu}$$

then $\zeta_X(s) = Z_X(q^{-s})$. Moreover

$$\begin{aligned}\frac{dZ_X}{Z_X} = d(\log Z_X) &= \sum_{\nu=1}^{\infty} (-M_\nu) \cdot \frac{-\nu t^{\nu-1}}{1 - t^\nu} \cdot dt\\ &= \frac{1}{t} \sum_{\nu=1}^{\infty} \nu M_\nu (t^\nu + t^{2\nu} + t^{3\nu} + \cdots) dt\\ &= \frac{1}{t} \sum_{\mu=1}^{\infty} N_\mu \cdot t^\mu \cdot dt.\end{aligned}$$

□

As an example, if $X = \mathbb{A}^n_{\mathbb{F}_q}$, then

$$N_\nu = \#\left((\mathbb{F}_{q^\nu})^n\right) = q^{\nu n}$$

hence

$$Z_{\left(\mathbb{A}^n_{\mathbb{F}_q}\right)}(t) = \frac{1}{1 - q^n t}.$$

Therefore by (3.3)

$$\zeta_{\left(\mathbb{A}^n_{\mathbb{F}_q}\right)}(s) = \frac{1}{1 - q^{n-s}}$$

and

$$\zeta_{\mathbb{A}^n_{\mathbb{Z}}}(s) = \prod_p \left(\frac{1}{1 - p^{n-s}} \right) = \zeta_0(s - n)$$

if $\zeta_0(s)$ is Riemann's zeta-function

$$\zeta_0(s) = \prod_p \left(\frac{1}{1 - p^{-1}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

From this, an easy consequence is:

PROPOSITION 3.4. *For all schemes X of finite type over $\mathbb{Z}$, $\zeta_X(s)$ converges if* $\mathrm{Re}(s) \gg 0$.

IDEA OF PROOF. First reduce to the case X affine and then by an affine embedding, reduce to the case of $\mathbb{A}^n$ using the fact that the Dirichlet series (1.3) for ζ_X has positive coefficients majorized by those for $\zeta_{\mathbb{A}^n}$. □

If X is a scheme over a field $\mathbb{F}_q$ again, a celebrated theorem of Dwork [**36**] states that Z_X is a rational function! If we then expand it in terms of its zeros and poles:

$$Z_X(t) = \frac{\prod_{i=1}^{N}(1 - \alpha_i t)}{\prod_{i=1}^{M}(1 - \beta_i t)}, \quad \alpha_i, \beta_i \in \mathbb{C}$$

it follows immediately that

$$\frac{dZ_X}{Z_X} = \sum_{\nu=1}^{\infty} \left(\sum_{i=1}^{M} \beta_i^{\nu} - \sum_{i=1}^{N} \alpha_i^{\nu} \right) t^{\nu-1} dt$$

and hence:

$$N_\nu = \sum_{i=1}^{M} \beta_i^{\nu} - \sum_{i=1}^{N} \alpha_i^{\nu}.$$

It seems most astonishing that the numbers N_ν of rational points should be such an elementary sequence! Even more remarkably, Deligne [**34**] has proven Weil's conjecture that for every i,

$$|\alpha_i|,\ |\beta_i| \in \{1, q^{1/2}, q, q^{3/2}, \ldots, q^{\dim X}\}.$$

I would like to give one very simple application of the fact that the frobenius $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}}$ has a graph:

PROPOSITION 3.5. *Let X be an $\mathbb{F}_q$-variety such that $\overline{X} \cong \mathbb{P}^1_{\overline{k}}$. Then X has at least one $\mathbb{F}_q$-rational point.*[4]

PROOF. If X has not $\mathbb{F}_q$-rational points, then $\mathbf{f}_X \colon \overline{X} \to \overline{X}$ has no fixed points. Let $\Gamma \subset \overline{X} \times_{\overline{k}} \overline{X}$ be the graph of $\mathbf{f}_X^{\mathrm{geom}}$. Then $\Gamma \cap \Delta = \emptyset$, $\Delta =$ diagonal. But now $\overline{X} \times_{\overline{k}} \overline{X} \cong \mathbb{P}^1_{\overline{k}} \times_{\overline{k}} \mathbb{P}^1_{\overline{k}}$ and via the Segre embedding this is isomorphic to a quadric in $\mathbb{P}^3_{\overline{k}}$. In fact, if $X_0,\ X_1$ (resp. $Y_0,\ Y_1$) are homogeneous coordinates, then

$$s \colon \mathbb{P}^1_{\overline{k}} \times_{\overline{k}} \mathbb{P}^1_{\overline{k}} \longrightarrow \mathbb{P}^3_{\overline{k}}$$

[4]We will see in Corollary VIII.1.8 that this implies $X \cong \mathbb{P}^1_k$ too. See Corollary VI.2.4 for a generalization of Proposition 3.5 to $\mathbb{P}^n$ over finite fields.

via

$$\begin{array}{ccccc} (X_0,X_1)\times(Y_0,Y_1) \longrightarrow (X_0Y_0, & X_0Y_1, & X_1Y_0, & X_1Y_1). \\ \| & \| & \| & \| \\ Z_0 & Z_1 & Z_2 & Z_3 \end{array}$$

The image of s is the quadric $Q = V(Z_0Z_3 - Z_1Z_2)$. But the point is that $s(\Delta) = Q \cap V(Z_1 - Z_2)$, so s maps $\mathbb{P}^1_{\bar{k}} \times_{\bar{k}} \mathbb{P}^1_{\bar{k}} \setminus \Delta$ isomorphically onto an *affine* variety $W = Q \cap [\mathbb{P}^3_{\bar{k}} \setminus V(Z_1 - Z_2)]$. So if $\Gamma \cap \Delta = \emptyset$, we get a closed immersion

$$\mathbb{P}^1_{\bar{k}} \cong \Gamma \longrightarrow W$$

of a complete variety in an affine one. But quite generally a morphism of a complete variety Γ to an affine variety W must be a constant map. If not, choose any function $a \in \Gamma(\mathcal{O}_W)$ which is not constant on the image of Γ and consider the composition

$$\Gamma \longrightarrow W \xrightarrow{a} \mathbb{A}^1_{\bar{k}} \subset \mathbb{P}^1_{\bar{k}}.$$

Γ complete $\implies$ image closed $\implies$ image is one point or the whole $\mathbb{P}^1_{\bar{k}}$. Since a is not constant on the image of Γ, the first is impossible and since $\infty \notin$ image, the second is impossible. $\square$

There are many other classes of varieties X which always have at least one rational point over a finite field $\mathbb{F}_q$: for instance, a theorem of Lang says that this is the case for any homogeneous space: cf. Theorem VI.2.1 and Corollary VI.2.5.

4. Flatness and specialization

In this section I would like to study morphisms $f\colon X \to S$ of finite type by considering them as families $\{f^{-1}(s)\}$ of schemes of finite type over fields, parametrized by the points s of a "base space" S. In particular, the most important case in many applications and for many proofs is when $S = \operatorname{Spec} R$, R a valuation ring. Our main goal is to explain how the concept "f is flat", defined via commutative algebra (cf. Definition 2.10), means intuitively that the fibres $f^{-1}(s)$ are varying "continuously".

We recall that flatness of a module M over a ring R is usually defined by the exactness property:

Definition 4.1. *M is a flat R-module if for all exact sequences*

$$N_1 \longrightarrow N_2 \longrightarrow N_3$$

of R-modules,

$$M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3$$

is exact.

By a simple analysis it is then checked that this very general property is in fact implied by the special cases where the exact sequence is taken to be

$$0 \longrightarrow \mathfrak{a} \longrightarrow R$$

($\mathfrak{a}$ an ideal in R), in which case it reads:

For all ideals $\mathfrak{a} \subset R$,

$$\mathfrak{a} \otimes_R M \longrightarrow M$$

is injective.

For basic facts concerning flatness, we refer the reader to Bourbaki [**27**, Chapter 1][5]. We list a few of these facts that we will use frequently, with some indication of proofs:

Proposition 4.2. *If M is presented in an exact sequence*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow M \longrightarrow 0$$

where N_2 is flat over R (e.g., N_2 is a free R-module), then M is flat over R if and only if

$$N_1/\mathfrak{a}N_1 \longrightarrow N_2/\mathfrak{a}N_2$$

is injective for all ideals $\mathfrak{a} \subset R$.

This is seen by "chasing" the diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & \text{Kernel?} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \mathfrak{a} \otimes_R N_1 & \longrightarrow & N_1 & \longrightarrow & N_1/\mathfrak{a}N_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{a} \otimes_R N_2 & \longrightarrow & N_2 & \longrightarrow & N_2/\mathfrak{a}N_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\text{Kernel?} & \longrightarrow & \mathfrak{a} \otimes_R M & \longrightarrow & M & \longrightarrow & M/\mathfrak{a}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

To link flatness of stalks of sheaves with flatness of the module of sections over an affine open set, we need:

Proposition 4.3. *If M is a B-module and B is an A-algebra via $i\colon A \to B$, then:*

$$\begin{aligned}
M \textit{ is flat over } A &\iff \forall \mathfrak{p} \textit{ prime ideals in } A,\ M \otimes_A A_\mathfrak{p} \textit{ is } A_\mathfrak{p}\textit{-flat.} \\
&\iff \forall \mathfrak{p} \textit{ prime ideals in } B,\ \textit{if } \mathfrak{p}_0 = i^{-1}(\mathfrak{p}),\ \textit{then } M_\mathfrak{p} \textit{ is } A_{\mathfrak{p}_0}\textit{-flat.} \\
&\iff \textit{The sheaf } \widetilde{M} \textit{ on } \operatorname{Spec} B \textit{ is flat over } \operatorname{Spec} A \textit{ (Definition 2.10).}
\end{aligned}$$

Proposition 4.4.

a) *If M is an A-module and B is an A-algebra, then*

$$M \textit{ flat over } A \Longrightarrow M \otimes_A B \textit{ flat over } B.$$

---

[5] (Added in publication) It would be worthwhile to point out that for an R-module M, the following are equivalent:

(i) M is flat over R.

(ii) (See, e.g., Bourbaki [**27**, Chap. I, §2.11, Corollary 1], Eisenbud [**38**, Corollary 6.5], Matsumura [**78**, Thoerem 7.6] and Mumford [**86**, Chap. III, §10, p. 295].) For elements $m_1, \ldots, m_l \in M$ and $a_1, \ldots, a_l \in R$ such that $\sum_{i=1}^l a_i m_i = 0$, there exist $m'_j \in M$ $(j = 1, \ldots, k)$ and $b_{ij} \in R$ $(i = 1, \ldots, l; j = 1, \ldots, k)$ such that

$$m_i = \sum_{j=1}^k b_{ij} m'_j \qquad \text{and} \qquad \sum_{i=1}^l b_{ij} a_i = 0.$$

(iii) For any R-homomorphism $\alpha\colon F \to M$ from a free R-module F of finite rank and for any finitely generated R-submodule $K \subset \operatorname{Ker}(\alpha)$, there exist a free R-module F' of finite rank and R-homomorphisms $\beta\colon F \to F'$ and $\gamma\colon F' \to M$ such that $\alpha = \gamma \circ \beta$ and that $K \subset \operatorname{Ker}(\beta)$.

The equivalence of (i) and (ii) is an easy consequence of Bourbaki [**27**, Chap. I, §2.11, Lemma 10]. In (iii), we may assume K to be generated by a single element. Then its equivalence to (ii) is obvious.

From this equivalence, we easily deduce that every flat R-module M is a (filtered) direct limit of free R-modules of finite rank, a result due independently to V. E. Govorov (1965) and D. Lazard (1964). (cf. Eisenbud [**38**, Theorem A6.6])

b) *If $\mathcal{F}$ is a quasi-coherent sheaf on X and we consider a fibre product diagram:*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "q"] & Y \end{tikzcd}$$

then (cf. Definition 2.10)

$$\mathcal{F} \text{ flat over } Y \Longrightarrow p^*\mathcal{F} \text{ flat over } Z.$$

c) *Especially*

$$f \text{ flat} \Longrightarrow g \text{ flat.}$$

Proposition 4.5.

a) *M flat over A $\Longrightarrow$ for all non-zero divisors $a \in A$, $M \xrightarrow{a} M$ is injective.*
b) *If A is a principal ideal domain or valuation ring, the converse is true.*

The point of (a) is that $A \xrightarrow{a} A$ injective implies $M \xrightarrow{a} M$ injective.

Proposition 4.6.

a) *If M is a B-module and B is an A-algebra, where A, B are noetherian and M is finitely generated then*

$$\begin{aligned} M \text{ flat over } A \Longrightarrow\ & \forall \mathfrak{p} \subset B, \text{ an associated primes of } M, \\ & \mathfrak{p} \cap A \text{ is an associated prime of } A. \end{aligned}$$

b) *$f \colon X \to Y$ a morphism of noetherian schemes, $\mathcal{F}$ a coherent sheaf on X, then*

$$\mathcal{F} \text{ flat over } \mathcal{O}_Y \Longrightarrow f(\mathrm{Ass}(\mathcal{F})) \subset \mathrm{Ass}(\mathcal{O}_Y).$$

Especially, f flat, $\eta \in X$ a generic point implies $f(\eta) \in Y$ is a generic point.

In fact, if $\mathfrak{p}_0 := \mathfrak{p} \cap A \notin \mathrm{Ass}(A)$, then there exists an element $a \in \mathfrak{p}_0 A_{\mathfrak{p}_0}$ which is a non-zero divisor. Then multiplication by a is injective in $M_{\mathfrak{p}}$, hence $\mathfrak{p} \notin \mathrm{Ass}(M)$.

Proposition 4.7. *If $\mathcal{F}$ is a coherent sheaf on a scheme X, then*

$$\mathcal{F} \text{ flat over } \mathcal{O}_X \Longleftrightarrow \mathcal{F} \text{ locally free.}$$

Proof. For each $x \in X$, there is a neighborhood U of x and a presentation

$$\mathcal{O}_U^s \xrightarrow{\alpha} \mathcal{O}_U^r \xrightarrow{\beta} \mathcal{F}|_U \longrightarrow 0.$$

Factor this through:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_U^r \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

We may assume that r is minimal, i.e., β induces an isomorphism

$$\Bbbk(x)^r \xrightarrow{\overline{\beta}} \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x.$$

By flatness of $\mathcal{F}_x$ over $\mathcal{O}_{x,X}$,

$$0 \longrightarrow \mathcal{K}_x/\mathfrak{m}_x\mathcal{K}_x \longrightarrow \Bbbk(x)^r \longrightarrow \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x \longrightarrow 0$$

is exact. Therefore $\mathcal{K}_x/\mathfrak{m}_x\mathcal{K}_x = (0)$ and $\mathcal{K}$ is trivial in a neighborhood of x by Proposition I.5.5 (Nakayama). $\square$

Another important general result is that a large class of morphisms are at least flat over an open dense subset of the image scheme:

THEOREM 4.8 (Theorem of generic flatness). *Let $f\colon X \to Y$ be a morphism of finite type between two irreducible reduced noetherian schemes, with $f(\eta_X) = \eta_Y$. Then there is a non-empty open $U \subset Y$ such that* $\operatorname{res} f\colon f^{-1}U \to U$ *is flat and surjective.*

PROOF. We can obviously replace Y by an affine open piece, and then covering X by affines $V_1, \dots, V_k$, if $\operatorname{res} f\colon V_i \cap f^{-1}U_i \to U_i$ is flat, then $\operatorname{res} f\colon f^{-1}(\bigcap U_i) \to \bigcap U_i$ is flat. So we may assume $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and we quote the very pretty lemma of Grothendieck. □

LEMMA 4.9 (SGA 1 [**4**, Exposé IV, Lemme 6.7, p. 102]). *Let A be a noetherian integral domain, B a finitely generated A-algebra, M a finitely generated B-module. Then there exists a non-zero $f \in A$ such that M_f is a free (hence flat) A_f-module.*

PROOF OF LEMMA 4.9. [6] Let K be the quotient field of A, so that $B \otimes_A K$ is a finitely generated K-algebra and $M \otimes_A K$ is a finitely generated module over it. Let n be the dimension of the support of this module and argue by induction on n. If $n < 0$, i.e., $M \otimes_A K = (0)$, then taking a finite set of generators of M over B, one sees that there exists an $f \in A$ which annihilates these generators, and hence M, so that $M_f = (0)$ and we are through. Suppose $n \geq 0$. One knows that the B-module M has a composition series whose successive quotients are isomorphic to modules $B/\mathfrak{p}_i$, $\mathfrak{p}_i \subset B$ prime ideals. Since an extension of free modules is free, one is reduced to the case where M itself has the form $B/\mathfrak{p}$, or even is identical to B, B being an integral domain. Applying Noether's normalization lemma (Zariski-Samuel [**119**, vol. 2, Chapter VII, §7, Theorem 25, p. 200]) to the K-algebra $B \otimes_A K$, one sees easily that there exists a non-zero $f \in A$ such that B_f is integral over a subring $A_f[t_1, \dots, t_n]$, where the t_i are indeterminates. Therefore one can already assume B integral over $C = A[t_1, \dots, t_n]$, so that it is a finitely generated torsion-free C-module. If m is its rank, there exists therefore an exact sequence of C-modules:

$$0 \longrightarrow C^m \longrightarrow B \longrightarrow M' \longrightarrow 0$$

where M' is a torsion C-module. It follows that the dimension of the support of the $C \otimes_A K$-module $M' \otimes_A K$ is strictly less than the dimension n of $C \otimes_A K$. By induction, it follows that, localizing by a suitable $f \in A$, one can assume M' is a free A-module. On the other hand C^m is a free A-module. Therefore B is a free A-module. □

In order to get at what I consider the intuitive content of "flat", we need first a deeper fact:

PROPOSITION 4.10. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let $B = A[X_1, \dots, X_n]_\mathfrak{p}$ where $\mathfrak{p} \cap A = \mathfrak{m}$. Let*

$$K \xrightarrow{u} L \xrightarrow{v} M$$

be finitely generated free B-modules and B-homomorphisms such that $v \circ u = 0$. If

$$K/\mathfrak{m}K \longrightarrow L/\mathfrak{m}L \longrightarrow M/\mathfrak{m}M$$

is exact, then

$$K \longrightarrow L \longrightarrow M$$

is exact and $M/v(L)$ is flat over A.

PROOF. Express u and v by matrices of elements of B and let A_0 be the subring of A generated over $\mathbb{Z}$ by the coefficients of these polynomials. Let $A_1 = (A_0)_{\mathfrak{m} \cap A_0}$. Then A_1 is a *noetherian* local ring and if $B_1 = A_1[X_1, \dots, X_n]_{\mathfrak{p} \cap A_1[X]}$, we may define a diagram

$$K_1 \xrightarrow{u_1} L_1 \xrightarrow{v_1} M_1$$

[6] Reproduced verbatim.

over B_1 such that $K \to L \to M$ arises from it by $\otimes_{B_1} B$ or equivalently by $\otimes_{A_1} A$ and then localizing at $\mathfrak{p}$. Let $\mathfrak{m}_1 = \mathfrak{m} \cap A_1$, $k_1 = A_1/\mathfrak{m}_1$, $k = A/\mathfrak{m}$. Then

$$K_1/\mathfrak{m}_1 K_1 \longrightarrow L_1/\mathfrak{m}_1 L_1 \longrightarrow M_1/\mathfrak{m}_1 M_1$$

is exact because $K/\mathfrak{m}K \to L/\mathfrak{m}L \to M/\mathfrak{m}M$ is exact and arises from it by $\otimes_{k_1} k$ (i.e., a non-exact sequence of k_1-vector spaces remains non-exact after $\otimes_{k_1} k$). Now if we prove the lemma for A_1, B_1, K_1, L_1 and M_1, it follows for A, B, K, L and M. In fact $M_1/v_1(L_1)$ flat over A_1 implies

$$M/v(L) \cong [(M_1/v_1(L_1)) \otimes_{A_1} A]_S \quad (S = \text{multiplicative system } A[X] \setminus \mathfrak{p})$$

is flat over A; and from the exact sequences:

$$K_1 \longrightarrow L_1 \longrightarrow v_1(L_1) \longrightarrow 0$$

$$0 \longrightarrow v_1(L_1) \longrightarrow M_1 \longrightarrow M_1/v_1(L_1) \longrightarrow 0$$

we deduce by $\otimes_{A_1} A$ and by localizing with respect to S that

$$K \longrightarrow L \longrightarrow (v_1(L_1) \otimes_{A_1} A)_S \longrightarrow 0$$

$$0 \longrightarrow (v_1(L_1) \otimes_{A_1} A)_S \longrightarrow M \longrightarrow M/\left(v_1(L_1) \otimes_{A_1} A\right)_S \longrightarrow 0$$

are exact, (using again $M_1/v_1(L_1)$ flat over A_1!), hence $K \to L \to M$ is exact. This reduces the lemma to the case A noetherian. In this case, we use the fact that B noetherian local with $\mathfrak{m} \subset$ maximal ideal of B implies

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n \cdot P = (0)$$

for any finitely generated B-module P (cf. Zariski-Samuel [**119**, vol. I, Chapter IV, Appendix, p. 253]). In particular

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n \cdot (L/u(K)) = (0)$$

or

$$\bigcap_{n=1}^{\infty} (\mathfrak{m}^n L + u(K)) = u(K).$$

So if $x \in \operatorname{Ker}(v)$ and $x \notin \operatorname{Image}(u)$ we can find an n such that $x \in \mathfrak{m}^n \cdot L + u(K)$, but $x \notin \mathfrak{m}^{n+1} \cdot L + u(K)$. Let $x = y + u(z)$, $y \in \mathfrak{m}^n \cdot L$, $z \in K$. The (u,v)-sequence induces by $\otimes\, \mathfrak{m}^n/\mathfrak{m}^{n+1}$ a new sequence:

$$\begin{array}{ccccc}
\mathfrak{m}^n K/\mathfrak{m}^{n+1} K & \xrightarrow{\quad u_n \quad} & \mathfrak{m}^n L/\mathfrak{m}^{n+1} L & \xrightarrow{\quad v_n \quad} & \mathfrak{m}^n M/\mathfrak{m}^{n+1} M \\
\sim\Vert & & \sim\Vert & & \sim\Vert \\
(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k K/\mathfrak{m}K & \longrightarrow & (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k L/\mathfrak{m}L & \longrightarrow & (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_k M/\mathfrak{m}M.
\end{array}$$

The bottom sequence is exact by hypothesis. On the other hand y maps to an element $\overline{y} \in \mathfrak{m}^n L/\mathfrak{m}^{n+1} L$ such that $v_n(\overline{y}) = 0$. Therefore $\overline{y} \in \operatorname{Image} u_n$, i.e., $y \in u(\mathfrak{m}^n K) + \mathfrak{m}^{n+1} L$, hence $x \in u(K) + \mathfrak{m}^{n+1} L$ — contradiction. This proves that the (u,v)-sequence is exact. Next, if $\mathfrak{a} \subset A$ is any ideal, the same argument applies to the sequence:

$$(*) \qquad K/\mathfrak{a} \cdot K \longrightarrow L/\mathfrak{a} \cdot L \longrightarrow M/\mathfrak{a} \cdot M$$

of $B/\mathfrak{a} \cdot B$-modules. Therefore all these sequences are exact. But from the exact sequences:

$$K \longrightarrow L \longrightarrow L/u(K) \longrightarrow 0$$

$$0 \longrightarrow L/u(K) \longrightarrow M \longrightarrow M/v(L) \longrightarrow 0,$$

we get in any case exact sequences:

$$(**) \qquad \begin{array}{l} K/\mathfrak{a}\cdot K \longrightarrow L/\mathfrak{a}\cdot L \longrightarrow (L/u(K))\otimes_A A/\mathfrak{a} \longrightarrow 0 \\ (L/u(K))\otimes_A A/\mathfrak{a} \longrightarrow M/\mathfrak{a}\cdot M \longrightarrow (M/v(L))\otimes_A A/\mathfrak{a} \longrightarrow 0 \end{array}$$

so the exactness of $(*)$ implies that $(**)$ is exact with (0) on the left too, i.e., by Proposition 4.2 $M/v(L)$ is flat over A. □

COROLLARY 4.11. *Let A be a local ring with maximal ideal $\mathfrak{m}$, and let $B = A[X_1, \dots, X_n]_{\mathfrak{p}}$ where $\mathfrak{p} \cap A = \mathfrak{m}$. Let $f_1, \dots, f_k \in B$. Then*

$$B/(f_1, \dots, f_k) \text{ is a flat } A\text{-algebra} \iff \left[\begin{array}{l} \forall \text{ syzygies } \sum_{i=1}^k \overline{g}_i \overline{f}_i = 0 \text{ in } B/\mathfrak{m}B, \\ \exists \text{ syzygy } \sum_{i=1}^k g_i f_i = 0 \text{ in } B \\ \text{with } g_i \text{ lifting } \overline{g}_i. \end{array}\right]$$

PROOF. $\Longleftarrow$: Since $B/\mathfrak{m}B$ is noetherian, the module of syzygies over $B/\mathfrak{m}B$ is finitely generated: let

$$\sum \overline{g}_{i,l}\overline{f}_i = 0, \qquad 1 \le l \le N$$

be a basis, and lift these to syzygies

$$\sum g_{i,l} f_i = 0.$$

Define homomorphisms:

$$B^N \xrightarrow{u} B^k \xrightarrow{v} B$$

$$u(a_1, \dots, a_N) = \left(\sum g_{1,l} a_l, \dots, \sum g_{k,l} a_l\right)$$
$$v(a_1, \dots, a_k) = \sum a_i f_i.$$

Then $v \circ u = 0$ and by construction

$$(B/\mathfrak{m}B)^N \xrightarrow{\overline{u}} (B/\mathfrak{m}B)^k \xrightarrow{\overline{v}} B/\mathfrak{m}B$$

is exact. Therefore $B/v(B^k) = B/(f_1, \dots, f_k)$ is A-flat by Proposition 4.10.

$\Longrightarrow$: Define v as above and call its kernel Syz, the module of syzygies so that we get:

$$0 \longrightarrow \mathrm{Syz} \longrightarrow B^k \xrightarrow{v} B \longrightarrow B/(f_1, \dots, f_k) \longrightarrow 0.$$

Split this into two sequences:

$$0 \longrightarrow \mathrm{Syz} \longrightarrow B^k \longrightarrow (f_1, \dots, f_k) \longrightarrow 0$$

$$0 \longrightarrow (f_1, \dots, f_k) \longrightarrow B \longrightarrow B/(f_1, \dots, f_k) \longrightarrow 0.$$

By the flatness of $B/(f_1, \dots, f_k)$, these give:

$$\mathrm{Syz}/\mathfrak{m}\cdot\mathrm{Syz} \longrightarrow (B/\mathfrak{m}B)^k \longrightarrow (f_1, \dots, f_k) \otimes_B B/\mathfrak{m}B \longrightarrow 0$$

$$0 \longrightarrow (f_1, \dots, f_k) \otimes_B B/\mathfrak{m}B \longrightarrow B/\mathfrak{m}B \longrightarrow B/(\mathfrak{m}B + (f_1, \dots, f_k)) \longrightarrow 0,$$

hence

$$\mathrm{Syz}/\mathfrak{m}\cdot\mathrm{Syz} \longrightarrow (B/\mathfrak{m}B)^k \xrightarrow{\overline{v}} B/\mathfrak{m}B \longrightarrow B/(\mathfrak{m}B + (f_1, \dots, f_k)) \longrightarrow 0$$

is exact. Since $\operatorname{Ker} \overline{v} =$ syzygies in $B/\mathfrak{m}B$, this shows that all syzygies in $B/\mathfrak{m}B$ lift to Syz. □

Putting it succinctly, *flatness means that syzygies for the fibres lift to syzygies for the whole scheme and hence restrict to syzygies for the other fibres*: certainly a reasonable continuity property.

The simplest case is when R is a valuation ring. We give this a name:

Definition 4.12. Let R be a valuation ring, and let η (resp. o) be the generic (resp. closed) point of $\operatorname{Spec} R$. Let $f\colon X \to \operatorname{Spec} R$ be a flat morphism of finite type (by Proposition 4.5, this means: $\mathcal{O}_X$ is a sheaf of torsion-free R-modules). Then we say that the closed fibre X_o of f is a specialization over R of the generic fibre X_η.

Note that the flatness of f is equivalent to saying that X_η is scheme-theoretically dense in X (Proposition II.3.11). In fact, if you start with any X of finite type over R, then define a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ by:

$$\mathcal{I}(U) = \operatorname{Ker}\left(\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X_\eta}(U \cap X_\eta)\right).$$

Then as in Proposition II.3.11 it follows easily that $\mathcal{I}$ is quasi-coherent and for all U affine, $\mathcal{I}(U)$ is just the ideal of R-torsion elements in $\mathcal{O}_X(U)$. If $\mathcal{O}_X/\mathcal{I}$ is the structure sheaf of the subscheme $\widetilde{X} \subset X$, then

a) $\widetilde{X}_\eta = X_\eta$

b) $\widetilde{X}_o$ is a specialization of X_η.

To give some examples of specializations, consider:

Example. 1.) If X is reduced and irreducible, with its generic point over η, then X_o is always a specialization of X_η.

Example. 2.) Denote by M the maximal ideal of R with the residue field $k = R/M$. The quotient field of R is denoted by K. If $f(X_1, \ldots, X_n)$ is a polynomial with coefficients in R and $\overline{f}$ is the same polynomial mod M, i.e., with coefficients in k, then the affine scheme $V(\overline{f}) \subset \mathbb{A}^n_k$ is a specialization of $V(f) \subset \mathbb{A}^n_K$ *provided that* $\overline{f} \not\equiv 0$. In fact, let $X = V(f) \subset \mathbb{A}^n_R$ and note that $R[X_1, \ldots, X_n]/(f)$ is torsion-free.

Example. 3.) If X is anything of finite type over R, and $Y_\eta \subset X_\eta$ is any closed subscheme, there is a unique closed subscheme $Y \subset X$ with generic fibre Y_η such that Y_o is a specialization of Y_η. (Proof similar to discussion above.)

It can be quite fascinating to see how this "comes out", i.e., given Y_η, guess what Y_o will be:

Example. 4.) In $\mathbb{A}^2_K$ with coordinates x, y, let Y_η be the union of the three distinct points

$$(0,0), \quad (0,\alpha), \quad (\alpha,0), \qquad \alpha \in M, \;\; \alpha \neq 0.$$

Look at the ideal:

$$I = \operatorname{Ker}\left(R[x,y] \xrightarrow{\phi} K \oplus K \oplus K\right)$$

where $\phi(f) = (f(0,0), f(0,\alpha), f(\alpha,0))$. I is generated by

$$xy, \quad x(x-\alpha), \quad y(y-\alpha),$$

hence reducing these mod M, we find

$$Y_o = \operatorname{Spec} k[x,y]/(x^2, xy, y^2)$$

the origin with "multiplicity 3". For other triples of points, what Y_o's can occur?

Example. 5.) (Hironaka) Take two skew lines in $\mathbb{A}^3_K$:

l_1 defined by $x = y = 0$

l_2 defined by $z = 0,\ x = \alpha, \quad (\alpha \in M, \alpha \neq 0)$.

Let $Y_\eta = l_1 \cup l_2$. To find Y_o, first compute:

$$I = \operatorname{Ker}\left(R[x,y,z] \longrightarrow \Gamma(\mathcal{O}_{l_1}) \oplus \Gamma(\mathcal{O}_{l_2})\right).$$

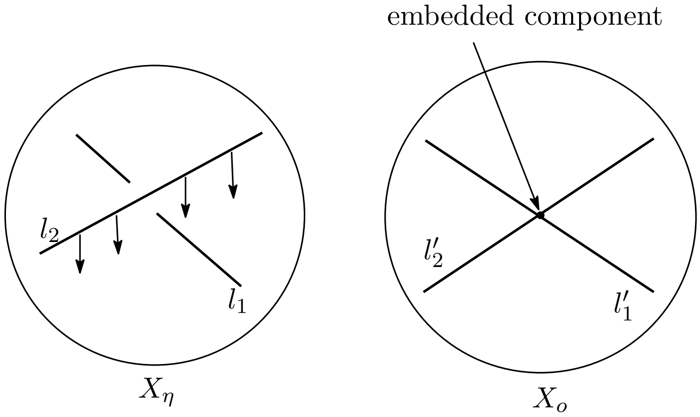

FIGURE IV.11. Specialization of two skew lines

One finds $I = (xz, yz, x(x-\alpha), y(x-\alpha))$. Reducing mod M, we find

$$X_o = \operatorname{Spec} k[x,y,z]/(xz, yz, x^2, xy).$$

Now

$$\sqrt{(x^2, xy, xz, yz)} = (x, yz)$$

which is the ideal of the two lines $l'_1 = V(x,y)$ and $l'_2 = V(x,z)$ which are the limits of l_1 and l_2 individually. But since

$$(x^2, xy, xz, yz) = (x, yz) \cap (x, y, z)^2$$

it follows that X_o has an embedded component where the two lines cross. The picture is as in Figure IV.11.

5. Dimension of fibres of a morphism

We would like to prove some general results on the dimensions of the fibres of a morphism. We begin with the case of a specialization:

THEOREM 5.1 (Dimension Theorem). *Let R be a valuation ring with quotient field K, residue field $k = R/M$, let $S = \operatorname{Spec} R$, and let X be a reduced, irreducible scheme of finite type over S with generic point over η. Then for every component W of X_o:*

$$\dim X_\eta = \dim W$$

$$\textit{i.e.,} \qquad \operatorname{trdeg}_K \mathbf{R}(X) = \operatorname{trdeg}_k \mathbf{R}(W).$$

PROOF. First of all, we may as well replace X by an affine open subset meeting W and not meeting any other components of X_o. This reduces us to the case where $X = \operatorname{Spec} A$ and X_o is irreducible (hence $\sqrt{M \cdot A}$ prime).

Next, the inequality $\dim X_o \leq \dim X_\eta$ is really simple: because if $r = \dim X_o$, then there exist $t_1, \ldots, t_r \in A$ such that $\bar{t}_1, \ldots, \bar{t}_r \in A/\sqrt{M \cdot A}$ are independent transcendentals over k. But if the t_i are dependent over K, let

$$\sum c_\alpha t^\alpha = 0$$

be a relation. Multiplying through by a suitable constant, since R is a valuation ring, we may assume $c_\alpha \in R$ and not all c_α are in M. Then $\sum \bar{c}_\alpha \bar{t}^\alpha = 0$ in $A/\sqrt{M \cdot A}$ is a non-trivial relation over k.

To get started in the other inequality, we will use:

LEMMA 5.2. *Let $k \subset K$ be any two fields and let X be a k-variety. Then if $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K = X_1 \cup \cdots \cup X_t$,*

$$\dim X = \dim X_i, \quad 1 \le i \le t.$$

PROOF OF LEMMA 5.2. If K is an algebraic extension of k, this follows from Theorem 2.4 by going up to $\overline{k}$ and down again. If K is purely transcendental over k, let $K = k(\ldots, t_\alpha, \ldots)$. Then if A is any integral domain containing k with quotient field L,

$$A \otimes_k K = (A[\ldots, t_\alpha, \ldots] \text{ localized with respect to } K \setminus (0))$$

is another integral domain and it contains K and has quotient field $L(\ldots, t_\alpha, \ldots)$. It follows that in this case $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K$ is reduced and irreducible and

$$\mathbf{R}(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K) = \mathbf{R}(X)(\ldots, t_\alpha, \ldots).$$

Therefore

$$\begin{aligned} \dim (X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, K) &= \mathrm{trdeg}_{k(\ldots, t_\alpha, \ldots)} \mathbf{R}(X)(\ldots, t_\alpha, \ldots) \\ &= \mathrm{trdeg}_k \mathbf{R}(X) \\ &= \dim X. \end{aligned}$$

Putting the two cases together, we get the general result. □

LEMMA 5.3. *Let R be any local integral domain (neither noetherian nor a valuation ring!), $S = \mathrm{Spec}\, R$, $\eta, o \in S$ as above. Let X be reduced and irreducible and let $\pi \colon X \to S$ be of finite type. Assume π has a section $\sigma \colon S \to X$. Then $\dim X_o = 0 \Longrightarrow \dim X_\eta = 0$.*

PROOF OF LEMMA 5.3. We can replace X by an affine neighborhood of $\sigma(o)$ and so reduce to the case $X = \mathrm{Spec}\, A$ for simplicity. On the ring level, we get

$$\begin{tikzcd} R \arrow[r, shift right, "\pi^*"'] & A \arrow[l, bend right, "\sigma^*"'] \end{tikzcd}$$

hence

$$A = R \oplus I, \quad \text{where } I = \mathrm{Ker}\, \sigma^*.$$

Consider the sequence of subschemes

$$\begin{tikzcd}[column sep=small] Y_n = \mathrm{Spec}\, A/I^n \arrow[dr] & \subset & X \arrow[dl] \\ & S. & \end{tikzcd}$$

If $x_1, \ldots, x_m$ generate A as a ring over R, let $x_i = a_i + y_i$, $a_i \in R$, $y_i \in I$. Then $y_1^{r_1} \cdots y_m^{r_m}$ with $0 \le \sum r_i < n$ generate A/I^n as a module over R. Being finitely generated over R at all, it follows by Nakayama's lemma that if $z_1, \ldots, z_n \in A/I^n$ generate $(A/I^n) \otimes_R (R/M)$ over R/M, they generate A/I^n over R. Thus

$$(*) \qquad \begin{aligned} \dim_k (A/I^n) \otimes_R k &= (\text{minimal number of generators of } A/I^n) \\ &\ge \dim_K (A/I^n) \otimes_R K. \end{aligned}$$

Now given any 0-dimensional scheme Z of finite type over a field L, then Z consists in a finite set of points $\{P_1, \ldots, P_t\}$, and the local rings $\mathcal{O}_{P_i,Z}$ are artinian. Then in fact $Z \cong \mathrm{Spec}\left(\prod_{i=1}^t \mathcal{O}_{P_i,Z}\right)$, hence is affine and a natural measure of its "size" is

$$\deg_L Z \underset{\text{def}}{=} \dim_L \Gamma(\mathcal{O}_Z).$$

In this language, $(*)$ says $\deg_k (Y_n)_o \ge \deg_K (Y_n)_\eta$. But $(Y_n)_o \subset X_o$ and X_o is itself 0-dimensional, so

$$\deg_k X_o \ge \deg_k (Y_n)_o \ge \deg_K (Y_n)_\eta.$$

This bound shows that $(Y_n)_\eta = (Y_{n+1})_\eta$ if $n \gg 0$. On the other hand, $(Y_n)_\eta$ is the subscheme of X_η consisting of the single point $x = \sigma(\eta)$ and defined by the ideal $\mathfrak{m}^n_{x,X_\eta}$. Thus $\mathfrak{m}^n_{x,X_\eta} = \mathfrak{m}^{n+1}_{x,X_\eta}$ if $n \gg 0$ and since $\mathcal{O}_{x,X_\eta}$ is noetherian, this means $\mathfrak{m}^n_{x,X_\eta} = (0)$ if $n \gg 0$. Thus $\mathcal{O}_{x,X_\eta}$ is in fact finite-dimensional over K, hence

$$\begin{aligned}\dim X_\eta &= \operatorname{trdeg}_K \mathbf{R}(X_\eta)\\ &= \operatorname{trdeg}_K \mathcal{O}_{x,X_\eta}\\ &= 0.\end{aligned}$$

□

Lemma 5.4. *Lemma 5.3 still holds even if a section doesn't exist.*

Proof of Lemma 5.4. Choose $x \in X_o$, let $S' = \operatorname{Spec} \mathcal{O}_{x,X}$, and consider

$$\begin{tikzcd} X \times_S S' \arrow[d] \\ S' \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

where $\sigma = (i, 1_{S'})$, $i\colon \operatorname{Spec} \mathcal{O}_{x,X} \to X$ being the canonical inclusion. Let X' be an irreducible component of $X \times_S S'$ containing $\sigma(S')$ with its reduced structure. Then

$$\begin{aligned}\dim X_o = 0 &\Longrightarrow \dim X'_o = 0 \text{ by Lemma 5.2}\\ &\Longrightarrow \dim X'_\eta = 0 \text{ by Lemma 5.3}\\ &\Longrightarrow \dim X_\eta = 0 \text{ by Lemma 5.2.}\end{aligned}$$

□

Going back to Theorem 5.1, we have now proven that $\dim X_o = 0 \iff \dim X_\eta = 0$. Suppose instead that both dimensions are positive. Choose $t \in A$ such that $\bar{t} \in A/\sqrt{M \cdot A}$ is transcendental over k and let

$$R' = (R[t] \text{ localized with respect to } \mathcal{S} = R[t] \setminus M \cdot R[t]) .$$

This is a new valuation ring with quotient field $K(t)$ and residue field $k(t)$ and π factors:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[ddr, "\pi"'] & \operatorname{Spec} A \arrow[d] & \operatorname{Spec} A_{\mathcal{S}} = X' \arrow[l, hook'] \arrow[d, "\pi'"] \\ & \operatorname{Spec} R[t] \arrow[d] & \operatorname{Spec} R' \arrow[l, hook'] \\ & \operatorname{Spec} R & \end{tikzcd}$$

Since t is transcendental in both A_K and $A/\sqrt{M \cdot A}$, π takes the generic points of both X_o and X_η into the subset $\operatorname{Spec} R'$ of $\operatorname{Spec} R[t]$, i.e., they lie in X'. Now $A_{\mathcal{S}}$ being merely a localization of A, X' has the same stalks as X. Therefore $\mathbf{R}(X'_\eta) = \mathbf{R}(X)$ and $\mathbf{R}(X'_o) = \mathbf{R}(X_o)$ and considering

X' over $S' = \operatorname{Spec} R'$:

$$\begin{aligned}\dim X'_\eta &= \operatorname{trdeg}_{K(t)} \mathbf{R}(X)\\ &= \operatorname{trdeg}_K \mathbf{R}(X) - 1\\ &= \dim X_\eta - 1\\ \dim X'_o &= \operatorname{trdeg}_{k(t)} \mathbf{R}(X_o)\\ &= \operatorname{trdeg}_k \mathbf{R}(X_o) - 1\\ &= \dim X_o - 1.\end{aligned}$$

Making an induction on $\min(\dim X_\eta, \dim X_o)$, this last step completes the proof. □

The dimension theorem (Theorem 5.1) has lots of consequences: first of all it has the following generalization to general morphisms of finite type:

Corollary 5.5. *Let $f\colon X \to Y$ be a morphism of finite type between two irreducible reduced schemes with f(generic point η_X) = generic point η_Y. Then for all $y \in Y$ and all components W of $f^{-1}(y)$:*

$$\dim W \geq \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X) = \dim f^{-1}(\eta_Y).$$

If f is flat over Y, equality holds.

Proof. We may as well assume $f^{-1}(y)$ is irreducible as otherwise we can replace X by an open subset to achieve this. Let $w \in f^{-1}(y)$. Choose a valuation ring R:

$$\mathcal{O}_{w,X} \subset R \subset \mathbf{R}(X)$$

with

$$\mathfrak{m}_{w,X} \subset \text{maximal ideal } M \text{ of } R.$$

Now form the fibre product:

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] & \operatorname{Spec} R. \arrow[l] \end{tikzcd}$$

Note that f' has a section $\sigma\colon \operatorname{Spec} R \to X'$ induced by the canonical map $\operatorname{Spec} R \to \operatorname{Spec} \mathcal{O}_{w,X} \to X$ (as in Lemma 5.4). Break up X'_η into its irreducible components and let their closures in X' with reduced structure be written $X^{(1)}, \ldots, X^{(n)}$. One of these, say $X^{(1)}$ contains the image of the section σ:

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"] & \supset & X^{(1)} \\ Y & \operatorname{Spec} R \arrow[l] \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

Let $o, \eta \in \operatorname{Spec} R$ be its closed and generic points: the various fibres are related by:

$$\begin{aligned} &X^{(1)}_\eta = \text{component of } X'_\eta, && X'_\eta = f^{-1}(\eta_Y) \times_{\operatorname{Spec} \mathbf{R}(Y)} \operatorname{Spec} K\\ &X^{(1)}_o \subset X'_o, && X'_o = f^{-1}(y) \times_{\operatorname{Spec} \mathbf{R}(Y)} \operatorname{Spec} R/M.\end{aligned}$$

Then:

$$\begin{aligned}\dim f^{-1}(y) &= \dim(\text{all components of } X'_o), \quad \text{by Lemma 5.2}\\ &\geq \dim(\text{any component of } X^{(1)}_o)\\ &= \dim X^{(1)}_\eta, \quad \text{by Theorem 5.1}\\ &= \dim f^{-1}(\eta_Y), \quad \text{by Lemma 5.2.}\end{aligned}$$

Now if X is flat over Y, then X' is flat over $\operatorname{Spec} R$, hence

$$X' = X^{(1)} \cup \cdots \cup X^{(n)}$$

(otherwise, let $U \subset X'$ be an open affine disjoint from $\bigcup X^{(i)}$ and if $U = \operatorname{Spec} A$, then $\operatorname{Spec}(A \otimes_R K) = U_\eta = \emptyset$, so A is a torsion R-module contradicting flatness). Therefore

$$X'_o = X_o^{(1)} \cup \cdots \cup X_o^{(n)}$$

hence for at least one i, $X_o^{(i)}$ = a component of X'_o and

$$\begin{aligned} \dim f^{-1}(y) &= \dim X_o^{(i)} \\ &= \dim X_\eta^{(i)} \\ &= \dim f^{-1}(\eta_Y). \end{aligned}$$

□

Combining Corollary 5.5 and Theorem 4.8 (generic flatness), we get:

Corollary 5.6. *Let $f\colon X \to Y$ be as in Corollary 5.5. Then there is an integer n and a non-empty open $U \subset Y$ such that for all $y \in U$ and all components W of $f^{-1}(y)$, $\dim W = n$.*

Combining these results and the methods of Part I [**87**, (3.16)], we deduce:

Corollary 5.7. *Let $f\colon X \to Y$ be any morphism of finite type with Y noetherian. Then the function*

$$x \longmapsto \max\{\dim W \mid W \text{ a component of } f^{-1}(f(x)) \text{ containing } x\}$$

is upper semi-continuous.

Another consequence of Theorem 5.1 is that we generalize Part I [**87**, (3.14)] to varieties over any ground field k:

Corollary 5.8. *Let k be a field and X a k-variety. If $t \in \Gamma(\mathcal{O}_X)$ and*

$$V(t) = \{x \in X \mid t(x) = 0\} \subsetneq X,$$

then for every component W of $V(t)$,

$$\dim W = \dim X - 1.$$

Proof. Let t define a morphism:

$$T\colon X \longrightarrow \mathbb{A}^1_k.$$

Then either T(generic point) = generic point, or T(generic point) = closed point a. In the second case $a \neq 0$ and $v(t) = \emptyset$ so there is nothing to prove. In the first case, $R = \mathcal{O}_{0,\mathbb{A}^1}$ is a valuation ring and making a base change:

$$\begin{tikzcd} X \arrow[d, "T"'] & X' \arrow[l] \arrow[d, "\pi"] \\ \mathbb{A}^1_k & \operatorname{Spec} R \arrow[l] \end{tikzcd}$$

we are in the situation of the dimension theorem. Now $\mathbf{R}(X) = \mathbf{R}(X')$, so

$$\begin{aligned} \dim\big(X'_\eta \text{ over quotient field of } R\big) &= \operatorname{trdeg}_{k(t)} \mathbf{R}(X) \\ &= \operatorname{trdeg}_k \mathbf{R}(X) - 1 \\ &= \dim X - 1 \end{aligned}$$

while W is a component of $T^{-1}(0)$, hence of $\pi^{-1}(0)$ and satisfies:

$$\begin{aligned}\dim(W \text{ over residue field of } R) &= \operatorname{trdeg}_k \mathbf{R}(W)\\ &= \dim W.\end{aligned}$$

$\square$

(Note that we have not used Krull's principal ideal theorem (Zariski-Samuel [**119**, vol. I, Chapter IV, §14, Theorem 29, p. 238]) in this proof.)

Up to this point, we have defined and discussed dimension only for varieties over various fields. There is a natural concept of dimension for arbitrary schemes which by virtue of the above corollary agrees with our definition for varieties:

Definition 5.9. If X is a scheme, then

$$\dim X = \left\{\begin{array}{l}\text{largest integer } n \text{ such that there exists a chain}\\ \text{of irreducible closed subsets: } \emptyset \neq Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n \subset X.\end{array}\right\}$$

If $Z \subset X$ is an irreducible closed set with generic point z, then

$$\begin{array}{l}\operatorname{codim}_X Z\\ \text{or}\\ \operatorname{codim}_X z\end{array} = \left\{\begin{array}{l}\text{largest integer } n \text{ such that there exists a chain}\\ \text{of irreducible closed subsets: } Z = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n \subset X\end{array}\right\}.$$

From the definition, one sees immediately that $\forall Z$ irreducible, closed:

$$\dim Z + \operatorname{codim}_X Z \leq \dim X.$$

But "$<$" can hold even for such spaces as $\operatorname{Spec} R$, R local noetherian integral domain! This pathology makes rather a mess of general dimension theory. The definition ties up with dimension in local ring theory as follows: if $Z \subset X$ is closed and irreducible, and $z \in Z$ is its generic point, then there is a bijection between closed irreducible $Z' \supset Z$ and prime ideals $\mathfrak{p} \subset \mathcal{O}_{z,X}$. Therefore:

$$\operatorname{codim}_X Z = \operatorname{Krull} \dim \mathcal{O}_{z,X}$$

where the Krull dim of a local ring is the maximal length of a chain of prime ideals: cf. Zariski-Samuel [**119**, vol. II, p. 288], or Atiyah-MacDonald [**20**, Chapter 11]. Moreover, in this language, Krull's principal ideal theorem (Zariski-Samuel [**119**, vol. I, Chapter IV, §14, Theorem 29, p. 238]) states:

If X is noetherian reduced and irreducible, $f \in \Gamma(\mathcal{O}_X)$, $f \neq 0$, then for all components W of $V(f)$,

$$\operatorname{codim}_X W = 1,$$

which generalizes Corollary 5.8.

Corollary 5.10. *Let k be a field and X a k-variety. Then the two definitions of dimension agree. More precisely, for every maximal chain*

$$\emptyset \neq Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n = X$$

we have:

$$\operatorname{trdeg}_k \mathbf{R}(Z_i) = i, \quad 0 \leq i \leq n.$$

In particular, X is "catenary", meaning that any two maximal chains have the same length. Therefore for all $Z \subset X$ closed irreducible, with generic point z:

$$\dim Z + \operatorname{codim}_X Z = \dim X$$

or

$$\operatorname{trdeg}_k \Bbbk(z) + \operatorname{Krull}\dim \mathcal{O}_{z,X} = \dim X.$$

Proof. Note on the one hand that a minimal irreducible closed subset Z_0 is just a closed point $Z_0 = \{z_0\}$, hence $\mathbf{R}(Z_0) = \Bbbk(z_0)$ is algebraic over k by Corollary II.2.11. On the other hand, a maximal proper closed irreducible subset $Z \subsetneqq X$ can be analyzed by Corollary 5.8. Let $U \subset X$ be an affine open set meeting Z and let $f \in \mathcal{O}_X(U)$ be a function 0 on $Z \cap U$ and 1 at some closed point $z' \in U \setminus (U \cap Z)$. Then

$$Z \cap U \subset V(f) \subsetneqq U,$$

hence $Z \cap U \subset W \subsetneqq U$, some component W of $V(f)$, hence $Z \subset \overline{W} \subsetneqq X$. By maximality of Z, $Z = \overline{W}$, hence

$$\begin{aligned} \operatorname{trdeg}_k \mathbf{R}(Z) &= \operatorname{trdeg}_k \mathbf{R}(W) \\ &= \operatorname{trdeg}_k \mathbf{R}(X) - 1, \quad \text{by Corollary 5.8.} \end{aligned}$$

These two observations prove Corollary 5.10. □

Corollary 5.11. *Let R be a Dedekind domain with an infinite number of prime ideals and quotient field K and let $\pi\colon X \to \operatorname{Spec} R$ be a reduced and irreducible scheme of finite type over R with $\pi(\eta_X) = \eta_R$, the generic point of* $\operatorname{Spec} R$. *Then every maximal chain looks like:*

$$\emptyset \neq Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_r \subsetneqq Y_{r+1} \subsetneqq \cdots \subsetneqq Y_{n+1} = X$$

where

a) $Z_r \subset \pi^{-1}(a)$ *for some closed point* $a \in \operatorname{Spec} R$ *and* $\operatorname{trdeg}_{\Bbbk(a)} \mathbf{R}(Z_i) = i$, $0 \le i \le r$
b) $\pi(Y_{r+1}) \ni \eta_R$ *and* $\operatorname{trdeg}_K \mathbf{R}(Y_{i+1}) = i$, $r \le i \le n$.

In particular $n = \operatorname{trdeg}_K \mathbf{R}(X)$, X *is catenary and*

$$\dim X = 1 + \operatorname{trdeg}_K \mathbf{R}(X).$$

Proof. This goes just like Corollary 5.10. By Chevalley's Nullstellensatz (Theorem II.2.9) a closed point $Z_0 = \{z_0\}$ of X lies over a closed point a of $\operatorname{Spec} R$ and $\Bbbk(z_0)$ is algebraic over $\Bbbk(a)$. And maximal proper closed irreducible $Z \subsetneqq X$ fall into two cases:
Case i): $Z_\eta \neq \emptyset$, so $Z_\eta \subsetneqq X_\eta$ is a maximal closed irreducible subset and so $\operatorname{trdeg}_K \mathbf{R}(Z) = \operatorname{trdeg}_K \mathbf{R}(X) - 1$;
Case ii): $Z_\eta = \emptyset$, so $Z \subset \pi^{-1}(a)$ in which case Z must be a component of $\pi^{-1}(a)$. Then by the Dimension Theorem (Theorem 5.1), $\operatorname{trdeg}_{\Bbbk(a)} \mathbf{R}(Z) = \operatorname{trdeg}_K \mathbf{R}(X)$. □

An important link between flatness and dimension theory is given by:

Proposition 5.12. *Let $f\colon X \to Y$ be a flat morphism of noetherian schemes and let $x \in X$, $y = f(x)$. Then:*

i) $\operatorname{Spec} \mathcal{O}_{x,X} \to \operatorname{Spec} \mathcal{O}_{y,Y}$ *is surjective,*
ii) $\operatorname{codim}_X(x) \ge \operatorname{codim}_Y(y)$.

Moreover if f is of finite type, then

iii) *for all open sets $U \subset X$, $f(U)$ is open in Y.*

The proof is straightforward using the fact that for all $Z \subset Y$

$$\operatorname{res} f\colon f^{-1}(Z) \longrightarrow Z$$

is still flat, and applying Theorem II.2.9 (Chevalley's Nullstellensatz) and Proposition 4.6.

6. Hensel's lemma

The most important situation for specialization is when the base ring R is a *complete* discrete valuation ring, such as $\mathbb{Z}_p$ or $k[[t]]$. One of the main reasons why this case is special is that Helsel's lemma holds. This "lemma" has many variants but we would like to put it as geometrically as possible:

LEMMA 6.1. (Hensel's lemma)[7]. *Let R be a complete local noetherian ring, $S = \operatorname{Spec} R$ and $\pi\colon X \to S$ a morphism of finite type. Suppose we have a decomposition of the closed fibre:*

$$X_o = Y_o \cup Z_o, \qquad \begin{array}{l} Y_o,\ Z_o \text{ open, disjoint} \\ Y_o = \{y\} \text{ a single point} \end{array}$$

Then we can decompose the whole scheme X:

$$X = Y \cup Z, \qquad \begin{array}{l} Y,\ Z \text{ open disjoint} \\ Y = \operatorname{Spec} B, \text{ finite and integral over } R \end{array}$$

so that $Y_o =$ closed fibre of Y, $Z_o =$ closed fibre of Z.

PROOF. Let $U \subset X$ be an affine open subset such that $U \cap X_o = \{y\}$. Let $U = \operatorname{Spec} B$, and consider the ideal

$$N = \bigcap_{n=1}^{\infty} M^n \cdot B, \quad \text{where } M = \text{maximal ideal of } R.$$

Now $\mathcal{O}_{y,X}$ is a localization $B_{\mathfrak{p}}$ of B and since $M \cdot B_{\mathfrak{p}} \subset \mathfrak{p} \cdot B_{\mathfrak{p}}$, by Krull's theorem (cf. Zariski-Samuel [**119**, vol. I, Chapter IV, §7, p. 216]):

$$N \cdot B_{\mathfrak{p}} \subset \bigcap_{n=1}^{\infty} M^n \cdot B_{\mathfrak{p}} \subset \bigcap_{n=1}^{\infty} (\mathfrak{p} B_{\mathfrak{p}})^n = (0).$$

Therefore, $\exists f \in B \setminus \mathfrak{p}$ such that $f \cdot N = (0)$. Now replace B by its localization B_f and U by U_f. Using this smaller neighborhood of Y, we can assume $\bigcap_{n=1}^{\infty} M^n \cdot B = (0)$. Now recall the algebraic fact:

> *If B is a module over a complete local ring (R, M) such that:*
> a) $\bigcap_{n=1}^{\infty} M^n \cdot B = (0)$
> b) *$B/M \cdot B$ is finite-dimensional over R/M,*
> *then B is a finitely generated R-module* (Zariski-Samuel [**119**, vol. II, Chapter VIII, §3, Theorem 7, p. 259]).

Since $\operatorname{Spec} B/M \cdot B = U_o = Y_o$ consists in one point, $\dim_{R/M} B/MB < +\infty$ and (a) and (b) hold. Therefore B is integrally dependent on R, and by Proposition II.6.5, $\operatorname{res} \pi\colon U \to S$ is a *proper* morphism. It follows that the inclusion i:

$$\begin{tikzcd} U \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

is proper, hence $U = \operatorname{Image}(i)$ is closed in X. Therefore if we set $Y = U$, $Z = X \setminus U$, we have the required decomposition. $\square$

[7]The lemma is also true whenever Y_o is proper over S: cf. EGA [**1**, Chapter III].

Note that in fact, since B is integrally dependent on R, all its maximal ideals contract to $M \subset R$; since $\operatorname{Spec} B$ has only one point, namely y, over the closed point $[M] \in \operatorname{Spec} R$, this means that B has only one maximal ideal, i.e., B is local. Therefore:

$$B = \mathcal{O}_{y,X}.$$

COROLLARY 6.2 (Classical Helsel's lemma). *Let R be a complete local noetherian ring with maximal ideal M and residue field $k = R/M$. Let $f(T)$ be a monic polynomial over R and let $\overline{f}$ be the reduced polynomial over k. Factor $\overline{f}$:*

$$\overline{f} = \prod_{i=1}^{n} g_i^{r_i}$$

where g_i are distinct, irreducible and monic. Then f factors:

$$f = \prod_{i=1}^{n} f_i$$

with $\overline{f}_i = g_i^{r_i}$.

PROOF. Apply Hensel's Lemma 6.1 to $X = \operatorname{Spec} R[T]/(f(T))$.

Then X_o consists in n points $[(g_i)] \in \mathbb{A}^1_k$, hence X decomposes into n disjoint pieces:

$$X = \bigcup_{i=1}^{n} X_i$$
$$X_i = \operatorname{Spec} R[T]/\mathfrak{a}_i$$
$$(X_i)_o = \operatorname{Spec} k[T]/(g_i^{r_i}).$$

If $d_i = \deg(g_i^{r_i})$, then $1, T, \ldots, T^{d_i-1}$ generate the R-module $(R[T]/\mathfrak{a}_i) \otimes_R k \cong k[T]/(g_i^{r_i})$, hence by Nakayama's lemma, they generate $R[T]/\mathfrak{a}_i$. Therefore $T^{d_i} \in \sum_{j=1}^{d_i-1} RT^j$ in $R[T]/\mathfrak{a}_i$, or $\mathfrak{a}_i$ contains a monic polynomial f_i of degree d_i. Then

a) $\overline{f}_i \in (g_i^{r_i})$, and since both are monic of the same degree, $\overline{f}_i = g_i^{r_i}$,
b) $\prod f_i$ is everywhere zero on X, so $\prod f_i \in (f)$, and since both are monic of the same degree, $\prod f_i = f$.

It follows easily that $\mathfrak{a}_i = (f_i)$ too, so that the decomposition of X into components and of f into factors are really equivalent! $\square$

COROLLARY 6.3. *Let R, M, k, $S = \operatorname{Spec} R$ be as before. Then for all finite separable field extension $k \subset L$, there is a unique flat morphism $\pi\colon X_L \to S$ of finite type*[8] *such that*

$$(*) \qquad \begin{array}{l} (X_L)_o \textit{ is reduced and consists in one point } x \\ \Bbbk(x) = L, \ X_L \textit{ connected.} \end{array}$$

In fact for all $p\colon Z \to S$ of finite type and α where:

$$(**) \qquad \begin{array}{l} Z_o = \textit{one point } z, \ Z \textit{ connected} \\ \alpha\colon L \hookrightarrow \Bbbk(z) \textit{ is } k\textit{-homomorphism,} \end{array}$$

there exists a unique S-morphism

$$f\colon Z \longrightarrow X_L$$

such that $f(z) = x$ and $f^*\colon \Bbbk(x) \to \Bbbk(z)$ is equal to α.

[8]In fact, $\pi\colon X_L \to S$ is *étale* in the sense to be defined in §V.3.

Proof. To construct X_L, write $L = k[X]/(\overline{f}(X))$, lift $\overline{f}$ to a polynomial f of the same degree over R and set $X_L = \operatorname{Spec} R[X]/(f(X))$. We prove next that any X_L flat over S with property $(*)$ has the universal property of Corollary 6.3 for all $p\colon Z \to S$ satisfying $(**)$. This will prove, in particular, that any two such X_L's are canonically isomorphic.

Consider

$$p_2\colon X_L \times_S Z \longrightarrow Z.$$

α induces a section $\overline{\sigma}$ of p_2 over $\{z\}$.

$$\begin{tikzcd}
\operatorname{Spec} L \times_S \operatorname{Spec} \Bbbk(z) \arrow[r, hook] & X_L \times_S Z \arrow[d] \\
\operatorname{Spec} \Bbbk(z) = \{z\} \arrow[u, "{(\operatorname{Spec}\alpha,1)}"] \arrow[r, hook] & Z.
\end{tikzcd}$$

By Hensel's Lemma 6.1, $Z = \operatorname{Spec} R'$, R' a finite local R-algebra, hence Hensel's Lemma 6.1 applies with S replaced by Z too: e.g., to p_2. It follows:

$$X_L \times_S Z = W_1 \cup W_2 \qquad \text{(disjoint)}$$
$$W_1 \cap p_2^{-1}(z) = \operatorname{Image} \overline{\sigma}$$
$$W_1 = \operatorname{Spec} R'',\ R'' \text{ a finite local } R'\text{-algebra.}$$

But p_2 is flat so R'' is flat over R', hence free (since R' is local and R'' is a finite R'-module). By assumption

$$(X_L)_o = \operatorname{Spec} L,$$
$$\text{so} \quad p_2^{-1}(z) = \operatorname{Spec}(L \otimes_k \Bbbk(z)).$$

Now L separable over k implies that $L \otimes_k \Bbbk(z)$ is a separable $\Bbbk(z)$-algebra — in particular it has no nilpotents. Thus:

$$p_2^{-1}(z) \cap W_1 \cong \operatorname{Spec} \Bbbk(z)$$

hence $R'' \otimes_{R'} \Bbbk(z) \cong \Bbbk(z)$ and $R'' \otimes_{R'} \Bbbk(z)$ has one generator. Therefore R'' is free over R' with one generator, i.e., $W_1 \cong Z$. This means that $\overline{\sigma}$ extends uniquely to a section σ of p_2:

$$\begin{tikzcd}
& X_L \times_S Z \arrow[d, "p_2"'] \\
\operatorname{Spec} \Bbbk(z) \arrow[ur, "\overline{\sigma}"] \arrow[r, hook] & Z \arrow[u, bend right, "\sigma"']
\end{tikzcd}$$

and $f = p_1 \circ \sigma$ has the required properties. ☐

Corollary 6.4. *Let R be a complete discrete valuation ring, $S = \operatorname{Spec} R$, $\pi\colon X \to S$ a morphism of finite type with X reduced and irreducible. Then:*

$$X_\eta = \textit{one point} \Longrightarrow X_o = \textit{zero or one point.}$$

This corollary allows us to define a very important map, the *specialization map* (to be used in §V.3):

Definition 6.5. Let X be of finite type over R: Let

$$\operatorname{Max}(X_\eta) = \text{set of closed points of } X_\eta$$
$$\operatorname{Max}(X_o) = \text{set of closed points of } X_o.$$

Let

$$\operatorname{Max}(X_\eta)^\circ = \text{set of } x \in \operatorname{Max}(X_\eta) \text{ such that } x \text{ is not closed in } X.$$

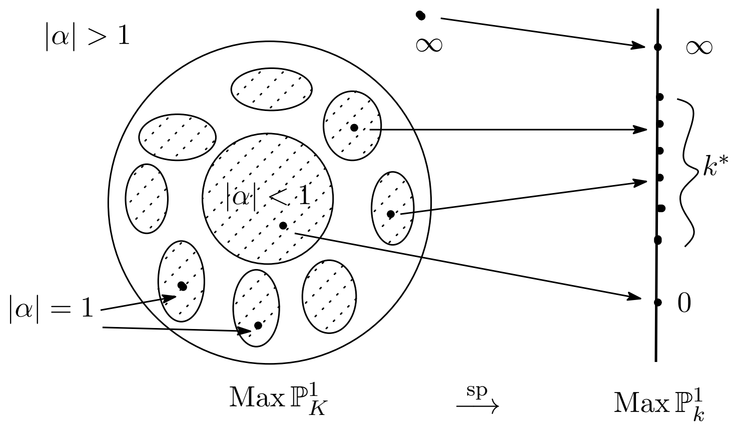

FIGURE IV.12. Specialization map for $\mathbb{P}^1_R$

Let

$$\text{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

be the map

$$x \longmapsto \overline{\{x\}} \cap X_o$$

(apply Corollary 6.4 to $\overline{\{x\}}$ with reduced structure; hence $\#(\overline{\{x\}} \cap X_o) = 0$ or 1). Note that if X is *proper* over S, then $\operatorname{Max}(X_\eta) = \operatorname{Max}(X_\eta)^\circ$ since $\pi(\overline{\{x\}})$ must be closed in S, hence $\overline{\{x\}} \cap X_o \neq \emptyset$.

The spaces $\operatorname{Max}(X_\eta)^\circ$ are the building blocks for the theory of "rigid analytic spaces" over K — cf. Tate [**112**].

EXAMPLE. $X = \mathbb{A}^1_R$. Then

$$\begin{aligned} \operatorname{Max}(\mathbb{A}^1_K) &= \text{set of conjugacy classes of algebraic elements over } K \\ \operatorname{Max}(\mathbb{A}^1_K)^\circ &= \text{those algebraic elements which are integral over } R \\ \operatorname{Max}(\mathbb{A}^1_k) &= \text{set of conjugacy classes of algebraic elements over } k \end{aligned}$$

and sp is the map:

if $x^n + a_1x^{n-1} + \cdots + a_n = 0$ is the irreducible equation for x, then $\operatorname{sp} x$ is a root of the equation $x^n + \overline{a}_1x^{n-1} + \cdots + \overline{a}_n = 0$, $\overline{a}_i = (a_i \bmod M)$.

More succinctly, R defines an absolute value

$$x \longmapsto |x|$$

on K making X into a complete topological field, via

$$|u \cdot \pi^n| = c^{-n}, \qquad \begin{array}{l}(\text{some fixed } c \in \mathbb{R},\ c > 1 \\ \text{all } u \in R^*,\ \pi = \text{generator of } M).\end{array}$$

Then this absolute value extends to $\overline{K}$ and $\operatorname{Max}(\mathbb{A}^1_K)^\circ$ is the unit disc:

$$\{x \text{ up to conjugacy } \mid |x| \leq 1\}.$$

On the other hand, if $X = \mathbb{P}^1_R$, then $\operatorname{Max}(\mathbb{P}^1_K)$ consists in $\{\infty\}$ plus $\operatorname{Max}(\mathbb{A}^1_K)$. And now since $\mathbb{P}^1_R$ is proper over S, sp is defined on the whole set $\operatorname{Max}(\mathbb{P}^1_K)$. It extends the above sp on $\operatorname{Max}(\mathbb{A}^1_K)^\circ$, and carries ∞ as well as the whole set

$$\operatorname{Max}(\mathbb{A}^1_K) \setminus \operatorname{Max}(\mathbb{A}^1_K)^\circ = \{x \text{ up to conjugacy } \mid |x| > 1\}$$

to ∞ in $\operatorname{Max}(\mathbb{P}^1_k)$. It looks like Figure IV.12

Proposition 6.6. *The map*

$$\mathrm{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

is surjective.

The proof goes by induction on $\dim X_o$. If $X_o = 0$, use Hensel's Lemma 6.1. If $x \in \operatorname{Max}(X_o)$ and $\dim X_o \geq 1$, choose $f \in \mathfrak{m}_{x,X}$ with $f \not\equiv 0$ on any component of X_o. Consider the subscheme $V(f)$ in a suitable neighborhood of x and apply Krull's principal ideal theorem (Zariski-Samuel [**119**, vol. I, Chapter IV, §14, Theorem 29, p. 238]). We leave the details to the reader.

Exercise—Addition needed

(1) If k is perfect, show that each k-form of $\overline{X}$ is determined up to k-isomorphism by the Galois action $\{\sigma_X \mid \sigma \in \operatorname{Gal}(\overline{k}/k)\}$ on $\overline{X}$ that it induces.

(2) In the situation of the previous problem, show that the k-forms of $\overline{X}$ over $\overline{k}$ up to k-isomorphism are in one-to-one correspondence with an appropriately defined set

$$H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Aut}_{\overline{k}}(\overline{X}))$$

of "1-group cohomology classes" of the Galois group $\operatorname{Gal}(\overline{k}/k)$ with respect to its natural action on the group $\operatorname{Aut}_{\overline{k}}(\overline{X})$ of $\overline{k}$-automorphisms of $\overline{X}$.

CHAPTER V

Singular vs. non-singular

1. Regularity

The purpose of this section is to translate some well-known commutative algebra into the language of schemes — as general references, see Zariski-Samuel [**119**, vol. I, Chapter IV and vol. II, Chapter VIII] and Atiyah-MacDonald [**20**, Chapter 11]. Consider:

a) $\mathcal{O} =$ local ring
b) $\mathfrak{m} =$ its maximal ideal
c) $k = \mathcal{O}/\mathfrak{m}$
d) $\mathfrak{m}/\mathfrak{m}^2$, a vector space over k
e) $\mathrm{gr}(\mathcal{O}) = \bigoplus_{n=1}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1}$, a graded k-algebra generated over k by $\mathfrak{m}/\mathfrak{m}^2$.

Lemma 1.1 (Easy lemma). *If $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$, then $\mathrm{gr}(\mathcal{O})$ integral domain $\implies$ $\mathcal{O}$ integral domain.*

Proof. If not, say $x, y \in \mathcal{O}$, $xy = 0$, $x \neq 0$, $y \neq 0$. Then $x \in \mathfrak{m}^l \setminus \mathfrak{m}^{l+1}$, $y \in \mathfrak{m}^{l'} \setminus \mathfrak{m}^{l'+1}$ for some l, l'; let $\overline{x} \in \mathfrak{m}^l/\mathfrak{m}^{l+1}$, $\overline{y} \in \mathfrak{m}^{l'}/\mathfrak{m}^{l'+1}$ be their images. Then $\overline{x} \cdot \overline{y} = 0$. ☐

f) Krull $\dim \mathcal{O} =$ length n of the longest chain of prime ideals:

$$\mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_n = \mathfrak{m}$$

g) If $\mathcal{O}$ is noetherian, then recall that

$$\begin{array}{lcl} \dim \mathcal{O} & = & \text{least } n \text{ such that } \exists x_1, \dots, x_n \in \mathfrak{m},\ \ \mathfrak{m} = \sqrt{(x_1, \dots, x_n)} \\ \text{OR} & = & \text{degree of Hilbert-Samuel polynomial } P \text{ defined by} \\ & & \qquad P(n) = l(\mathcal{O}/\mathfrak{m}^n), \quad n \gg 0. \qquad (l \text{ denotes the length.}) \end{array}$$

Definition 1.2. Note that by (g)[1], $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim \mathcal{O}$. Then $\mathcal{O}$ is regular if it is noetherian and equivalently,

$$\mathrm{gr}(\mathcal{O}) = \text{symmetric algebra generated by } \mathfrak{m}/\mathfrak{m}^2$$

OR

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim \mathcal{O}.$$

Note that

$$\mathcal{O} \text{ regular} \implies \mathcal{O} \text{ integral domain}$$

by the Easy Lemma 1.1.

Definition 1.3. Let X be a scheme, $x \in X$. Then

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \underset{\text{def}}{=} \text{Zariski-cotangent space at } x \text{, denoted } T^*_{x,X}$$

$$\mathrm{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \Bbbk(x)) \underset{\text{def}}{=} \text{Zariski-tangent space at } x \text{, denoted } T_{x,X}.$$

[1]Since if $x_1, \dots, x_n \in \mathfrak{m}$ span $\mathfrak{m}/\mathfrak{m}^2$ over k, then by Nakayama's lemma, they generate $\mathfrak{m}$ as an ideal, hence $\dim \mathcal{O} \leq n$.

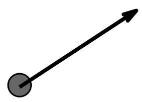

FIGURE V.1. Disembodied tangent vector

Note that we can embed $T_{x,X}$ as the set of $\mathbb{k}(x)$-rational points in an affine space over $\mathbb{k}(x)$:

$$\mathbf{T}_{x,X} = \operatorname{Spec}(\operatorname{Symm}^*(\mathfrak{m}_x/\mathfrak{m}_x^2)) \underset{\text{non-canonically}}{\cong} \mathbb{A}^n_{\mathbb{k}(x)}$$

if $n = \dim_{\mathbb{k}(x)} \mathfrak{m}_x/\mathfrak{m}_x^2$ and $\operatorname{Symm}^* =$ symmetric algebra.

In some cases, the tangent space at a point $x \in X$ has a pretty functorial definition: Suppose X is a scheme over a field k and x is a k-rational point. Then

$$T_{x,X} \cong \left\{ \begin{array}{c} \text{set of all morphisms } \tau \text{ such that} \\ \begin{tikzcd} \operatorname{Spec} k[\epsilon]/(\epsilon^2) \arrow[rr, "\tau"] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd} \\ \text{commutes and Image}\,\tau = \{x\} \end{array} \right\}.$$

In fact, by Proposition I.3.10, the set of such τ is isomorphic to the set of local k-algebra homomorphisms:

$$\tau^* \colon \mathcal{O}_{x,X} \longrightarrow k[\epsilon]/(\epsilon^2).$$

Then $\tau^*(\mathfrak{m}_{x,X}) \subset k \cdot \epsilon$ and $\tau^*(\mathfrak{m}_{x,X}^2) = (0)$. Since $\mathcal{O}_{x,X}$ is a local k-algebra with residue field k:

$$\mathcal{O}_{x,X}/\mathfrak{m}_{x,X}^2 \cong k \oplus \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2,$$

hence τ^* is given by a k-linear map

$$\operatorname{res} \tau^* \colon \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \longrightarrow k \cdot \epsilon$$

and any such map defines a τ^*. But the set of such maps is $T_{x,X}$. Because of this result, one often visualizes $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$ as a sort of disembodied tangent vector as in Figure V.1.

Given a morphism $f \colon X \to Y$, let $x \in X$ and $y = f(x)$. Then f induces maps on the Zariski tangent and cotangent spaces:

i) $f^* \colon \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$ induces a homomorphism of $\mathbb{k}(x)$-vector spaces:

$$df_x^* \colon \left(\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2\right) \otimes_{\mathbb{k}(y)} \mathbb{k}(x) \longrightarrow \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2$$

ii) Dualizing, this gives a morphism

$$df_x \colon T_{x,X} \longrightarrow T_{y,Y} \widehat{\otimes}_{\mathbb{k}(y)} \mathbb{k}(x)$$

(where $\widehat{\ }$ on $\otimes$ comes in only in case $\mathfrak{m}_{y,Y}/\mathfrak{m}_{y,Y}^2$ is infinite dimesional! — in which case $T_{y,Y}$ has a natural linear topology, and one must complete $T_{y,Y} \otimes_{\mathbb{k}(y)} \mathbb{k}(x)$, etc.)

DEFINITION 1.4. The tangent cone to X at x is $\mathbf{TC}_{x,X} = \operatorname{Spec}(\operatorname{gr}(\mathcal{O}_{x,X}))$. Since $\operatorname{gr}(\mathcal{O}_{x,X})$ is a quotient of the symmetric algebra $\operatorname{Symm}(\mathfrak{m}_x/\mathfrak{m}_x^2)$, we get a closed immersion:

$$\mathbf{TC}_{x,X} \subset \mathbf{T}_{x,X}.$$

DEFINITION 1.5. x is a regular point of X if $\mathcal{O}_{x,X}$ is a regular local ring, i.e., if $\mathbf{TC}_{x,X} = \mathbf{T}_{x,X}$. X is regular if it is locally noetherian and all its points are regular.

We will see in §4 below that a complex projective variety X is regular at a point x if and only if it is non-singular at x as defined in Part I [**87**, Chapter I]. Thus the concept of regularity can be viewed as a generalization to arbitrary schemes of the concept of non-singularity (but **n.b.** the remarks in §4 below on Sard's lemma and the examples). Many of the concepts introduced in Part I [**87**] for non-singular varieties go over to general regular schemes. For instance, a basic theorem in commutative algebra is that a regular local ring is a UFD (cf. Zariski-Samuel [**119**, vol. II, Appendix 7]; or Kaplansky [**64**, §4-2]). As we saw in §III.6, this means that we have a classical theory of divisors on a regular scheme, i.e.,

X regular $\Longrightarrow$

$$\left\{\begin{array}{l}\text{Group of Cartier divisors}\\ \mathrm{Div}(X)\text{ on }X\end{array}\right\} \cong \left\{\begin{array}{l}\text{Group of cycles formed from irreducible}\\ \text{codimension one closed subsets}\end{array}\right\}.$$

More generally, it is on a regular scheme X that there is a good intersection theory of cycles whatever their codimension. Recall that a closed irreducible subset $Z \subset X$ is said to have codimension r if the local ring $\mathcal{O}_{\eta_Z,X}$ at its generic point η_Z has Krull dimension r: hence if $z \in Z$ is any point, the prime ideal

$$\mathfrak{p}(Z) \subset \mathcal{O}_{z,X}$$

defining Z has *height* r (i.e., since, by definition, height$(\mathfrak{p}(Z)) =$ length of greatest chain of prime ideals:

$$(0) \subset \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_h = \mathfrak{p}(Z),$$

which equals the Krull dimension of $(\mathcal{O}_{z,X})_{\mathfrak{p}(Z)} \cong \mathcal{O}_{\eta_Z,X}$). Then another basic theorem in commutative algebra is:

1.6.

Algebraic form	*If $\mathcal{O}$ is a regular local ring, $\mathfrak{p}_1, \mathfrak{p}_2 \subset \mathcal{O}$ are prime ideals, and $\mathfrak{p}'$ is a minimal prime ideal containing $\mathfrak{p}_1 + \mathfrak{p}_2$, then* height$(\mathfrak{p}') \leq$ height$(\mathfrak{p}_1) +$ height$(\mathfrak{p}_2)$ *(Serre* [**101**, p. V-18]).

Geometrically, this means:

1.7.

Geometric form	*If X is a regular scheme, and $Z_1, Z_2 \subset X$ are irreducible closed subsets, then for every component W of $Z_1 \cap Z_2$:* $\operatorname{codim} W \leq \operatorname{codim} Z_1 + \operatorname{codim} Z_2$.

Moreover, when equality holds, there is a natural concept of the *intersection multiplicity* of Z_1 and Z_2 along W: see Serre [**101**, Chapter V]. This is defined using the functors Tor_i and allows one to define an associative, commutative, distributive product between cycles which intersect properly (i.e., with no components of too high dimension). (See also §VII.5.) There is, however, one big difficulty in this theory. One of the key methods used in Part I [**87**] in our discussion of intersections in the classical case of X over $\operatorname{Spec}\mathbb{C}$ is the "reduction to the diagonal Δ": instead of intersecting Z_1, Z_2 in X, we formed the intersection of $Z_1 \times_{\operatorname{Spec}\mathbb{C}} Z_2$ and Δ in $X \times_{\operatorname{Spec}\mathbb{C}} X$, and used the fact that Δ is a local complete intersection in $X \times_{\operatorname{Spec}\mathbb{C}} X$. This reduction works equally well for a regular variety X over any algebraically closed field k, and can be extended to all equi-characteristic X, but fails for regular schemes like $\mathbb{A}^n_{\mathbb{Z}}$ with mixed characteristic local rings (residue field of characteristic p, quotient field of characteristic 0). The problem is that the product

$$\mathbb{A}^{2n}_{\mathbb{Z}} = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$$

has dimension $2n+1$ which is less than $2(\dim \mathbb{A}^n_{\mathbb{Z}}) = 2n+2$: for instance, at the point $P \in \mathbb{A}^n_{\mathbb{Z}}$ where $X_1 = \cdots = X_n = 0$ over $[p] \in \operatorname{Spec}\mathbb{Z}$, the cotangent space to $\mathbb{A}^n_{\mathbb{Z}}$ has a basis

$$dX_1, \ldots, dX_n, dp.$$

And at the point $(P, P) \in \mathbb{A}^{2n}_{\mathbb{Z}}$, if we let X_i and Y_i be coordinates in the two factors,

$$dX_1, \ldots, dX_n, dY_1, \ldots, dY_n, dp$$

is a basis of the cotangent space. Thus it is not like a product in the arithmetic direction. One finds, e.g., that $Z_1, Z_2 \subset \mathbb{A}^n_{\mathbb{Z}}$ may intersect properly, while $Z_1 \times_{\operatorname{Spec}\mathbb{Z}} Z_2$, $\Delta \subset \mathbb{A}^{2n}_{\mathbb{Z}}$ don't; that Z_1, Z_2 may be regular while $Z_1 \times_{\operatorname{Spec}\mathbb{Z}} Z_2$ is not. Nonetheless, Serre managed to show that intersection theory works except for one property: it is still unknown whether the intersection multiplicity $i(Z_1, Z_2; W)$ is always positive![2]

For intersection theory on non-singular varieties of arbitrary characteristic, see Samuel [**95**]. A basic fact from commutative algebra that makes it work is the following:

PROPOSITION 1.8. *Let R be a regular local ring of dimension r, with maximal ideal $\mathfrak{m}$, residue field k and quotient field K. Let M be a finitely generated R-module. Then there is a Hilbert-Samuel polynomial $P(t)$ of degree at most r such that*

$$P(n) = l(M/\mathfrak{m}^n M) \quad \text{if } n \gg 0. \qquad (l \text{ denotes the length.})$$

Let

$$P(t) = e\frac{t^r}{r!} + \text{lower terms.}$$

Then

$$e = \dim_K(M \otimes_R K).$$

Proof left to the reader.

2. Kähler differential

Again we begin with algebra: let B be an A-algebra:

[2] (Added in publication) Let P and Q be prime ideals in a regular local ring such that $R/(P+Q)$ has finite length (hence $\dim(R/P) + \dim(R/Q) \leq \dim(R)$). Serre defined the intersection number to be

$$\chi(R/R, R/Q) := \sum_{i=0}^{\infty} (-1)^i \operatorname{length}_R \operatorname{Tor}_i^R(R/P, R/Q),$$

and conjectured

- (non-negativity) $\chi(R/P, R/Q) \geq 0$,
- (positivity) $\chi(R/P, R/Q) > 0$ if and only if $\dim(R/P) + \dim(R/Q) = \dim(R)$.

Serre proved the assertions when R contains a field (equi-characteristic case) using reduction to the diagonal. For the mixed characteristic case, the vanishing (the "only if" part of the positivity conjecture) was proved in 1985 by Roberts [**93**] and independently by Gillet-Soulé [**41**]. The non-negativity conjecture was proved by O. Gabber in the middle of 1990's. The positivity conjecture in the mixed characteristic case is still open.

2.1.

$$\begin{aligned}\Omega_{B/A} \underset{def}{=} & \textit{free } B\textit{-module on elements } db, \textit{ for all } b \in B, \\ & \textit{modulo the relations:} \\ & \quad d(b_1 + b_2) = db_1 + db_2 \\ & \quad d(b_1 b_2) = b_1 \cdot db_2 + b_2 \cdot db_1, \\ & \quad d(a) = 0, \ \textit{for all } a \in A.\end{aligned}$$

In other words, the map

$$d\colon B \longrightarrow \Omega_{B/A}$$

is an A-derivation and $(\Omega_{B/A}, d)$ is universal — i.e., for all B-module M and all maps

$$D\colon B \longrightarrow M$$

such that

$$\begin{aligned}D(b_1 + b_2) &= Db_1 + Db_2 \\ D(b_1 b_2) &= b_1 \cdot Db_2 + b_2 \cdot Db_1 \\ Da &= 0, \text{ all } a \in A,\end{aligned}$$

there is a unique B-module homomrophism $\phi\colon \Omega_{B/A} \to M$ such that $D = \phi \circ d$.

Proposition 2.2. *If*

$$I = \operatorname{Ker}(B \otimes_A B \ni b_1 \otimes b_2 \longmapsto b_1 b_2 \in B),$$

then I/I^2 is a $(B \otimes_A B)/I$-module, i.e., a B-module, and

$$\Omega_{B/A} \cong I/I^2 \qquad (\textit{as } B\textit{-module}).$$

d goes over to the map

$$\begin{aligned}B &\longrightarrow I \\ b &\longmapsto 1 \otimes b - b \otimes 1.\end{aligned}$$

Proof. I) check that $b \mapsto b \otimes 1 - 1 \otimes b$ is an A-derivation from B to I/I^2. Therefore it extends to a B-module homomorphism $\Omega_{B/A} \to I/I^2$.

II) Define a ring $E = B \oplus \Omega_{B/A}$, where B acts on $\Omega_{B/A}$ through the module action and the product of two elements of $\Omega_{B/A}$ is always 0. Define an A-bilinear map $B \times B \to E$ by $(b_1, b_2) \mapsto (b_1 b_2, b_1 db_2)$. By the universal mapping property of $\otimes$, it factors

$$B \times B \longrightarrow B \otimes_A B \xrightarrow{\phi} E$$

and it follows immediately that $\phi(I) \subset \Omega_{B/A}$. Therefore $\phi(I^2) = (0)$ and ϕ gives $\overline{\phi}\colon I/I^2 \to \Omega_{B/A}$.

III) These maps are easily seen to be inverse to each other. □

Some easy properties of Ω are:

2.3. *If B and C are A-algebras, then:*

$$\Omega_{(B \otimes_A C)/C} \cong \Omega_{B/A} \otimes_A C.$$

2.4. *If $\mathfrak{a} \subset B$ is an ideal then there is a natural map*

$$\begin{aligned}\mathfrak{a}/\mathfrak{a}^2 &\longrightarrow \Omega_{B/A} \otimes_B (B/\mathfrak{a}) \\ a &\longmapsto da \otimes 1\end{aligned}$$

and the cokernel is isomorphic to $\Omega_{(B/\mathfrak{a})/A}$.

2.5. *If B is an A-algebra and C is a B-algebra, then there is a natural exact sequence*

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0.$$

EXAMPLE. 1: Let $A = k$, $B = k[X_1, \ldots, X_n]$. Then $\Omega_{B/A}$ is a free B-module with generators $dX_1, \ldots, dX_n$, and

$$dg = \sum_{i=1}^{n} \frac{\partial g}{\partial X_i} \cdot dX_i, \quad \text{all } g \in B.$$

More generally, if

$$B = k[X_1, \ldots, X_n]/(f_1, \ldots, f_m),$$

then $\Omega_{B/A}$ is generated, as B-module, by $dX_1, \ldots, dX_n$, but with relations:

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} \cdot dX_j = 0.$$

EXAMPLE. 2: What happens when A and B are fields, i.e., $\Omega_{K/k} = ?$. The dual K-vector space $\mathrm{Hom}_K(\Omega_{K/k}, K)$ is precisely the vector space $\mathrm{Der}_k(K, K)$ of k-derivations from K to K. Then it is well known:

a) $\mathrm{Der}_k(K, K) = (0) \Longleftrightarrow K/k$ is separable algebraic.
b) If $\{f_\alpha\}_{\alpha \in S}$ is a transcendence basis of K over k and K is separable over $k(\ldots, f_\alpha, \ldots)$, then a k-derivation D can have any values on the f_α and is determined by its values on the f_α's.
c) If characteristic $k = p$, then any k-derivation D kills $k \cdot K^p$. If $p^s = [K : k \cdot K^p]$ and we write $K = kK^p(b_1^{1/p}, \ldots, b_s^{1/p})$, $(b_i \in k \cdot K^p)$, and $a_i = b_i^{1/p}$, then a k-derivation D can have any values on the a_i and is determined by its values on the a_i's.

We conclude:

a′) $\Omega_{K/k} = (0) \Longleftrightarrow K/k$ is separable algebraic.
(More generally, if R is a finitely generated k-algebra, then it is not hard to show that

$$\Omega_{R/k} = (0) \Longleftrightarrow R \text{ is a direct sum of separable algebraic field extensions.)}$$

b′) If K is finitely generated and separable over k, then $\forall f_1, \ldots, f_n \in K$,

$$\begin{bmatrix} df_1, \ldots, df_n \text{ are} \\ \text{a basis of } \Omega_{K/k} \end{bmatrix} \Longleftrightarrow \begin{bmatrix} f_1, \ldots, f_n \text{ are a separating transcendence} \\ \text{basis of } K \text{ over } k \end{bmatrix}.$$

c′) If K is finitely generated over k and $\mathrm{char}(K) = p$ and $p^s = [K : k \cdot K^p]$, then $\forall f_1, \ldots, f_s \in K$,

$$\begin{bmatrix} f_1, \ldots, f_s \text{ are a } p\text{-basis of } K \text{ over } k, \\ \text{i.e., } K = k \cdot K^p(f_1, \ldots, f_s) \end{bmatrix} \Longleftrightarrow \begin{bmatrix} df_1, \ldots, df_s \text{ are a} \\ \text{basis of } \Omega_{K/k} \end{bmatrix}.$$

It follows easily too that if $f_1, \ldots, f_s$ are a p-basis then $\mathrm{Der}_{k(f_1, \ldots, f_s)}(K, K) = (0)$, hence K is separable algebraic over $k(f_1, \ldots, f_s)$. Thus

$$s \geq \mathrm{trdeg}_k K$$

with equality if and only if K is separable over k.

For details here, cf. for example, Zariski-Samuel [**119**, vol. I, Chapter 2, §17].

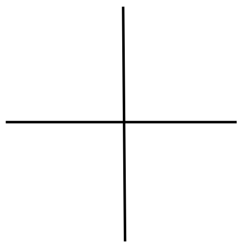

FIGURE V.2. Crossing lines

EXAMPLE. 3: Let $A = k$, $B = k[X, Y]/(XY)$. Then by Example 1, dX and dY generate $\Omega_{B/A}$ with the one relation $XdY + YdX = 0$.

Consider the element $\omega = XdY = -YdX$. Then $X\omega = Y\omega = 0$, so the submodule M generated by ω is $k\omega$, a one-dimensional k-space. On the other hand, in Ω/M we have $XdY = YdX = 0$, so $\Omega/M \cong B \cdot dX \oplus B \cdot dY$. Note that $B \cdot dX \cong \Omega_{B_X/k}$, where $B_X = B/(Y) \cong k[X]$; likewise, $B \cdot dY \cong \Omega_{B_Y/k}$. That is, the module of differentials on $\operatorname{Spec} B$ (which looks like that in Figure V.2) is the module of differentials on the horizontal and vertical lines separately extended by a torsion module. (One can even check that the extension is non-trivial, i.e., does not split.)

All this is easy to globalize. Let $f\colon X \to Y$ be any morphism. The closed immersion

$$\Delta\colon X \longrightarrow X \times_Y X$$

"globalizes" the multiplication homomorphism $\delta\colon B \otimes_A B \to B$. Let $\mathcal{I}$ be the quasi-coherent $\mathcal{O}_{X\times_Y X}$-ideal defining the closed subscheme $\Delta(X)$. Then $\mathcal{I}^2$ is also a quasi-coherent $\mathcal{O}_{X\times_Y X}$-ideal and $\mathcal{I}/\mathcal{I}^2$ is a quasi-coherent $\mathcal{O}_{X\times_Y X}$-module. It is also a module over $\mathcal{O}_{X\times_Y X}/\mathcal{I}$, which is $\mathcal{O}_{\Delta(X)}$ extended by zero. As all its stalks off $\Delta(X)$ are 0, $\mathcal{I}/\mathcal{I}^2$ is actually a sheaf of $(\Delta(X), \mathcal{O}_{\Delta(X)})$-modules, quasi-coherent in virtue of the nearly tautologous:

LEMMA 2.6. *If $S \subset T$ are a scheme and a closed subscheme, and if $\mathcal{F}$ is an $\mathcal{O}_S$-module, then $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module on S if and only if $\mathcal{F}$, extended by (0) on $T \setminus S$, is a quasi-coherent $\mathcal{O}_T$-module on T.*

DEFINITION 2.7. $\Omega_{X/Y}$ is the quasi-coherent $\mathcal{O}_X$-module obtained by carrying $\mathcal{I}/\mathcal{I}^2$ back to X by the isomorphism $\Delta\colon X \xrightarrow{\sim} \Delta(X)$.

Clearly, for all $U = \operatorname{Spec}(B) \subset X$ and $V = \operatorname{Spec}(A) \subset Y$ such that $f(U) \subset V$, the restriction of $\Omega_{X/Y}$ to U is just $\widetilde{\Omega_{B/A}}$. Therefore we have globalized our affine construction.

The following properties are easy to check:

2.8. *The stalks of $\Omega_{X/Y}$ are given by:*

$$(\Omega_{X/Y})_x = (\Omega_{\mathcal{O}_{x,X}/\mathcal{O}_{y,Y}})^{\sim} \qquad (\textit{if } y = f(x)).$$

2.9.

$$\Omega_{(X\times_S Y)/Y} \cong \Omega_{X/S} \otimes_{\mathcal{O}_S} \mathcal{O}_Y.$$

2.10. *If $Z \subset X$ is a closed subscheme defined by the sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$, then $\exists$ a canonical map:*

$$(*) \qquad \begin{aligned} \mathcal{I}/\mathcal{I}^2 &\longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \\ a &\longmapsto da \otimes 1 \end{aligned}$$

and the cokernel is isomorphic to $\Omega_{Z/Y}$.

2.11. *If X is of finite type over Y, then $\Omega_{X/Y}$ is finitely generated.*

2.12. *If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{F}) \cong \{\textit{sheaf of derivations from } \mathcal{O}_X \textit{ to } \mathcal{F} \textit{ over } \mathcal{O}_Y\}.$$

(2.10) allows us to compare the Zariski-cotangent space at $x \in X$ and $\Omega_{X/Y}$. In fact, if you let $Z = \overline{\{x\}}$ with reduced structure, and look at the stalks of $(*)$ at x, you get the $\Bbbk(x)$-linear homomorphism:

$$\begin{aligned} \mathfrak{m}_x/\mathfrak{m}_x^2 &\longrightarrow (\Omega_{X/Y})_x \otimes_{\mathcal{O}_{x,X}} \Bbbk(x) \\ a &\longmapsto da \otimes 1 \end{aligned}$$

and the cokernel is

$$(\Omega_{Z/Y})_x \cong \Omega_{\Bbbk(x)/\mathcal{O}_{y,Y}} \cong \Omega_{\Bbbk(x)/\Bbbk(y)}.$$

Moreover $\mathfrak{m}_y \cdot \mathcal{O}_x$ is in the kernel since $da = 0$, $\forall a \in \mathcal{O}_y$. Now in reasonably geometric cases such as when X and Y are of finite type over an algebraically closed k, and x and y are closed points, then $\Bbbk(x) = \Bbbk(y) = k$, so $\Omega_{\Bbbk(x)/\Bbbk(y)} = (0)$; and it turns out that the induced map

$$T^*_{x,f^{-1}(y)} \cong \mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x) \longrightarrow (\Omega_{X/Y})_x \otimes_{\mathcal{O}_x} \Bbbk(x)$$

is injective too, i.e., the quasi-coherent sheaf $\Omega_{X/Y}$ essentially results from glueing together the separate vector spaces $\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x)$ — which are nothing but the cotangent spaces to the fibres $f^{-1}(y)$ at various points x.

To prove this and see what happens in nasty cases, first define:

Definition 2.13 (Grothendieck). If $K \supset k$ are two fields, let

$$\Upsilon_{K/k} = \operatorname{Ker}(\Omega_{k/\mathbb{Z}} \otimes_k K \longrightarrow \Omega_{K/\mathbb{Z}})$$

called the module of imperfection.

This is a K-vector space and its dual is

$$\{\text{space of derivations } D\colon k \to K\}/\{\text{restrictions of derivations } D\colon K \to K\}$$

which is well known to be 0 iff K is separable over k (cf. Zariski-Samuel [**119**, vol. I, Chapter II, §17, Theorem 42, p. 128]).

Theorem 2.14. *For all $f\colon X \to Y$ and all $x \in X$, if $y = f(x)$, there is a canonical 5-term exact sequence:*

$$0 \longrightarrow \Upsilon^{\Bbbk(x)}_{(\mathcal{O}_{x,X} \otimes_{\mathcal{O}_{y,Y}} \Bbbk(y))/\Bbbk(y)} \longrightarrow \Upsilon_{\Bbbk(x)/\Bbbk(y)} \longrightarrow T^*_{x,f^{-1}(y)} \longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_{x,X}} \Bbbk(x) \longrightarrow \Omega_{\Bbbk(x)/\Bbbk(y)} \longrightarrow 0,$$

where

$$\Upsilon^{\Bbbk(y)}_{(\mathcal{O}_{x,X} \otimes_{\mathcal{O}_{y,Y}} \Bbbk(y))/\Bbbk(y)} := \operatorname{Ker}\left(\Omega_{\Bbbk(y)/\mathbb{Z}} \otimes_{\Bbbk(y)} \Bbbk(x) \longrightarrow \Omega_{(\mathcal{O}_{x,X} \otimes_{\mathcal{O}_{y,Y}} \Bbbk(y))/\mathbb{Z}} \otimes_{\mathcal{O}_{x,X}} \Bbbk(x)\right).$$

Proof. By (2.9), none of the terms change if we make a base change:

$$\begin{tikzcd} X \arrow[d, "f"'] & f^{-1}(y) \arrow[l] \arrow[d] \\ Y & \operatorname{Spec} \Bbbk(y). \arrow[l] \end{tikzcd}$$

Therefore we may assume $Y = \operatorname{Spec} k$, $k = \Bbbk(y)$ a field. But now $(\Omega_{X/Y})_x = \Omega_{\mathcal{O}_{x,X}/k}$ and note that if $R = \mathcal{O}_{x,X}/\mathfrak{m}_x^2$

$$\Omega_{\mathcal{O}_{x,X}/k} \otimes \Bbbk(x) \cong \Omega_{R/k} \otimes \Bbbk(x)$$

(by (2.4) applied with $\mathfrak{a} = \mathfrak{m}_x^2$). We are reduced to the really elementary:

Lemma. *Let R be a local k-algebra, with maximal ideal M, residue field $K = R/M$. Assume $M^2 = (0)$. There is a canonical exact sequence:*

$$\Upsilon_{K/k} \longrightarrow M \longrightarrow \Omega_{R/k} \otimes_R K \longrightarrow \Omega_{K/k} \longrightarrow 0.$$

Proof of Lemma. By (2.4) we have an exact sequence:

$$M \xrightarrow{\alpha} \Omega_{R/\mathbb{Z}} \otimes_R K \longrightarrow \Omega_{K/\mathbb{Z}} \longrightarrow 0.$$

Now by Cohen's structure theorem (Zariski-Samuel [**119**, vol. II, Chapter VIII, §12, Theorem 27, p. 304]), as a ring (but not necessarily as k-algebra), $R \cong K \oplus M$. Using such a direct sum decomposition, it follows that the projection of R onto M is a derivation of R into the $K = R/M$-module M, hence it factors:

$$\begin{tikzcd} R \arrow[rr, "\text{projection}"] \arrow[dr, "d"'] & & M \\ & \Omega_{R/\mathbb{Z}} \otimes_R K \arrow[ur, "\beta"'] & \end{tikzcd}$$

It is easy to see that $\beta \circ \alpha = 1_M$ and this proves that α is injective! Now the homomorphism $k \to R$ gives rise to an exact sequence $\Omega_{k/\mathbb{Z}} \otimes_k R \to \Omega_{R/\mathbb{Z}} \to \Omega_{R/k} \to 0$, hence to:

$$\begin{tikzcd} & 0 \arrow[d] & & \\ & M \arrow[d] & & \\ \Omega_{k/\mathbb{Z}} \otimes_k K \arrow[r] & \Omega_{R/\mathbb{Z}} \otimes_R K \arrow[r] \arrow[d] & \Omega_{R/k} \otimes_k K \arrow[r] & 0 \\ & \Omega_{K/\mathbb{Z}} \arrow[d] & & \\ & 0 & & \end{tikzcd}$$

It follows from this diagram that there is a natural map from $\mathrm{Ker}(\Omega_{k/\mathbb{Z}} \otimes_k K \to \Omega_{K/\mathbb{Z}})$, i.e., $\Upsilon_{K/k}$, to M and that the image is $\mathrm{Ker}(M \to \Omega_{R/k} \otimes_k K)$. This plus (2.4) proves the lemma. $\square$

$\square$

Corollary 2.15. *If $\Bbbk(x)$ is separable algebraic over $\Bbbk(y)$, then*

$$\mathfrak{m}_x/(\mathfrak{m}_x^2 + \mathfrak{m}_y \cdot \mathcal{O}_x) \longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x)$$

is an isomorphism.

Example. 4: A typical case where inseparability enters is:

$$\begin{array}{ll} Y = \operatorname{Spec} k, & k \text{ imperfect and } a \in k \setminus k^p \\ X = \mathbb{A}^1_k, & x = \text{point corresponding to prime ideal } (t^p - a) \subset k[t] \\ & \text{i.e., } x = \text{point with coordinate } a^{1/p}. \end{array}$$

Then

$$\begin{aligned} \Bbbk(x) &= k(a^{1/p}) \\ \mathfrak{m}_x/\mathfrak{m}_x^2 &= (\text{free rank one } \Bbbk(x)\text{-module generated by } t^p - a) \\ \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x) &= (\text{free rank one } \Bbbk(x)\text{-module generated by } dt) \end{aligned}$$

and the map works out:

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow \in \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x)$$

$$t^p - a \longmapsto \tfrac{d}{dt}(t^p - a) \cdot dt = 0$$

hence is 0.

An interesting example of the global construction of Ω is given by the projective bundles introduced in Chapter III:

EXAMPLE. 5: Let S be a scheme and let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_S$-modules. Recall that we constructed $\pi\colon \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E})_S \to S$ by $\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_S(\mathrm{Symm}^\bullet \mathcal{E})$. Let $\mathcal{K}$ be the kernel of the canonical homomorphism α:

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi^*\mathcal{E} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \longrightarrow 0.$$

Then I claim:

2.16.

$$\Omega_{\mathbb{P}(\mathcal{E})/S} \cong \mathcal{K}(-1) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}}(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1), \mathcal{K}).$$

We will prove this locally when $S = \mathrm{Spec}\, R$ is affine and $\mathcal{E}$ is free, leaving to the reader to check that the isomorphism is independent of the choice of basis hence globalizes. Assume

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_S \cdot t_i.$$

Let

$$\begin{aligned} U_i &= \text{open subset } \mathbb{P}(\mathcal{E})_{t_i} \\ &\cong \mathrm{Spec}\, R\left[\frac{t_0}{t_i}, \dots, \frac{t_n}{t_i}\right]. \end{aligned}$$

To avoid confusion, introduce an alias e_i for t_i in

$$\pi^*\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}(\mathcal{E})} \cdot e_i$$

leaving the t_i to denote the induced global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Then

$$\alpha(e_i) = t_i, \qquad 0 \le i \le n$$

and $\mathcal{K} = \mathrm{Ker}(\alpha)$ has a basis on U_j:

$$e_i - \frac{t_i}{t_j} e_j, \qquad 0 \le i \le n,\ i \ne j.$$

Therefore $\mathcal{K}(-1)$ has a basis on U_j:

$$\frac{t_j \otimes e_i - t_i \otimes e_j}{t_j^2}, \qquad 0 \le i \le n,\ i \ne j.$$

On the other hand

$$\Omega_{\mathbb{P}(\mathcal{E})/S}|_{U_j} = \bigoplus_{i=0}^{n} \mathcal{O}_{U_j} \cdot d\left(\frac{t_i}{t_j}\right).$$

Define $\beta\colon \Omega_{\mathbb{P}(\mathcal{E})/S}|_{U_j} \to \mathcal{K}(-1)|_{U_j}$ by

$$\beta\left(d\left(\frac{t_i}{t_j}\right)\right) = \frac{t_j \otimes e_i - t_i \otimes e_j}{t_j^2}.$$

Heuristically, if we expand

$$d\left(\frac{t_i}{t_j}\right) = \frac{t_j dt_i - t_i dt_j}{t_j^2}$$

then β is given by the simple formula

$$\beta(dt_i) = e_i$$

which makes it clear why the definition of β is independent of the choice of basis.

REMARK. (Added in publication) (cf. Example I.8.9) For a locally free $\mathcal{O}_S$-module $\mathcal{E}$ and a positive integer r, let $\pi\colon \operatorname{Grass}^r(\mathcal{E}) \to S$ be the Grassmannian scheme over S, whose Z-valued points for each S-scheme Z are in one-to-one correspondence with the $\mathcal{O}_Z$-locally free quotients $\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E} \to \mathcal{G} \to 0$ of rank r. Let $\alpha\colon \pi^*\mathcal{E} \to \mathcal{Q} \to 0$ be the universal quotient with $\mathcal{Q}$ a locally free $\mathcal{O}_{\operatorname{Grass}^r(\mathcal{E})}$-module of rank r. Let $\mathcal{K} = \operatorname{Ker}(\alpha)$ so that we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \pi^*\mathcal{E} \xrightarrow{\alpha} \mathcal{Q} \longrightarrow 0.$$

Then generalizing the case $r = 1$ in (2.16) above, we have

$$\Omega_{\operatorname{Grass}^r(\mathcal{E})/S} = \mathcal{H}om_{\mathcal{O}_{\operatorname{Grass}^r(\mathcal{E})/S}}(\mathcal{Q}, \mathcal{K}).$$

3. Smooth morphisms

DEFINITION 3.1. First of all, the canonical morphism:

$$\begin{array}{c} X = \operatorname{Spec} R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r) \\ \Big\downarrow{\scriptstyle f} \\ Y = \operatorname{Spec} R \end{array}$$

is called smooth of relative dimension n at a point $x \in X$ whenever the Jacobian matrix evaluated at x:

$$\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{\substack{1\le i\le r\\ 1\le j\le n+r}}$$

has maximal rank, i.e., r. Secondly, an arbitrary morphism $f\colon X \to Y$ is smooth of relative dimension n at a point $x \in X$ if there exist affine open neighborhoods $U \subset X$, $V \subset Y$ of x and y such that $f(U) \subset V$ and $\exists$ a diagram:

$$\begin{tikzcd} U \arrow[r, hook, "\text{open immersion}"'] \arrow[d, "\operatorname{res} f"'] & \operatorname{Spec} R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r) \arrow[d, "g"] \\ V \arrow[r, hook, "\text{open immersion}"'] & \operatorname{Spec} R \end{tikzcd}$$

with g of above type, i.e., $\operatorname{rk}((\partial f_i/\partial X_j)(x)) = r$. f is smooth of relative dimension n if this holds for all $x \in X$. f is étale if it is smooth of relative dimension 0.

REMARK. (Added in publication)

(1) The smoothness of $f\colon X \to Y$ at $x \in X$ does not depend on the choice of the presentation

$$\operatorname{Spec}(R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)) \quad \text{with} \quad \operatorname{rk}\left(\frac{\partial f_i}{\partial X_j}(x)\right) = r.$$

See, for instance, the proof of Proposition 3.6 below.

(2) Smooth morphisms are flat as will be shown in Proposition 3.19 below. An alternative proof can be found in Mumford [**86**, Chap. III, §10, p. 305]. Theorem 3′ there states:

> Let $f\colon X \to Y$ be a morphism of finite type. Then f is smooth of relative dimension k if and only if f is flat and its geometric fibres are disjoint unions of k-dimensional non-singular varieties.

This statement will be given in this book as Criterion 4.8 below. The proof of flatness in Mumford [**86**, Chap. III, §10] successively uses the following ([**86**, Chap. III, §10, p. 297]):

> Let M be a B-module, and B an algebra over A. Let $f \in B$ have the property that for all maximal ideals $\mathfrak{m} \subset A$, multiplication by f is injective in $M/\mathfrak{m}M$. Then M flat over A implies M/fM flat over A.

This very concrete definition has lots of easy consequences:

Proposition 3.2. *If $f\colon X \to Y$ is smooth at $x \in X$, then it is smooth in a neighborhood U of x.*

Proof. If in some affine $U \subset X$ where f is presented as above, δ is the $r \times r$-minor of $(\partial f_i/\partial X_j)$ which is non-zero at x, then f is smooth in the distinguished open subset U_δ of U. $\square$

Proposition 3.3. *If $f\colon X \to Y$ is smooth of relative dimension n, then for all $Y' \to Y$, the canonical morphism*

$$X \times_Y Y' \longrightarrow Y'$$

is smooth of relative dimension n. In particular,

i) *for all $y \in Y$ the fibre $f^{-1}(y)$ is smooth of relative dimension n over $\Bbbk(y)$,*
ii) *if $Y = \operatorname{Spec} k$, $Y' = \operatorname{Spec} \overline{k}$, $\overline{k}$ an algebraic closure of k, then*

$$X \text{ smooth over } k \Longrightarrow \overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k} \text{ smooth over } \overline{k}.$$

Proof. Obvious. $\square$

Proposition 3.4. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are smooth morphisms at $x \in X$ and $y = f(x) \in Y$ respectively, then $g \circ f\colon X \to Z$ is smooth at x.*

Proof. Obvious. $\square$

Proposition 3.5. *A morphism $f\colon X \to Y$ is smooth of relative dimension n at x if and only if it factors in a neighborhood U of x:*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, phantom, "\cap"] & Y \times \mathbb{A}^n \arrow[r, "p_1"] & Y \\ X \arrow[urr, "f"'] & & \end{tikzcd}$$

where g is étale.

Proof. "if" follows from the last result. As for "only if", it suffices to consider the case $X = \operatorname{Spec} R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$, $Y = \operatorname{Spec} R$. Say $\det\left((\partial f_i/\partial X_{n+j})\right)_{1 \le i,j \le r} \neq 0$. Let the homomorphism

$$R[X_1, \dots, X_n] \longrightarrow R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$$

define g. Then g is étale near x and $f = p_1 \circ g$. $\square$

Proposition 3.6. *If $f\colon X \to Y$ is smooth of relative dimension n at $x \in X$, then $\exists$ a neighborhood U of x such that $\Omega_{X/Y}|_U = \mathcal{O}_X^n|_U$. Especially, if f is étale, then $\Omega_{X/Y}|_U = (0)$.*

Proof. It suffices to show that if $S = R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r)$ and $\delta = \det(\partial f_i/\partial X_j)_{1 \le i,j \le r}$, then $(\Omega_{S/R}) \otimes_S S_\delta$ is a free S_δ-module of rank n. But $\Omega_{S/R}$ is generated over S by $dX_1, \dots, dX_{n+r}$ with relations $\sum_{j=1}^{n+r} (\partial f_i/\partial X_j) dX_j = 0$, $1 \le i \le r$. Writing these relations

$$\sum_{j=1}^{r} \frac{\partial f_i}{\partial X_j} dX_j = -\sum_{j=r+1}^{n+r} \frac{\partial f_i}{\partial X_j} dX_j$$

and letting $(\xi_{ij})_{1\le i,j\le r} \in M_r(S_\delta)$ be the inverse of the matrix $(\partial f_i/\partial X_j)_{1\le i,j\le r}$, it follows that in $(\Omega_{S/R}) \otimes_S S_\delta$,

$$dX_l = -\sum_{i=1}^{r}\sum_{j=r+1}^{r+n} \xi_{li} \cdot \frac{\partial f_i}{\partial X_j} \cdot dX_j, \qquad 1 \le l \le r$$

and that these are the only relations among the dX_i's. Therefore $dX_{r+1}, \dots, dX_{r+n}$ are a free basis of $(\Omega_{S/R}) \otimes_S S_\delta$. $\square$

Definition 3.7. If $f\colon X \to Y$ is smooth, let $\Theta_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$, called the relative tangent sheaf of X over Y. Note that it is locally free and if $x \in X$, $y = f(x)$ and $\mathbb{k}(x)$ is separable algebraic over $\mathbb{k}(y)$, then

$$(\Theta_{X/Y})_x \otimes \mathbb{k}(x) \cong T_{x,f^{-1}(y)}, \quad \text{the Zariski-tangent space to the fibre.}$$

Moreover, by (2.12), $\Theta_{X/Y}$ is isomorphic to the sheaf $\mathcal{D}er_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ of derivations from $\mathcal{O}_X$ to itself killing $\mathcal{O}_Y$.

Note moreover that according to the proof of Proposition 3.6, X can be covered by affine open sets U in which there are functions $X_1, \dots, X_n$ such that:

1) any differential $\omega \in \Omega_{X/Y}(U)$ can be uniquely expanded
$$\omega = \sum_{i=1}^{n} a_i \cdot dX_i, \qquad a_i \in \mathcal{O}_X(U),$$
2) any derivation $D \in \Theta_{X/Y}(U)$ can be uniquely expanded
$$D = \sum_{i=1}^{n} a_i \cdot \frac{\partial}{\partial X_i}, \qquad a_i \in \mathcal{O}_X(U)$$
$(\partial/\partial X_i$ dual to $dX_i)$.

When $Y = \operatorname{Spec}\mathbb{C}$, it is easy at this point to identify the sheaves $\Omega_{X/\mathbb{C}}$ and $\Theta_{X/\mathbb{C}}$ with the sheaves of holomorphic differential forms and holomorphic vector fields on X with "polynomial coefficients"; or alternatively, with the sheaves of polynomial sections of the cotangent vector bundle and tangent vector bundle to X. We will discuss this in §VIII.3.

I would like to examine next the relationship between the local rings $\mathcal{O}_{x,X}$ and $\mathcal{O}_{y,Y}$ when there is smooth morphism $f\colon X \to Y$ with $f(x) = y$. When there is no residue field extension, the completions of these rings are related in the simplest possible way:

Proposition 3.8. *If $f\colon X \to Y$ is smooth of relative dimension n at x and if the natural map:*

$$\mathbb{k}(y) \xrightarrow{\approx} \mathbb{k}(x), \qquad \textit{where } y = f(x)$$

is an isomorphism, then the formal completions are related by:

$$\widehat{\mathcal{O}}_{x,X} \cong \widehat{\mathcal{O}}_{y,Y}[[t_1, \dots, t_n]].$$

Proof. The problem being local, we may assume

$$\begin{aligned} X &= \operatorname{Spec} R[X_1, \dots, X_{n+r}]/(f_1, \dots, f_r) \\ Y &= \operatorname{Spec} R, \quad R \text{ local ring, } y = \text{closed point of } Y, \end{aligned}$$
$$\text{with } \det\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{1\le i,j\le r} \ne 0.$$

Now if $x = [\mathfrak{p}]$, $\mathfrak{p} \subset R[X_1, \dots, X_{n+r}]$, then we have inclusions:

$$\mathbb{k}(y) = R/(R \cap \mathfrak{p}) \subset R[X_1, \dots, X_{n+r}]/\mathfrak{p} \subset \mathbb{k}(x).$$

Since all these are equal, $\exists a_1, \ldots, a_{n+r} \in R$ such that $X_i - a_i \in \mathfrak{p}$; more succinctly, x is the point over $y \in Y$ where $X_1 = a_i, \ldots, X_{n+r} = a_{n+r}$. Then $\mathfrak{p} \supset (\mathfrak{p} \cap R + (X_1 - a_1, \ldots, X_{n+r} - a_{n+r}))$ and in fact equality must hold because the ideal on the right is already maximal. Now we may as well change coordinates replacing $X_i - a_i$ by X_i so that x is at the origin, i.e., $\mathfrak{p} = \mathfrak{p} \cap R + (X_1, \ldots, X_{n+r})$. Now if $Z = Y \times \mathbb{A}^{n+r}$, we have

$$\begin{aligned} \mathcal{O}_{x,X} \cong& \mathcal{O}_{x,Z}/(f_1, \ldots, f_r), \\ \mathcal{O}_{x,Z} =& \text{localization of } \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}] \text{ at the} \\ & \quad \text{maximal ideal } \mathfrak{m}_{y,Y} + (X_1, \ldots, X_{n+r}), \end{aligned}$$

hence

$$\begin{aligned} \widehat{\mathcal{O}}_{x,X} &\cong \widehat{\mathcal{O}}_{x,Z}/(f_1, \ldots, f_r), \\ \widehat{\mathcal{O}}_{x,Z} &\cong \widehat{\mathcal{O}}_{y,Y}[[X_1, \ldots, X_{n+r}]]. \end{aligned}$$

Using the hypothesis that f is smooth at x, everything now follows (with $R = \widehat{\mathcal{O}}_{y,Y}[[X_1, \ldots, X_n]]$, $Y_i = X_{n+i}$) from:

Theorem 3.9 (Formal Implicit Function Theorem). *Let R be a ring complete in the $\mathfrak{a}$-adic topology for some ideal $\mathfrak{a} \subset R$. Suppose $f_1, \ldots, f_r \in R[[Y_1, \ldots, Y_r]]$ satisfy*

a) $f_i(0) \in \mathfrak{a}$
b) $\det(\partial f_i / \partial Y_j)(0) \in R^*$.

Then there are unique elements $g_i \in \mathfrak{a}$, $1 \leq i \leq r$, such that

a) $Y_i - g_i \in$ *ideal generated by* $f_1, \ldots, f_r$ *in* $R[[Y]]$
b) $f_i(g_1, \ldots, g_r) = 0$, $1 \leq i \leq r$;

equivalently, (a) *and* (b) *say that the following maps are well-defined isomorphisms inverse to each other:*

$$R \underset{\substack{\text{substitution} \\ h(Y) \mapsto h(g)}}{\overset{\text{inclusion}}{\rightleftarrows}} R[[Y_1, \ldots, Y_r]]/(f_1, \ldots, f_r).$$

Proof of Theorem 3.9. The matrix $(\partial f_i / \partial Y_j)(0)$ is invertible in $M_r(R)$, so changing coordinates by its inverse, we may assume

$$f_i = a_i + Y_i + (\text{terms of degree } \geq 2 \text{ in } Y\text{'s}).$$

Then making induction on r, it is enough to show $\exists g(Y_1, \ldots, Y_{r-1})$ so that:

$$R[[Y_1, \ldots, Y_{r-1}]] \underset{\substack{\text{substitution} \\ \text{of } g \text{ for } Y_r}}{\overset{\text{canonical map}}{\rightleftarrows}} R[[Y_1, \ldots, Y_r]]/(f_r)$$

are well-defined inverse isomorphisms. Letting $R' = R[[Y_1, \ldots, Y_{r-1}]]$, $\mathfrak{a}' = \mathfrak{a} \cdot R + (Y_1, \ldots, Y_{r-1})$, we reduce the proof to the case $r = 1$! We then have merely the linear case of the Weierstrass Preparation Theorem: $f(0) \in \mathfrak{a}$, $f'(0) = 1$, then $\exists$ a unit $u \in R[[Y]]$ and $a \in \mathfrak{a}$ such that $f(Y) = u(Y) \cdot (Y - a)$. This is proven easily by successive approximations:

$$\begin{aligned} a_1 &= 0 \\ a_{n+1} &= a_n - f(a_n) \\ a &= \lim_{n \to \infty} a_n. \end{aligned}$$

One checks by induction that $f(a_n) \in \mathfrak{a}^n$, hence $f(a) = 0$. Making the substitution $Z = Y - a$, $g(Z) = f(Y+a)$ has no constant terms, so $g(Z) = Z \cdot \widetilde{g}(Z)$, so $f(Y) = g(Y-a) = (Y-a) \cdot \widetilde{g}(Y-a)$. Let $u(Y) = \widetilde{g}(Y-a)$. Since $\widetilde{g}(0) = f'(a) \equiv f'(0) \pmod{\mathfrak{a}}$, $u(0) \in R^*$, hence $u \in R[[Y]]^*$. □

□

Unfortunately, there is no such simple structure theorem for $\widehat{\mathcal{O}}_{x,X}$ as $\widehat{\mathcal{O}}_{y,Y}$-algebra in general. If $\Bbbk(x)$ is separable algebraic over $\Bbbk(y)$, then one can still say something: let

$$\widetilde{\mathcal{O}} = \text{the unique finite free } \widehat{\mathcal{O}}_{y,Y}\text{-algebra with } \widetilde{\mathcal{O}}/\mathfrak{m}_{y,Y}\widetilde{\mathcal{O}} \cong \Bbbk(x)$$

as defined in Corollary IV.6.3. Note that $\operatorname{Spec}\widetilde{\mathcal{O}}$ is, in fact étale over $\operatorname{Spec}\mathcal{O}_{y,Y}$: if $\Bbbk(x) \cong \Bbbk(y)[T]/(\overline{f}(T))$ and f lifts $\overline{f}$ and has the same degree, then

$$\widetilde{\mathcal{O}} \cong \widehat{\mathcal{O}}_{y,Y}[T]/(f(T))$$

and

$$\Big(\text{Image in } \widetilde{\mathcal{O}}/\mathfrak{m}_{y,Y}\widetilde{\mathcal{O}} \text{ of } f'(T)\Big) = \overline{f'}(T) \not\equiv 0$$

since $\Bbbk(x)$ is separable over $\Bbbk(y)$. Then it can be proven that

$$\widehat{\mathcal{O}}_{x,X} \cong \widetilde{\mathcal{O}}[[t_1, \ldots, t_n]].$$

If X is étale over Y, this follows directly from the universal property Corollary IV.6.3 of $\widetilde{\mathcal{O}}$. In general, choose the lift f of $\overline{f}$ to have coefficients in $\mathcal{O}_{y,Y}$ and replacing Y by a neighborhood of y, we get a diagram:

$$\begin{array}{ccc} X & \xleftarrow{\;q\;} & \operatorname{Spec}\mathcal{O}_X[T]/(f(T)) = X' \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi'} \\ Y & \xleftarrow[\;p\;]{} & \operatorname{Spec}\mathcal{O}_Y[T]/(f(T)) = Y' \end{array}$$

There is one point $y' \in Y'$ over $y \in Y$ and $\Bbbk(y') \cong \Bbbk(x)$; then we get a point $x' \in X'$ over x and y' as the image of

$$\operatorname{Spec}\Bbbk(x) \longrightarrow \operatorname{Spec}\big(\Bbbk(x) \otimes_{\Bbbk(y)} \Bbbk(y')\big) \longrightarrow X \times_Y Y' = X'.$$

Applying Proposition 3.8 to the smooth φ' and the étale q, we find:

$$\widehat{\mathcal{O}}_{x,X} \cong \widehat{\mathcal{O}}_{x',X'} \cong \widehat{\mathcal{O}}_{y',Y'}[[t_1, \ldots, t_n]] \cong \widetilde{\mathcal{O}}[[t_1, \ldots, t_n]].$$

At any point of a smooth morphism, there is a simple structure theorem for $\operatorname{gr}\mathcal{O}_{x,X}$ as $\operatorname{gr}\mathcal{O}_{y,Y}$-algebra, hence for $\mathbf{TC}_{X,x}$ as a scheme over $\mathbf{TC}_{Y,y}$:

PROPOSITION 3.10. *If $f\colon X \to Y$ is smooth at x of relative dimension n and $y = f(x)$, then $\operatorname{gr}\mathcal{O}_x$ is a polynomial ring in n variables over $\operatorname{gr}(\mathcal{O}_y) \otimes_{\Bbbk} \Bbbk(x)$ — more precisely, $\exists t_1, \ldots, t_n \in \mathfrak{m}_x/\mathfrak{m}_x^2$ such that*

$$\mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1} \cong \bigoplus_{l=0}^{\nu} \bigoplus_{\left(\substack{\text{multi-indices} \\ \alpha,\ |\alpha|=\nu-l}\right)} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1} \otimes_{\Bbbk(y)} \Bbbk(x)\Big) \cdot t^\alpha$$

Thus

$$\mathbf{TC}_{X,x} \cong \mathbf{TC}_{y,Y} \times_{\operatorname{Spec}\Bbbk(y)} \mathbb{A}^n_{\Bbbk(x)}.$$

PROOF. There are two cases to consider: adding a new variable and dividing by a new equation. The first is:

LEMMA 3.11. *Let $x \in Y \times \mathbb{A}^1$, let t be the variable in $\mathbb{A}^1$ and let $y = p_1(x) \in Y$. Note that $p_1^{-1}(y) \cong \mathbb{A}^1_{\Bbbk(y)}$. Either:*

1) *x is the generic point of $\mathbb{A}^1_{\Bbbk(y)}$ in which case $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_x$, $\Bbbk(x) \cong \Bbbk(y)(t)$ and*
$$(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}) \otimes_{\Bbbk(y)} \Bbbk(x) \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$$
is an isomorphism,
2) *x is a closed point of $\mathbb{A}^1_{\Bbbk(y)}$ in which case $\exists$ a monic polynomial $f(t)$ such that $\mathfrak{m}_x = \mathfrak{m}_y \cdot \mathcal{O}_x + f \cdot \mathcal{O}_x$, $\Bbbk(x) \cong \Bbbk(y)[t]/(\overline{f})$, and*
$$\bigoplus_{l=0}^{\nu} \Big((\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1}) \otimes_{\Bbbk(y)} \Bbbk(x)\Big) \cdot f_1^{\nu-l} \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$$
is an isomorphism (here f_1 = image of f in $\mathfrak{m}_x/\mathfrak{m}_x^2$).

Proof of Lemma 3.11. In the first case,
$$\mathcal{O}_x = \text{localization of } \mathcal{O}_y[t] \text{ with respect to prime ideal } \mathfrak{m}_y \cdot \mathcal{O}_y[t].$$
Then $\mathfrak{m}_x$ is generated by $\mathfrak{m}_y \cdot \mathcal{O}_y[t]$, hence by $\mathfrak{m}_y$, and:
$$\begin{aligned}
\mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1} &\cong \Big((\mathfrak{m}_y \cdot \mathcal{O}_y[t])^\nu \,/\, (\mathfrak{m}_y \cdot \mathcal{O}_y[t])^{\nu+1}\Big) \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x \\
&\cong \big((\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}) \otimes_{\Bbbk(y)} \Bbbk(y)[t]\big) \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x \\
&\cong \big(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}\big) \otimes_{\Bbbk(y)} \big(\Bbbk(y)[t] \otimes_{\mathcal{O}_y[t]} \mathcal{O}_x\big) \\
&\cong \big(\mathfrak{m}_y^\nu/\mathfrak{m}_y^{\nu+1}\big) \otimes_{\Bbbk(y)} \Bbbk(y)(t)
\end{aligned}$$
Taking $\nu = 0$, this shows that $\Bbbk(x) \cong \Bbbk(y)(t)$ and putting this back in the general case, we get what we want.

In the second case,
$$\begin{aligned}
\mathcal{O}_x = \ &\text{localization of } \mathcal{O}_y[t] \text{ with respect to maximal ideal } \mathfrak{p} \\
&\text{where } \mathfrak{p} = \text{inverse image of principal ideal } (\overline{f}) \subset \Bbbk(y)[t], \\
&\overline{f} \text{ monic and irreducible of some degree } d.
\end{aligned}$$
Lift $\overline{f}$ to a monic $f \in \mathcal{O}_y[t]$. Then $\mathfrak{p} = \mathfrak{m}_y \cdot \mathcal{O}_y[t] + f \cdot \mathcal{O}_y[t]$, hence $\mathfrak{m}_x = \mathfrak{p} \cdot \mathcal{O}_x = \mathfrak{m}_y \cdot \mathcal{O}_x + f \cdot \mathcal{O}_x$. Now since $\mathfrak{p}$ is maximal, $\mathcal{O}_y[t]/\mathfrak{p}^{\nu+1} \xrightarrow{\approx} \mathcal{O}_x/\mathfrak{m}_x^{\nu+1}$ for all ν, hence $\mathfrak{p}^\nu/\mathfrak{p}^{\nu+1} \xrightarrow{\approx} \mathfrak{m}_x^\nu/\mathfrak{m}_x^{\nu+1}$. On the other hand, $\mathcal{O}_y[t]/(f^{\nu+1})$ is a free $\mathcal{O}_y$-module with basis:
$$1, t, \ldots, t^{d-1}, f, ft, \ldots, ft^{d-1}, \ldots, f^\nu, f^\nu t, \ldots, f^\nu t^{d-1}.$$
In terms of this basis:
$$\mathfrak{p}^m/(f^{\nu+1}) = \bigoplus_{l=0}^{m} \bigoplus_{i=1}^{d-1} \mathfrak{m}_y^l \cdot f^{m-l} \cdot t^i,$$
hence
$$\mathfrak{p}^\nu/\mathfrak{p}^{\nu+1} \cong \bigoplus_{l=0}^{\nu} \bigoplus_{i=0}^{d-1} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1}\Big) \cdot f_1^{\nu-l} \cdot t^i.$$
Now $\Bbbk(x) \cong \Bbbk(y)[t]/(\overline{f}) = \bigoplus_{i=0}^{d-1} \Bbbk(x) \cdot t^i$, so in this direct sum decomposition,
$$\bigoplus_{i=0}^{d-1} \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1}\Big) \cdot f_1^{\nu-l} \cdot t^i = \Big(\mathfrak{m}_y^l/\mathfrak{m}_y^{l+1}\Big) \otimes_{\Bbbk(y)} \Bbbk(x) \cdot f_1^{\nu-l}$$
and (2) follows. ∎

By induction, Proposition 3.10 follows for the case $X = Y \times \mathbb{A}^n$, $f = p_1$. Now every smooth morphism is locally of the form:

$$\begin{tikzcd} X = V(f_1,\dots,f_r) \arrow[dr] & \subset & Y \times \mathbb{A}^{n+r} \arrow[dl, "p_1"] & \text{: call this scheme } Z \\ & Y & & \end{tikzcd}$$

Consider the homomorphism:

$$\mathfrak{m}_{x,Z}/\left(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}\right) \longrightarrow \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \Bbbk(x).$$

$\Omega_{Z/Y}$ is a free $\mathcal{O}_Z$-module with basis $dX_1, \dots, dX_{n+r}$ and the canonical map takes:

$$f \bmod \mathfrak{m}_{x,Z}^2 \longrightarrow \sum_{j=1}^{n+r} \frac{\partial f_i}{\partial X_j} \cdot dX_j.$$

By smoothness, the images of the f_i in $\Omega_{Z/Y} \otimes \Bbbk(x)$ are independent over $\Bbbk(x)$, hence the f_i in $\mathfrak{m}_{x,Z}/\left(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}\right)$ are independent over $\Bbbk(x)$. Proposition 3.10 now follows by induction on r using:

Lemma 3.12. *Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a local homomorphism of local rings such that $\operatorname{gr}\mathcal{O}_2$ is a polynomial ring in r variables over $\operatorname{gr}\mathcal{O}_1$. Let $f \in \mathfrak{m}_2$ have non-zero image in $\mathfrak{m}_2/\left(\mathfrak{m}_2^2 + \mathfrak{m}_x \cdot \mathcal{O}_2\right)$. Then*

$$\operatorname{gr}(\mathcal{O}_2/f\cdot\mathcal{O}_2) \cong \operatorname{gr}(\mathcal{O}_2)/f_1 \cdot \operatorname{gr}(\mathcal{O}_2) \qquad (f_1 = \textit{image of } f \textit{ in } \mathfrak{m}_2/\mathfrak{m}_2^2)$$

and is a polynomial ring in $r-1$ variables over $\operatorname{gr}\mathcal{O}_1$.

Proof of Lemma 3.12. By induction, $\operatorname{gr}(\mathcal{O}_2/f\cdot\mathcal{O}_2)$ is the quotient of $\operatorname{gr}\mathcal{O}_2$ by the leading forms of all elements $f \cdot g$ of $f \cdot \mathcal{O}_2$. If $g \in \mathfrak{m}_2^l \setminus \mathfrak{m}_2^{l+1}$, its leading form $\overline{g}$ is in $\mathfrak{m}_2^l/\mathfrak{m}_2^{l+1}$. The hypothesis on f means that f_1 can be taken as one of the variables in the presentation of $\operatorname{gr}\mathcal{O}_2$ as a polynomial ring, hence f_1 is a non-zero-divisor in $\operatorname{gr}\mathcal{O}_2$. Therefore $f_1 \cdot \overline{g} \neq 0$, i.e., $f \cdot g \notin \mathfrak{m}_2^{l+2}$ and the leading form of $f \cdot g$ is equal to $f_1 \cdot \overline{g}$. Thus $\operatorname{gr}(\mathcal{O}_2/f\cdot\mathcal{O}_2) \cong (\operatorname{gr}\mathcal{O}_2)/f_1 \cdot \operatorname{gr}\mathcal{O}_2$ as required. □

□

Corollary 3.13. *If $f\colon X \to Y$ is smooth at x of relative dimension n and $y = f(x)$, then*

$$df_x^*\colon T_{y,Y}^* \otimes_{\Bbbk(y)} \Bbbk(x) \longrightarrow T_{x,X}^* \quad \textit{is injective,}$$

hence $df_x\colon T_{x,X} \to T_{y,Y} \widehat{\otimes}_{\Bbbk(y)} \Bbbk(x)$ is surjective.

Corollary 3.14. *If $f\colon X \to Y$ is smooth at x and $y = f(x)$ is a regular point of Y, then x is a regular point of X.*

Corollary 3.15. *If a K-variety X is smooth of relative dimension n over K at some point $x \in X$, then $n = \dim X$.*

Proof. Apply Proposition 3.10 to the generic point $\eta \in X$. □

Corollary 3.16. *If $f\colon X \to Y$ is smooth of relative dimension n, then its fibres $f^{-1}(y)$ are reduced and all components are n-dimensional.*

Proof. Combine Lemma 1.1, Proposition 3.3 and Corollary 3.14. □

COROLLARY 3.17. *If $f\colon X \to \operatorname{Spec} k$ is smooth at $x \in X$ and we write:*

$$\mathcal{O}_{x,X} = k[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}/(f_1, \ldots, f_r)$$

as usual, then the module of syzygies

$$\sum_{i=1}^{r} g_i f_i = 0, \qquad g_i \in k[X]_{\mathfrak{p}}$$

is generated by the trivial ones:

$$(f_j) \cdot f_i + (-f_i) \cdot f_j = 0, \qquad 1 \le i < j \le r.$$

PROOF. Let $B = k[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}$ and $K = B/\mathfrak{p} \cdot B$. We have seen in the proof of Proposition 3.10 that $\operatorname{gr} B$ is a graded polynomial ring over K in which $\overline{f}_1, \ldots, \overline{f}_r \in \mathfrak{p}B/(\mathfrak{p}B)^2$ are independent linear elements. We apply:

LEMMA 3.18. *Let A be any ring. Over $A[T_1, \ldots, T_r]$, the module of syzygies*

$$\sum_{i=1}^{r} g_i T_i = 0, \qquad g_i \in A[T]$$

is generated by the trivial ones:

$$(T_j) \cdot T_i + (-T_i) \cdot T_j = 0, \qquad 1 \le i < j \le r.$$

(Proof is a direct calculation which we leave to the reader.)

Therefore we know the syzygies in $\operatorname{gr} B$! Now let Syz be the module of all syzygies:

$$0 \longrightarrow \operatorname{Syz} \longrightarrow B^r \xrightarrow{v} B$$
$$v(a_1, \ldots, a_r) = \sum a_i f_i$$

and let Triv be the submodule of Syz generated by the "trivial" ones. Now

$$\bigcap_{\nu=1}^{\infty} \mathfrak{p}^{\nu}(B^r/\operatorname{Triv}) = (0)$$

so

$$\operatorname{Triv} = \bigcap_{\nu=1}^{\infty} \left((\mathfrak{p}^{\nu}B)^r + \operatorname{Triv}\right).$$

Therefore if $\operatorname{Syz} \supsetneqq \operatorname{Triv}$, we can find a syzygy $(g_1, \ldots, g_r)$ with $g_i \in \mathfrak{p}^{\nu}B$ such that for no trivial syzygy $(h_1, \ldots, h_r)$ are all $g_i + h_i \in \mathfrak{p}^{\nu+1}B$. Let $\overline{g}_i =$ image of g_i in $\mathfrak{p}^{\nu}B/\mathfrak{p}^{\nu+1}B$. Then

$$\sum \overline{g}_i \overline{f}_i = 0$$

is a syzygy in $\operatorname{gr} B$. By Lemma 3.18,

$$(\overline{g}_1, \ldots, \overline{g}_r) = \sum_{1 \le i < j \le r} \overline{a}_{ij}(0, \ldots, \underset{\text{place}}{\overset{i\text{-th}}{\overline{f}_j}}, \ldots, \underset{\text{place}}{\overset{j\text{-th}}{-\overline{f}_i}}, \ldots, 0).$$

Lifting the $\overline{a}_{ij}$ to B, this gives a contradiction. □

Combining Corollary 3.17 with Proposition IV.4.10 now shows (See Proposition VII.5.7 for a strengthening.):

PROPOSITION 3.19. *Let $f\colon X \to Y$ be a smooth morphism. Then f is flat and for every $x \in X$, if*

$$\mathcal{O}_{x,X} = \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}/(f_1, \ldots, f_r)$$

as usual, then the module of syzygies:

$$\sum_{i=1}^{r} g_i f_i = 0, \qquad g_i \in \mathcal{O}_{y,Y}[X]_{\mathfrak{p}}$$

is generated by the trivial ones.

PROOF. Let $A = \mathcal{O}_{y,Y}$, $B = \mathcal{O}_{y,Y}[X]_{\mathfrak{p}}$ and apply Proposition IV.4.10 to the sequence:

$$\begin{aligned} &B^{r(r-1)/2} \xrightarrow{u} B^r \xrightarrow{v} B \\ &u(\ldots, a_{ij}, \ldots) = (\ldots, -\sum_{l<i} f_l \cdot a_{li} + \sum_{i<l} f_l \cdot a_{il}, \ldots) \\ &v(a_1, \ldots, a_r) = \sum a_i f_i. \end{aligned}$$

By Corollary 3.17, it is exact after $\otimes_{\mathcal{O}_{y,Y}} \Bbbk(y)$ so it is exact as it stands and $\operatorname{Coker} v$ is A-flat. $\square$

In fact, it can be shown[3] that if $f\colon X \to Y$ is any morphism which can be expressed locally as

$$\operatorname{Spec} A[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \longrightarrow \operatorname{Spec} A$$

where all fibres have dimension n, then f has the two properties of Proposition 3.19, i.e., f is flat and the syzygies among the f_i are trivial. Such a morphism f is called a *relative local complete intersection*. The property of the syzygies being generated by the trivial ones is an important one in homological algebra; in particular when it holds, it implies that one can explicitly *resolve* $B/(f_1, \ldots, f_r)$ as B-module, i.e., give all higher order syzygies as well: we will prove this later — §VII.5.

An interesting link can be made between the concept of smoothness and the theory of schemes over *complete* discrete valuation rings (§IV.6). In fact, let R be a complete discrete valuation ring, $S = \operatorname{Spec} R$, $k = R/M$, $K =$ fraction field of R. Let

$$f\colon X \longrightarrow S$$

be a smooth morphism of relative dimension n. Consider the specialization:

$$\operatorname{sp}\colon \operatorname{Max}(X_\eta)^\circ \longrightarrow \operatorname{Max}(X_o)$$

introduced in §IV.6. Let $x \in X_o$ be a k-rational point. Then the smoothness of f allows one to construct *analytic coordinates* on X near x, so that

$$\begin{aligned} \operatorname{sp}^{-1}(x) &\cong \text{open } n\text{-dimensional polycylinder in } \mathbb{A}^n_K \\ \text{i.e.} \quad &\cong \{x \in \operatorname{Max}(\mathbb{A}^n_K) \mid |p_i(x)| < 1, \ \text{all } i\}\,. \end{aligned}$$

[3]One need only generalize Corollary 3.17 and this follows from the Cohen-Macaulay property of $k[X_1, \ldots, X_n]$: cf. Zariski-Samuel [**119**, vol. II, Appendix 6].

4. Criteria for smoothness

In this section, we will present four important criteria for the smoothness of a morphism f. The first concerns when a variety X over a field k is smooth over $\operatorname{Spec} k$. But it holds equally well for any reduced and irreducible scheme X of finite type over a *regular* scheme Y:

CRITERION 4.1. *Let Y be a regular irreducible scheme and $f\colon X \to Y$ a morphism of finite type. Assume X is reduced and irreducible and that $f(\eta_X) = \eta_Y$. Let $r = \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X)$. Then $\forall x \in X$*

a) $\dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x)\right) \geq r$
b) *equality holds if and only if f is smooth at x in which case the relative dimension must be r and $\Omega_{X/Y} \cong \mathcal{O}_X^r$ in a neighborhood of x.*

PROOF. Let $\eta \in X$ be its generic point. Then

$$(\Omega_{X/Y})_\eta \cong \Omega_{\mathbf{R}(X)/\mathbf{R}(Y)}.$$

This $\mathbf{R}(X)$-vector space is dual to the vector space of $\mathbf{R}(Y)$-derivations from $\mathbf{R}(X)$ into itself. But by Example 2 in §2, the dimension of this space is $\geq \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X)$. Now since f is of finite type, $\Omega_{X/Y}$ is a finitely generated $\mathcal{O}_X$-module, hence by Proposition I.5.5 (Nakayama), $\forall x \in X$

$$\dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right) \geq \dim_{\mathbf{R}(X)} (\Omega_{X/Y})_\eta \geq \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X) = r.$$

Now if f is smooth at any $x \in X$, it is smooth at η and then by Corollary 3.15 its relative dimension must be r, hence $\Omega_{X/Y} \cong \mathcal{O}_X^r$ near x, hence

$$\dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right) = r.$$

Now assume conversely that $r = \dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right)$. To prove f is smooth at x, we replace X and Y by affine neighborhoods of x and y, so we have:

$$\begin{aligned} X &= \operatorname{Spec} R[X_1, \ldots, X_n]/(f_1, \ldots, f_l) \\ Y &= \operatorname{Spec} R. \end{aligned}$$

Then

$$\begin{aligned} &\Omega_{X/Y} \\ &\quad \cong \bigoplus_{i=1}^n \mathcal{O}_X \cdot dX_i \Bigg/ \left(\text{modulo relations } \sum_{j=1}^n \frac{\partial f_i}{\partial X_j} \cdot dX_j = 0, \ \ 1 \leq i \leq l \right) \end{aligned}$$

hence

$$\begin{aligned} &\Omega_{x,X/Y} \otimes \Bbbk(x) \\ &\quad \cong \bigoplus_{i=1}^n \Bbbk(x) \cdot dX_i \Bigg/ \left(\text{modulo relations } \sum_{j=1}^n \frac{\partial f_i}{\partial X_j}(x) \cdot dX_j = 0, \ \ 1 \leq i \leq l \right). \end{aligned}$$

The matrix $(\partial f_i / \partial X_j)$ is known as the Jacobian matrix for the above presentation of X. It follows that

$$\dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right) = n - \operatorname{rk} \left(\frac{\partial f_i}{\partial X_j}(x) \right).$$

Therefore in our case $(\partial f_i / \partial X_j(x))$ has rank $n - r$. Pick out $f_{i_1}, \ldots, f_{i_{n-r}}$ such that

$$\operatorname{rk} \left(\frac{\partial f_{i_l}}{\partial X_j}(x) \right) = n - r$$

and hence define

$$\widetilde{X} = \operatorname{Spec} R[X_1, \ldots, X_n]/(f_{i_1}, \ldots, f_{i_{n-r}}).$$

Then we get a diagram

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr, "f"'] & & \widetilde{X} \arrow[dl, "\widetilde{f}"] \\ & Y & \end{tikzcd}$$

and find that $\widetilde{f}$ is smooth of relative dimension r at x. But then by Corollary 3.14, $\mathcal{O}_{x,\widetilde{X}}$ is a regular local ring. In particular it is an integral domain and $\widetilde{X}$ has a unique component $\widetilde{X}_\circ$ containing x. By Corollary 3.15 applied to the generic point of $\widetilde{X}_\circ$,

$$r = \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(\widetilde{X}_\circ).$$

In other words, both $\mathcal{O}_{x,\widetilde{X}}$ and its quotient $\mathcal{O}_{x,X} = \mathcal{O}_{x,\widetilde{X}}/(\text{other } f_i\text{'s})$ are integral domains of the same transcendence degree over $\mathbf{R}(Y)$! This is only possible if they are equal (cf. Part I [**87**, Proposition (1.14)]). So $\mathcal{O}_{x,X} = \mathcal{O}_{x,\widetilde{X}}$, hence $X = \widetilde{X}$ in a neighborhood of x and X is smooth over Y at x. $\square$

COROLLARY 4.2 (Jacobian Criterion for Smoothness). *If in the situation of Criterion* 4.1, $Y = \operatorname{Spec} R$, $X = \operatorname{Spec} R[X_1, \ldots, X_n]/(f_1, \ldots, f_l)$, *then*

$$f \textit{ is smooth at } x \iff \operatorname{rk}\left(\frac{\partial f_i}{\partial X_j}(x)\right) = n - r.$$

COROLLARY 4.3. *In the situation of Criterion* 4.1,

$$\begin{bmatrix} \exists x \in X \textit{ such that} \\ f \textit{ is smooth (resp. étale) at } x \end{bmatrix} \iff \begin{bmatrix} \mathbf{R}(X) \textit{ is separable} \\ \textit{(resp. separable algebraic)} \\ \textit{over } \mathbf{R}(Y) \end{bmatrix}.$$

PROOF. If f is smooth somewhere, it is smooth at η; and the criterion at η is:

$$\dim(\text{vector space of } \mathbf{R}(Y)\text{-derivations of } \mathbf{R}(X) \text{ to } \mathbf{R}(X)) = \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X).$$

By Example 2 in §2, this is equivalent to $\mathbf{R}(X)$ being separable over $\mathbf{R}(Y)$. $\square$

COROLLARY 4.4. *If $f\colon X \to Y$ is étale, then for all $y \in Y$, the fibre $f^{-1}(y)$ is a finite set of reduced points each of which is* $\operatorname{Spec} K$, *K separable algebraic over $\Bbbk(y)$.*

PROOF. Proposition 3.3 and Corollary 4.3. $\square$

COROLLARY 4.5. *In the situation of Criterion* 4.1 *if $x \in X$, $y = f(x)$, then f is smooth over Y at x if and only if the fibre $f^{-1}(y)$ is smooth of relative dimension r over* $\operatorname{Spec}\Bbbk(y)$ *at x (n.b. one must assume the two r's are the same, i.e.,* $\dim f^{-1}(y) = \operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X)$*).*

A slightly more general version of Criterion 4.1 is sometimes useful:

CRITERION. 4.1$^+$ *Let Y be a regular irreducible scheme and let $f\colon X \to Y$ be a morphism of finite type. Let*

$$X = X_1 \cup \cdots \cup X_t$$

be the components of X and assume $f(\eta_{X_i}) = \eta_Y$, $1 \le i \le t$. Let

$$r = \min_{1 \le i \le t} \left(\operatorname{trdeg}_{\mathbf{R}(Y)} \mathbf{R}(X_{i,\mathrm{red}})\right).$$

Then for all $x \in X$:

a) $\dim_{\Bbbk(x)} \Omega_{X/Y} \otimes_{\mathcal{O}_x} \Bbbk(x) \ge r$
b) *equality holds if and only if f is smooth of relative dimension r at x.*

In some cases, we can give a criterion for smoothness via Zariski-tangent spaces (as in the theory of differential geometry):

CRITERION 4.6. *Let $f\colon X \to Y$ be as in the previous criterion. Assume further that $\Bbbk(x)$ is separable over $\Bbbk(y)$. Then*

$$f \text{ is smooth at } x \iff \begin{bmatrix} x \text{ is a regular point of } X \text{ and} \\ df_x\colon T_{x,X} \to T_{y,Y} \otimes_{\Bbbk(y)} \Bbbk(x) \text{ is surjective} \end{bmatrix}.$$

PROOF. "$\implies$" was proven in Corollaries 3.13 and 3.14. To go backwards, use the lemma:

LEMMA 4.7. *Let X be a noetherian scheme and $X' \subset X$ a closed subscheme. Suppose $x \in X'$ is a point which is simultaneously regular on both X and X' and suppose $r = \dim \mathcal{O}_{x,X} - \dim \mathcal{O}_{x,X'}$. Then $\exists$ a neighborhood $U \subset X$ of x and $f_1, \dots, f_r \in \mathcal{O}_X(U)$ such that the ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ defining X' is given by*

$$\mathcal{I}|_U = \sum_{i=1}^{r} f_i \cdot \mathcal{O}_X$$

and moreover $\overline{f}_1, \dots, \overline{f}_r \in \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2$ are independent over $\Bbbk(x)$.

PROOF OF LEMMA 4.7. We know $\mathcal{O}_{x,X'} \cong \mathcal{O}_{x,X}/\mathcal{I}_x$, hence

$$\mathrm{gr}(\mathcal{O}_{x,X'}) \cong \mathrm{gr}(\mathcal{O}_{x,X})\,/(\text{ideal generated by leading forms of elements of } \mathcal{I}_x)\,.$$

Both "gr" are graded polynomial rings, the former in $m+r$ variables, the latter in m variables for some m. This is only possible if the ideal of leading forms is generated by r independent linear forms $\overline{f}_1, \dots, \overline{f}_r$. Lift these to $f_1', \dots, f_r' \in \mathcal{I}_x$, hence to $f_1, \dots, f_r \in \mathcal{I}(U)$ for some open $U \subset X$. New $\sum f_i \cdot \mathcal{O}_{x,X} \subset \mathcal{I}_x$ so we get three rings:

$$\mathcal{O}_{x,X} \xrightarrow{\alpha} \mathcal{O}_{x,X}/\textstyle\sum f_i \cdot \mathcal{O}_{x,X} \xrightarrow{\beta} \mathcal{O}_{x,X}/\mathcal{I}_x = \mathcal{O}_{x,X'}.$$

These induces:

$$\mathrm{gr}(\mathcal{O}_{x,X}) \xrightarrow{\mathrm{gr}(\alpha)} \mathrm{gr}\left(\mathcal{O}_{x,X}/\textstyle\sum f_i \cdot \mathcal{O}_{x,X}\right) \xrightarrow{\mathrm{gr}(\beta)} \mathrm{gr}(\mathcal{O}_{x,X'}).$$

But by construction, $\mathrm{Ker}(\mathrm{gr}(\beta) \circ \mathrm{gr}(\alpha)) \subset \mathrm{Ker}(\mathrm{gr}(\alpha))$, so $\mathrm{gr}(\beta)$ is an isomorphism. Then β is an isomorphism too, hence $\mathcal{I}_x = \sum f_i \cdot \mathcal{O}_{x,X}$. Now because X is noetherian, the two sheaves $\mathcal{I}|_U$ and $\sum f_i \cdot \mathcal{O}_X|_U$ are both finitely generated and have the same stalks at x: hence they are equal in some open $U' \subset U$. □

Now whenever $f\colon X \to Y$ is a morphism of finite type, Y is noetherian, $x \in X$ is a regular point and $y = f(x) \in Y$ is a regular point, factor f locally:

$$\begin{tikzcd} X = V(f_1, \dots, f_l) \arrow[r, hook] \arrow[dr, "f"'] & Y \times \mathbb{A}^n = Z \arrow[d, "p_1"] \\ & Y \end{tikzcd}$$

and note that $\mathcal{O}_{x,X} \cong \mathcal{O}_{x,Z}/(f_1, \dots, f_l)$ where $\mathcal{O}_{x,X}$ and $\mathcal{O}_{x,Z}$ are both regular. It follows from Lemma 4.7 that in some neighborhood of x, $X = V(f_1, \dots, f_s)$ where $\overline{f}_1, \dots, \overline{f}_s \in \mathfrak{m}_{x,Z}/\mathfrak{m}_{x,Z}^2$ are independent. Now df_x surjective means dually that

$$\begin{tikzcd} (\mathfrak{m}_y/\mathfrak{m}_y^2) \otimes_{\Bbbk(y)} \Bbbk(x) \arrow[r] & \mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \arrow[d, equal, "\sim"] \\ & \mathfrak{m}_{x,Z} \Big/ \left(\mathfrak{m}_{x,Z}^2 + \sum_{i=1}^{s} \overline{f}_i \cdot \Bbbk(x)\right) \end{tikzcd}$$

is injective. This implies that $\overline{f}_1, \ldots, \overline{f}_s$ are also independent in $\mathfrak{m}_{x,Z}/(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z})$. Since $\Bbbk(x)$ is separable over $\Bbbk(Y)$, $\Upsilon_{\Bbbk(x)/\Bbbk(y)} = (0)$, hence

$$\mathfrak{m}_{x,Z}/(\mathfrak{m}_{x,Z}^2 + \mathfrak{m}_y \cdot \mathcal{O}_{x,Z}) \longrightarrow \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \Bbbk(x)$$

is injective by Theorem 2.14. Therefore finally $df_1, \ldots, df_s \in \Omega_{Z/Y} \otimes_{\mathcal{O}_Z} \Bbbk(x)$ are independent, which is precisely the condition that $V(f_1, \ldots, f_s)$ is smooth over Y at x. $\square$

The most important case for these results is when $Y = \operatorname{Spec} k$, X a k-variety. There are then in fact two natural notions of "non-singularity" for a point $x \in X$.

a) x a regular point,
b) $X \to \operatorname{Spec} k$ smooth at x.

Our results show that they almost coincide! In fact:

$$x \text{ a regular point} \Longleftarrow x \text{ a smooth point}, \qquad \text{by Corollary 3.14}$$

and if $\Bbbk(x)$ is separable over k, then:

$$x \text{ a regular point} \Longleftrightarrow x \text{ a smooth point}, \qquad \text{by Criterion 4.6.}$$

But by the Jacobian Criterion 4.2, if $\overline{k} =$ algebraic closure of k, and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ and $\overline{x} \in \overline{X}$ lies over x, then

$$x \text{ smooth on } X \Longleftrightarrow \overline{x} \text{ smooth on } \overline{X}.$$

Putting this together:

$$\begin{aligned} x \text{ regular on } X \Longleftarrow\ & x \text{ smooth on } X \\ \Longleftrightarrow\ & \overline{x} \text{ smooth on } \overline{X} \\ \Longleftrightarrow\ & \overline{x} \text{ regular on } \overline{X}. \end{aligned}$$

The pathological situation where these are not all equivalent occurs only over an imperfect field k and is quite interesting. It stems from the geometric fact that over an algebraically closed ground field in characteristic p, Sard's lemma fails abysmally:

Example. Let k be an algebraically closed field of characteristic $p \neq 0$. There exist morphisms $f\colon \mathbb{A}_k^m \to \mathbb{A}_k^n$ such that every fibre $f^{-1}(x)$ (x closed point) is singular.

a) $f\colon \mathbb{A}_k^1 \to \mathbb{A}_k^1$ given by $f(a) = a^p$. Then if $b \in \mathbb{A}_k^1$ is a closed point and $b = a^p$, the scheme-theoretic fibre is:

$$\begin{aligned} f^{-1}(b) &= \operatorname{Spec} k[X]/(X^p - b) \\ &= \operatorname{Spec} k[X]/(X-a)^p \\ &\cong \operatorname{Spec} k[X']/(X'^p), \qquad (\text{if } X' = X - a) \end{aligned}$$

none of which are reduced. Similarly, the differential

$$df\colon T_{a,\mathbb{A}^1} \longrightarrow T_{a^p,\mathbb{A}^1}$$

is everywhere 0 and f is nowhere étale.

b) $f\colon \mathbb{A}_k^2 \to \mathbb{A}_k^1$ given by $f(a,b) = a^2 - b^p$. Then if $d \in \mathbb{A}_k^1$ is a closed point and $d = c^p$, the scheme-theoretic fibre is:

$$\begin{aligned} f^{-1}(d) &= \operatorname{Spec} k[X,Y]/(X^2 - Y^p - d) \\ &= \operatorname{Spec} k[X,Y]/(X^2 - (Y+c)^p) \\ &\cong \operatorname{Spec} k[X,Y']/(X^2 - Y'^p), \qquad \text{if } Y' = Y + c. \end{aligned}$$

Thus the fibre $f^{-1}(d)$ is again a k-variety, in fact a plane curve, but with a singularity at $X = Y' = 0$ as in Figure V.3:

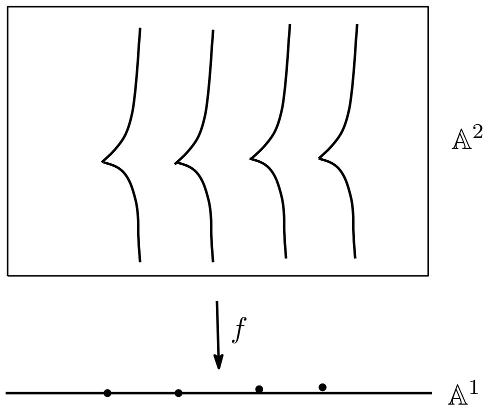

FIGURE V.3. Every fibre is singular.

c) Now if t is the coordinate on $\mathbb{A}^1_k$, then $\mathbf{R}(\mathbb{A}^1_k) = k(t)$: a non-perfect field of characteristic p. Consider the *generic fibre* $f^{-1}(\eta)$ of the previous example. It is a 1-dimensional $k(t)$-variety equal to:

$$\begin{aligned}\mathbb{A}^2_k \times_{\mathbb{A}^1_k} \operatorname{Spec} k(t) &= \operatorname{Spec} k[X,Y] \otimes_{k[t]} k(t) \\ &= \operatorname{Spec} k(t)[X,Y]/(X^2 - Y^p - t),\end{aligned}$$

i.e., it is the plane curve $X^2 = Y^p + t$. But now $t \notin k(t)^p$, so this curve is not isomorphic over $k(t)$ to $X^2 = (Y')^p$. In fact, $k[X,Y] \otimes_{k[t]} k(t)$ is a localization of $k[X,Y]$, so the local rings of $f^{-1}(\eta)$ are all local rings of $\mathbb{A}^2_k$ too, hence they are all regular, i.e., $f^{-1}(\eta)$ is a regular scheme! But the Jacobian matrix of the defining equations of this curve is:

$$\begin{aligned}\frac{\partial}{\partial X}(X^2 - Y^p - t) &= 2X \\ \frac{\partial}{\partial Y}(X^2 - Y^p - t) &= 0\end{aligned}$$

so all 1×1-minors vanish at the point $x = V(X, Y^p + t) \in f^{-1}(\eta)$. Thus $f^{-1}(\eta)$ is not smooth over $k(t)$ at x.

The third and fourth criteria for smoothness are more general and do not assume that the base scheme Y is regular.

CRITERION 4.8. *Consider a finitely presented morphism $f\colon X \to Y$. Take a point $x \in X$ and let $y = f(x)$. Then*

$$f \text{ is smooth at } x \iff \left[\begin{array}{l} f \text{ is flat at } x \text{ and the fibre} \\ f^{-1}(y) \text{ is smooth over } \Bbbk(y) \text{ at } x. \end{array}\right]$$

PROOF. $\Longrightarrow$ was proven in Propositions 3.3 and 3.19. To prove the converse, we may assume $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A[X_1, \dots, X_n]/(f_1, \dots, f_r)$. Then let $x = [\mathfrak{p}]$, where $\mathfrak{p}$ is a prime ideal in $A[X_1, \dots, X_n]$ and let $\mathfrak{q} = \mathfrak{p} \cap A$ and $k =$ (quotient field of $A/\mathfrak{q}$) $\cong \Bbbk(y)$. Note that the fibre $f^{-1}(y)$ equals

$$\operatorname{Spec} k[X_1, \dots, X_n]/(\overline{f}_1, \dots, \overline{f}_r).$$

If s is the dimension of $f^{-1}(y)$ at x, it follows that

$$\operatorname{rk}\left(\frac{\partial \overline{f}_i}{\partial X_j}(x)\right) = n - s.$$

Thus $n-s \leq r$ and renumbering, we may assume that:

$$\det_{1\leq i,j\leq n-s}\left(\frac{\partial \overline{f}_i}{\partial X_j}(x)\right) \neq 0.$$

Consider the diagram:

$$\begin{tikzcd}[column sep=tiny]
X = \operatorname{Spec} A[X]/(f_1,\dots,f_r) \arrow[dr, "f", "(\text{flat at } x)"'] & \subset & \operatorname{Spec} A[X]/(f_1,\dots,f_{n-s}) = X' \arrow[dl, "f'"', "(\text{smooth at } x)"] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

Then the fibres: $f^{-1}(y) \subset (f')^{-1}(y)$ over y are both smooth of dimension s at x, hence they are equal in a neighborhood of x. I claim that in fact X and X' are equal in a neighborhood of x, hence f is smooth at x. To prove this, it suffices to show

$$(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}} = (f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}$$

or, by Nakayama's lemma, to show

$$\frac{(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}}}{(f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k = (0).$$

But consider the exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow \dfrac{(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}}}{(f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}} \longrightarrow & \dfrac{A[X]_{\mathfrak{p}}}{(f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}} & \longrightarrow & \dfrac{A[X]_{\mathfrak{p}}}{(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}}} & \longrightarrow 0. \\
& \sim\| & & \sim\| & \\
& \mathcal{O}_{x,X'} & & \mathcal{O}_{x,X} &
\end{array}$$

The last ring is flat over A, so

$$0 \longrightarrow \frac{(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}}}{(f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k \,—$$

$$\begin{array}{ccccc}
\longrightarrow \dfrac{A[X]_{\mathfrak{p}}}{(f_1,\dots,f_{n-s})\cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k & \longrightarrow & \dfrac{A[X]_{\mathfrak{p}}}{(f_1,\dots,f_r)\cdot A[X]_{\mathfrak{p}}} \otimes_{A_{\mathfrak{q}}} k \longrightarrow 0 \\
\sim\| & & \sim\| \\
\mathcal{O}_{x,(f')^{-1}(y)} & & \mathcal{O}_{x,f^{-1}(y)}
\end{array}$$

is exact. But $\mathcal{O}_{x,(f')^{-1}(y)} \xrightarrow{\approx} \mathcal{O}_{x,f^{-1}(y)}$, so the module on the left is (0). ☐

Corollary 4.9. *Let $f\colon X \to Y$ be a finitely presented morphism. Then for all $x \in X$, $y = f(x)$,*

$$f \text{ is étale at } x \Longleftrightarrow \left[\begin{array}{l} f \text{ is flat at } x, \text{ the fibre } f^{-1}(y) \text{ is reduced} \\ \text{at } x \text{ and } \Bbbk(x) \text{ is separable algebraic over } \Bbbk(y). \end{array}\right]$$

The last criterion is a very elegant idea due to Grothendieck. It is an infinitesimal criterion involving A-valued points of X and Y when A is an artin local ring. We want to consider a *lifting* for such point described by the diagram:

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r, "\psi_0"] \arrow[d, phantom, "\cap"] & X \arrow[d, "f"] \\
\operatorname{Spec} A \arrow[r, "\phi_1"'] \arrow[ur, dashed, "\psi_1"] & Y
\end{tikzcd}$$

This means that we have an A-valued point ϕ_1 of Y and a lifting ψ_0 of the induced (A/I)-valued point (I is any ideal in A). Then the problem is to lift ϕ_1 to an A-valued point ψ_1 of X extending ψ_0. The criterion states:

CRITERION 4.10. *Let $f\colon X \to Y$ be any morphism of finite type where Y is a noetherian scheme. Then f is smooth if and only if:*

For all artin local rings A, ideals $I \subset A$, and all A-valued points ϕ_1 of Y and (A/I)-valued points ψ_0 of X such that:

$$f \circ \psi_0 = \textit{restriction of } \phi_1 \textit{ to } \operatorname{Spec} A/I$$

there is an A-valued point ψ_1 of X such that

$$\begin{aligned} f \circ \psi_1 &= \phi_1 \\ \psi_0 &= \textit{restriction of } \psi_1 \textit{ to } \operatorname{Spec} A/I. \end{aligned}$$

(See diagram.)

$f\colon X \to Y$ satisfying the lifting property in Criterion 4.10 is said to be *formally smooth* in EGA [**1**, Chapter IV, §17]. This criterion plays crucial roles in deformation theory (cf. §VIII.5).

PROOF. Suppose first that f is smooth and ψ_0, ϕ_1 are given. Look at the induced morphism f_1:

$$\begin{tikzcd} & X_1 = X \times_Y \operatorname{Spec} A \arrow[d, "f_1"] \\ \operatorname{Spec} A/I \arrow[ur, "\psi_0'"] \arrow[r, phantom, "\subset"] & \operatorname{Spec} A \end{tikzcd}$$

which is smooth by Proposition 3.3. Then ψ_0 defines a *section* ψ_0' of f_1 over the subscheme $\operatorname{Spec} A/I$ of the base which we must extend to a section of f_1 over the whole of $\operatorname{Spec} A$. Let $y \in \operatorname{Spec} A$ be its point and let $x \in X_1$ be the image of ψ_0'. Then $\Bbbk(x) = \Bbbk(y)$, so by Proposition 3.8

$$\widehat{\mathcal{O}}_{x,X_1} \cong A[[t_1, \dots, t_n]].$$

If the section ψ_0' is given by

$$(\psi_0')^*(t_i) = \overline{a}_i \in A/I,$$

choose $a_i \in A$ over $\overline{a}_i$. Then define a section ψ_i' of f_1 by

$$(\psi_1')^*(t_i) = a_i.$$

Now suppose f satisfies the lifting criterion. Choose $x \in X$. We will verify the definition of smoothness directly, i.e., find a local presentation of f near x as

$$\operatorname{Spec} R[T_1, \dots, T_n]/(f_1, \dots, f_l) \longrightarrow \operatorname{Spec} R$$

where $\det(\partial f_i/\partial X_j) \neq 0$. To start, let f be presented locally by

$$\operatorname{Spec} R[T_1, \dots, T_n]/I \longrightarrow \operatorname{Spec} R$$

and let

$$r = \dim_{\Bbbk(x)} \left(\Omega_{X/Y} \otimes \Bbbk(x)\right).$$

We may replace X by $\operatorname{Spec} R[T_1, \dots, T_n]/I$ and Y by $\operatorname{Spec} R$ if we wish. Since $\Omega_{X/Y} \otimes \Bbbk(x)$ is generated by $dT_1, \dots, dT_n$ with relations $df = 0$, $f \in I$, we can choose $f_1, \dots, f_{n-r} \in I$ such that

$$\Omega_{X/Y} \otimes \Bbbk(x) \cong \left(\bigoplus_{i=1}^{n} \Bbbk(x) \cdot dT_i \right) \Big/ \langle df_1, \dots, df_{n-r} \rangle$$

and in particular

$$\det\left(\frac{\partial f_i}{\partial T_j}(x)\right) \neq 0.$$

This allows us to *factor* f locally through a smooth morphism:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & X_1 = \operatorname{Spec} R[T_1, \ldots, T_n]/(f_1, \ldots, f_{n-r}) \arrow[dl, "f_1"] \\ Y & \end{tikzcd}$$

where f_1 is smooth at x and

$$\Omega_{X_1/Y} \otimes \Bbbk(x) \longrightarrow \Omega_{X/Y} \otimes \Bbbk(x) \tag{4.11}$$

is an isomorphism.

We now apply the lifting property to the artin local rings $A_\nu = \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1}$ and the ideals $I_\nu = I_1 \cap \mathfrak{m}^{\nu-1}_{x,X_1} + \mathfrak{m}^\nu_{x,X_1}$, where I_1 is the image of I under

$$R[T_1, \ldots, T_n] \to R[T_1, \ldots, T_n]/(f_1, \ldots, f_{n-r}).$$

We want to define by induction on ν morphisms r_ν:

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{x,X}/\mathfrak{m}^\nu_{x,X} \arrow[rr, hook] \arrow[d, phantom, "\cap"] & & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1} \arrow[r, hook] \arrow[urr, dashed, "r_\nu"] & X_1 \arrow[r, "f_1"'] & Y \end{tikzcd}$$

which extend each other. Given r_ν, r_ν plus the canonical map

$$\operatorname{Spec} \mathcal{O}_{x,X_1}/(I_1 + \mathfrak{m}^{\nu+1}_{x,X_1}) = \operatorname{Spec} \mathcal{O}_{x,X}/\mathfrak{m}^{\nu+1}_{x,X} \hookrightarrow X$$

induce a map

$$\operatorname{Spec} \mathcal{O}_{x,X_1}/(I_1 \cap \mathfrak{m}^\nu_{x,X_1} + \mathfrak{m}^{\nu+1}_{x,X_1}) \longrightarrow X.$$

(This is because $\mathcal{O}_{x,X_1}/(I_1 \cap \mathfrak{m}^\nu_{x,X_1} + \mathfrak{m}^{\nu+1}_{x,X_1})$ can be identified with the subring of $(\mathcal{O}_{x,X_1}/(I_1 + \mathfrak{m}^{\nu+1}_{x,X_1})) \oplus \mathcal{O}_{x,X_1}/\mathfrak{m}^\nu_{x,X_1}$ of pairs both members of which have the same image in $\mathcal{O}_{x,X_1}/(I_1 + \mathfrak{m}^\nu_{x,X_1})$.) Apply the lifting property to find $r_{\nu+1}$. Now the whole family $\{r_\nu\}$ defines a morphism r:

$$\begin{tikzcd} & \operatorname{Spec} \widehat{\mathcal{O}}_{x,X_1} \arrow[dl, "r"'] \arrow[d, phantom, "\cap"] \\ X \arrow[r, hook] \arrow[dr] & X_1 \arrow[d] \\ & Y \end{tikzcd}$$

which is in effect a retraction of a formal neighborhood of X in X_1 onto X, all over Y. Ring-theoretically, this means

$$\widehat{\mathcal{O}}_{x,X_1} \cong \widehat{\mathcal{O}}_{x,X} \oplus J$$

and where the R-algebra structure of $\widehat{\mathcal{O}}_{x,X_1}$ is given by the R-algebra structure of $\widehat{\mathcal{O}}_{x,X}$. It follows that

$$\Omega_{X_1/Y} \otimes \widehat{\mathcal{O}}_{x,X} \cong (\Omega_{X/Y} \otimes \widehat{\mathcal{O}}_{x,X}) \oplus (J/J^2).$$

But, then applying (4.11), we find

$$(J/J^2) \otimes \Bbbk(x) = (0),$$

hence by Nakayama's lemma, $J = (0)$. Thus $\widehat{\mathcal{O}}_{x,X_1} \cong \widehat{\mathcal{O}}_{x,X}$, hence $\mathcal{O}_{x,X_1} \cong \mathcal{O}_{x,X}$ and $X \cong X_1$ in a neighborhood of x. $\square$

5. Normality

Recall that in §III.6 we defined a scheme X to be *normal* if its local rings $\mathcal{O}_{x,X}$ are integral domains integrally closed in their quotient field. In particular, if $X = \operatorname{Spec} R$ is affine and integral, then

$$\begin{aligned} X \text{ is normal} &\iff R_{\mathfrak{p}} \text{ integrally closed in } \mathbf{R}(X),\ \forall \mathfrak{p} \\ &\iff R \text{ integrally closed in } \mathbf{R}(X) \end{aligned}$$

(using the facts (i) that a localization of an integrally closed domain is integrally closed and (ii) $R = \bigcap_{\mathfrak{p}} R_{\mathfrak{p}}$.) An important fact is that regular schemes are normal. This can be proven either using the fact that regular local rings are UFD's (cf. Zariski-Samuel [**119**, vol. II, Appendix 7]; or Kaplansky [**64**, §4-2]) and that all UFD's are integrally closed (Zariski-Samuel [**119**, vol. I, p. 261]); or one can argue directly that for a noetherian local ring $\mathcal{O}$, $\operatorname{gr}\mathcal{O}$ integrally closed $\Longrightarrow$ $\mathcal{O}$ integrally closed (Zariski-Samuel [**119**, vol. II, p. 250]). As we saw in §III.6, normality for noetherian rings is really the union of two distinct properties, each interesting in its own right. We wish to globalize this. First we must find how to express globally the condition:

$$R = \bigcap_{\mathfrak{p} \text{ non-zero minimal prime}} R_{\mathfrak{p}}.$$

(Added in publication) We use the following terminology: A point x of a locally noetherian scheme X is *not an embedded point* if the natural map $\mathcal{O}_{x,X} \to \Gamma(\operatorname{Spec}(\mathcal{O}_{x,X}) \setminus \{x\})$ is injective. Equivalently, x is an *embedded point* of X if $\dim(\mathcal{O}_{x,X}) \geq 1$ and x is an associated point of $\mathcal{O}_{x,X}$.

PROPOSITION-DEFINITION 5.1. *Let X be a noetherian scheme with no embedded components and let $x \in X$ be a point of codimension at least 2. Say $\eta_1, \ldots, \eta_n$ are the generic points of the components of X containing x. The following are equivalent:*

a) *$\forall$ neighborhoods U of x, and $f \in \mathcal{O}_X\Big(U \setminus (\overline{\{x\}} \cap U)\Big)$, there is a neighborhood $U' \subset U$ of x such that f extends to $f' \in \mathcal{O}_X(U')$.*

a′)
$$\mathcal{O}_{x,X} = \bigcap_{\substack{y \in X \text{ with} \\ x \in \overline{\{y\}} \\ x \neq y}} \mathcal{O}_{y,X}$$
(all these rings being subrings of the total quotient ring $\bigoplus_{i=1}^n \mathcal{O}_{\eta_i,X}$).

b) *$\forall f \in \mathfrak{m}_{x,X}$ with $f(\eta_i) \neq 0$ all i, x is not an embedded point of the subscheme $V(f)$ defined near x.*

b′) *$\exists f \in \mathfrak{m}_{x,X}$ with $f(\eta_i) \neq 0$ all i, and x not an embedded point of $V(f)$.*

Points with these properties we call proper points*; others are called* improper[4]*. If all points are proper, X is said to have Property* S2.

PROOF. It is easy to see (a) $\iff$ (a′), and (b) $\Longrightarrow$ (b′) is obvious. To see (b′) $\Longrightarrow$ (a), take

$$g \in \mathcal{O}_X\Big(U \setminus (\overline{\{x\}} \cap U)\Big), \qquad U \text{ affine}$$

and let $f \in \mathfrak{m}_{x,X}$ be such that $V(f)$ has no embedded components. Then the distinguished open set U_f of U is inside $U \setminus (\overline{\{x\}} \cap U)$, hence we can write:

$$g = g_1/f^m, \quad g_1 \in \mathcal{O}_X(U).$$

[4] This is not standard terminology; it is suggested by an old Italian usage: cf. Semple-Roth [**98**, Chapter 13, §6.4].

We now prove by induction on l that $g_1/f^l \in \mathcal{O}_{x,X}$, starting with $l=0$ where we know it, and ending at $l=m$ where it proves that $g \in \mathcal{O}_{x,X}$, hence $g \in \mathcal{O}_X(U')$ some $U' \subset U$. Namely, if $l<m$, and $h = g_1/f^l \in \mathcal{O}_{x,X}$, consider the function $\overline{h}$ induced by h on $V(f)$ in a neighborhood of x. Since $h = f^{m-l}\cdot g$, it follows that $\overline{h}=0$ on $V(f)\setminus(\overline{\{x\}}\cap V(f))$, i.e., $\operatorname{Supp}\overline{h} \subset \overline{\{x\}}\cap V(f)$. Since x is not an embedded component of $V(f)$, $\overline{h}=0$ at x too, i.e., $g_1/f^{l+1} = h/f \in \mathcal{O}_{x,X}$.

To see (a$'$) $\Longrightarrow$ (b), suppose $f \in \mathfrak{m}_{x,X}$, $f(\eta_i) \neq 0$ and suppose $g \in \mathcal{O}_{x,X}$ restricts to a function $\overline{g}$ on $V(f)$ whose support is contained in $\overline{\{x\}}\cap V(f)$. Then for all $y \in X$ with $y \in \overline{\{x\}}$, $x \neq y$, $\overline{g}$ is 0 in $\mathcal{O}_{y,V(f)}$, i.e., $g \in f\cdot\mathcal{O}_{y,X}$. Then

$$g/f \in \bigcap_{\substack{y\in X\\ x\in\overline{\{y\}}\\ x\neq y}} \mathcal{O}_{y,X} = \mathcal{O}_{x,X},$$

hence $\overline{g}=0$. □

Criterion 5.2 (Basic criterion for normality (Krull-Serre)). *Let X be a reduced noetherian scheme. Then*

$$X \textit{ is normal} \Longleftrightarrow \begin{cases} \text{a) } \forall x \in X \textit{ of codimension } 1,\ X \textit{ is regular at } x \\ \text{b) } X \textit{ has Property } \mathrm{S2}. \end{cases}$$

In particular (a) *and* (b) *imply that the components of X are disjoint.*

Proof. If X is affine and irreducible, say $X = \operatorname{Spec} R$, then Property S2, in form (a$'$), implies immediately:

$$\forall \mathfrak{p} \text{ prime ideal in } R: \qquad R_{\mathfrak{p}} = \bigcap_{\substack{\mathfrak{q} \text{ non-zero minimal prime}\\ \mathfrak{q}\subset\mathfrak{p}}} R_{\mathfrak{q}}.$$

Since

$$R = \bigcap_{\text{all } \mathfrak{p}} R_{\mathfrak{p}},$$

the criterion reduces to Krull's result (Theorem III.6.5). Everything in the criterion being local, it remains to prove (a) + (b) $\Longrightarrow$ all components of X are disjoint. Let

$$S = \{x \in X \mid x \text{ is in at least two components of } X\},$$

and let x be some generic point of S. Then $\mathcal{O}_{x,X}$ is not a domain so by (a), $\operatorname{codim} x \geq 2$. Then consider the function e which is 1 on one of the components through x, 0 on all the others. Clearly

$$e \in \bigcap_{\substack{y\in X\\ x\in\overline{\{y\}}\\ x\neq y}} \mathcal{O}_{y,X}, \qquad e \notin \mathcal{O}_{x,X}$$

which contradicts S2. Thus $S=\emptyset$. □

Here is an example of how this criterion is used:

Proposition 5.3. *Assume X is a regular irreducible scheme and $Y \subsetneqq X$ is a reduced and irreducible codimension* 1 *subscheme. Then Y has Property* S2.

Proof. Let $y \in Y$ be a point of codimension ≥ 2 and let $f \in \mathcal{O}_{y,X}$ be a local equation for Y. Take any $g \in \mathfrak{m}_{y,Y} \setminus (f\mathcal{O}_{y,X} \cup \mathfrak{m}_{y,X}^2)$. Let $\overline{g}$ be the image of g in $\mathcal{O}_{y,Y}$, let Z be the subscheme of X defined by $g = 0$ near y, and let $\overline{f}$ be the image of f in $\mathcal{O}_{y,Z}$. Then

$$\begin{aligned} y \text{ is a proper point of } Y &\iff \{y\} \text{ not embedded component of } V(\overline{g}) \subset Y \\ &\iff \{y\} \text{ not embedded component of } V(f,g) \subset X \\ &\iff \{y\} \text{ not embedded component of } V(\overline{f}) \subset Z \\ &\iff y \text{ is a proper point of } Z. \end{aligned}$$

But $\mathcal{O}_{y,Z} = \mathcal{O}_{y,X}/g \cdot \mathcal{O}_{y,X}$ is regular (since $g \notin \mathfrak{m}_{y,X}^2$), hence Z is normal at y hence every point is proper. □

Corollary 5.4. *If X is regular, irreducible, $Y \subsetneq X$ is reduced irreducible of codimension 1, then if Y itself is regular at all points of codimension 1, Y is normal.*

Another application of the basic criterion is:

Proposition 5.5. *Let $f\colon Y \to X$ be a smooth morphism, where X is a normal noetherian scheme. Then Y is normal (and locally noetherian).*

Proof. As X is the disjoint union of its components, we can replace X by one of these and so assume X irreducible with generic point η. Note that since $\mathcal{O}_{\eta,X} =$ the field $\mathbf{R}(X)$, the local rings of any $y \in f^{-1}(\eta)$ on the fibre $f^{-1}(\eta)$ and on Y are the same.

a) Y is reduced: in fact f flat implies

$$f(\mathrm{Ass}(\mathcal{O}_Y)) \subset \mathrm{Ass}(\mathcal{O}_X) = \{\eta\}.$$

For for all $y \in \mathrm{Ass}(\mathcal{O}_Y)$,

$$\mathcal{O}_{y,Y} = \mathcal{O}_{y,f^{-1}(\eta)}$$

is an integral domain, since $f^{-1}(\eta)$ is smooth over $\mathrm{Spec}\,\mathbf{R}(X)$, hence is regular.

b) If $y \in Y$ has codimension ≤ 1, then by Corollary IV.5.10, $f(y)$ has codimension 0 or 1, hence X is regular at $f(y)$. Since f is smooth, Y is regular at y by Corollary 3.14.

c) If $y \in Y$ has codimension > 1, we seek some $g \in \mathcal{O}_{y,Y}$ with $g(y) = 0$, $g \not\equiv 0$ on any component of Y through y, and such that $V(g)$ has no embedded components through y. There are two cases:

c_1)

$$\begin{aligned} f(y) = \eta \Longrightarrow & \mathcal{O}_{y,Y} = \mathcal{O}_{y,f^{-1}(\eta)} \text{ regular, hence normal} \\ \Longrightarrow & \text{any } g \in \mathfrak{m}_{y,Y},\ g \neq 0 \text{ has this property} \\ & \text{by the Basic Criterion 5.2.} \end{aligned}$$

c_2) $f(y) = x$ has codimension ≥ 1 in X. But then since X is normal, there is a $g \in \Gamma(U_x, \mathcal{O}_X)$, U_x some neighborhood of x, such that $g(x) = 0$, $g(\eta) \neq 0$ and $V(g)$ has no embedded components. Then $f^*(g) \in \Gamma(f^{-1}U_x, \mathcal{O}_Y)$ is not zero at any generic points of Y while $f^*(g)(y) = 0$. Moreover,

$$V(f^*(g)) \cong V(g) \times_X Y,$$

so $V(f^*(g))$ is smooth over $V(g)$. We get:

$$\begin{aligned} y \in \mathrm{Ass}(\mathcal{O}_{V(f^*(g))}) &\Longrightarrow f(y) \in \mathrm{Ass}(\mathcal{O}_{V(g)}) \\ &\Longrightarrow f(y) = \text{generic point of } V(g) \\ &\Longrightarrow \text{codimension of } f(y) \text{ is } 1 \\ &\Longrightarrow X \text{ regular at } f(y) \\ &\Longrightarrow \mathcal{O}_{y,Y} \text{ regular, hence normal} \\ &\Longrightarrow V(f^*(g)) \text{ has no embedded} \\ &\qquad \text{components through } y. \end{aligned}$$

□

In particular this shows that a smooth scheme over a normal scheme is locally irreducible and if one looks back at the proof of Criterion 4.1 for smoothness, one sees that it now extends *verbatim* to the case where the image scheme is merely assumed normal, i.e., (as generalized in Criterion 4.1^+):

CRITERION 5.6. *Let X be an irreducible normal noetherian scheme and $f\colon Y \to X$ a morphism of finite type. Assume all components Y_i of Y dominate X and let*

$$r = \min \operatorname{trdeg}_{\mathbf{R}(X)} \mathbf{R}(Y_{i,\mathrm{red}}).$$

Then $\forall y \in Y$

a) $\dim_{\Bbbk(y)} \Omega_{Y/X} \otimes_{\mathcal{O}_x} \Bbbk(y) \geq r$
b) *equality holds if and only if f is smooth at y of relative dimension r.*

EXAMPLE. The simplest way to get non-normal schemes is to start with any old scheme and "collapse" the tangent space at a point or "identify" two distinct points. To be precise, let

$$X = \operatorname{Spec} R$$

be a k-variety.

a) If $x = [\mathfrak{m}]$ is a k-rational point, so that $R \cong k + \mathfrak{m}$, consider

$$X_0 = \operatorname{Spec}(k + \mathfrak{m}^2).$$

The natural morphism:

$$\pi\colon X \longrightarrow X_0$$

is easily seen to be bijective, but if $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, the f is integrally dependent on $k + \mathfrak{m}^2$, but $\notin k + \mathfrak{m}^2$. So X_0 is not normal.

b) If $x_i = [\mathfrak{m}_i]$, $i = 1, 2$ are two k-rational points, let

$$\begin{aligned} R_0 &= \{f \in R \mid f(x_1) = f(x_2)\} \\ &= k + \mathfrak{m}_1 \cap \mathfrak{m}_2 \\ X_0 &= \operatorname{Spec} R_0. \end{aligned}$$

The natural morphism

$$\pi\colon X \longrightarrow X_0$$

is bijective except that x_1, x_2 have the same image. Moreover, if $f \in R$, then f satisfies the equation:

$$(X - f(x_1))(X - f(x_2)) = a, \quad \text{where } a = (f - f(x_1))(f - f(x_2)) \in R_0.$$

So X_0 is not normal. Moreover, one can check that $\Omega_{X/X_0} = (0)$ but π is not étale in this case so this morphism illustrates the fact that Criterion 4.1^+ does not extend to non-normal Y's.

One of the major reasons why normal varieties play a big role in algebraic geometry is that all varieties can be "normalized", i.e., there is a canonical process modifying them only slightly leading to a normal variety. If there were a similar easy canonical process leading from a general variety to a regular one, life would be much simpler!

PROPOSITION-DEFINITION 5.7. *Let X be a reduced and irreducible scheme. Let $L \supset \mathbf{R}(X)$ be a finite algebraic extension. Then there is a unique quasi-coherent sheaf of $\mathcal{O}_X$-algebra:*

$$\mathcal{O}_X \subset \mathcal{A} \subset \textit{constant sheaf } L$$

such that for all affine U:

$$\mathcal{A}(U) = \textit{integral closure of } \mathcal{O}_X(U) \textit{ in } L.$$

We set

$$\begin{aligned} X_L &= \operatorname{Spec}_X(\mathcal{A}) \\ &\underset{def}{=} \textit{union of affines } \operatorname{Spec}\mathcal{A}(U), \\ &\qquad \textit{as } U \textit{ runs over affines in } X, \end{aligned}$$

and call this the normalization of X in L. In particular, if $L = \mathbf{R}(X)$, we call this the normalization of X. X_L is normal and irreducible with function field L.

To see that this works, use (I.5.9), and check that if $U = \operatorname{Spec} R$ is an affine in X and U_f is a distinguished open set, then $\mathcal{A}(U_f) = \mathcal{A}(U) \otimes_R R_f$. This is obvious.

Note for instance that in the two examples above, normalization just undoes the clutching or identification: X is the normalization of X_0.

Sadly, normalization is seriously flawed as a tool by the very unfortunate fact that even for some of the nicest schemes X you could imagine — e.g., regular affine and 1-dimensional — there are cases where X_L is not of finite type over X. This situation has been intensively studied, above all by Nagata (cf. his book [**89**] and Matsumura [**78**, Chapter 12]). We have no space to describe the rather beautiful pathology that he revealed and the way he "explained" it. Suffices it to recall that:

5.8.

- *X noetherian normal L separable over $\mathbf{R}(X) \Longrightarrow X_L$ of finite type over X.*
- *X itself of finite type over a field $\Longrightarrow X_L$ of finite type over X (cf. Zariski-Samuel* [**119**, vol. I, Chapter V, §4]*).*
- *X itself of finite type over $\mathbb{Z} \Longrightarrow X_L$ of finite type over X (cf. Nagata* [**89**, (37.5)]*).*

We conclude with a few miscellaneous remarks on normalization. The schemes $\operatorname{Proj} R$ can be readily normalized by taking the integral closure of R:

PROPOSITION 5.9. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded integral domain with $R_1 \neq (0)$ and let*

$$\begin{aligned} K_0 &= \textit{field of elements } f/g,\ f,g \in R_n \textit{ for some } n,\ g \neq 0 \\ &= \mathbf{R}(\operatorname{Proj} R). \end{aligned}$$

Then if $t =$ *any fixed element of* R_1*, the quotient field of* R *is isomorphic to* $K_0(t)$*. Let* $L_0 \supset K_0$ *be a finite algebraic extension and let*

$$S = \textit{integral closure of } R \textit{ in } L_0(t).$$

Then S *is graded and* $\operatorname{Proj} S$ *is the normalization of* $\operatorname{Proj} R$ *in* L_0*.*

PROOF. Left to the reader. □

An interesting relation between normalization and associated points is given by:

PROPOSITION 5.10. *Let* X *be a reduced and irreducible noetherian scheme and let*

$$\pi\colon \widetilde{X} \longrightarrow X, \qquad \widetilde{X} = \operatorname{Spec}_X(\mathcal{A})$$

be its normalization. Assume π *is of finite type hence* $\mathcal{A}$ *is coherent. Then for all* $y \in X$ *of codimension at least* 2*:*

$$y \textit{ is an improper point} \Longleftrightarrow y \in \operatorname{Ass}(\mathcal{A}/\mathcal{O}_X).$$

The proof is easy using the fact that every point of $\widetilde{X}$ is proper.

One case in which normalization does make a scheme regular is when its dimension is one. This can be used to prove:

PROPOSITION 5.11. *Let* k *be a field,* $K \supset k$ *a finitely generated extension of transcendence degree* 1*. Then there is one and (up to isomorphism) only one regular complete* k*-variety* X *with function field* K*, and it is projective over* k*.*

PROOF. Let $R^0 \subset K$ be a finitely generated k-algebra with quotient field K, let $X^0 = \operatorname{Spec} R^0$ and embed X^0 in $\mathbb{A}^n_k$ for some n using generators of R^0. Let $\overline{X^0}$ be the closure of X^0 in $\mathbb{P}^n_k$ and write it as $\operatorname{Proj} R'$. Let R'' be the integral closure of R' in its quotient field. Then by Proposition 5.9, $X'' = \operatorname{Proj} R''$ is normal. Since it has dimension 1, it is regular and has the properties required. Uniqueness is easy using Proposition II.4.8, and the fact that the local rings of the closed points of X'' are valuation rings, hence *maximal* proper subrings of K. □

6. Zariski's Main Theorem

A second major reason why normality is important is that Zariski's Main Theorem holds for general normal schemes. To understand this in its natural context, first consider the classical case: $k = \mathbb{C}$, X a k-variety, and x is a closed point of X. Then we have the following two sets of properties:

N1) X *formally normal* at x, i.e., $\widehat{\mathcal{O}}_{x,X}$ an integrally closed domain.

N2) X *analytically normal* at x, i.e., $\mathcal{O}_{x,X,\mathrm{an}}$, the ring of germs of holomorphic functions at x, is an integrally closed domain.

N3) X *normal* at x.

N4) *Zariski's Main Theorem* holds at x, i.e., $\forall f\colon Z \to X$, f birational and of finite type with $f^{-1}(x)$ finite, then $\exists U \subset X$ Zariski-open with $x \in U$ and

$$\operatorname{res} f\colon f^{-1}U \longrightarrow U$$

an isomorphism.

U1) X *formally unibranch* at x, i.e., $\operatorname{Spec}\left(\widehat{\mathcal{O}}_{x,X}\right)$ irreducible.

U2) X *analytically unibranch* at x, i.e., $\operatorname{Spec}(\mathcal{O}_{x,X,\mathrm{an}})$ irreducible, or equivalently, the germ of analytic space defined by X at x is irreducible.

U3) X *unibranch* at x, i.e., if $X' =$ normalization of X in $\mathbf{R}(X)$, $\pi\colon X' \to X$ the canonical morphism, then $\pi^{-1}(x) =$ one point.

U4) X *topologically unibranch* at x. (Recall that in Part I [**87**, (3.9)], an irreducible algebraic variety X over $\mathbb{C}$ was defined to be *topologically unibranch* at a point $x \in X(\mathbb{C})$ if for every closed subvariety $Y \subsetneqq X$ and every open subset $V \ni x$ in the classical topology, there exists a classical open neighborhood $U \ni x$ contained in V such that $U \setminus (U \cap Y(\mathbb{C}))$ is connected in the classical topology.)

U5) The *Connectedness Theorem* holds at x, i.e., $\forall f\colon Z \to X$, f proper, Z integral, $f(\eta_Z) = \eta_X$ and $\exists U \subset X$ Zariski-open dense with $f^{-1}(y)$ connected for all $y \in U$, then $f^{-1}(x)$ is connected too.

6.1. *I claim:*

i) all properties N are equivalent,
ii) all properties U are equivalent,
iii) N $\Longrightarrow$ U.

Modulo two steps for which we refer the reader to Zariski-Samuel [**119**] and Gunning-Rossi [**54**], this is proven as follows:

N1 $\Longleftrightarrow$ N2 $\Longleftrightarrow$ N3**:** We have inclusions:

$$\mathcal{O}_{x,X} \subset \mathcal{O}_{x,X,\mathrm{an}} \subset \widehat{\mathcal{O}}_{x,X}$$

and

$$\mathcal{O}_{x,X,\mathrm{an}} \cap \mathbf{R}(X) = \mathcal{O}_{x,X}$$
$$\widehat{\mathcal{O}}_{x,X} \cap \begin{pmatrix} \text{total quotient} \\ \text{ring of } \mathcal{O}_{x,X,\mathrm{an}} \end{pmatrix} = \mathcal{O}_{x,X,\mathrm{an}}$$

(This follows from the fact that if $f, g \in \mathcal{O}$, $\mathcal{O}$ noetherian local, then $f|g$ in $\mathcal{O}$ iff $f|g$ in $\widehat{\mathcal{O}}$: cf. Part I [**87**, §1D].) Therefore the implications

$$\begin{aligned} \widehat{\mathcal{O}}_{x,X} \text{ integrally closed domain} &\Longrightarrow \mathcal{O}_{x,X,\mathrm{an}} \text{ integrally closed domain} \\ &\Longrightarrow \mathcal{O}_{x,X} \text{ integrally closed domain} \end{aligned}$$

are obvious. The fact:

$$\mathcal{O}_{x,X} \text{ integrally closed domain} \Longrightarrow \widehat{\mathcal{O}}_{x,X} \text{ integrally closed domain}$$

is a deep Theorem of Zariski (cf. Zariski-Samuel [**119**, vol. II, p. 320]). He proved this for all points x on k-varieties X, for all perfect fields k. It was later generalized by Nagata to schemes X of finite type over any field k or over $\mathbb{Z}$ (cf. Nagata [**89**, (37.5)]). Although this step appears quite deep, note that if we strengthen the hypothesis and assume $\mathcal{O}_{x,X}$ actually regular, then since regularity is a property of $\mathrm{gr}(\mathcal{O}_{x,X})$ and $\mathrm{gr}(\mathcal{O}_{x,X}) \cong \mathrm{gr}(\widehat{\mathcal{O}}_{x,X})$, it follows very simply that $\widehat{\mathcal{O}}_{x,X}$ is also regular, hence is an integrally closed domain!

N1 $\Longrightarrow$ U1**:** Obvious.

U1 $\Longrightarrow$ U2**:** Obvious because

$$\mathcal{O}_{x,X,\mathrm{an}}/\sqrt{(0)} \subset \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)},$$

so if the latter is a domain, so is the former.

U2 $\Longrightarrow$ U4**:** See Gunning-Rossi [**54**, p. 115].

U4 $\Longrightarrow$ U5**:** This was proven in Part I [**87**, (3.24)] for projective morphisms f. The proof generalizes to any proper f.

(Added in publication) The proof uses a basic fact that the proper morphism $f\colon Z \to X$ induces a *topological proper* map $f_{\mathbb{C}}\colon Z(\mathbb{C}) \to X(\mathbb{C})$, that is, the inverse image of any compact subset of $X(\mathbb{C})$ is compact, hence the image of any closed subset of $Z(\mathbb{C})$ is closed.

Suppose that $f^{-1}(x)$ were the disjoint union of two non-empty closed subvarieties Y_1 and Y_2. Then there exist disjoint classical open subsets $W_1 \supset Y_1(\mathbb{C})$ and $W_2 \supset Y_2(\mathbb{C})$ in $Z(\mathbb{C})$. Let

$$V_0 := X(\mathbb{C}) \setminus f(Z(\mathbb{C}) \setminus (W_1 \cup W_2)),$$

an open neighborhood of x. Since X is topologically unibranch at x, there exists a classical open neighborhood V of x in V_0 such that $V \cap U(\mathbb{C})$ is connected (with U in the statement of U5). Since $V \subset V_0$, we get

$$f^{-1}(U(\mathbb{C}) \cap V) \subset f^{-1}(V) \subset W_1 \cup W_2.$$

Since each fibre $f^{-1}(y)$ is connected for $y \in U(\mathbb{C}) \cap V$ and f is surjective, we deduce that

$$U(\mathbb{C}) \cap V \subset [(U(\mathbb{C}) \cap V) \setminus f(Z(\mathbb{C}) \setminus W_1)] \cup [(U(\mathbb{C}) \cap V) \setminus f(Z(\mathbb{C}) \setminus W_2)],$$

and the right hand side is a disjoint union of two open subsets of the connected open subset $U(\mathbb{C}) \cap V$ in the classical topology. Hence one of the two open subsets is equal to $U(\mathbb{C}) \cap V$, say

$$(U(\mathbb{C}) \cap V) \setminus f(Z(\mathbb{C}) \setminus W_1) = U(\mathbb{C}) \cap V.$$

This implies that $f(W_1) \cap U(\mathbb{C}) = \emptyset$, or equivalently, $W_1 \subset f^{-1}(X \setminus U)(\mathbb{C})$. This is impossible because $f^{-1}(X \setminus U)$ is a proper subvariety of the irreducible variety Z and W_1 is an open subset of $Z(\mathbb{C})$.

U5 $\Longrightarrow$ U3: Let $\pi\colon X' \to X$ be the normalization of X in $\mathbf{R}(X)$. π is of finite type by (5.8), hence it is proper by Proposition II.6.5. π is birational, hence an isomorphism over some non-empty $U \subset X$. Therefore U5 applies to π and $\pi^{-1}(x)$ is connected. But since $X' = \operatorname{Spec} \mathcal{A}$, $\mathcal{A}$ coherent, $\pi^{-1}(x) = \operatorname{Spec}(\mathcal{A}_x/\mathfrak{m}_x \cdot \mathcal{A}_x)$ and $\mathcal{A}_x/\mathfrak{m}_x \cdot \mathcal{A}_x$ is finite-dimensional over $\mathbb{C}$; thus $\pi^{-1}(x)$ is a finite set too, hence it consists in one point.

U3 $\Longrightarrow$ U1: Let $\mathcal{O}'_{x,X}$ be the integral closure of $\mathcal{O}_{x,X}$ in $\mathbf{R}(X)$: it is a local ring and a finite $\mathcal{O}_{x,X}$-module. By flatness of $\widehat{\mathcal{O}}_{x,X}$ over $\mathcal{O}_{x,X}$, we find

$$\widehat{\mathcal{O}}_{x,X} \subset \mathcal{O}'_{x,X} \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}}_{x,X}$$

and by finiteness of $\mathcal{O}'_{x,X}$,

$$\mathcal{O}'_{x,X} \otimes_{\mathcal{O}_{x,X}} \widehat{\mathcal{O}}_{x,X} \cong \text{completion } \widehat{\mathcal{O}'}_{x,X} \text{ of } \mathcal{O}'_{x,X} \text{ in its } \mathfrak{m}_x\text{-adic topology.}$$

By N3 $\Longrightarrow$ N1, $\widehat{\mathcal{O}'}_{x,X}$ is a domain, so therefore $\widehat{\mathcal{O}}_{x,X}$ is a domain and U1 is proven.

N3 $\Longrightarrow$ N4: (Zariski's Main Theorem) We use the fact already proven that N3 $\Longrightarrow$ N1 $\Longrightarrow$ U1 $\Longrightarrow$ U5 and prove N3+U5 $\Longrightarrow$ N4. This is quite easy using Chow's lemma (Theorem II.6.3). Let $f\colon Z \to X$ be a birational morphism of finite type with $f^{-1}(x)$ finite. Then we can find a diagram:

$$\begin{tikzcd}
Z' \arrow[r, hook, "\substack{\text{open}\\ \text{dense}}"'] \arrow[d, "g"'] & \overline{Z'} \arrow[r, hook] & \mathbb{P}^n \times X \arrow[ddll, "p_2"] \\
Z \arrow[d, "f"'] & & \\
X & &
\end{tikzcd}$$

where g is proper and birational, $\overline{Z'} =$ closure of Z' in $\mathbb{P}^n \times X$ with reduced structure. Now if we write $f^{-1}(x) = \{y_1, \dots, y_t\}$, then since f is of finite type, each y_i is open in $f^{-1}(x)$ and proper over $\mathbb{C}$. Then if $Y_i = g^{-1}(y_i)$, each Y_i is open in $(f \circ g)^{-1}(x)$ and proper over $\mathbb{C}$. Let $h =$ restriction of p_2 to $\overline{Z'}$. Then $(f \circ g)^{-1}(x)$ is open in $h^{-1}(x)$, hence each Y_i is open in $h^{-1}(x)$. But being proper over $\mathbb{C}$, Y_i must also be closed in $h^{-1}(x)$:

$$h^{-1}(x) = Y_1 \cup \dots \cup Y_t \cup \big(h^{-1}(x) \setminus (f \circ g)^{-1}(x)\big)$$

is a decomposition of $h^{-1}(x)$ into open and closed pieces. So the Connectedness Theorem implies $t = 1$ and $x \notin h(\overline{Z'} \setminus Z')$. But h is proper so $h(\overline{Z'} \setminus Z')$ is closed in X. Replacing X by $X \setminus h(\overline{Z'} \setminus Z')$, we can therefore assume $Z' = \overline{Z'}$, i.e., Z' is proper over X. It follows that Z is proper over X, and $f^{-1}(x) =$ one point y.

Next replacing X by a smaller neighborhood U of x and Z by $f^{-1}(U)$, we can assume Z and X are affine: to see this, let V be any affine neighborhood of y. Since f is proper, $f(Z \setminus V)$ is closed. Let U be an affine neighborhood of x contained in $X \setminus f(Z \setminus V)$. Then $f^{-1}(U) \subset V$ and $f^{-1}(U)$ is affine by Proposition II.4.5.

Now if $X = \operatorname{Spec} R$, $Z = \operatorname{Spec} R[x_1, \dots, x_n]$, where $x_i \in \mathbf{R}(X)$, consider the morphism $[x_i] \colon Z \to \mathbb{A}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}}$. This induces

$$([x_i], f) \colon Z \longrightarrow \mathbb{P}^1_{\mathbb{C}} \times_{\operatorname{Spec}(\mathbb{C})} X$$

which is proper since f is proper. Let Γ_i be its image. Then Γ_i is closed and $(\infty, x) \notin \Gamma_i$. Therefore there is some expression:

$$\begin{aligned} p(t) =& a_m t^m + a_{m-1} t^{m-1} + \dots + a_0 \\ & a_i \in \mathcal{O}_{x,X} \\ & t = \text{coordinate on } \mathbb{P}^1_{\mathbb{C}} \\ & p(t) \equiv 0 \text{ on } \Gamma_i \\ & t^{-m} p(t) \neq 0 \text{ at } (\infty, x). \end{aligned}$$

Thus $a_m \notin \mathfrak{m}_{x,X}$, and x_i, as an element of $\mathbf{R}(X)$, satisfies $g(x_i) = 0$. In other words, x_i is integrally dependent on $\mathcal{O}_{x,X}$. So $x_i \in \mathcal{O}_{x,X}$, hence $x_i \in \mathcal{O}_X(U_i)$ for some neighborhood U_i of x. It follows that f is an isomorphism over $U_1 \cap \dots \cap U_n$.

N4 $\Longrightarrow$ N3**:** Let $\pi \colon X' \to X$ be the normalization of X in $\mathbf{R}(X)$ and apply Zariski's Main Theorem with $f = \pi$.

Remark. (Added in publication) (Chai) It is easy to give an example of a complex algebraic variety X and a point $x \in X$ that is unibranch but not normal: Take $X = \operatorname{Spec} R$ with $R = \mathbb{C} + t^2\mathbb{C}[t]$, and let x correspond to the quotient of R by the maximal ideal $t^2\mathbb{C}[t]$ of R. The normalization of R is the polynomial ring $\mathbb{C}[t]$, and $\mathbb{A}^1 \to X$ is a homeomorphism.

Now consider the same situation for general integral noetherian[5] schemes. N2, U2 and U4 do not make sense, but N1, N3, N4, U1, U3 and U5 do.

We need modify U5 however to read:

$\widetilde{\text{U5}}$) The *Connectedness Theorem* holds at x, i.e., $\forall f \colon Z \to X$, f proper, Z integral, $f(\eta_Z) = \eta_X$ and the *geometric generic fibre* of f connected (i.e., if $\Omega =$ an algebraic closure of $\mathbf{R}(X)$, then via the canonical

$$i \colon \operatorname{Spec} \Omega \longrightarrow X,$$

$Z \times_X \operatorname{Spec} \Omega$ should be connected), then $f^{-1}(x)$ is connected too.

[5]N3 $\Longrightarrow$ N4 is proved even for non-noetherian X in EGA [**1**, Chapter IV, (8.12.10)].

Remark. (Added in publication) (Chai) In the statement of $\widetilde{\text{U5}}$, one cannot strengthen the conclusion to "$f^{-1}(x)$ is geometrically connected". Here is an example: Let $Z = \operatorname{Spec} \mathbb{C}[t]$, the affine line over $\mathbb{C}$. Let $\mathbb{R} + t\mathbb{C}[t]$ be the ring of all polynomials $g(t) \in \mathbb{C}[t]$ such that $g(0) \in \mathbb{R}$. We have an isomorphism

$$\alpha\colon \mathbb{R}[u,v]/(u^2+v^2) \xrightarrow{\sim} R, \qquad u \mapsto t, \quad v \mapsto \sqrt{-1}t.$$

Let $X = \operatorname{Spec} R$. It is easy to see that $\mathbb{C}[t]$ is the integral closure of R in the fraction field $\mathbb{C}(t)$ of R, $f\colon Z \to X$ is a homeomorphism, and f is an isomorphism outside the closed point $x := \operatorname{Spec}(R/t\mathbb{C}[t]) \cong \operatorname{Spec} \mathbb{R}$. However, $f^{-1}(x) \cong \operatorname{Spec} \mathbb{C}$, which is connected, but not geometrically connected over $x \cong \operatorname{Spec} \mathbb{R}$.

6.2. Then Zariski (for k-varieties) and Grothendieck (in general) have shown:

$$\begin{array}{ccccc} \text{N1} & \Longrightarrow & \text{N3} & \Longleftrightarrow & \text{N4} \\ \Downarrow & & \Downarrow & & \\ \text{U1} & \Longrightarrow & \text{U3} & \Longleftrightarrow & \widetilde{\text{U5}} \end{array}$$

but Nagata [**89**, Appendix A1] has given counterexamples to N3 $\Longrightarrow$ N1, U3 $\Longrightarrow$ U1.

(Note that we do have these implications when X is *excellent*. Examples of excellent rings are fields, $\mathbb{Z}$, complete local rings and Dedekind domains of generic characteristic 0. Finitely generated algebras over excellent rings are excellent.)

To prove these implications, first note that N1 $\Longrightarrow$ U1 and N3 $\Longrightarrow$ U3 are obvious; that N1 $\Longrightarrow$ N3 is proven just as above. Moreover, N4 $\Longrightarrow$ N3 and $\widetilde{\text{U5}}$ $\Longrightarrow$ U3 are proven as above, except that since the normalization $\pi\colon X' \to X$ may not be of finite type, N4 and $\widetilde{\text{U5}}$ should be applied to partial normalizations, i.e., $\operatorname{Spec} R[a_1,\dots,a_n] \to \operatorname{Spec} R$, a_i integrally dependent on R. Moreover, N3 + $\widetilde{\text{U5}}$ $\Longrightarrow$ N4 is proven as above. Therefore it remains to prove U1 $\Longrightarrow$ U3 and U3 $\Longrightarrow$ $\widetilde{\text{U5}}$.

U1 $\Longrightarrow$ U3: This is an application of Hensel's lemma (Lemma IV.6.1). If $\pi^{-1}(x)$ has more than one point, it is easy to see that we can find an element $a \in \mathbf{R}(X)$ integrally dependent on $\mathcal{O}_{x,X}$ such that already in the morphism:

$$\widetilde{\pi}\colon \operatorname{Spec} \mathcal{O}_{x,X}[a] \longrightarrow \operatorname{Spec} \mathcal{O}_{x,X}$$

$\pi^{-1}(x)$ consists in more than one point. Consider the three rings:

$$\mathcal{O}_{x,X} \subset \mathcal{O}_{x,X}[a] \subset \mathbf{R}(X).$$

Tensoring with $\widehat{\mathcal{O}}_{x,X}$, we get:

$$\widehat{\mathcal{O}}_{x,X} \subset \widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X}[a] \subset \widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathbf{R}(X).$$

Dividing all three rings by their nilpotents, we get

$$\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)} \subset \left(\widehat{\mathcal{O}}_{x,X} \otimes_{\mathcal{O}_{x,X}} \mathcal{O}_{x,X}[a]\right)/\sqrt{(0)} \subset \left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbf{R}(X)\right)/\sqrt{(0)}.$$

By U1, $\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$ is a domain, and since $\mathbf{R}(X)$ is a localization of $\mathcal{O}_{x,X}$, $\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbf{R}(X)\right)/\sqrt{(0)}$ is a localization of $\widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$, i.e.,

$$\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathbf{R}(X)\right)/\sqrt{(0)} \subset \text{quotient field of } \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}.$$

This implies that $\left(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a]\right)/\sqrt{(0)}$ is a domain hence $\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a])$ is irreducible. Now look at

$$\widehat{\pi}\colon \operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a]) \longrightarrow \operatorname{Spec} \widehat{\mathcal{O}}_{x,X}.$$

But since $\widehat{\pi}^{-1}(\text{closed point}) \cong \widetilde{\pi}^{-1}(x)$, which has more than one point, by Hensel's lemma (Lemma IV.6.1), $\operatorname{Spec}(\widehat{\mathcal{O}}_{x,X} \otimes \mathcal{O}_{x,X}[a])$ is not irreducible!

U3 $\Longrightarrow$ $\widetilde{\text{U5}}$: (i.e., Unibranch implies the Connectedness Theorem.) We follow Zariski's idea (cf. Zariski [**118**]) and deduce this as an application of the fundamental theorem of "holomorphic functions" (cf. [**118**, Chapter VIII]. See also "GFGA" in §VIII.2.):

6.3 (Fundamental theorem of "holomorphic functions"). *$\forall f\colon Z \to X$ proper, X noetherian, then $f_*\mathcal{O}_Z$ is a coherent sheaf of $\mathcal{O}_X$-algebras and for all $x \in X$*

$$\varprojlim_{\nu} (f_*\mathcal{O}_Z)_x/\mathfrak{m}_x^{\nu} \cdot (f_*\mathcal{O}_Z)_x \cong \varprojlim_{\nu} \Gamma\left(f^{-1}(x), \mathcal{O}_Z/\mathfrak{m}_x^{\nu} \cdot \mathcal{O}_Z\right).$$

To apply this to the situation of $\widetilde{\text{U5}}$, suppose $f^{-1}(x) = W_1 \cup W_2$, W_i open disjoint. Then define idempotents:

$$e_\nu \in \Gamma(f^{-1}(x), \mathcal{O}_Z/\mathfrak{m}_x^{\nu} \cdot \mathcal{O}_Z)$$
$$e_\nu = 0 \text{ on } W_1, \quad e_\nu = 1 \text{ on } W_2.$$

These define an element $\widehat{e}$ in the limit: approximating this with an element $e \in (f_*\mathcal{O}_Z)_x \bmod \mathfrak{m}_x \cdot (f_*\mathcal{O}_Z)_x$, it follows that $e = 0$ on W_1, $e = 1$ on W_2. Let e extend to a section of $f_*\mathcal{O}_Z$ in an affine neighborhood $U = \operatorname{Spec} R$ of x.

Next, for all open $U \subset X$,

$$f_*\mathcal{O}_Z(U) \underset{\text{def}}{=} \Gamma(f^{-1}(U), \mathcal{O}_Z) \subset \Gamma(f^{-1}(\eta_X), \mathcal{O}_{f^{-1}(\eta_X)}).$$

The generic fibre $f^{-1}(\eta_X)$ of f is a complete variety over the field $\mathbf{R}(X)$, hence

$$L = \Gamma(f^{-1}(\eta_X), \mathcal{O}_{f^{-1}(\eta_X)})$$

is a field, finite and algebraic over $\mathbf{R}(X)$. Applying the theory of §IV.2, $f^{-1}(\eta_X)$ is also a variety over L and passing to the algebraic closure $\overline{\mathbf{R}(X)}$ of $\mathbf{R}(X)$, we find that the geometric scheme:

$$\overline{f^{-1}(\eta_X)} = f^{-1}(\eta_X) \times_{\operatorname{Spec}\mathbf{R}(X)} \operatorname{Spec}\overline{\mathbf{R}(X)} \longrightarrow \operatorname{Spec}\overline{\mathbf{R}(X)}$$

in fact lies over $\operatorname{Spec}(L \otimes_{\mathbf{R}(X)} \overline{\mathbf{R}(X)})$. All points of the latter are conjugate, so $\overline{f^{-1}(\eta_X)}$ maps onto $\operatorname{Spec}(L \otimes_{\mathbf{R}(X)} \overline{\mathbf{R}(X)})$. By assumption $\overline{f^{-1}(\eta_X)}$ is connected, hence $\operatorname{Spec}(L \otimes_{\mathbf{R}(X)} \overline{\mathbf{R}(X)})$ consists in one point, hence L is purely inseparable over $\mathbf{R}(X)$. So we may assume $L^{p^l} \subset \mathbf{R}(X)$. In particular $e^{p^l} \in \mathbf{R}(X)$.

Since $f_*\mathcal{O}_Z(U)$ is a finite R-module, e^{p^l} is integrally dependent on R too. Let R' be the integral closure of R in $\mathbf{R}(X)$ and we can factor the restriction of f to $f^{-1}(U)$ via the function e^{p^l}:

$$\begin{tikzcd}
Z \arrow[dd, "f"'] & \supset & f^{-1}U \arrow[dd, "\operatorname{res} f"'] \arrow[dr] & & \operatorname{Spec} R' \arrow[dl, "g"'] \\
& & & \operatorname{Spec} R[e^{p^l}] \arrow[d, "f'"'] & \\
X & \supset & U \arrow[r, equal] & \operatorname{Spec} R &
\end{tikzcd}$$

Since e^{p^l} takes on values 0 and 1 on $f^{-1}(x)$, it follows that $(f')^{-1}(x)$ consists in at least two points! But R' integral over $R[e^{p^l}]$ so g is surjective by the going-up theorem (Zariski-Samuel [**119**, vol. I, Chapter V, §2, Theorem 3, p. 257]).

An elementary proof that N1 $\Longrightarrow$ N4 can be given along the lines of the proof that U1 $\Longrightarrow$ U3. We sketch this: Given $f\colon Z \to X$ as in N4, form the diagram:

$$\begin{tikzcd} Z \arrow[d, "f"'] & \left(\operatorname{Spec} \widehat{\mathcal{O}}_{x,X}\right) \times_X Z = Z' \arrow[l] \arrow[d, "f'"] \\ X & \operatorname{Spec} \widehat{\mathcal{O}}_{x,X} = X' \arrow[l] \end{tikzcd}$$

Decompose Z' via Hensel's lemma (Lemma IV.6.1). Then it follows that Z'_{red} has a component Z'' which projects by a finite birational morphism to X'. This means that $Z' = \operatorname{Spec} R'$, where R' is a local domain finite over the normal local domain $\widehat{\mathcal{O}}_{x,X}$ and is contained in the fraction field of $\widehat{\mathcal{O}}_{x,X}$. It follows that $Z'' \xrightarrow{\sim} X'$. Hence f' has a section. Using

$$\widehat{\mathcal{O}}_{x,X} \cap \mathbf{R}(X) = \mathcal{O}_{x,X},$$

it follows easily that f is a local isomorphism.

Remark. (Added in publication) (Chai) A *local* ring R is said to be *unibranch* if R_{red} is an integral domain whose integral closure in its fraction field is a local ring. If in addition the residue field of the integral closure of R_{red} is a purely inseparable extension of the residue field of R, then we say that R is *geometrically unibranch*. A scheme X is said to be unibranch or geometrically unibranch at a point x if so is the local ring $\mathcal{O}_{x,X}$.

Consider the following properties for a pair (X, x), where X is a noetherian integral scheme.

GU3) X is geometrically unibranch at x.

GU5) (Strong form of Zariski's Connectedness Theorem) For every proper morphism $f\colon Z \to X$ with Z integral and $f(\eta_Z) = \eta_X$, if the generic fiber of f is geometrically connected, then $f^{-1}(x)$ is geometrically connected, too.

Then we have the following implications.

$$\begin{tikzcd} \mathrm{N3} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathrm{GU3} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & \mathrm{U3} \arrow[d, Leftrightarrow] \\ \mathrm{N4} & \mathrm{GU5} \arrow[r, Leftrightarrow] & \widetilde{\mathrm{U5}} \end{tikzcd}$$

There is yet another statement that Grothendieck calls "Zariski's Main Theorem" which generalizes the statement we have used so far. This is the result:

Theorem 6.4 (Zariski-Grothendieck "Main Theorem"). *Let X be any quasi-compact scheme and suppose*

$$f\colon Z \longrightarrow X$$

is a morphism of finite type with finite fibres. Then there exists a factorization of f:

$$Z \overset{i}{\hookrightarrow} \operatorname{Spec}_X \mathcal{A} \xrightarrow{\ \pi\ } X$$

where i is an open immersion and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras such that for all affine $U \subset X$, $\mathcal{A}(U)$ is finitely generated and integral over $\mathcal{O}_X(U)$.

The proof can be found in EGA: (in [**1**, Chapter III, (4.4.3)] for X noetherian f quasi-projective; in [**1**, Chapter IV, (8.12.6)] for f of finite presentation; in [**1**, Chapter IV, (18.12.13)] in the general case!) We will not use this result in this book. Theorem 6.4 has the following important corollaries which we will prove and use (for X noetherian):

Corollary 6.5. *Let $f\colon Z \to X$ be a morphism. Then the following are quivalent:*

a) f *is proper with finite fibres,*
b) f *is finite* (Definition II.6.6)*, i.e., the sheaf* $\mathcal{A} = f_*\mathcal{O}_Z$ *is quasi-coherent, for all* $U \subset X$ *affine* $\mathcal{A}(U)$ *is finitely generated as algebra and integral over* $\mathcal{O}_X(U)$*, and the natural morphism* $Z \to \mathrm{Spec}_X(\mathcal{A})$ *is an isomorphism.*

PROOF USING THEOREM 6.4. (b) $\Longrightarrow$ (a) is elementary: use Proposition II.6.5. As for (a) $\Longrightarrow$ (b), everything is local over X so we may assume $X = \mathrm{Spec}\, R$. Then by Theorem 6.4 f factors:

$$Z \hookrightarrow \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, R.$$

Since Z is proper over $\mathrm{Spec}\, R$, the image of Z in $\mathrm{Spec}\, B$ is closed as well as open, hence $Z \cong \mathrm{Spec}\, B/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Then $f_*\mathcal{O}_Z \cong \widetilde{B/\mathfrak{a}} \cong \mathrm{Spec}\, f_*\mathcal{O}_Z$. □

COROLLARY 6.6 (Characterization of normalizations). *Let* X *be an integral scheme,* Z *a normal, integral scheme and* $f\colon Z \to X$ *a proper surjective morphism with finite fibres. Then* $\mathbf{R}(Z)$ *is a finite algebraic extension of* $\mathbf{R}(X)$ *and* Z *is isomorphic to the normalization of* X *in* $\mathbf{R}(Z)$*.*

PROOF. Straightforward. □

COROLLARY 6.7. *Let* X *be a normal noetherian scheme,* $f\colon Z \to X$ *a proper étale morphism with* Z *connected. Then* Z *is isomorphic to the normalization of* X *in some finite separable field extension* $L \supset \mathbf{R}(X)$*.*

PROOF. This reduces to Corollary 6.6 because of Proposition 5.5. □

INDEPENDENT PROOF OF COROLLARY 6.5 WHEN X IS NOETHERIAN. Assume $f\colon Z \longrightarrow X$ given, proper with finite fibres. Let $\mathcal{A} = f_*\mathcal{O}_Z$. Then by the fundamental theorem of "holomorphic functions" (6.3), $\mathcal{A}$ is an $\mathcal{O}_X$-module of finite type, hence $\mathcal{A}(U)$ is finitely generated as algebra and integral over $\mathcal{O}_X(U)$ for all affine U. Let $Y = \mathrm{Spec}_X\, \mathcal{A}$ so that we have a factorization:

$$\begin{tikzcd} Z \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Note that Y is noetherian, h is proper with finite fibres and now $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$. We claim that under these hypotheses, h is an isomorphism, which will prove Corollary 6.5. First of all, h is surjective: in fact h proper implies $h(Z)$ closed and if $h(Z) \subsetneqq Y$, then $h_*\mathcal{O}_Z$ would be annihilated by some power of the ideal of $h(Z)$, hence would not be isomorphic to $\mathcal{O}_Y$. Secondly, h is injective: if $h^{-1}(y)$ consisted in more than one point, we argue as in the proof that U3 $\Longrightarrow$ $\widetilde{\text{U5}}$ and find a non-trivial idempotent in

$$\varprojlim_{\nu} (h_*\mathcal{O}_Z)_y / \mathfrak{m}_y^\nu \cdot (h_*\mathcal{O}_Z)_y.$$

But since $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$, this is just the completion $\widehat{\mathcal{O}}_{y,Y}$ which is a local ring. The only idempotent in local rings are 0 and 1 so this is a contradiction. Thus h is bijective and closed, hence it is a homeomorphism. Since $h_*\mathcal{O}_Z \cong \mathcal{O}_Y$, h even sets up an isomorphism of the ringed space $(Z, \mathcal{O}_Z)$ with $(Y, \mathcal{O}_Y)$, i.e., $Z \cong Y$ as schemes. □

(7.2)

$$\begin{tikzcd}
 & \operatorname{Spec}\widehat{\mathcal{O}}_{y,Y} \arrow[dl] \arrow[ddd] & \operatorname{Spec}\left(\widehat{\mathcal{O}}_{y,Y}/N\cdot\widehat{\mathcal{O}}_{y,Y}\right) \arrow[l, hook'] \arrow[d, "\approx"'] & & \\
 & & \operatorname{Spec}(\mathcal{O}_{y',Y'}) \cup Y_2' \arrow[d, equal] & & \\
Y \arrow[d, "f"'] & & Y \times_X \operatorname{Spec} R \arrow[ll] \arrow[d] & & \operatorname{Spec}(\mathcal{O}_{y',Y'} \otimes_R K) \cup \cdots \\
X & \operatorname{Spec}\widehat{\mathcal{O}}_{x,X} \arrow[l] & \operatorname{Spec}\widehat{\mathcal{O}}_{x,X}/N = R \arrow[l, hook'] & Y \times_X \operatorname{Spec} K \arrow[ul] \arrow[d] \arrow[ur, equal] \arrow[dr, equal] & \\
 & \operatorname{Spec}\mathbf{R}(X) = \{\eta_X\} \arrow[ul] & & \operatorname{Spec} K \arrow[ll] \arrow[ul] & f^{-1}(\eta_X) \times_{\operatorname{Spec}\mathbf{R}(X)} \operatorname{Spec} K
\end{tikzcd}$$

FIGURE V.4

7. Multiplicities following Weil

We can generalize to the case of schemes the concept of multiplicity of a point for a finite morphism introduced for complex varieties by topological means in Part I [**87**, (3.12), (4.19)]:

DEFINITION 7.1 (à la Weil). Let X be a noetherian integral scheme, $x \in X$ a formally unibranch point. Let $f\colon Y \to X$ be a morphism of finite type and let y be an isolated point of $f^{-1}(x)$. Then we define $\operatorname{mult}_y(f)$ as follows: Let $R = \widehat{\mathcal{O}}_{x,X}/\sqrt{(0)}$: By assumption this is an integral domain. Let $K =$ quotient field of R. Form the fibre product:

$$\begin{tikzcd}
Y \arrow[d, "f"'] & Y' \arrow[l] \arrow[d] \\
X & \operatorname{Spec} R \arrow[l]
\end{tikzcd}$$

Let $y' \in Y'$ be the unique point over y. By Hensel's lemma (Lemma IV.6.1):

$$\begin{aligned} Y' &= Y_1' \cup Y_2' \qquad \text{(disjoint)} \\ Y_1' &= \operatorname{Spec}\mathcal{O}_{y',Y'}, \text{ being finite over } \operatorname{Spec} R. \end{aligned}$$

Define

$$\operatorname{mult}_y f = \dim_K\left(\mathcal{O}_{y',Y'} \otimes_R K\right).$$

If we write down all the schemes that this interesting definition suggests, we get the diagram in Figure V.4 which needs to be pondered (we let $N = \sqrt{(0)}$ in $\widehat{\mathcal{O}}_{x,X}$): This shows that to get $\operatorname{mult}_y f$, we take the generic fibre of f, extend it to the bigger ground field $K \supset \mathbf{R}(X)$, split this K-scheme into two disjoint pieces in some sense by specializing from η_X to x, and then measure the size of one of these pieces!

A few comments on this definition:

7.3. $[\Bbbk(y) : \Bbbk(x)]_s$ *divides* $\operatorname{mult}_y f$, *hence we write*

$$\operatorname{mult}_y(f) = [\Bbbk(y) : \Bbbk(x)]_s \cdot \operatorname{mult}_y^{\circ}(f).$$

PROOF. Let $L \subset \Bbbk(y)$ be the subfield of elements separable over $\Bbbk(x)$ and let $\widetilde{\mathcal{O}}$ be the finite étale extension of $\widehat{\mathcal{O}}_{x,X}$ with residue field L, as in Corollary IV.6.3 (see also §3 of the present chapter). Then by Corollary IV.6.3, $\mathcal{O}_{y',Y'}$ is an $\widetilde{\mathcal{O}}$-algebra, hence if $\widetilde{K}$ is the quotient field of

$\widetilde{\mathcal{O}}$, $\mathcal{O}_{y',Y'} \otimes_R K$ is a vector space over $\widetilde{K}$. Therefore $[\widetilde{K} : K] | \operatorname{mult}_y f$. But

$$\begin{aligned}[\widetilde{K} : K] &= \text{rank of } \widetilde{\mathcal{O}} \text{ as free } \widehat{\mathcal{O}}_{x,X}\text{-module} \\ &= [L : \Bbbk(x)] \\ &= [\Bbbk(y) : \Bbbk(x)]_s.\end{aligned}$$

□

7.4. $\operatorname{mult}_y f \geq 1$ *if and only if* Y *has a component* Y_1 *through* y *dominating* X *(i.e.,* $\eta_{Y_1} \mapsto \eta_X$*).*

PROOF. If Y has no such component, there will be some non-zero $a \in \mathcal{O}_{x,X}$ such that $f^*a = 0$ in $\mathcal{O}_{y,Y}$. Therefore $f^*a = 0$ in $\mathcal{O}_{y',Y'}$ and $\mathcal{O}_{y',Y'} \otimes_R K = (0)$. To prove the converse, use generic flatness (Theorem IV.4.8): there is a non-zero $a \in \mathcal{O}_{x,X}$ such that the localization $(\mathcal{O}_{y,Y})_a$ is flat over $(\mathcal{O}_{x,X})_a$. Making the base change, it follows that Y_1' is flat over $\operatorname{Spec} R$ over the open set R_a. But then

$$\begin{aligned}\operatorname{mult}_y f = 0 &\Longrightarrow \mathcal{O}_{y',Y'} \otimes_R R_a = (0) \\ &\Longrightarrow a^l = 0 \text{ in } \mathcal{O}_{y',Y'} \text{ for some } l \\ &\Longrightarrow a^l = 0 \text{ in } \widehat{\mathcal{O}}_{y,Y}/N \cdot \widehat{\mathcal{O}}_{y,Y} \text{ (see diagram in Figure V.4)} \\ &\Longrightarrow a^m = 0 \text{ in } \widehat{\mathcal{O}}_{y,Y} \text{ for some } m \\ &\Longrightarrow a^m = 0 \text{ in } \mathcal{O}_{y,Y} \\ &\Longrightarrow \text{no component of } Y \text{ through } y \text{ dominates } X.\end{aligned}$$

□

7.5. *Assume* X *is formally normal at* x *and that all associated points of* Y *lie over* η_X*. Then*

$$\operatorname{mult}_y^\circ f = 1 \text{ if and only if } f \text{ is étale at } y.$$

PROOF. If f is étale, then f is flat, hence $Y_1' \to \operatorname{Spec} R$ is flat, hence $\mathcal{O}_{y',Y'}$ is a free R-module of some rank n. But on the one hand,

$$n = \dim_K \mathcal{O}_{y',Y'} \otimes_R K = \operatorname{mult}_y f$$

and on the other hand:

$$n = \dim_{\Bbbk(x)} \mathcal{O}_{y',Y'} \otimes_R \Bbbk(x) = \dim_{\Bbbk(x)} \mathcal{O}_{y,f^{-1}(x)}.$$

But $f^{-1}(x)$ is zero-dimensional and reduced at y because f is étale, hence $\mathcal{O}_{y,f^{-1}(x)} = \Bbbk(y)$, hence $n = [\Bbbk(y) : \Bbbk(x)]$. But f étale also implies $\Bbbk(y)$ separable over $\Bbbk(x)$, so $\operatorname{mult}_y^\circ f = 1$. Conversely, if $\operatorname{mult}_y^\circ f = 1$, then using the notation of the proof of (7.3), $\mathcal{O}_{y',Y'} \otimes_R K \cong \widetilde{K}$. Now $\widetilde{\mathcal{O}}$ is étale over $\widehat{\mathcal{O}}_{x,X}$ which we have assumed is an integrally closed domain. Therefore $\widetilde{\mathcal{O}}$ is an integrally closed domain. But if $\mathfrak{a} = \{a \in \mathcal{O}_{y',Y'} \mid a \cdot b = 0 \text{ for some } b \in R,\ b \neq 0\}$, then $\mathcal{O}_{y',Y'}/\mathfrak{a}$ is an $\widetilde{\mathcal{O}}$-algebra, integrally dependent on $\widetilde{\mathcal{O}}$ and contained in $\mathcal{O}_{y',Y'} \otimes_R K = \widetilde{K}$. Thus $\mathcal{O}_{y',Y'}/\mathfrak{a} = \widetilde{\mathcal{O}}$. Using generic flatness of f as in (7.4), we find $a \in \mathcal{O}_{x,X}$ such that $(\mathcal{O}_{y',Y'})_a$ is flat over R_a. Since this means $(\mathcal{O}_{y',Y'})_a$ is torsion-free as R_a-module, $\mathfrak{a}_a = (0)$ or $a^l \cdot \mathfrak{a} = (0)$, some l. But now by hypothesis $a \neq 0$ at any associated point of Y so

$$\mathcal{O}_{y,Y} \xrightarrow{a} \mathcal{O}_{y,Y}$$

is injective. Since $Y \times_X \operatorname{Spec} R$ is flat over Y,

$$\mathcal{O}_{y',Y'} \xrightarrow{a} \mathcal{O}_{y',Y'}$$

is injective too. Therefore $\mathfrak{a} = (0)$, and $\mathcal{O}_{y',Y'} \cong \widetilde{\mathcal{O}}$. Therefore

$$\begin{aligned}(\Omega_{Y/X})_y \otimes_{\mathcal{O}_{y,Y}} \Bbbk(y) &\cong (\Omega_{Y'_1/\operatorname{Spec} R}) \otimes_{\mathcal{O}_{y',Y'}} \Bbbk(y)\\ &\cong (\Omega_{\operatorname{Spec}\widetilde{\mathcal{O}}/\operatorname{Spec} R}) \otimes_{\widetilde{\mathcal{O}}} L\\ &= (0)\end{aligned}$$

so Y is étale over X at y by Criterion 4.1$^+$. ☐

The most famous result about multiplicities is the formula $n = \sum e_i f_i$ (cf. Zariski-Samuel [**119**, vol. I, p. 287]). In our language, the result is:

THEOREM 7.6. *Let $f\colon Y \to X$ be a finite surjective morphism between integral schemes, and assume X formally irreducible at x. Then if $f^{-1}(x) = \{y_1, \ldots, y_t\}$:*

$$[\mathbf{R}(Y) : \mathbf{R}(X)] = \sum_{i=1}^{t} \operatorname{mult}^\circ_y(f) \cdot [\Bbbk(y) : \Bbbk(x)]_s.$$

PROOF. This follows immediately from the big diagram in Figure V.4: in fact,

$$Y \times_X \operatorname{Spec} R = \bigcup_{i=1}^{t} Y'_i \qquad \text{(disjoint)}$$

where Y'_i has one closed point y'_i lying over $y_i \in Y$. Then

$$\operatorname{Spec}(\mathbf{R}(Y) \otimes_{\mathbf{R}(X)} K) = f^{-1}(\eta_X) \times_{\operatorname{Spec}\mathbf{R}(X)} \operatorname{Spec} K = \bigcup_{i=1}^{t} \operatorname{Spec}(\mathcal{O}_{y'_i,Y'_i} \otimes_R K),$$

hence

$$\mathbf{R}(Y) \otimes_{\mathbf{R}(X)} K \cong \bigoplus_{i=1}^{t} \left[\mathcal{O}_{y'_i,Y'_i} \otimes_R K\right].$$

Therefore

$$\begin{aligned}[\mathbf{R}(Y) : \mathbf{R}(X)] &= \dim_K \mathbf{R}(Y) \otimes_{\mathbf{R}(X)} K\\ &= \sum_{i=1}^{t} \dim_K \left(\mathcal{O}_{y'_i,Y'_i} \otimes_R K\right)\\ &= \sum_{i=1}^{t} \operatorname{mult}_{y_i} f.\end{aligned}$$

☐

Exercise—Modifications needed

For some of the notions and terminology in the following, the reader is referred to Part I [**87**].

(1) When x is a regular point of X, use Exercise 1, §4A with $R = \widehat{\mathcal{O}}_{x,X}$ to prove that

$$\operatorname{mult}_y(f) = e(\mathfrak{m}_{x,X} \cdot \mathcal{O}_{y,Y}; \mathcal{O}_{y,Y}).$$

Use this to give a second proof of the equality of the "results" of Part I and Part II in case X is non-singular at x.

(2) In the definition of $\mathrm{mult}_y(f)$, say $\widetilde{X}$ is any intermediate integral scheme:

$$\begin{array}{ccccc} X & \longleftarrow & \widetilde{X} & \longleftarrow & \operatorname{Spec} R \\ \eta_X & \longleftarrow\!\shortmid & \eta_{\widetilde{X}} & \longleftarrow\!\shortmid & [(0)] \end{array}$$

such that the decomposition of Y' is induced by a decomposition already over $\widetilde{X}$:

$$Y \times_X \widetilde{X} = \widetilde{Y}_1 \cup \widetilde{Y}_2.$$

Let $\widetilde{y} =$ image of y' in $\widetilde{Y}_1$, $\widetilde{x} =$ image of x' in $\widetilde{X}$ and $\widetilde{K} = \mathbf{R}(\widetilde{X})$. Then show

$$\mathrm{mult}_y f = \dim_{\widetilde{K}} \left(\mathcal{O}_{\widetilde{y},\widetilde{Y}_1} \otimes_{\mathcal{O}_{\widetilde{x},\widetilde{X}}} \widetilde{K} \right).$$

Now if X is of finite type over $\mathbb{C}$, take

$$\widetilde{X} = \operatorname{Spec} \mathcal{O}_{x,X,\mathrm{an}}.$$

Using the fact that $\mathcal{O}_{y,Y,\mathrm{an}}$ is a finite $\mathcal{O}_{x,X,\mathrm{an}}$-module, show that $Y \times_X \widetilde{X}$ as above decomposes and that $\widetilde{Y}_1 = \operatorname{Spec} \mathcal{O}_{y,Y,\mathrm{an}}$. Deduce that the multiplicity of (7.1) is equal to the multiplicity of Part I [**87**, (4.19)].

(3) Referred to in §VIII.3 (Kummer theory) Let X be a noetherian scheme with $1/n, \zeta \in \Gamma(\mathcal{O}_X)$, $\zeta =$ primitive n-th root of unity, and consider pairs (π, ϕ):

$$\begin{array}{c} Y \circlearrowleft \phi \\ \pi \big\downarrow \\ X \end{array}$$

π étale and proper, $\pi = \pi \circ \phi$, $\phi^n = 1_Y$ and for all geometric points:

$$\lambda\colon \operatorname{Spec} k \longrightarrow X, \qquad k \text{ algebraically closed},$$

we assume

$$Y \times_X \operatorname{Spec} k = n \text{ points permuted cyclically by } \phi \times 1_y.$$

We call this an *n-cyclic étale covering* of X. Prove that $\exists$ an invertible sheaf $\mathcal{L}$ on X and an isomorphism $\alpha\colon \mathcal{L}^n \xrightarrow{\approx} \mathcal{O}_X$ such that

$$\begin{aligned} &Y = \mathrm{Spec}_X \mathcal{A} \\ &\mathcal{A} = \mathcal{O}_X \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{n-1} \\ &\qquad \text{with multiplication} \\ &\qquad \mathcal{L}^i \times \mathcal{L}^j \longrightarrow \begin{cases} \mathcal{L}^{i+j} & i+j<n \\ \mathcal{L}^{i+j-n} & i+j \geq n \text{ via } \alpha. \end{cases} \end{aligned}$$

Hint: Write $Y = \mathrm{Spec}_X \mathcal{A}$ (cf. Proposition-Definition I.7.3) and show that $\mathcal{A}$ decomposes into eigensheaves under the action of ϕ^*:

$$\mathcal{A} = \bigoplus_{\nu=0}^{n-1} \mathcal{L}_\nu, \qquad \phi^*(x) = \zeta^\nu \cdot x, \quad x \in \mathcal{L}_\nu(U).$$

Use the fact: flat + finite presentation over a local ring $\Longrightarrow$ free to deduce that the $\mathcal{L}_\nu$ are locally free. Then show by computing geometric fibres that $\operatorname{rk} \mathcal{L}_\nu = 1$ and multiplication induces an isomorphism $\mathcal{L}_i \otimes \mathcal{L}_j \xrightarrow{\approx} \mathcal{L}_{i+j}$ or $\mathcal{L}_{i+j-n}$. Show conversely that for any $\mathcal{L}$, α, we obtain an n-cyclic étale covering Y. Deduce that if X is a complete variety over an algebraically closed field k, then:

$$\{\text{Set of } n\text{-cyclic étale coverings}\} \cong \{\lambda \in \operatorname{Pic}(X) \mid n\lambda = 0\}.$$

(See Theorem VIII.4.2 for the case $n = \operatorname{char} k$.)

(4) (cf. Remark at the end of §2) For simplicity, let $S = \operatorname{Spec}(k)$ with a field k. For a finite dimensional k-vector space E, consider the Grassmannian scheme $\operatorname{Grass}^r(E)$ over k. Let

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\operatorname{Grass}^r(E)} \otimes_k E \xrightarrow{\alpha} \mathcal{Q} \longrightarrow 0$$

be the universal exact sequence on $\operatorname{Grass}^r(E)$. A k-rational point $x \in \operatorname{Grass}^r(E)$ corresponds to an exact sequence of k-vector spaces

$$0 \longrightarrow \mathcal{K}(x) \longrightarrow E \longrightarrow \mathcal{Q}(x) \longrightarrow 0,$$

where $\mathcal{K}(x)$ and $\mathcal{Q}(x)$ are the fibres at x of $\mathcal{K}$ and $\mathcal{Q}$, respectively. Using the description of the tangent space in terms of $k[\epsilon]/(\epsilon^2)$ in §1, show

$$T_{x,\operatorname{Grass}^r(E)} = \operatorname{Hom}_k(\mathcal{K}(x), \mathcal{Q}(x)),$$

hence

$$T^*_{x,\operatorname{Grass}^r(E)} = \operatorname{Hom}_k(\mathcal{Q}(x), \mathcal{K}(x)).$$

(5) (The tensor product $\mathcal{L}^n$ of a line bundle $\mathcal{L}$ is denoted $\mathcal{L}^{\otimes}$ here, to avoid confusion with the direct sum $\mathcal{L}^{\oplus n}$.) Let X be a noetherian integral scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, and $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ a global section of $\mathcal{L}^{\otimes n}$, $n \geq 2$. Let $B \subset X$ be the Cartier divisor defined by f, so that f defines an isomorphism $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(B)$. Let $\mathbb{L} := \operatorname{Spec}\left(\bigoplus_{m\geq 0} \mathcal{L}^{\otimes(-m)}\right) \xrightarrow{\widetilde{\pi}} X$, thought of as the total space of the line bundle over X whose sheaf of germs of sections is $\mathcal{L}$. Denote by T the tautological global section of $\widetilde{\pi}^*\mathcal{L}$, corresponding to the canonical element

$$\mathbf{1} \in \Gamma(X, \mathcal{L}^{\otimes(-1)} \otimes \mathcal{L}) \subset \bigoplus_{m\geq 0} \Gamma(X, \mathcal{L}^{\otimes(-m)} \otimes \mathcal{L}) = \Gamma(\mathbb{L}, \widetilde{\pi}^*\mathcal{L}).$$

The *cyclic covering* of order n of X attached to the triple $(X, \mathcal{L}, f)$ is by definition the divisor $Y \subset \mathbb{L}$ of the section $T^n - \pi^* f \in \Gamma(\mathbb{L}, \pi^*\mathcal{L}^{\otimes n})$. Let $\pi\colon Y \to X$ be the finite locally free morphism induced by $\widetilde{\pi}$. Let $B_1 \subset Y$ be the Cartier divisor in Y attached to the $T|_Y \in \Gamma(Y, \pi^*\mathcal{L})$, the image in in $\Gamma(Y, \pi^*\mathcal{L})$ of the tautological section of $\widetilde{\pi}^*\mathcal{L}$.

(i) Show that $\pi_*\mathcal{O}_Y$ is isomorphic to $\oplus_{0\leq m\leq n-1}\mathcal{L}^{\otimes(-m)}$ as an $\mathcal{O}_X$-module.
(ii) If n is invertible in $\mathcal{O}_X$, then $\pi\colon \pi^{-1}(Y \setminus B_1) \to X \setminus B$ is finite étale.
(iii) Verify that B_1 is the inverse image of B in Y, and we have a natural isomorphism $\pi^*\mathcal{L} \cong \mathcal{O}_Y(B_1)$. Consequently $\pi^*\mathcal{O}_X(B) \cong \mathcal{O}_Y(B_1)^{\otimes n}$.
(iv) Suppose that n is invertible in $\mathcal{O}_X$ and X is smooth over a scheme S. Then the canonical sheaf $\mathcal{K}_{Y/S} := \Omega_{Y/S}$ for Y/S is isomorphic to $\pi^*\left(\mathcal{K}_{X/S} \otimes \mathcal{L}^{\otimes(n-1)}\right)$.

(6) Work over an algebraically closed field k of characteristic $\neq 2$. Let $B \subset \mathbb{P}^2$ be a smooth conic curve defined by a homogeneous quadratic polynomial $f(x, y, z)$. Let $\pi\colon Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ attached to the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), f)$, a smooth projective surface.

(i) Show that $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is an ample invertible $\mathcal{O}_Y$-module. Moreover the complete linear system $|\Gamma(Y, \mathcal{L})|$ is base point free.
(ii) Show that the canonical sheaf $\mathcal{K}_Y := \Omega^2_Y$ is isomorphic to $\mathcal{L}^{\otimes -2}$, and $(\mathcal{K}_Y)^2 = 8$.
(iii) If l is a line in $\mathbb{P}^2$ meeting B at two distinct points, then $\pi^{-1}(l)$ is a smooth curve in Y and $\deg(\mathcal{L}|_{\pi^{-1}(l)}) = 2$.
(iv) If l is a tangent line to B, then $\pi^{-1}(l)$ is the union $\widetilde{l_1} \cup \widetilde{l_2}$ of two smooth curves in Y meeting transversally at a point. Moreover $\deg(\mathcal{L}|_{\widetilde{l_i}}) = 1$ for $i = 1, 2$.
(v) Show that B is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

(vi) Show that $\dim \Gamma(Y, \mathcal{L}) = 4$, $\dim \Gamma(Y, \mathcal{L}^{\otimes 2}) = 9$, and $\dim \mathrm{Symm}^2(\Gamma(Y, \mathcal{L})) = 10$. Conclude that the image of the morphism $\phi_{\mathcal{L}} \colon Y \to \mathbb{P}^3$ defined by the complete linear system $|\Gamma(Y, \mathcal{L})|$ is contained in a quadric hypersurface in $\mathbb{P}^3$.

(vii) Show that $\phi_{\mathcal{L}}$ is a closed embedding.

(7) Work over a field k of characteristic $\neq 2$. Let $B \subset \mathbb{P}^2$ be a smooth curve defined by a homogeneous polynomial $f(x, y, z)$ of degree 4. Let $\pi \colon Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ attached to the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2), f)$, a smooth projective surface. Denote by B_1 the ramification locus of π in Y. We know by construction that π induces an isomorphism $B_1 \xrightarrow{\sim} B$. Moreover the anticanonical sheaf $\mathcal{K}_Y^{\otimes -1}$ is ample and isomorphic to $\pi^* \mathcal{O}_{\mathbb{P}^2}(1)$; denote it by $\mathcal{L}$. (Note: It is known that Y is a Del Pezzo surface of degree 2, i.e., $Y \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, k^{\mathrm{alg}}$ is isomorphic to the blow up of $\mathbb{P}^2$ whose center is the union of 7 distinct points of $\mathbb{P}^2$, no three of which lie on one line and no six of which line on one conic.)

(i) Suppose that l is a line in $\mathbb{P}^2$ intersecting B transversally at 4 distinct points, i.e., l is not a tangent line to B. Let $D_1 = \pi^{-1}(l)$ be the inverse image of l in Y. Show that D_1 is a smooth curve, $\deg_D(\mathcal{K}_Y) = -2$, $(D)_Y^2 = 2$, and D is a curve of genus 1.

(ii) Show that the inverse image of any line in $\mathbb{P}^2$ tangent to B is a singular divisor in Y. Here the inverse image of a line l in Y means $\mathrm{Spec}(\mathcal{O}_Y / \pi^* \mathcal{I} \cdot \mathcal{O}_Y)$, where $\mathcal{I}$ is the ideal of $\mathcal{O}_{\mathbb{P}^2}$ which defines the line l.

(iii) Suppose that l is line in $\mathbb{P}^2$ that is tangent to B at a point x_0, and l intersects B transversally at two points $x_1 \neq x_2$ different from x_0. Let $D_2 = \pi^{-1}(l)$ be the inverse image of l in Y, and y_0, y_1, y_2 the three points of D_2 above x_0, x_1 and x_2, respectively. Show that D_2 is an irreducible divisor on Y with $(D_2)_Y^2 = 2$, $(\mathcal{K}_Y \cdot D_2)_Y = -2$. The curve D_2 is smooth at y_1 and y_2, and has an ordinary double point at y_0. Moreover D_2 is a rational curve.

(iv) Suppose that l is a line in $\mathbb{P}^2$ that intersects B at a point x_0 with multiplicity 3. Let $D_3 = \pi^{-1}(l)$ be the inverse image of l in Y. Show that D_3 is an irreducible rational curve with a cusp, with $(D_3)_Y^2 = 2$, $(\mathcal{K}_Y \cdot D_3)_Y = -2$.

(v) Suppose that l is a line in $\mathbb{P}^2$ that is tangent to B at two distinct points x_1 and x_2. Assume moreover that every element of k has a square root. Show that the inverse image of l in Y is a union of two smooth curves C_1 and C_2 meeting transversally at the two points y_1, y_2 above x_1 and x_2, respectively, and the map π induces an isomorphism $C_i \xrightarrow{\sim} l$ for $i = 1, 2$. We have $\deg_{C_i}(\mathcal{L}) = ((C_1 + C_2) \cdot C_i)_Y = 1$, $\deg_{C_i}(\mathcal{K}_Y) = -1$, $(C_i)_Y^2 = -1$, $(C_1 \cdot C_2)_Y = 2$.
Hint: Here is a sample calculation. After a linear change of variables, we may assume that the equation of the tangent line is $y = 0$, and the affine equation of the plane curve B is of the form $f(x, y) = y g(x, y) + a(x - b_1)^2 (x - b_2)^2$, with $g(x, y) \in k[x, y]$, $a, b_1, b_2 \in k$, $a \neq 0$, $b_1 \neq b_2$, where $(x, y) = (b_i, 0)$ corresponds to the point x_i. Then over the affine open in question, the inverse image of l in Y is $\mathrm{Spec}\left(k[u, x]/(u^2 - a(x - b_1)^2 (x - b_2)^2)\right)$.

(vi) Suppose that l is a line in $\mathbb{P}^2$ that intersects B at a point x_0 with multiplicity 4. Show that the inverse image of l in Y is a union of two smooth rational curves C_1 and C_2 on Y meeting at the point y_0 above x_0 with multiplicity 2. We have $\deg_{C_i}(\mathcal{L}) = ((C_1 + C_2) \cdot C_i)_Y = 1$, $\deg_{C_i}(\mathcal{K}_Y) = -1$, $(C_i)_Y^2 = -1$, $(C_1 \cdot C_2)_Y = 2$, same as in (v).

(8) Work over a field k of characteristic $\neq 2$. Let $B \subset \mathbb{P}^2$ be a smooth curve defined by a homogeneous polynomial $f(x, y, z)$ of degree 6. Let $\pi\colon Y \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ attached to the triple $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2), f)$, a smooth projective surface. Denote by B_1 the ramification locus of π in Y. We know by construction that π induces an isomorphism $B_1 \xrightarrow{\sim} B$. Moreover the anticanonical sheaf $\mathcal{K}_Y$ is trivial. Let $\mathcal{L} := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, an ample invertible $\mathcal{O}_Y$-module of degree 2.

(i) Show that $H^0(Y, \mathcal{O}_Y) = (0)$. (Note: The polarized surface $(Y, \mathcal{L})$ is a K3 surface of degree 2.)

(ii) Let l be a line in $\mathbb{P}^2$, and let $C := Y \times_{\mathbb{P}^2} l$ be the scheme theoretic inverse image of l in Y. Show that if l intersects B transversally at 6 distinct points, then C is a smooth curve of genus 2, and the ramification locus in l of the projection $C \to l$ is $l \cap B$.

(iii) Notation as in (ii). Discuss all possibilities of the configuration of C, including the following.

(a) If l is tangent to B at x_0 and intersects B at four distinct points x_1, x_2, x_3, x_4 different from x_0, then C is irreducible, $(C)^2 = 2$, C has an ordinary double point at the point y_0 above x_0, and the normalization $\widetilde{C}$ of C is a smooth curve of genus 1. (Write down the j-invariant of $\widetilde{C}$ in terms of the cross ratio of the four points x_1, x_2, x_3, x_4 on l.)

(b) If l is tangent to B at two distinct points x_1 and x_2, and l meets B at two distinct points x_3, x_4 other than x_1 and x_2, then C has two ordinary double points at the two points y_1, y_2 above x_1, x_2, and the normalization of C is a smooth curve of genus 0.

(c) If l is tangent to B at three distinct points x_1, x_2, x_3, then C is the disjoint union of two smooth rational curves E_1, E_2 meeting transversally at the three points y_1, y_2, y_3 above x_1, x_2, x_3, with $(E_1)^2 = (E_2)^2 = -2$, $(E_1 \cdot E_2) = 3$.

(A degenerate case of (c) is: l meets B at x_0 with multiplicity four. and at x_1 with multiplicity two; then E_1 meets E_2 with multiplicity 2 at y_0. A degenerate case of (a) is: l meets B at x_0 with multiplicity three and also at three other distinct points x_1, x_2, x_3; then C has a cusp at the point y_0 above x_0, and the normalization of C is a smooth curve of genus 1.)

(iv) Show that there are only a finite number of complete smooth curves of genus 0 on the surface Y.

(9) A *double six* in $\mathbb{P}^3$ is a pair of sextuples of disjoint lines $(l_1, \ldots, l_6)$, $(m_1, \ldots, m_6)$ such that $l_i \cap m_i = \emptyset$ for all i and l_i meets m_j at a point if $i \neq j$. Find a double six on the Fermat cubic. (Find the number of all double six's if you feel adventurous.)

(10) Find all lines on the Fermat cubic surface in $\mathbb{P}^3$.

Let $X = G(2, 4)$ be the Grassmanina of lines in $\mathbb{P}^3$. Let $\mathcal{S} \to X$ be the tautological rank two subbundle of the trivial rank four vector bundle on X, and let $\mathcal{S}^\vee$ be the dual of $\mathcal{S}$. Let $\mathcal{E} := \mathrm{Symm}^3(\mathcal{S}^\vee)$ be the third symmetric product of $\mathcal{S}^\vee$, a rank four vector bundle on X. We want to compute the Chern number $c_4(\mathcal{E})$, i.e., the pairing of $c_4(\mathcal{E})$ with the fundamental class of X. This number is the "expected number of lines" on a generic cubic surface in $\mathbb{P}^3$, because any cubic form $f(x_0, x_1, x_2, x_3)$ defines a section s_f of $\mathcal{E}$, and the zero locus of this section corresponds to lines in the cubic surface defined by $f(x_0, x_1, x_2, x_3)$.

First we express $c_4(\mathcal{E})$ in terms of $c_1(\mathcal{S}^\vee)$ and $c_2(\mathcal{S}^\vee)$. This is an exercise in symmetric functions in two variables, i.e., we will get a formula for $c_4(\mathrm{Symm}^3 \mathcal{F})$ for every rank two vector bundle $\mathcal{F}$. Apply the splitting principle: assume that $\mathcal{F} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$. Then

$$\begin{aligned} c(\operatorname{Symm}^3 \mathcal{F}) &= (1 + 3c_1(\mathcal{L}_1)) \cdot (1 + 2c_1(\mathcal{L}_1) + c_2(\mathcal{L}_2)) \times \\ &\qquad \times (1 + c_1(\mathcal{L}_1) + 2c_2(\mathcal{L}_2)) \cdot (1 + 3c_1(\mathcal{L}_2)) \end{aligned}$$

and we get

$$\begin{aligned} c_4(\operatorname{Symm}^3 \mathcal{F}) &= 9c_1(\mathcal{L}_1)c_2(\mathcal{L}_2) \cdot (2c_1(\mathcal{L}_1) + c_2(\mathcal{L}_2)) \cdot (c_1(\mathcal{L}_1) + 2c_2(\mathcal{L}_2) \\ &= 9c_2(\mathcal{F})^2 + 18c_1(\mathcal{F})^2 c_2(\mathcal{F}) \end{aligned}$$

from the identity

$$9t_1t_2(2t_1 + t_2)(t_1 + 2t_2) = 9(t_1t_2)^2 + 18t_1t_2(t_1 + t_2)^2.$$

Applying the general identity to $\mathcal{F} = \mathcal{S}^\vee$, we get

$$c_4(\mathcal{E}) = 9c_2(\mathcal{S}^\vee)^2 + 18c_1(\mathcal{S}^\vee)^2 c_2(\mathcal{S}^\vee).$$

To evaluate the Chern number $c_4(\mathcal{E})$, first recall that

$$c_1(\mathcal{S}^\vee) = \sigma_{1,0}, \quad c_2(\mathcal{S}^\vee) = \sigma_{1,1},$$

where $\sigma_{1,0}$ and $\sigma_{1,1}$ are Schbert cycles on X; see Griffiths-Harris [**44**]. The rest is an exercise in the Schubert calculus for $G(2,4)$. There are four Schubert cycles whose dimensions are between 1 and 3: $\sigma_{1,0}$, $\sigma_{2,0}$, $\sigma_{1,1}$, $\sigma_{2,1}$, of dimensions 3, 2, 2, 1, respectively. Their products are given by

$$\sigma_{1,0} \cdot \sigma_{1,0} = \sigma_{2,0} + \sigma_{1,1}, \quad \sigma_{1,0} \cdot \sigma_{2,0} = \sigma_{2,1}, \quad \sigma_{1,0} \cdot \sigma_{1,1} = \sigma_{2,1},$$

$$\sigma_{2,0} \cdot \sigma_{2,0} = 1, \quad \sigma_{1,0} \cdot \sigma_{2,1} = 1, \quad \sigma_{1,1} \cdot \sigma_{1,1} = 1, \quad \sigma_{2,0} \cdot \sigma_{1,1} = 0.$$

So we get

$$\begin{aligned} c_2(\mathcal{S}^\vee)^2 &= \sigma_{1,1} \cdot \sigma_{1,1} = 1, \\ c_1(\mathcal{S}^\vee)^2 c_2(\mathcal{S}^\vee) &= \sigma_{1,0} \cdot \sigma_{1,0} \cdot \sigma_{1,1} = 1 \end{aligned}$$

and

$$c_4(\mathcal{E}) = 9c_2(\mathcal{S}^\vee)^2 + 18c_1(\mathcal{S}^\vee)^2 c_2(\mathcal{S}^\vee) = 9 + 18 = 27.$$

CHAPTER VI

Group schemes and applications

1. Group schemes

Definition 1.1. Let $f\colon G \to S$ be an S-scheme. Then G is a group scheme over S if we are given three S-morphisms:

$$\begin{array}{ll} \mu\colon G \times_S G \longrightarrow G & (\text{“multiplication”}) \\ \iota\colon G \longrightarrow G & (\text{“inverse”}) \\ \epsilon\colon S \longrightarrow G & (\text{“identity”}) \end{array}$$

such that the following diagrams commute:

a) (“associativity”)

$$\begin{tikzcd} & G \times_S G \arrow[dr, "\mu"] & \\ G \times_S (G \times_S G) \arrow[ur, "1_G \times \mu"] \arrow[d, equal, "\sim"'] & & G \\ (G \times_S G) \times_S G \arrow[dr, "\mu \times 1_G"'] & & \\ & G \times_S G \arrow[uur, "\mu"'] & \end{tikzcd}$$

b) (“left and right identity laws”)

$$\begin{tikzcd} G \times_S S \arrow[r, "1_G \times \epsilon"] \arrow[d, equal, "\sim"'] & G \times_S G \arrow[dr, "\mu"] & \\ G \arrow[rr, "1_G"] & & G \\ S \times_S G \arrow[u, equal, "\sim"] \arrow[r, "\epsilon \times 1_G"'] & G \times_S G \arrow[ur, "\mu"'] & \end{tikzcd}$$

c) (“left and right inverse laws”)

$$\begin{tikzcd} & G \times_S G \arrow[r, "1_G \times \iota"] & G \times_S G \arrow[dr, "\mu"] & \\ G \arrow[ur, "\Delta"] \arrow[rr, "f"] \arrow[dr, "\Delta"'] & & S \arrow[r, "\epsilon"] & G \\ & G \times_S G \arrow[r, "\iota \times 1_G"'] & G \times_S G \arrow[ur, "\mu"'] & \end{tikzcd}$$

To relate this to the usual idea of a group, let $p\colon T \to S$ be any scheme over S and consider $\mathrm{Hom}_S(T, G)$, the set of T-valued points of G over S! Then:

a′) via μ, get a law of composition in $\mathrm{Hom}_S(T, G)$:
$\forall f, g \in \mathrm{Hom}_S(T, G)$, define $f \cdot g$ to be the composition:

$$T \xrightarrow{(f,g)} G \times_S G \xrightarrow{\mu} G$$

(this is associative by virtue of (a)),

b′) via ϵ, get a distinguished element $\epsilon \circ p \in \mathrm{Hom}_S(T, G)$ which is a two-sided identity for this law of composition by virtue of (b), c′) via ι, get a map $f \mapsto f^{-1}$ of $\mathrm{Hom}_S(T,G)$, $f^{-1} = \iota \circ f$, which is a two-sided inverse for this law of composition by virtue of (c).

Summarizing, (μ, ϵ, ι) make $\mathrm{Hom}_S(T,G)$ into an ordinary group for every T over S: For instance, if $S = \operatorname{Spec} k$, then the set of k-rational points of G is a group, and if k is algebraically closed and G is of finite type over k, this means that the set of closed points of G is a group. If you think about it, this is really what one should expect: for instance suppose you want to consider $\mathbb{A}^n_k$ as a group via vector addition. If $\mathbb{A}^n_k = \operatorname{Spec} k[X_1, \ldots, X_n]$, then for any two k-valued points P', P'' their sum is defined by:

$$X_i(P' + P'') = X_i(P') + X_i(P'');$$

thus if $\mu(P', P'') = P' + P''$, then the pull-back of the function X_i is computed via:

$$\begin{aligned} \mu^*(X_i) &= X_i(\mu(P', P'')) \\ &= X_i(P') + X_i(P'') \\ &= (X_i \circ p_1)(P', P'') + (X_i \circ p_2)(P', P'') \\ &= (p_1^* X_i + p_2^* X_i)(P', P''). \end{aligned}$$

Thus the law of composition:

$$\begin{array}{ccc} \mathbb{A}^n_k \times_{\operatorname{Spec} k} \mathbb{A}^n_k & \xrightarrow{\qquad\qquad \mu \qquad\qquad} & \mathbb{A}^n_k \\ \| & & \| \\ \operatorname{Spec} k[p_1^* X_1, \ldots, p_1^* X_n, p_2^* X_1, \ldots, p_2^* X_n] & & \operatorname{Spec} k[X_1, \ldots, X_n] \end{array}$$

is defined by $\mu^* X_i = p_1^* X_i + p_2^* X_i$. Similarly, define ι and ϵ via $\iota^* X_i = -X_i$ and $\epsilon^* X_i = 0$. Now if $\eta \in \mathbb{A}^n_k$ is the generic point, then to try to add η to itself, one would choose a point $\zeta \in \mathbb{A}^n_k \times \mathbb{A}^n_k$ such that $p_1(\zeta) = p_2(\zeta) = \eta$ and define $\eta + \eta$ to be $\mu(\zeta)$. But, taking $n = 1$ for instance, then one could take

$$\zeta = \begin{cases} \text{generic point of } \mathbb{A}^1_k \times \mathbb{A}^1_k \\ \text{or} \\ \text{generic point of line } p_1^* X = -p_2^* X + a, \qquad (a \in k). \end{cases}$$

In the first case, one sees that $\mu(\zeta) =$ generic point of $\mathbb{A}^1_k$, and in the second case, $\mu(\zeta) =$ (the point $X = a$)! The moral is that $\eta + \eta$ is not well-defined.

Another standard group scheme is: define

$$\begin{aligned} \mathrm{GL}_{n,k} &= \operatorname{Spec}\left(k[X_{11}, \ldots, X_{nn}] \left[\frac{1}{\det(X_{ij})} \right] \right) \\ \mu^*(X_{ij}) &= \sum_{k=1}^{n} p_1^* X_{ik} \cdot p_2^* X_{kj} \\ \epsilon^*(X_{ij}) &= \delta_{ij} \\ \iota^*(X_{ij}) &= (-1)^{i+j} \cdot ((j,i)\text{-th minor of } (X_{ij})) \frac{1}{\det(X_{i'j'})}. \end{aligned}$$

More elegantly, all the group schemes $\mathrm{GL}_{n,k}$ (resp. $\mathbb{A}^n_k$) over various base schemes $\operatorname{Spec} k$ are "induced" from one single group scheme $\mathrm{GL}_{n,\mathbb{Z}}$ (resp. $\mathbb{A}^n_{\mathbb{Z}}$) over $\operatorname{Spec} \mathbb{Z}$. One checks readily that if $f\colon G \to S$ is a group scheme over S, and $p\colon T \to S$ is any morphism, then $p_2\colon G \times_S T \to T$ is a group scheme over T in a canonical way. And one can define "universal" general linear and affine group scheme by:

$$\begin{aligned} \mathrm{GL}_{n,\mathbb{Z}} &= \operatorname{Spec}\left(\mathbb{Z}[X_{11},\dots,X_{nn}]\left[\frac{1}{\det(X_{ij})}\right]\right) \\ \mathbb{A}^n_{\mathbb{Z}} &= \operatorname{Spec}\mathbb{Z}[X_1,\dots,X_n] \\ \mu^*, \epsilon^*, \iota^* &\text{ given by the same formulae as before.} \end{aligned}$$

(Added in publication)

In terms of the way we defined S-schemes as representable functors $(\mathrm{Sch}/S)^\circ \to (\mathrm{Sets})$ in §I.8, we can formulate group schemes over S as follows:

Denote by (Groups) the category consisting of groups and homomorphisms of groups. Then group schemes G over S are exactly those S-schemes such that the functors h_G they represent are *group functors*, that is, factor through the functor (Groups) $\to$ (Sets) (that sends a group to its underlying set and a homomorphism to the underlying map)

$$h_G\colon (\mathrm{Sch}/S)^\circ \longrightarrow (\mathrm{Groups}) \longrightarrow (\mathrm{Sets}).$$

Here are some examples:

Example 1.2. (cf. Example I.8.4) $\mathbb{G}_{a,S} = \operatorname{Spec}_S(\mathcal{O}_S[T])$ is a commutative group scheme over S with the additive group

$$\operatorname{Hom}_S(Z,\mathbb{G}_{a,S}) = \Gamma(Z,\mathcal{O}_Z) \quad \text{for} \quad Z \in (\mathrm{Sch}/S)$$

and with an obvious homomorphism $f^*\colon \Gamma(Z,\mathcal{O}_Z) \to \Gamma(Z',\mathcal{O}_{Z'})$ for every S-morphism $f\colon Z' \to Z$.

More generally, we have:

Example 1.3. (cf. Example I.8.5) Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module on S. Then the relatively affine S-scheme

$$\operatorname{Spec}_S(\operatorname{Symm}(\mathcal{F})),$$

where $\operatorname{Symm}(\mathcal{F})$ is the symmetric algebra of $\mathcal{F}$ over $\mathcal{O}_S$, represents the additive group functor G defined as follows:

$$G(Z) = \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_Z) \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with the obvious homomorphism

$$G(f) = f^* : \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{F}_Z,\mathcal{O}_Z) \to \operatorname{Hom}_{\mathcal{O}_{Z'}}(\mathcal{O}_{Z'} \otimes_{\mathcal{O}_S} \mathcal{F}, \mathcal{O}_{Z'}) = \operatorname{Hom}_{\mathcal{O}_{Z'}}(f^*(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{F}), f^*\mathcal{O}_Z)$$

for $f \in \operatorname{Hom}_S(Z', Z)$.

Similarly to Example 1.2, we have:

Example 1.4. (cf. Example I.8.6) $\mathbb{G}_{m,S} = \operatorname{Spec}_S(\mathcal{O}_S[T,T^{-1}])$ is a commutative group scheme over S with the multiplicative group

$$\operatorname{Hom}_S(Z,\mathbb{G}_{m,S}) = \Gamma(Z,\mathcal{O}_Z)^* \quad \text{for } Z \in (\mathrm{Sch}/S),$$

where the asterisk denotes the set of invertible elements, with the obvious homomorphism

$$f^*\colon \Gamma(Z,\mathcal{O}_Z)^* \to \Gamma(Z,\mathcal{O}_{Z'})^*$$

for each $f \in \operatorname{Hom}_S(Z', Z)$.

More generally:

Example 1.5. (cf. Example I.8.7) Let n be a positive integer.

$$\mathrm{GL}_{n,S} = \mathrm{Spec}_S\left(\mathcal{O}_S\left[T_{11}, \ldots, T_{nn}, \frac{1}{\det(T)}\right]\right),$$

where $T = (T_{ij})$ is the $n \times n$-matrix with indeterminates T_{ij} as entries, is a relatively affine S-group scheme representing the multiplicative group functor

$$\mathrm{Hom}_S(Z, \mathrm{GL}_{n,S}) = \mathrm{GL}_n(\Gamma(Z, \mathcal{O}_Z)), \qquad \text{for } Z \in (\mathrm{Sch}/S),$$

the set of invertible $n \times n$-matrices with entries in $\Gamma(Z, \mathcal{O}_Z)$, with obvious homomorphisms corresponding to S-morphisms. Clearly, $G_{m,S} = \mathrm{GL}_{1,S}$.

Even more generally, we have (cf. EGA [**1**, Chapter I, revised, Proposition (9.6.4)]):

Example 1.6. (cf. Example I.8.8) Let $\mathcal{E}$ be a locally free $\mathcal{O}_S$-module of finite rank (cf. Definition 5.3). The multiplicative group functor G defined by

$$G(Z) = \mathrm{Aut}_{\mathcal{O}_Z}(\mathcal{O}_Z \otimes_{\mathcal{O}_S} \mathcal{E}) \quad \text{for } Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms is represented by a relatively affine group S-scheme $\mathrm{GL}(\mathcal{E})$. Example 1.5 is a special case with

$$\mathrm{GL}_{n,S} = \mathrm{GL}(\mathcal{O}_S^{\oplus n}).$$

Example 1.7. For a positive integer n and a scheme S, the "multiplicative group of n-th roots of unity" $\mu_{n,S}$ is the multiplicative group scheme over S defined by

$$\mu_{n,S}(Z) = \{\zeta \in \Gamma(Z, \mathcal{O}_Z)^* \mid \zeta^n = 1\}, \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$. It is represented by the S-scheme

$$\mu_{n,S} = \mathrm{Spec}_S(\mathcal{O}_S[t]/(t^n - 1)).$$

Example 1.8. Let S be a scheme of prime characteristic p (that is, $p = 0$ in $\mathcal{O}_S$, e.g., $S = \mathrm{Spec}(k)$ for a field k of characteristic $p > 0$). $\alpha_{p,S}$ is an additive group scheme over S defined by

$$\alpha_{p,S}(Z) = \{\xi \in \Gamma(Z, \mathcal{O}_Z) \mid \xi^p = 0\}, \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$. It is represented by the S-scheme

$$\alpha_{p,S} = \mathrm{Spec}_S(\mathcal{O}_S[t]/(t^p)).$$

For $\nu \geq 2$, we can define $\alpha_{p^\nu,S}$ similarly.

Example 1.9. The relative Picard functor in Example I.8.12 is the commutative group functor

$$\mathrm{Pic}_{X/S}\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Groups})$$

defined by

$$\mathrm{Pic}_{X/S}(Z) = \mathrm{Coker}[\varphi^*\colon \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(X \times_S Z)] \quad \text{for each } S\text{-scheme } \varphi\colon Z \to S$$

and the homomorphism $f^*\colon \mathrm{Pic}_{X/S}(Z) \to \mathrm{Pic}_{X/S}(Z')$ induced by the inverse image by each S-morphism $f\colon Z' \to Z$. The "sheafified" version of the relative Picard functor $\mathrm{Pic}_{X/S}$ when representable thus gives rise to a commutative group scheme over S called the *relative Picard scheme* of X/S. The reader is again referred to FGA [**2**, exposés 232, 236] as well as Kleiman's account on the interesting history (before and after FGA [**2**]) in FAG [**3**, Chapter 9]. See also Bosch, Lütkebohmert and Raynaud [**26**] and Mumford [**84**]. It is not hard to see that

$$\mathrm{Lie}(\mathrm{Pic}_{X/k}) = H^1(X, \mathcal{O}_X)$$

in the sense to be defined in Proposition-Definition 1.12 below.

Murre [**88**] gave a general criterion for the representability of commutative group functors over $S = \mathrm{Spec}(k)$ with a field k.

EXAMPLE 1.10. (FGA [**2**, exposés 195, 221]) Let X be a scheme over S. The automorphism functor of X/S is the multiplicative group functor $G\colon (\mathrm{Sch}/S)^\circ \to (\mathrm{Groups})$ defined by

$$G(Z) = \mathrm{Aut}_Z(X \times_S Z), \quad \text{for} \quad Z \in (\mathrm{Sch}/S)$$

and an obvious homomorphism $G(Z) \to G(Z')$ induced by the base extension by each S-morphism $f\colon Z' \to Z$.

If it is representable, then the S-scheme representing it is denoted $\mathrm{Aut}_S(X)$ and called the *automorphism group scheme* of X/S. It is not hard to see that over a field k,

$$\mathrm{Lie}(\mathrm{Aut}_k(X)) = H^0(X, \Theta_X) = \text{the tangent space of } \mathrm{Aut}_k(X) \text{ at } \mathrm{id}_X$$

in the sense to be defined in Proposition-Definition 1.12.

For instance, if $X = \mathbb{P}^n_S$, then $\mathrm{Aut}_S(\mathbb{P}^n_S) = \mathrm{PGL}_{n+1,S}$ (cf. Mumford [**83**, Chapter 0, §5, p.20]), where

$$\mathbb{P}^n_S = \mathrm{Proj}_S(\mathcal{O}_S[X_0, \ldots, X_n]) = \mathbb{P}^n_{\mathbb{Z}} \times S, \qquad \mathrm{PGL}_{n+1,S} = \mathrm{PGL}_{n+1,\mathbb{Z}} \times S$$

$$\mathrm{PGL}_{n+1} = \mathrm{PGL}_{n+1,\mathbb{Z}} \text{ open subset of } \mathrm{Proj}(\mathbb{Z}[A_{00}, \ldots, A_{nn}]) \text{ with } \det(A_{ij}) \neq 0.$$

Matsumura-Oort [**79**] gave a general criterion for the representability of group functors over $S = \mathrm{Spec}(k)$ with a field k, generalizing the commutative case dealt with by Murre [**88**].

THEOREM 1.11 (Cartier [**28**]). *Any group scheme G of finite type over a field k of characteristic 0 is smooth, hence, in particular, reduced.*

PROOF. We reproduce the proof in [**85**, Chapter III, §11, Theorem, p.101]. Denote by $e \in G$ the image of the identity morphism $\epsilon\colon \mathrm{Spec}(k) \to G$. Obviously e is a k-rational point, that is, $\Bbbk(e) = k$. For simplicity, we denote

$$\mathcal{O} = \mathcal{O}_{e,G}, \qquad \mathfrak{m} = \mathfrak{m}_{e,G}.$$

By what we saw in §V.4, it suffices to show that $\mathcal{O}$ is a regular local ring, since the argument works for the base extension $G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ to the algebraic closure $\overline{k}$, and the translation by $\mathrm{Spec}(\overline{k})$-valued points of G are isomorphisms sending e to the other closed points of $G \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$.

Choose $x_1, \ldots, x_n \in \mathfrak{m}$ so that their images form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$. Thus we obtain a continuous surjective k-algebra homomorphism from the formal power series ring to the completion of $\mathcal{O}$:

$$\alpha\colon k[[t_1, \ldots, t_n]] \longrightarrow \widehat{\mathcal{O}}, \qquad \alpha(t_i) = x_i.$$

As we show immediately after this proof (cf. Proposition-Definition 1.12), the map

$$\mathrm{Der}_k(\mathcal{O}) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) = T_{e,G}$$

sending a local vector field $D \in \mathrm{Der}_k(\mathcal{O})$ at e to the tangent vector of G at e sending $f \in \mathfrak{m}$ to $(Df)(e)$ is surjective. Hence we can choose $D_1, \ldots, D_n \in \mathrm{Der}_k(\mathcal{O})$ such that

$$D_i(x_j) = \delta_{ij}.$$

The D_i's obviously induce derivations of the completion $\widehat{\mathcal{O}}$ so that we get the Taylor expansion map (k is of characteristic 0!)

$$\begin{aligned} \beta\colon \widehat{\mathcal{O}} &\longrightarrow k[[t_1, \dots, t_n]] \\ f &\longmapsto \sum_{\substack{\nu_1,\dots,\nu_n \\ \nu_i \geq 0}} \frac{(D_1^{\nu_1} \cdots D_n^{\nu_n} f)(e)}{\nu_1! \cdots \nu_n!} t_1^{\nu_1} \cdots t_n^{\nu_n}, \end{aligned}$$

which is a continuous k-algebra homomorphism. β is surjective since $\beta(x_i) \equiv t_i \mod (t_1, \dots, t_n)^2$. Consequently, $\beta \circ \alpha$ is a surjective k-algebra homomorphism of $k[[t_1, \dots, t_n]]$ onto itself, hence is an automorphism. Thus α is injective as well so that

$$\alpha\colon k[[t_1, \dots, t_n]] \xrightarrow{\sim} \widehat{\mathcal{O}},$$

and $\widehat{\mathcal{O}}$ is regular, hence so is $\mathcal{O}$. □

In general, let G be a scheme over a field k, and e a k-rational point of G. Denote by $\mathrm{Der}_k(\mathcal{O}_G)$ the space of global k-derivations of $\mathcal{O}_G$ into itself, that is, the space of *vector fields* on G.

Introduce the k-algebra of "dual numbers"

$$\Lambda = k[\delta]/(\delta^2) = k \oplus k\delta.$$

Then the vector fields $D \in \mathrm{Der}_k(\mathcal{O}_G)$ are in one-to-one correspondence with the Λ-algebra automorphisms

$$\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$$

inducing the identity automorphism modulo δ by

$$\widetilde{D}(a + b\delta) = a + ((Da) + b)\delta, \qquad a, b \in \mathcal{O}_G.$$

Likewise, the tangent vectors $t \in \mathrm{Der}_k(\mathcal{O}_{e,G}, k)$ of G at e are in one-to-one correspondence with the Λ-algebra homomorphisms

$$\widetilde{t}\colon \mathcal{O}_{e,G} \otimes_k \Lambda \longrightarrow \Lambda$$

inducing the canonical surjection $\mathcal{O}_{e,G} \to k \mod \mathfrak{m}_{e,G}$ by

$$\widetilde{t}(a + b\delta) = a(e) + (t(a) + b(e))\delta, \qquad a, b \in \mathcal{O}_{e,G}.$$

Proposition-Definition 1.12. *Let G be a group scheme over a field k. A vector field $D \in \mathrm{Der}_k(\mathcal{O}_G)$ is said to be* left invariant *if*

$$\begin{tikzcd} \mathcal{O}_G \arrow[r, "D"] \arrow[d, "\mu^*"'] & \mathcal{O}_G \arrow[d, "\mu^*"] \\ \mathcal{O}_{G \times_k G} \arrow[r, "1 \otimes_k D"] & \mathcal{O}_{G \times_k G} \end{tikzcd}$$

is a commutative diagram. The k-vector space $\mathrm{Lie}(G)$ *of left invariant vector fields on G is called the* Lie algebra *of G. We have a natural isomorphism of k-vector spaces*

$$\mathrm{Lie}(G) \xrightarrow{\sim} T_{e,G},$$

where $e \in G$ is the image of the identity morphism $\epsilon\colon$ $\mathrm{Spec}(k) \to G$.

PROOF. Let

$$\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$$

be the the Λ-algebra automorphism corresponding to a vector field $D \in \mathrm{Der}_k(\mathcal{O}_G)$. Then the left invariance of D is equivalent to the commutativity of the following diagram

$$(*) \qquad \begin{array}{ccc} G \times_k G \times_k \operatorname{Spec}\Lambda & \xrightarrow{1_G \times \widetilde{D}} & G \times_k G \times_k \operatorname{Spec}\Lambda \\ \Big\downarrow{\scriptstyle \mu \times 1_\Lambda} & & \Big\downarrow{\scriptstyle \mu \times 1_\Lambda} \\ G \times_k \operatorname{Spec}\Lambda & \xrightarrow[\widetilde{D}]{} & G \times_k \operatorname{Spec}\Lambda, \end{array}$$

where we use the same symbol $\widetilde{D}$ for the $(\operatorname{Spec}\Lambda)$-automorphism $G \times_k \operatorname{Spec}\Lambda \xrightarrow{\sim} G \times_k \operatorname{Spec}\Lambda$ induced by $\widetilde{D}\colon \mathcal{O}_G \otimes_k \Lambda \xrightarrow{\sim} \mathcal{O}_G \otimes_k \Lambda$, etc.

If we denote

$$D' = p_1 \circ \widetilde{D}\colon G \times_k \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G \times_k \operatorname{Spec}\Lambda \xrightarrow{p_1} G,$$

then the commutativity of the diagram $(*)$ is equivalent to

$$D'(x \cdot y, l) = x \cdot D'(y, l), \qquad \forall x, y \in G(Z), \quad \forall l \in (\operatorname{Spec}\Lambda)(Z) \quad (Z\text{-valued points})$$

for any k-scheme Z, or equivalently,

$$D'(x, l) = x \cdot D'(\epsilon, l), \qquad \forall x \in G(Z), \quad \forall l \in (\operatorname{Spec}\Lambda)(Z)$$

for any k-scheme Z. If we denote

$$\widetilde{t} = p_1 \circ \widetilde{D} \circ (\epsilon, 1_\Lambda)\colon\ \operatorname{Spec}\Lambda \longrightarrow G \times_k \operatorname{Spec}\Lambda \xrightarrow{\widetilde{D}} G \times_k \operatorname{Spec}\Lambda \xrightarrow{p_1} G,$$

then $\widetilde{D}$ is the right multiplication by $\widetilde{t} \in G(\operatorname{Spec}\Lambda)$. Thus the Λ-valued points $\widetilde{t}$ of G are in one-to-one correspondence with the automorphisms $\widetilde{D}$ of $G \times_k \operatorname{Spec}\Lambda$ over $\operatorname{Spec}\Lambda$ such that the diagram $(*)$ commutes by the correspondence

$$p_1 \circ \widetilde{D} \circ (\epsilon, 1_\Lambda) = \widetilde{t}.$$

Thus the left invariant vector fields $D \in \mathrm{Der}_k(\mathcal{O}_G)$ are in one-to-one correspondence with the tangent vectors

$$t \in \mathrm{Der}_k(\mathcal{O}_{e,G}, k) = T_{e,G}.$$

□

REMARK. When $S = \operatorname{Spec}(k)$ with a field k of characteristic $p > 0$, the additive group scheme

$$\alpha_{p^\nu,S} = \operatorname{Spec}(k[t]/(t^{p^\nu}))$$

is not reduced with only one point! If n is divisible by p, the "multiplicative group of roots of unity" $\mu_{n,S}$ is not reduced either. Indeed, if $n = p^\nu \times n'$ with n' not divisible by p, then

$$\mu_{n,S} = \operatorname{Spec}(k[t]/(t^n - 1)) = \operatorname{Spec}(k[t]/(t^{n'} - 1)^{p^\nu}).$$

DEFINITION 1.13. An S-morphism $f\colon H \to G$ is a *homomorphism* of group schemes over S if the map

$$f(Z)\colon H(Z) \longrightarrow G(Z), \qquad \forall Z \in (\mathrm{Sch}/S)$$

is a group homomorphism. The *kernel* $\mathrm{Ker}(f)$ is then defined as the group functor

$$\mathrm{Ker}(f)(Z) = \mathrm{Ker}(f(Z)\colon H(Z) \longrightarrow G(Z)), \qquad \forall Z \in (\mathrm{Sch}/S)$$

with obvious homomorphisms corresponding to S-morphisms $Z' \to Z$.

Obviously, $\mathrm{Ker}(f)$ is a group scheme over S represented by the fibre product

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r] \arrow[d] & S \arrow[d, "\epsilon_G"] \\ H \arrow[r, "f"'] & G, \end{tikzcd}$$

where ϵ_G is the identity morphism for G.

EXAMPLE 1.14. If G is a commutative group scheme over S with the group law written additively, the morphism $n\,\mathrm{id}_G$ for any positive integer n defined by

$$G(Z) \ni \xi \longmapsto n\,\mathrm{id}_G(\xi) = n\xi = \underbrace{\xi + \cdots + \xi}_{n \text{ times}} \in G(Z), \qquad \forall Z \in (\mathrm{Sch}/S)$$

is obviously a homomorphism of group schemes over S. Very often we denote ${}_nG = \mathrm{Ker}(n\,\mathrm{id}_G)$. For example

$$\mu_{n,S} = {}_n\mathbb{G}_{m,S}.$$

There is an important homomorphism peculiar to characteristic $p > 0$.

DEFINITION 1.15. Let S be a scheme of prime characteristic p (that is, $p = 0$ in $\mathcal{O}_S$, e.g., $S = \mathrm{Spec}(k)$ with a field k of characteristic $p > 0$). As in Definition IV.3.1 denote by

$$\phi_S \colon S \longrightarrow S$$

the morphism that is set-theoretically the identity map while $\phi_S^*(a) = a^p$ for all open $U \subset S$ and for all $a \in \Gamma(U, \mathcal{O}_S)$. For any S-group scheme $\pi\colon G \to S$, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\phi_G"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ S \arrow[r, "\phi_S"'] & S, \end{tikzcd}$$

hence a morphism, called the *Frobenius morphism*,

$$F\colon G \longrightarrow G^{(p)} = G^{(p/S)} := (S, \phi_S) \times_S G,$$

where (S, ϕ_S) denotes the S-scheme $\phi_S\colon S \to S$. By the commutativity of the diagrams involving ϕ's, F is easily seen to be a homomorphism of group schemes over S, and is called the *Frobenius homomorphism*.

We define the iterated Frobenius homomorphism

$$F^\nu\colon G \to G^{(p^\nu)} = G^{(p^\nu/S)}$$

similarly.

EXAMPLE 1.16. We have

$$\alpha_{p,S} = \mathrm{Ker}(F\colon \mathbb{G}_{a,S} \longrightarrow \mathbb{G}_{a,S}^{(p)}).$$

For the following result, we restrict ourselves to the affine case $S = \mathrm{Spec}(k)$ with a commutative ring k with 1 for simplicity.

THEOREM 1.17 (Cartier duality). *Let $G = \mathrm{Spec}(A)$ be a commutative finite locally free group scheme over a commutative ring k with 1. Then the group functor*

$$\widehat{G}\colon (k\text{-algebras}) \longrightarrow (\text{Groups})$$

defined for every k-algebra R by

$$\widehat{G}(R) := \mathrm{Hom}_{R\text{-groupscheme}}(G_R, \mathbb{G}_{m,R}),$$
$$\textit{with } G_R := G \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R),\ \mathbb{G}_{m,R} := \mathbb{G}_m \times \mathrm{Spec}(R)$$

and an obvious homomorphism $\widehat{G}(R_1) \to \widehat{G}(R_2)$ for every k-algebra homomorphism $R_1 \to R_2$, is represented by a commutative finite locally free k-group scheme $\mathrm{Spec}(A')$ *with*

$$A' := \mathrm{Hom}_{k\text{-module}}(A, k)$$

endowed with an appropriate commutative co-commutative Hopf algebra structure over k. There is a canonical isomorphism

$$G \xrightarrow{\sim} \widehat{\widehat{G}}.$$

$\widehat{G}$ is called the Cartier dual *of G.*

Proof. Since $G = \mathrm{Spec}(A)$ is a commutative finite locally free group scheme over k, the k-algebra A is a projective k-module of finite rank endowed with the following k-linear maps

(unity)	$i\colon k \to A$
(multiplication)	$m\colon A \otimes_k A \to A$
(inverse)	$\tau\colon A \to A$
(co-unity)	$\epsilon\colon A \to k$
(co-multiplication)	$\mu\colon A \to A \otimes_k A$

satisfying the axioms for a commutative co-commutative Hopf algebra over k.

Let $A' := \mathrm{Hom}_{k-\mathrm{module}}(A, k)$. Dualizing the structure maps for A, we get k-linear maps

(unity)	$i'\colon k \to A'$
(multiplication)	$m'\colon A \otimes_k A' \to A'$
(inverse)	$\tau'\colon A' \to A'$
(co-unity)	$\epsilon'\colon A' \to k$
(co-multiplication)	$\mu'\colon A' \to A' \otimes_k A'$,

making A' a commutative co-commutative Hopf algebra over k. Here i' is the transpose of ϵ, m' is the transpose of μ, τ' is the transpose of τ, ϵ' is the transpose of i and μ' is the transpose of m.

Let $\widehat{G} = \mathrm{Spec}(A')$ be the commutative finite locally free group scheme attached to A'.

For every commutative k-algebra R, we use a subscript R to denote the base-changed objects like $A_R := A \otimes_k R$, $A'_R = A' \otimes_k R = \mathrm{Hom}_{R-\mathrm{module}}(A_R, R)$ and for morphisms like $\mu'_R\colon A'_R \to A'_R \otimes_R A'_R$, $\epsilon'_R\colon A'_R \to R$.

The set of R-valued points

$$G(R) = \mathrm{Hom}_{k\text{-alg}}(A, R) \hookrightarrow \mathrm{Hom}_{k\text{-module}}(A, R) = \mathrm{Hom}_{R\text{-module}}(A_R, R) = A'_R$$

of G is identified with the set of all $\widehat{\phi} \in A'_R$ satisfying the following properties (i) and (ii):

(i) $\mu'_R(\widehat{\phi}) = \widehat{\phi} \otimes \widehat{\phi} \in A'_R \otimes_R A'_R$

(ii) $\epsilon'_R(\widehat{\phi}) = 1 \in R$.

(ii′) $\widehat{\phi} \in (A'_R)^*$.

Note that (i) says that the R-linear map $\phi\colon A_R \to R$ corresponding to $\widehat{\phi} \in A'_R$ respects multiplication, while (ii) says that $\phi \circ i_R = \mathrm{id}_R$. So (i) and (ii) say that ϕ is a homomorphism of k-algebras.

On the other hand, the set

$$\mathrm{Hom}_{R\text{-groupscheme}}(\widehat{G} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R), \mathbb{G}_m \times \mathrm{Spec}(R))$$

of all R-homomorphisms of R-group schemes from $\widehat{G} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ to $\mathbb{G}_m \times \mathrm{Spec}(R)$ is naturally identified with the set of all elements $\widehat{\phi} \in A'_R$ satisfying the conditions (i) and (ii′).

LEMMA 1.18. *Suppose that* $\widehat{\phi} \in A'_R$ *satisfies* (i). *Then* (ii) *and* (ii′) *are equivalent. In other words, one has a natural bijection*

$$G(R) \xrightarrow{\sim} \mathbf{Hom}_{k\text{-groupscheme}}(\widehat{G}, \mathbb{G}_{m,k})(R).$$

PROOF OF LEMMA. (ii′)$\Longrightarrow$(ii). Applying the identity

$$(\epsilon'_R \otimes \epsilon'_R) \circ \mu'_R = \epsilon'_R$$

(corresponding to $1 \cdot 1 = 1$ in A_R) to $\widehat{\phi}$, we get

$$\epsilon'_R(\widehat{\phi})^2 = \epsilon'_R(\widehat{\phi})$$

Hence $\epsilon'_R(\widehat{\phi}) = 1$ because $\epsilon'_R(\widehat{\phi})$ is a unit in A'_R by (ii′).

(ii)$\Longrightarrow$(ii′). Applying the identity (for the inverse in $\widehat{G}_R := \mathrm{Spec}(A'_R)$)

$$m'_R \circ (1_{A'} \otimes \tau'_R) \circ \mu'_R = i'_R \circ \epsilon'_R$$

to $\widehat{\phi}$, we get $\widehat{\phi} \cdot \tau'_R(\widehat{\phi}) = 1$ in A'_R. So $\widehat{\phi}$ is a unit in A'_R. □

Applying the above lemma to $\widehat{G}$, we see that the commutative k-group functor

$$\mathbf{Hom}_{k\text{-groupscheme}}(G, \mathbb{G}_{m,k})$$

is representable, and naturally identified with $\widehat{G} = \mathrm{Spec}(A')$ (as schemes at this point). One can reformulate this as a morphism

$$\mathrm{can}_G \colon G \times_{\mathrm{Spec}(k)} \widehat{G} \longrightarrow \mathbb{G}_{m,k}$$

obtained from the above lemma applied to the tautological element $\widehat{\phi} = \mathrm{id}_A \in G(A)$ when $R = A$. This morphism corresponds to the k-algebra homomorphism

$$k[T, T^{-1}] \longrightarrow A \otimes_k A',$$

which sends T to the "*diagonal element*" $\delta \in A \otimes_k A'$ that corresponds to id_A. Since δ also corresponds to $\mathrm{id}_{A'}$, the canonical morphism can_G is naturally identified with $\mathrm{can}_{\widehat{G}}$. Moreover, the lemma tells us that

$$(\dagger) \qquad \mu'_A(\delta) = \delta \otimes_A \delta \in A \otimes_k A' \otimes_k A', \quad \epsilon'_A(\delta) = i(1) \in A \quad \text{and} \quad \delta \cdot \tau'_A(\delta) = 1.$$

The same argument (because δ also corresponds to the tautological element in $\widehat{G}(A')$) gives

$$(\ddagger) \qquad \mu_{A'}(\delta) = \delta \otimes_{A'} \delta \in A \otimes_k A \otimes_k A', \quad \epsilon_{A'}(\delta) = i'(1) \in A', \quad \text{and} \quad \delta \cdot \tau_{A'}(\delta) = 1.$$

Note that $\delta \otimes_A \delta$ is the product of $p_{12}(\delta)$ and $p_{13}(\delta)$ in $A \otimes_k A' \otimes_k A'$, and $\delta \otimes_{A'} \delta$ is the product of $p_{13}(\delta)$ and $p_{23}(\delta)$ in $A \otimes_k A \otimes_k A'$. The formulas (†) and (‡) give the multiplicative inverse of δ in $A \otimes_k A'$, namely, $\tau'_A(\delta) = \tau_{A'}(\delta)$. More importantly they also show that the canonical map

$$\mathrm{can} \colon G \times_{\mathrm{Spec}(k)} \widehat{G} \to \mathbb{G}_{m,k}$$

is bi-multiplicative. □

EXAMPLE 1.19. Let H be an abstract commutative finite group. Write k^H for the set of all k-valued functions on H, and let $k[H]$ be the group algebra of H over k. The delta functions δ_h at $h \in H$ form a k-basis of k^H, and we have $\delta_x \cdot \delta_y = \delta(x, y)\delta_x$ for all $x, y \in H$, where $\delta(x, y)$ denotes Kronecker's symbol. The co-multiplication, co-unit and inverse in k^H are given by

$$\mu \colon \delta_h \ \mapsto \sum_{x,y \in H,\ x \cdot y = h} \delta_x \otimes \delta_y, \qquad \epsilon \colon \delta_x \mapsto \delta(x, 0)\, \delta_x, \qquad \tau \colon \delta_x \mapsto \delta_{-x}.$$

The group algebra $k[H]$ is best thought of as the convolution algebra of all k-valued measures on H, where the basis element $[h]$ corresponding to an element $h \in H$ is "evaluation at h". The co-multiplication, co-unit and inverse are given by

$$\mu' \colon [x] \mapsto [x] \otimes [x], \qquad \epsilon' \colon [x] \mapsto 1, \qquad \tau' \colon [x] \mapsto [-x]\,.$$

Some samples of the equalities in (†) and (‡) are:

$$\begin{aligned} \mu_{k[H]}\left(\sum_{x\in H} \delta_x \otimes [x]\right) &= \sum_{y,z\in H} \delta_y \otimes \delta_z \otimes [y+z] \\ &= \left(\sum_{y\in H} \delta_y \otimes 1 \otimes [y]\right) \cdot \left(1 \otimes \sum_{z\in H} \delta_z \otimes 1 \otimes [z]\right) \end{aligned}$$

in $k^H \otimes_k k^H \otimes_k k[H]$,

$$\epsilon_{k^H}\left(\sum_{x\in H} \delta_x \otimes [x]\right) = \sum_{x\in H} \delta_x = i_{k^H}(1)$$

in k^H, and

$$\begin{aligned} \left(\sum_{x\in H} \delta_x \otimes [x]\right) \cdot \left(\sum_{y\in H} \delta_y \otimes [y]\right) &= \sum_{x,y\in H} \delta(x,y)\, \delta_x \otimes [x-y] \\ &= \left(\sum_{x} \delta_x\right) \otimes [0] = i_{k^H \otimes k[H]}(1) \end{aligned}$$

in $k^H \otimes_k k[H]$.

When $H = \mathbb{Z}/n\mathbb{Z}$ we have $\mathrm{Spec}(k[\mathbb{Z}/n\mathbb{Z}]) = \mathrm{Spec}(k[T]/(T^n - 1)) = \mu_n \times \mathrm{Spec}(k)$.

EXAMPLE 1.20. Let p be a prime number, $k \supset \mathbb{F}_p$ a field, and $G = \alpha_p = \alpha_p \times \mathrm{Spec}(k) = \mathrm{Spec}(k[X]/(X^p))$. Let $x \in A = k[X]/(X^p)$ be the image of X in A. The co-multiplication and co-unity are determined by

$$\mu \colon x \mapsto x \otimes 1 + 1 \otimes x \qquad \text{and} \qquad \epsilon \colon x \mapsto 0.$$

Let $y_0, y_1, \ldots, y_{p-1} \in A' = \mathrm{Hom}_{k\text{-module}}(A, k)$ be the basis dual to $\{1, x, x^2, \ldots, x^{p-1}\}$. Then we have

$$\mu' \colon y_i \mapsto \sum_{0\le a\le i} y_a \otimes y_{i-a}, \qquad y_1^i = i!y_i, \quad \forall i = 0, 1, \ldots, p-1, \quad y^p = 0\,.$$

Then $x \mapsto y_1$ establishes an isomorphism $A \cong A'$ of Hopf-algebras. The diagonal element

$$\delta = \sum_{i=0}^{p-1} x^i \otimes y_i \in A \otimes_k A'$$

is equal to

$$\exp(x \otimes y_1) = 1 + x \otimes y_1 + \frac{x^2 \otimes y_1^2}{2!} + \cdots + \frac{x^{p-1} \otimes y_1^{p-1}}{(p-1)!} = E_p(xy),$$

where $E_p(T)$ is the truncated exponential

$$E_p(T) := 1 + T + \frac{T^2}{2!} + \cdots + \frac{T^{p-1}}{(p-1)!} \in k[T].$$

In other words, the formula $(x, y) \mapsto E(xy)$ gives an auto-duality pairing

$$\alpha_a \times \alpha_p \longrightarrow \mathbb{G}_m$$

which identifies α_p with its own Cartier dual.

EXAMPLE 1.21. Let $k \supset \mathbb{F}_p$ be a field of characteristic p. Let

$$\alpha_{p^n} = \mathrm{Ker}\left(F^n\colon \mathbb{G}_{a,k} \to \mathbb{G}_{a,k}^{(p^n)}\right) = \mathrm{Spec}(k[X]/(X^{p^n})).$$

We have short exact sequences

$$0 \longrightarrow \alpha_{p^n} \xrightarrow{j_{n,n+m}} \alpha_{p^{n+m}} \xrightarrow{\beta_{n+m,m}} \alpha_{p^m} \longrightarrow 0$$

for positive integers m, n, where $j_{n,n+m}$ is the natural inclusion and $\beta_{n+m,m}$ is induced by the Frobenius homomorphism F^n.

Write $A := k[X]/(X^{p^n})$ and let x be the image of X in A. Let $y_0, y_1, \ldots, y_{p^n-1}$ be the k-basis in $A' := \mathrm{Hom}_{k\text{-module}}(A, k)$ dual to the k-basis $1, x, x^2, \ldots, x^{p^n-1}$ of A. The co-multiplication on A is given by

$$\mu\colon x \mapsto x \otimes 1 + 1 \otimes x.$$

The co-multiplication, unity and co-unity on A' are given by

$$\mu'\colon y_i \mapsto \sum_{0\le a\le i} y_a \otimes y_{i-a}, \quad i = 0, 1, \ldots, p^n - 1; \qquad i'\colon 1 \mapsto y_0, \qquad \epsilon'\colon y_i \mapsto 0, \quad \forall i > 0.$$

It is straightforward to deduce from $\mu(x) = x \otimes 1 + 1 \otimes x$ that

$$y_1^2 = 2\, y_2, y_1^3 = 3! y_3, \ldots, y_1^{p-1} = (p-1)! \cdot y_{p-1}, y_1^p = 0.$$

Similarly we have

$$y_{p^a}^j = j! \cdot y_{jp^a} \quad \text{and} \quad y_{p^a}^p = 0, \quad \forall a = 0, 1, \ldots, n-1, \quad \forall j = 0, 1, \ldots, p-1.$$

More generally, for every positive integer i with $0 \le i \le p^n - 1$, written in p-adic expansion in the form $i = \sum_{0\le a\le n-1} j_a p^a$,

$$y_i = y_{j_0 + j_1 p + \cdots + j_{n-1}p^{n-1}} = \prod_{0\le a\le n-1} \frac{y_{p^a}^{j_a}}{j_a!}.$$

So A' is isomorphic to

$$k[Z_0, Z_1, \ldots, Z_{n-1}]/(Z_0^p, Z_1^p, Z_2^p, \ldots, Z_{n-1}^p)$$

as k-algebras, such that y_{p^a} corresponds to the image of Z_a for $a = 0, 1, \ldots, n-1$. The diagonal element

$$\delta = \sum_{i=0}^{p^n-1} x^i \otimes y_i \in A \otimes_k A' \cong k[X, Z_0, Z_1, \ldots, Z_{n-1}]/(X^{p^n}, Z_0^p, Z_1^p, \ldots, Z_{n-1}^p)$$

can be written in terms of the truncated exponential $E_p(T) = 1 + T + T^2/2! + \cdots + T^{p-1}/(p-1)!$ as the image of the polynomial

$$\delta(X, \underline{Z}) = \delta(X, Z_0, Z_1, \ldots, Z_{n-1}) = \prod_{a=0}^{n-1} E_p(X^{p^a} \cdot Z_a)$$

in $k[X, Z_0, Z_1, \ldots, Z_{n-1}]/(X^{p^n}, Z_0^p, Z_1^p, \ldots, Z_{n-1}^p)$. The group law of the Cartier dual $\widehat{\alpha_{p^{n-1}}}$ of α_{p^n} is completely determined by the polynomial $\delta(X, \underline{Z})$ as follows: Using $Z_0, Z_1, \ldots, Z_{n-1}$ as the coordinates on $\widehat{\alpha_{p^{n-1}}}$, the sum of two points in $\widehat{\alpha_{p^n}}$ with coordinates $\underline{z} = (z_0, z_1, \ldots, z_{n-1})$ and $\underline{w} = (w_0, w_1, \ldots, w_{n-1})$ is the point with coordinates $\underline{\Phi}(\underline{z}, \underline{w})$, where

$$\begin{aligned}&\underline{\Phi}(\underline{Z},\ \underline{W})\\ &\quad = \quad (\Phi_0(\underline{Z},\ \underline{W}), \ldots, \Phi_{n-1}(\underline{Z}, \underline{W}))\\ &\quad \in \quad \Big(k[Z_0, Z_1, \ldots, Z_{n-1}, W_0, W_1, \ldots, W_{n-1}]/(Z_0^p, \ldots, Z_{p-1}^p, W_0^p, \ldots, W_{n-1}^p)\Big)^n\end{aligned}$$

is determined by the equation

$$\prod_{a=0}^{n-1} E_p\left(X^{p^a}\Phi_a(\underline{Z}, \underline{W})\right) = \prod_{a=1}^{n-1} E_p\left(X^{p^a} \cdot Z_a\right) \cdot \prod_{a=1}^{n-1} E_p\left(X^{p^a} \cdot W_a\right)$$

in $k[X, Z_0, Z_1, \ldots, Z_{n-1}, W_0, W_1, \ldots, W_{n-1}]/(X^{p^n}, Z_0^p, \ldots, Z_{p-1}^p, W_0^p, \ldots, W_{n-1}^p)$. Notice that

$$E_p(XZ + XW) \equiv E_p(XZ) \cdot E_p(XW) \pmod{(X^p, Z^p, W^p)},$$

but

$$\begin{aligned}E_p(X(Z_0 + W_0) + X^p(Z_1 + W_1)) \quad &\not\equiv \quad E_p(XZ_0 + X^pZ_1) \cdot E_p(XW_0 + X^pW_1)\\ &\qquad \pmod{(X^{p^2}, Z_0^p, Z_1^p, W_0^p, W_1^p)}.\end{aligned}$$

So the usual "exponential rule" does not hold for the truncated exponential when applied to rings like $k[X, Z_0, Z_1, W_0, W_1]/(X^{p^2}, Z_0^p, Z_1^p, W_0^p, W_1^p)$.

The Cartier dual of the homomorphism $\beta_{n+m,m}\colon \alpha_{p^{n+m}} \to \alpha_{p^m}$ induced by $F^n\colon x \mapsto x^{p^n}$, the n-th power of the Frobenius, corresponds to the homomorphism

$$\beta'_{n+m,m}\colon k[Y_0, \ldots, Y_{n+m-1}]/(Y_0^p, \ldots, Y_{n+m-1}^p) \longrightarrow k[Y_0, \ldots, Y_{m-1}]/(Y_0^p, \ldots, Y_{m-1}^p)$$

of Hopf algebras such that

$$\beta'_{n+m,m}\colon Y_0, \ldots, Y_{n-1} \mapsto 0; \qquad \beta'_{n+m,m}\colon Y_{n+a} \mapsto Y_a, \quad a = 0, \ldots m-1\,.$$

Similarly, the natural immersion $j_{n,n+m}\colon \alpha_{p^n} \hookrightarrow \alpha_{p^{n+m}}$ corresponds to the homomorphism

$$j'_{n,n+m}\colon k[Y_0, \ldots, Y_{n-1}]/(Y_0^p, \ldots, Y_{n-1}^p) \longrightarrow k[Y_0, \ldots, Y_{n+m-1}]/(Y_0^p, \ldots, Y_{n+m-1}^p)$$

of Hopf algebras which sends each Y_a to Y_a for all $a = 0, 1, \ldots, n-1$. Using the maps j'_{m_1,m_2}, one easily sees that for each positive integer a with $0 \le a \le n$, the a-th component $\Phi_a(\underline{Z}, \underline{W})$ of the group law comes from a unique polynomial in $\mathbb{F}_p[Z_0, \ldots, Z_a, W_0, \ldots, W_a]$ independent of n whose degree in each variable is $\le p-1$. For instance

$$\Phi_0(\underline{Z}, \underline{W}) = Z_0 + W_0, \qquad \Phi_1(\underline{Z}, \underline{W}) = Z_1 + W_1 + \sum_{i=1}^{p-1} \frac{Z_0^i}{i!} \cdot \frac{W_0^{p-i}}{(p-i)!}.$$

These formulas could most easily be understood in terms of Witt vectors. (See, for instance, Mumford [**84**, Lecture 26 by G. Bergman].)

EXAMPLE 1.22. An S-group scheme $\pi\colon X \to S$ is called an *abelian scheme* if π is smooth and proper with connected geometric fibres. X turns out to be commutative (at lease when S is noetherian). (cf. Mumford [**83**, Corollary 6.6, p.117])

When $S = \mathrm{Spec}(k)$ with a field k, an abelian scheme X over S is called an *abelian variety* over k. Thus X is a geometrically connected group scheme proper and smooth over k. In this case, the commutativity is shown in two different ways in Mumford [**85**, pp.41 and 44]. X is also shown to be *divisible*, that is, $n\,\mathrm{id}_X$ is surjective for any positive integer n.

When $k = \mathbb{C}$, the set $X(\mathbb{C})$ of $\mathbb{C}$-valued points of an abelian variety X over $\mathbb{C}$ turns out to be a complex torus.

EXAMPLE 1.23. An *algebraic group* G is a *smooth* group scheme of finite type over a field k. An algebraic group G over k is *affine* if and only if it can be realized as a *linear group*, that is, as a closed subgroup of a general linear group $\mathrm{GL}_{n,k}$.

DEFINITION 1.24. Suppose $\phi\colon H \to G$ is a homomorphism of S-group schemes. A pair $(G/H, \pi)$ of an S-scheme G/H and an S-morphism $\pi\colon G \to G/H$ is said to be the *quotient* of G by H, if it is universal for all pairs (Y, f) of an S-scheme Y and an S-morphism $f\colon G \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} G \times_S H \arrow[r, "\mu_G \circ (1_G \times_S \phi)"] \arrow[d, "p_1"'] & G \arrow[d, "f"] \\ G \arrow[r, "f"'] & Y, \end{tikzcd}$$

that is, there exists a unique S-morphism $f'\colon G/H \to Y$ such that $f = f' \circ \pi$. If H is a *normal* S-subgroup scheme of G with ϕ the canonical monomorphism so that $H(Z)$ is a normal subgroup of $G(Z)$ for any $Z \in (\mathrm{Sch}/S)$, then G/H inherits a unique structure of S-group scheme such that $\pi\colon G \to G/H$ is an S-homomorphism with $\mathrm{Ker}(\pi) = H$. In this case G/H is called the *quotient* group scheme.

We certainly need conditions for the existence of G/H.

- FGA [**2**, exposé 212, Corollaries 7.3 and 7.4] shows the existence in the case where S is the spectrum of an artinian ring (in particular, a field): Suppose G is of finite type and flat over S and that H is an S-subgroup scheme of G with H flat over S. Then G/H exists with $\pi\colon G \to G/H$ flat and surjective. Moreover, the quotient is shown to commute with base changes $S' \to S$.
- Demazure-Gabriel [**35**, Chapter III, §3] and SGA3 [**6**, exposés $\mathrm{VI_A}$ and $\mathrm{VI_B}$] deal with the quotient in terms of the "sheafification" of the contravariant functor
$$(\mathrm{Sch}/S) \ni Z \longmapsto G(Z)/H(Z) \in (\mathrm{Sets}).$$
- (cf. Borel [**24**, Chapter II, Theorem 6.8]) If G is an algebraic group over a field k and H is a closed algebraic subgroup over k, then G/H exists (Weil 1955 and Rosenlicht 1956) and is a smooth quasi-projective (cf. Definition II.5.8) algebraic variety over k (Chow 1957). See Raynaud [**92**] for the corresponding results in the case of more general base schemes S.

More generally, an action $G \times_S X \to X$ of a group scheme G over S on an S-scheme X will be defined in Definition 2.3 below.

EXAMPLE 1.25. $\mathrm{PGL}_{n+1} = \mathrm{GL}_{n+1}/\mathbb{G}_m$ where $\mathbb{G}_m \subset \mathrm{GL}_{n+1}$ is the normal subgroup scheme of "invertible scalar matrices".

We just mention the following basic results:

THEOREM 1.26 (Chevally 1953). (See Rosenlicht [**94**, Theorem 16] and Chevalley [**29**].A "modern" proof can be found in Conrad [**33**].) *A connected algebraic group G over a perfect field k has a closed connected affine normal subgroup L such that G/L is an abelian variety. Such L is unique and contains all other closed connected affine subgroups of G.*

THEOREM 1.27 (Chevalley). (cf. Demazure-Gabriel [**35**, Chapter III, §3.5], SGA3 [**6**, $\mathrm{VI_B}$, Theorem 11.17, p.408] and Humphreys [**49**]) *If G is an affine algebraic group and H is a closed normal algebraic subgroup, then G/H is an affine algebraic group.*

Borel subgroups (cf. Borel [**24**]) of affine algebraic groups play crucial roles in the classification of semi-simple affine algebraic groups. Basic references are Chevalley [**31**], SGA3 [**6**] and Demazure-Gabriel [**35**].

2. Lang's theorems over finite fields

We can combine the geometric frobenius morphism (Definition IV.3.2) with ideas of smoothness to give a very pretty result due to Lang [**76**].

THEOREM 2.1 (Lang). *Let $k = \mathbb{F}_q$, $\overline{k}$ = an algebraic closure of k.*

a) *Let G be a connected reduced group scheme of finite type over* $\operatorname{Spec} k$ *and let $\overline{G} = G \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. Then $\overline{G}$ will be regular (smooth over $\overline{k}$) and irreducible.*
b) *Let*

$$\mathbf{f}_G = \mathbf{f}_G^{\text{geom}} \colon \overline{G} \longrightarrow \overline{G}$$

be the geometric frobenius morphism (cf. Definition IV.3.2*). Define a $\overline{k}$-morphism $\psi\colon \overline{G} \to \overline{G}$ on closed points by*

$$x \longmapsto \psi(x) = x \cdot \mathbf{f}_G(x)^{-1}$$

and in general by the composition:

$$\psi\colon \overline{G} \xrightarrow{\Delta} \overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \xrightarrow{(1_{\overline{G}} \times (\iota \circ \mathbf{f}_G))} \overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \xrightarrow{\mu} \overline{G}.$$

Then ψ is finite étale and surjective.

c) *Moreover the group $G(k)$ of k-rational points of G is finite and if we let each $a \in G(k)$ act on $\overline{G}$ by right translation R_a, then*
   1) *$\forall a \in G(k)$, $\psi \circ R_a = \psi$*
   2) *$\forall x, y \in \overline{G}$, $\psi(x) = \psi(y) \iff \exists a \in G(k)$ such that $x = R_a(y)$.*

PROOF. According to Theorem IV.2.4, $\overline{G}$ is reduced because $\mathbb{F}_q$ is perfect. Therefore the set of regular (smooth over $\overline{k}$) points $U \subset \overline{G}$ is dense (cf. Jacobian criterion in Corollary V.4.2). But if $x, y \in \overline{G}$ are any two closed points, right translation by $x^{-1} \cdot y$ is an automorphism of $\overline{G}$ taking x to y. So if $x \in U$, then $y \in U$ too. Therefore U contains every closed point, hence $U = \overline{G}$. But then the components of $\overline{G}$ are disjoint. Now the identity point $e = \text{Image}(\epsilon)$ is a k-rational point of $\overline{G}$, hence it is $\text{Gal}(\overline{k}/k)$-invariant. Therefore the component $\overline{G}_\circ$ of $\overline{G}$ containing e as well as $\overline{G} \setminus \overline{G}_\circ$ are Gal-invariant open sets. By Theorem IV.2.3, this implies that G is disconnected too, unless $\overline{G} = \overline{G}_\circ$. This proves (a).

Next note that $\mathbf{f}_G\colon \overline{G} \to \overline{G}$ is a homomorphism of $\overline{k}$-group schemes, i.e.,

$$\begin{tikzcd}
\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \arrow[r, "\mu"] \arrow[d, "\mathbf{f}_G \times \mathbf{f}_G"'] & \overline{G} \arrow[d, "\mathbf{f}_G"] \\
\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G} \arrow[r, "\mu"] & \overline{G}
\end{tikzcd}$$

commutes. This is because if you write $\overline{G} = G \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then μ equals $\mu' \times 1_{\overline{k}}$ where $\mu'\colon G \times_{\operatorname{Spec} k} G \to G$ is multiplication for G; but by definition $\mathbf{f}_G = \phi_G^\nu \times 1_{\overline{k}}$ (if $q = p^\nu$) and for any morphism $g\colon X \to Y$ in characteristic p, $\phi_X \circ g = g \circ \phi_X$ (cf. Definition IV.3.1). Then for all closed points $x \in \overline{G}$, $a \in G(k)$

$$\begin{aligned}\psi \circ R_a(x) &= \psi(x \cdot a)\\ &= x \cdot a \cdot \mathbf{f}_G(x \cdot a)^{-1}\\ &= x \cdot a \cdot \mathbf{f}_G(a)^{-1} \cdot \mathbf{f}_G(x)^{-1}\\ &= x \cdot a \cdot a^{-1} \cdot \mathbf{f}_G(x)^{-1}\\ &= \psi(x)\end{aligned}$$

and for all closed points $x, y \in \overline{G}$:

$$\begin{aligned}\psi(x) = \psi(y) &\iff x \cdot \mathbf{f}_G(x)^{-1} = y \cdot \mathbf{f}_G(y)^{-1}\\ &\iff y^{-1} \cdot x = \mathbf{f}_G(y)^{-1} \cdot \mathbf{f}_G(x)\\ &\iff y^{-1} \cdot x = \mathbf{f}_G(y^{-1} \cdot x)\\ &\iff y^{-1} \cdot x \text{ is } \mathrm{Gal}(\overline{k}/k)\text{-invariant}\\ &\iff y^{-1} \cdot x = a \in G(k)\\ &\iff x = R_a(y) \text{ for some } a \in G(k).\end{aligned}$$

But now for any scheme X of finite type over k, $X(k)$ is finite. The last result shows that the two closed subsets of $\overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{G}$, namely

$$\bigcup_{a \in G(k)} (\text{Graph of } R_a) \text{ and the fibre product: } \quad \begin{tikzcd}[column sep=small] & \overline{G} \times_{\overline{G}} \overline{G} \arrow[dl] \arrow[dr] & \\ \overline{G} \arrow[dr, "\psi"'] & & \overline{G} \arrow[dl, "\psi"] \\ & \overline{G} & \end{tikzcd}$$

have the same closed points. Therefore these sets are equal.! This proves (c).

Now we come to the main point — (b). We prove first that ψ is étale using Criterion V.4.6: $\forall x \in \overline{G}$ closed, $d\psi_x \colon T_{x,\overline{G}} \to T_{\psi(x),\overline{G}}$ is an isomorphism. We use:

Lemma 2.2. *If X is a scheme over $k = \mathbb{F}_q$ and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, then the $\overline{k}$-morphism $\mathbf{f}_X = \mathbf{f}_X^{\mathrm{geom}} \colon \overline{X} \to \overline{X}$ induces the zero map*

$$\mathbf{f}_X^* \colon \Omega_{\overline{X}/\overline{k}} \longrightarrow \Omega_{\overline{X}/\overline{k}}.$$

Proof of Lemma 2.2. We may as well assume X affine, say $= \operatorname{Spec} R$. Then $\overline{X} = \operatorname{Spec}(R \otimes_k \overline{k})$ and $\mathbf{f}_X$ is induced by the homomorphism

$$\begin{aligned}R \otimes_k \overline{k} &\longrightarrow R \otimes_k \overline{k}\\ \sum a_i \otimes b_i &\longmapsto \sum a_i^q \otimes b_i.\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{f}_X^* \Big(d(\sum a_i \otimes b_i) \Big) &= d(\sum a_i^q \otimes b_i)\\ &= \sum d(a_i^q) \otimes b_i + \sum a_i^q \otimes db_i\\ &= 0.\end{aligned}$$

$\square$

By Chapter V, this means that for all closed points of $x \in \overline{G}$,

$$(d\mathbf{f}_X)_x \colon T_{x,\overline{G}} \longrightarrow T_{\mathbf{f}_X(x),\overline{G}}$$

is zero. To compute $d\psi_x\colon T_{x,\overline{G}} \to T_{\psi(x),\overline{G}}$, use the identification of $T_{x,\overline{G}}$ with the set of $\overline{k}[\delta]$-valued points $t\colon \operatorname{Spec}\overline{k}[\delta] \to \overline{G}$ of $\overline{G}$ with Image$(t) = \{x\}$, where $\overline{k}[\delta]$ is the $\overline{k}$-algebra of dual numbers (cf. §1 and §V.1). In terms of this identification, if $t \in T_{x,\overline{G}}$, then $d\psi_x(t)$ is nothing but $\psi \circ t$. Hence using the group law in the set of $\overline{k}[\delta]$-valued points of $\overline{G}$:

$$d\psi_x(t) = t \cdot \mathbf{f}_X(t)^{-1}.$$

But if O_y is the 0 tangent vector at y, i.e.,

$$\operatorname{Spec}\overline{k}[\delta] \longrightarrow \operatorname{Spec}\overline{k} \longrightarrow \overline{G} \text{ with image } y,$$

then Lemma 2.2 showed that $\mathbf{f}_X(t) = O_{\mathbf{f}(x)}$, hence

$$d\psi_x(t) = t \cdot O_{\mathbf{f}_X(x)^{-1}}, \qquad \forall t \in T_{x,X}.$$

The map $t \mapsto t \cdot O_{\mathbf{f}_X(x)}$ is then an inverse to $d\psi_x$ so $d\psi_x$ is an isomorphism.

Next, ψ is surjective. In fact for all closed points $a \in \overline{G}$ we can introduce a new morphism $\psi^{(a)}$ given on closed points by:

$$\psi^{(a)}(x) = x \cdot a \cdot \mathbf{f}_X(x)^{-1}.$$

The same argument given for ψ also shows that $\psi^{(a)}$ is étale. Therefore $\psi^{(a)}$ is flat (cf. Corollary V.4.9) and by Proposition IV.5.12 Image$(\psi^{(a)})$ is open. Therefore

$$\operatorname{Image}(\psi) \cap \operatorname{Image}(\psi^{(a)}) \neq \emptyset,$$

i.e., $\exists$ closed points $b_1, b_2 \in \overline{G}$ such that

$$b_1 \cdot \mathbf{f}_X(b_1)^{-1} = b_2 \cdot a \cdot \mathbf{f}_X(b_2)^{-1}.$$

Then one calculates immediately that $\psi(b_2^{-1} \cdot b_1) = a$.

Finally ψ is finite: by Exercise (2) in Chapter II (possibly moved to another location?), $\exists$ a non-empty open $U \subset \overline{G}$ such that $\operatorname{res}\psi\colon \psi^{-1}U \to U$ is finite. But if L_a is left translation by a, then for all closed points $a \in \overline{G}$, consider the commutative diagram:

$$\begin{tikzcd}
\overline{G} \arrow[r, "L_a", "\approx"'] \arrow[d, "\psi"'] & \overline{G} \arrow[d, "\psi"] \\
\overline{G} \arrow[r, "L_a \circ R_{\mathbf{f}(a)^{-1}}", "\approx"'] & \overline{G}
\end{tikzcd}$$

It follows that $\operatorname{res}\psi$ is finite from $L_a(\psi^{-1}U)$ to $L_a(R_{\mathbf{f}(a)^{-1}}(U))$ too. Since $\overline{G}$ is covered by the open sets $L_a(\psi^{-1}U)$, ψ is everywhere finite. □

For example, applied to $\mathbb{A}^1_{\overline{k}}$, the theorem gives the Artin-Schreier homomorphism:

$$\psi\colon \mathbb{A}^1_{\overline{k}} \longrightarrow \mathbb{A}^1_{\overline{k}}$$
$$\psi(x) = x - x^q$$
$$\operatorname{Ker}\psi = \mathbb{F}_q \subset \mathbb{A}^1_{\overline{k}}.$$

On $\mathbb{G}_m(\overline{k}) = \mathrm{GL}_1(\overline{k})$, ψ is the homomorphism

$$\psi(x) = x^{1-q}$$
$$\operatorname{Ker}\psi = \mathbb{F}_q^* \subset \mathbb{G}_m(\overline{k}) = \mathrm{GL}_1(\overline{k}),$$

while on $\mathrm{GL}_2(\overline{k})$, ψ is given by:

$$\begin{aligned}\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a^q & b^q \\ c^q & d^q \end{pmatrix}^{-1} \\ &= \frac{1}{(ad-bc)^q} \begin{pmatrix} ad^q - bc^q & -ab^q + ba^q \\ cd^q - dc^q & -cb^q + da^q \end{pmatrix}.\end{aligned}$$

Lang invented this theorem because of its remarkable application to homogeneous spaces for G over k. We need another definition to explain this:

DEFINITION 2.3. Let $f\colon G \to S$ plus (μ, ϵ, ι) be a group scheme and let $p\colon X \to S$ be any scheme over S. Then an action of G on X is an S-morphism:

$$\sigma\colon G \times_S X \longrightarrow X$$

such that the following diagrams commute:

a) ("associativity")

$$\begin{tikzcd} (G \times_S G) \times_S X \arrow[r, "\mu \times 1_X"] \arrow[d, equal, "\sim"'] & G \times_S X \arrow[dr, "\sigma"] & \\ & & X \\ G \times_S (G \times_S X) \arrow[r, "1_G \times \sigma"'] & G \times_S X \arrow[ur, "\sigma"'] & \end{tikzcd}$$

b) ("identity acts by identity")

$$\begin{tikzcd} S \times_S X \arrow[r, "\epsilon \times 1_X"] \arrow[d, equal, "\sim"'] & G \times_S X \arrow[d, "\sigma"] \\ X \arrow[r, "1_X"] & X \end{tikzcd}$$

COROLLARY 2.4. *Let G be a connected reduced group scheme of finite type over $k = \mathbb{F}_q$ and let X be a scheme of finite type over k on which G acts via σ. Let Σ be a set of subschemes of $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ such that:*

a) *$\forall Z \in \Sigma$, $a \in \overline{G}$ closed, $\sigma(a, Z) \in \Sigma$, and $\forall Z_1, Z_2 \in \Sigma$, $\exists a \in \overline{G}$ closed such that $\sigma(a)(Z_1) = Z_2$*[1]
b) *if $\mathbf{f}_X^{\mathrm{arith}}\colon \overline{X} \to \overline{X}$ is the frobenius automorphism (cf. Definition* IV.3.2*), then $\forall Z \in \Sigma$, $\mathbf{f}_X^{\mathrm{arith}}(Z) \in \Sigma$.*

Then Σ contains at least one subscheme Z of the form $Z' \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$, Z' a subscheme of X.

PROOF. Start with any $Z \in \Sigma$ and combine (a) and (b) to write

$$\mathbf{f}_X^{\mathrm{arith}}(Z) = \sigma(a)(Z), \quad a \in \overline{G} \text{ closed}.$$

By Lang's theorem (Theorem 2.1),

$$a^{-1} = b \cdot \mathbf{f}_G(b)^{-1}, \quad b \in \overline{G} \text{ closed}.$$

Now on closed points, $\mathbf{f}_G^{\mathrm{geom}} = \mathbf{f}_G^{\mathrm{arith}}$, so we deduce

$$\mathbf{f}_X^{\mathrm{arith}}(Z) = \sigma(\mathbf{f}_G^{\mathrm{arith}} b)(\sigma(b^{-1})(Z)),$$

[1] $\sigma(a)$ is short for the automorphism of $\overline{X}$:

$$\overline{X} = \operatorname{Spec} \overline{k} \times_{\operatorname{Spec} \overline{k}} \overline{X} \xrightarrow{\{a\} \times \overline{X}} \overline{G} \times_{\operatorname{Spec} \overline{k}} \overline{X} \xrightarrow{\sigma} \overline{X}.$$

hence since σ is defined over k:

$$\begin{aligned}\mathbf{f}_X^{\text{arith}}\left(\sigma(b^{-1})(Z)\right) &= \sigma(\mathbf{f}_X^{\text{arith}} b^{-1})(\mathbf{f}_X^{\text{arith}}(Z)) \\ &= \sigma(b^{-1})(Z).\end{aligned}$$

Therefore $\sigma(b^{-1})(Z) \in \Sigma$ and is invariant under $\mathrm{Gal}(\overline{k}/k)$. So by Theorem IV.2.9, $\sigma(b^{-1})(Z) = Z' \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ for some subscheme Z' of X. □

COROLLARY 2.5. *Let G, X be as above over $k = \mathbb{F}_q$. Assume the group of closed points of $\overline{G}$ acts transitively on the set of closed points of $\overline{X}$. Then $X(k) \neq \emptyset$.*

PROOF. Apply Corollary 2.4 with $\Sigma =$ the closed points of $\overline{X}$. □

If X is a smooth quadric hypersurface in $\mathbb{P}^n_k$, or a smooth cubic curve in $\mathbb{P}^2_k$ with $k = \mathbb{F}_q$, it can be shown that such a G always exists, hence X has a k-rational point! For some conics in $\mathbb{P}^2_k$, the next corollary tells us more:

COROLLARY 2.6. *Let Y be a scheme of finite type over $k = \mathbb{F}_q$ such that*

$$\overline{Y} \cong \mathbb{P}^n_{\overline{k}} \text{ over } \overline{k}.$$

Then

$$Y \cong \mathbb{P}^n_k \text{ over } k.$$

PROOF. Take the X in Corollary 2.4 to be $Y \times_{\mathrm{Spec}\, k} \mathbb{P}^n_k$. Let Σ be the set of graphs of $\overline{k}$-isomorphisms from $\mathbb{P}^n_{\overline{k}}$ to $\overline{Y}$. Let $G = \mathrm{GL}_{n+1,k}$ and let G act on X by acting trivially on Y and acting in the usual fashion on $\mathbb{P}^n_k$ (one should check that this action is a morphism). Recall that every $\overline{k}$-automorphism of $\mathbb{P}^n_{\overline{k}}$ is induced by the action of some $g \in \mathrm{GL}_{n+1}(\overline{k}) =$ the closed points of G (cf. Example 1.10): this shows that the closed points of $\overline{G}$ act transitively on Σ. It follows that the graph Γ_f of some $f\colon \mathbb{P}^n_{\overline{k}} \xrightarrow{\approx} \overline{Y}$ is defined over k, hence $f = f' \times 1_{\overline{k}}$, where f' is a k-isomorphism of $\mathbb{P}^n_k$ and Y. □

REMARK. See Proposition IV.3.5 and Corollary VIII.1.8 for $\mathbb{P}^1$ over finite fields.

Exercise—Addition needed

(1) Let k be a field, and V a finite dimensional vector space over k. Let $\rho\colon\ \mathrm{GL}_n \to \mathrm{GL}(V)$ be a k-linear rational representation of GL_n on V, i.e., the homomorphism ρ is a k-morphism of group schemes over k. Suppose that $v \in V$ is a vector fixed by the subgroup B of all upper-triangular elements in GL_n. Prove that v is fixed by GL_n.
Hint: The quotient variety GL_n / B is proper over k.

CHAPTER VII

The cohomology of coherent sheaves

1. Basic Čech cohomology

We begin with the general set-up.

(i) X any topological space

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in S} \text{ an open covering of } X$$
$$\mathcal{F} \text{ a presheaf of abelian groups on } X.$$

Define:

(ii)

$$\begin{aligned} C^i(\mathcal{U}, \mathcal{F}) &= \text{group of } i\text{-cochains with values in } \mathcal{F} \\ &= \prod_{\alpha_0, \dots, \alpha_i \in S} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}). \end{aligned}$$

We will write an i-cochain $s = \{s(\alpha_0, \dots, \alpha_i)\}$, i.e.,

$$s(\alpha_0, \dots, \alpha_i) = \text{the component of } s \text{ in } \mathcal{F}(U_{\alpha_0} \cap \cdots U_{\alpha_i}).$$

(iii) $\delta \colon C^i(\mathcal{U}, \mathcal{F}) \to C^{i+1}(\mathcal{U}, \mathcal{F})$ by

$$\delta s(\alpha_0, \dots, \alpha_{i+1}) = \sum_{j=0}^{i+1} (-1)^j \operatorname{res} s(\alpha_0, \dots, \widehat{\alpha}_j, \dots, \alpha_{i+1}),$$

where res is the restriction map

$$\mathcal{F}(U_\alpha \cap \cdots \cap \widehat{\mathcal{U}}_{\alpha_j} \cap \cdots \cap U_{\alpha_{i+1}}) \longrightarrow \mathcal{F}(U_{\alpha_0} \cap \cdots U_{\alpha_{i+1}})$$

and $\widehat{\quad}$ means "omit". For $i = 0, 1, 2$, this comes out as

$$\begin{aligned} \delta s(\alpha_0, \alpha_1) &= s(\alpha_1) - s(\alpha_0) \text{ if } s \in C^0 \\ \delta s(\alpha_0, \alpha_1, \alpha_2) &= s(\alpha_1, \alpha_2) - s(\alpha_0, \alpha_2) + s(\alpha_0, \alpha_1) \text{ if } s \in C^1 \\ \delta s(\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= s(\alpha_1, \alpha_2, \alpha_3) - s(\alpha_0, \alpha_2, \alpha_3) + s(\alpha_0, \alpha_1, \alpha_3) - s(\alpha_0, \alpha_1, \alpha_2) \text{ if } s \in C^2. \end{aligned}$$

One checks very easily that the composition δ^2:

$$C^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^{i+1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^{i+2}(\mathcal{U}, \mathcal{F})$$

is 0. Hence we define:

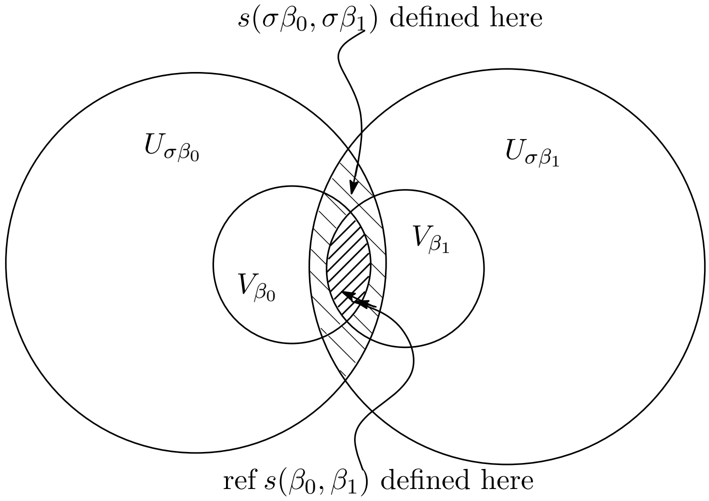

FIGURE VII.1

(iv)

$$\begin{aligned}
Z^i(\mathcal{U}, \mathcal{F}) &= \operatorname{Ker}\left[\delta\colon C^i(\mathcal{U}, \mathcal{F}) \longrightarrow C^{i+1}(\mathcal{U}, \mathcal{F})\right] \\
&= \text{group of } i\text{-cocycles}, \\
B^i(\mathcal{U}, \mathcal{F}) &= \operatorname{Image}\left[\delta\colon C^{i-1}(\mathcal{U}, \mathcal{F}) \longrightarrow C^i(\mathcal{U}, \mathcal{F})\right] \\
&= \text{group of } i\text{-coboundaries} \\
H^i(\mathcal{U}, \mathcal{F}) &= Z^i(\mathcal{U}, \mathcal{F})/B^i(\mathcal{U}, \mathcal{F}) \\
&= i\text{-th Čech-cohomology group with respect to } \mathcal{U}.
\end{aligned}$$

For $i = 0, 1$, this comes out:

$$\begin{aligned}
H^0(\mathcal{U}, \mathcal{F}) =& \text{group of maps } \alpha \mapsto s(\alpha) \in \mathcal{F}(U_\alpha) \text{ such that} \\
& \quad s(\alpha_1) = s(\alpha_0) \text{ in } \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \\
\cong& \Gamma(X, \mathcal{F}) \textit{ if } \mathcal{F} \textit{ is a sheaf}. \\
H^1(\mathcal{U}, \mathcal{F}) =& \text{group of cochains } s(\alpha_0, \alpha_1) \text{ such that} \\
& \quad s(\alpha_0, \alpha_2) = s(\alpha_0, \alpha_1) + s(\alpha_1, \alpha_2) \\
& \quad \text{modulo the cochains of the form} \\
& \quad s(\alpha_0, \alpha_1) = t(\alpha_0) - t(\alpha_1).
\end{aligned}$$

Next suppose $\mathcal{U} = \{U_\alpha\}_\alpha$ and $\mathcal{V} = \{V_\beta\}_{\beta \in T}$ are two open coverings and that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$, i.e., for all $V_\beta \in \mathcal{V}$, $V_\beta \subset U_\alpha$ for some $\alpha \in S$. Fixing a map $\sigma\colon T \to S$ such that $V_\beta \subset U_{\sigma(\beta)}$, define

(v) the refinement homomorphism

$$\mathrm{ref}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F})$$

by the homomorphism on i-cochains:

$$\mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}}(s)(\beta_0, \dots, \beta_i) = \operatorname{res} s(\sigma\beta_0, \dots, \sigma\beta_i)$$

(using $\operatorname{res}\colon \mathcal{F}(U_{\sigma\beta_0} \cap \cdots \cap U_{\sigma\beta_i}) \to \mathcal{F}(V_{\beta_0} \cap \cdots \cap V_{\beta_i})$ and checking that $\delta \circ \mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}} = \mathrm{ref}^\sigma_{\mathcal{U},\mathcal{V}} \circ \delta$, so that ref on cochains induces a map ref on cohomology groups.) (cf. Figure VII.1)

Now one might fear that the refinement map depends on the choice of $\sigma\colon T \to S$, but here we encounter the first of a series of nice identities that make cohomology so elegant — although "ref" on cochains depends on σ, "ref" on cohomology does not.

(vi) Suppose $\sigma, \tau\colon T \to S$ satisfy $V_\beta \subset U_{\sigma\beta} \cap U_{\tau\beta}$. Then

a) for all 1-cocycles s for the covering $\mathcal{U}$,

$$\begin{aligned}
\mathrm{ref}^{\sigma}_{\mathcal{U},\mathcal{V}}\, s(\alpha_0,\alpha_1) &= s(\sigma\alpha_0,\sigma\alpha_1)\\
&= s(\sigma\alpha_0,\tau\alpha_1) - s(\sigma\alpha_1,\tau\alpha_1)\\
&= \{s(\sigma\alpha_0,\tau\alpha_0) + s(\tau\alpha_0,\tau\alpha_1)\} - s(\sigma\alpha_1,\tau\alpha_1)\\
&= s(\tau\alpha_0,\tau\alpha_1) + s(\sigma\alpha_0,\tau\alpha_0) - s(\sigma\alpha_1,\tau\alpha_1)\\
&= \mathrm{ref}^{\tau}_{\mathcal{U},\mathcal{V}}\, s(\alpha_0,\alpha_1) + s(\sigma\alpha_0,\tau\alpha_0) - s(\sigma\alpha_1,\tau\alpha_1)
\end{aligned}$$

i.e., the two ref's differ by the coboundary δt, where

$$t(\alpha) = \mathrm{res}\, s(\sigma\alpha,\tau\alpha) \in \mathcal{F}(V_\alpha).$$

More generally, one checks easily that

b) if $s \in Z^i(\mathcal{U},\mathcal{F})$, then

$$\mathrm{ref}^{\sigma}_{\mathcal{U},\mathcal{V}}\, s - \mathrm{ref}^{\tau}_{\mathcal{U},\mathcal{V}}\, s = \delta t$$

where

$$t(\alpha_0,\ldots,\alpha_{i-1}) = \sum_{j=0}^{i-1} (-1)^j s(\sigma\alpha_0,\ldots,\sigma\alpha_j,\tau\alpha_j,\ldots,\tau\alpha_{i-1}).$$

For general presheaves $\mathcal{F}$ and topological spaces X, one finally passes to the limit via ref over finer and finer coverings and defines:

(vii) [1]

$$H^i(X,\mathcal{F}) = \varinjlim_{\mathcal{U}} H^i(\mathcal{U},\mathcal{F}).$$

Here are three important variants of the standard Čech complex. The first is called the *alternating cochains*:

$C^i_{\mathrm{alt}}(\mathcal{U},\mathcal{F})$ = group of i-cochains s as above such that:

a) $s(\alpha_0,\ldots,\alpha_n) = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$

b) $s(\alpha_{\pi 0},\ldots,\alpha_{\pi n}) = \mathrm{sgn}(\pi)\cdot s(\alpha_0,\ldots,\alpha_n)$ for all permutations π.

For $i = 1$, one sees that every 1-cocycle is automatically alternating; but for $i > 1$, this is no longer so. One checks immediately that $\delta(C^i_{\mathrm{alt}}) \subset C^{i+1}_{\mathrm{alt}}$, hence we can form the cohomology of the complex $(C^\bullet_{\mathrm{alt}},\delta)$. By another beautiful identity, it turns out that the cohomology of the subcomplex $C^\bullet_{\mathrm{alt}}$ and the full complex $C^\bullet$ are exactly the same!

The following proof was modified in publication.

For the proof Serre [**99**, §3, No. 20, Prop. 2] refers to Eilenberg-Steenrod [**37**, Chap. VI, §5 and Thm. 6.10] in constructing an endomorphism ${}^t h = ({}^t h_i)$ of the cochain complex $C^\bullet = C^\bullet(\mathcal{U},\mathcal{F})$ (hence ${}^t h_i\delta = \delta\,{}^t h_{i-1}$ for all i), which is a retract from $C^\bullet$ to the subcomplex $C^\bullet_{\mathrm{alt}}$ (i.e., ${}^t h$ restricts to the identity on $C^\bullet_{\mathrm{alt}}$) together with a homomorphism

$${}^t k_i\colon C^{i+1} \longrightarrow C^i, \qquad i = 0,1,\ldots$$

[1] This group, the Čech cohomology, is often written $\check{H}^i(X,\mathcal{F})$ to distinguish it from the "derived functor" cohomology. In most cases they are however canonically isomorphic and as we will not define the latter, we will not use the ˇ .

such that

$$\mathrm{id} - {}^t h_i = {}^t k_i \delta + \delta {}^t k_{i-1}, \quad i = 0, \dots, \text{ with } {}^t k_{-1} = 0,$$

hence ${}^t h$ induces the inverse to the canonical homomorphism

$$H^\cdot(\text{complex } C^\cdot_{\text{alt}}) \longrightarrow H^\cdot(\text{complex } C^\cdot).$$

${}^t h_i$ and ${}^t k_i$ are the "transpose" of h_i and k_i for the chain complex $K_\cdot$ built out of the set S of open sets $\mathcal{U}_\alpha$: For all $i \geq 0$, let K_i be the free abelian group generated by the ordered sequences $(\alpha_0, \dots, \alpha_i)$ of elements in S with $\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_i} \neq \emptyset$. The boundary map $\partial\colon K_{i+1} \to K_i$ is defined by

$$\partial(\alpha_0, \dots, \alpha_{i+1}) := \sum_{j=0}^{i+1} (-1)^j (\alpha_0, \dots, \widehat{\alpha_j}, \dots, \alpha_{i+1}).$$

Endowing S with a total order, define an endomorphism $h_i\colon K_i \to K_i$ by

$$h_i(\alpha_0, \dots, \alpha_i) = 0$$

if $\alpha_0, \dots, \alpha_i$ are not distinct, while

$$h_i(\alpha_0, \dots, \alpha_i) = \operatorname{sgn}(\sigma) h_i(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(i)})$$

if $\alpha_0, \dots, \alpha_i$ are distinct and σ is the permutation of $\{0, 1, \dots, i\}$ such that $\alpha_{\sigma(0)} < \alpha_{\sigma(1)} < \dots < \alpha_{\sigma(i)}$.

It is easy to show that $\partial h_i = h_{i-1}\partial$ for all $i \geq 1$ so that $h = (h_i)$ is an endomorphism of the chain complex $K_\cdot$. Moreover, the "transpose" ${}^t h_i\colon C^i \to C^i$ obviously induces the identity map on $C^i_{\text{alt}} \subset C^i$ and has the property ${}^t h_i(C^i) = C^i_{\text{alt}}$.

Eilenberg-Steenrod [**37**, Chap. VI, §5] constructs a homotopy

$$k_i\colon K_i \longrightarrow K_{i+1}, \qquad i = 0, 1, \dots$$

such that

$$\mathrm{id} - h_i = \partial k_i + k_{i-1}\partial, \qquad i = 0, 1, \dots \text{ with } k_{-1} = 0$$

as follows: Let

$$k_0 = 0.$$

For $i \geq 1$, let

$$k_i(\alpha_0, \dots, \alpha_i) := \Psi_{\alpha_0}\big((\mathrm{id} - h_i)(\alpha_0, \dots, \alpha_i) - k_{i-1}\partial(\alpha_0, \dots, \alpha_i)\big),$$

where Ψ_{α_0} is defined as follows: For $l \leq i-1$ and $(\beta_0, \dots, \beta_l) \in K_l$ with $\{\beta_0, \dots, \beta_l\} \subset \{\alpha_0, \dots, \alpha_i\}$,

$$\Psi_{\alpha_0}(\beta_0, \dots, \beta_l) := (\alpha_0, \beta_0, \dots, \beta_l) \in K_{l+1}.$$

Clearly, we have

$$\partial\Psi_{\alpha_0}(\beta_0, \dots, \beta_l) = (\beta_0, \dots, \beta_l) - \Psi_{\alpha_0}\partial(\beta_0, \dots, \beta_l).$$

We now show

$$(*) \qquad \mathrm{id} - h_i = \partial k_i + k_{i-1}\partial$$

by induction on $i \geq 0$. Note that $k_{-1} = k_0 = 0$ and $(*)$ holds for $i = 0$. For $i \geq 1$,

$$\begin{aligned}
&\partial k_i(\alpha_0, \ldots, \alpha_i) \\
&\quad= \partial\Psi_{\alpha_0}\big((\mathrm{id} - h_i)(\alpha_0, \ldots, \alpha_i) - k_{i-1}\partial(\alpha_0, \ldots, \alpha_i)\big) \\
&\quad= (\mathrm{id} - h_i)(\alpha_0, \ldots, \alpha_i) - k_{i-1}\partial(\alpha_0, \ldots, \alpha_i) \\
&\qquad -\Psi_{\alpha_0}\Big(\partial\big((\mathrm{id} - h_i)(\alpha_0, \ldots, \alpha_i) - k_{i-1}\partial(\alpha_0, \ldots, \alpha_i)\big)\Big) \\
&\quad= (\mathrm{id} - h_i)(\alpha_0, \ldots, \alpha_i) - k_{i-1}\partial(\alpha_0, \ldots, \alpha_i) \\
&\qquad -\Psi_{\alpha_0}\Big((\mathrm{id} - h_{i-1})\partial(\alpha_0, \ldots, \alpha_i) - \partial k_{i-1}\partial(\alpha_0, \ldots, \alpha_i)\Big) \\
&\quad= (\mathrm{id} - h_i)(\alpha_0, \ldots, \alpha_i) - k_{i-1}\partial(\alpha_0, \ldots, \alpha_i),
\end{aligned}$$

since

$$\partial k_{i-1}\partial = ((\mathrm{id} - h_{i-1}) - k_{i-2}\partial)\,\partial$$

by the induction hypothesis.

Here is the explicit formula for $i = 1$:

$$(\mathrm{id} - h_1)(\alpha_0, \alpha_1) = \begin{cases} (\alpha_0, \alpha_1) & \text{if } \alpha_0 = \alpha_1 \\ 0 & \text{if } \alpha_0 < \alpha_1 \\ (\alpha_0, \alpha_1) + (\alpha_1, \alpha_0) & \text{if } \alpha_0 > \alpha_1, \end{cases}$$

hence

$$\begin{aligned}
k_1(\alpha_0, \alpha_1) &= \Psi_{\alpha_0}\left((\mathrm{id} - h_1)(\alpha_0, \alpha_1) - 0\right) \\
&= \begin{cases} (\alpha_0, \alpha_0, \alpha_1) & \text{if } \alpha_0 = \alpha_1 \\ 0 & \text{if } \alpha_0 < \alpha_1 \\ (\alpha_0, \alpha_0, \alpha_1) + (\alpha_0, \alpha_1, \alpha_0) & \text{if } \alpha_0 > \alpha_1. \end{cases}
\end{aligned}$$

Consequently, ${}^t\!k_1 \colon C^2 \to C^1$ sends $s \in C^2$ to ${}^t\!k_1 s \in C^1$ with

$$({}^t\!k_1 s)(\alpha_0, \alpha_1) = \begin{cases} s(\alpha_0, \alpha_0, \alpha_1) & \text{if } \alpha_0 = \alpha_1 \\ 0 & \text{if } \alpha_0 < \alpha_1 \\ s(\alpha_0, \alpha_0, \alpha_1) + s(\alpha_0, \alpha_1, \alpha_0) & \text{if } \alpha_0 > \alpha_1. \end{cases}$$

The second variant is *local cohomology.* Suppose $Y \subset X$ is a closed subset and that the covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ has the property:

$$X \setminus Y = \bigcup_{\alpha \in S_0} U_\alpha \quad \text{for a subset } S_0 \subset S.$$

Consider the subgroups:

$$C^i_{S_0}(\mathcal{U}, \mathcal{F}) = \{s \in C^i(\mathcal{U}, \mathcal{F}) \mid s(\alpha_0, \ldots, \alpha_i) = 0 \text{ if } \alpha_0, \ldots, \alpha_i \in S_0\}.$$

One checks that $\delta(C^i_{S_0}) \subset C^{i+1}_{S_0}$, hence one can define $H^i_{S_0}(\mathcal{U}, \mathcal{F})$ = cohomology of complex $(C^\cdot_{S_0}, \delta)$. Passing to a limit with refinements ($\mathcal{V}$, T_0 refines $\mathcal{U}$, S_0 if $\exists \rho \colon T \to S$ such that $V_\beta \subset U_{\rho\beta}$ and $\rho(T_0) \subset S_0$), one gets $H^i_Y(X, \mathcal{F})$ much as above.

The third variation on the same theme is the hypercohomology of a complex of presheaves:

$$\mathcal{F}^\cdot \colon 0 \longrightarrow \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \mathcal{F}^m \longrightarrow 0$$

(i.e., $d_{i+1} \circ d_i = 0$, for all i). If $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ is an open covering, we get a double complex

$$C^{ij} = C^i(\mathcal{U}, \mathcal{F}^j)$$

where

$$\delta_1 \colon C^{ij} \longrightarrow C^{i+1,j} \text{ is the Čech coboundary}$$

$$\delta_2 \colon C^{ij} \longrightarrow C^{i,j+1} \text{ is given by applying } d_j \text{ to the cochain.}$$

Then $\delta_1\delta_2 = \delta_2\delta_1$ and if we set

$$C^{(n)} = \sum_{i+j=n} C^{i,j}$$

and use $d = \delta_1 + (-1)^i\delta_2 \colon C^{(n)} \to C^{(n+1)}$ as differential, then $d^2 = 0$. This is called the associated "total complex". Define

$$\mathbb{H}^n(\mathcal{U}, \mathcal{F}^{\boldsymbol{\cdot}}) = n\text{-th cohomology group of complex } (C^{(\cdot)}, d).$$

Passing to a limit with refinements, one gets $\mathbb{H}^n(X, \mathcal{F}^{\boldsymbol{\cdot}})$. This variant is very important in the De Rham theory (cf. §VIII.3 below).

The most important property of Čech cohomology is the *long exact cohomology sequence.* Suppose

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of *presheaves* (which means that

$$0 \longrightarrow \mathcal{F}_1(U) \longrightarrow \mathcal{F}_2(U) \longrightarrow \mathcal{F}_3(U) \longrightarrow 0$$

is exact for every open U). Then for every covering $\mathcal{U}$, we get a big diagram relating the cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C^{i-1}(\mathcal{U}, \mathcal{F}_1) & \to & C^{i-1}(\mathcal{U}, \mathcal{F}_2) & \to & C^{i-1}(\mathcal{U}, \mathcal{F}_3) & \to & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \longrightarrow & C^{i}(\mathcal{U}, \mathcal{F}_1) & \longrightarrow & C^{i}(\mathcal{U}, \mathcal{F}_2) & \longrightarrow & C^{i}(\mathcal{U}, \mathcal{F}_3) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \\
0 & \to & C^{i+1}(\mathcal{U}, \mathcal{F}_1) & \to & C^{i+1}(\mathcal{U}, \mathcal{F}_2) & \to & C^{i+1}(\mathcal{U}, \mathcal{F}_3) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

with exact rows, i.e., a short exact sequence of complexes of abelian groups. By a standard fact in homological algebra, this always leads to a long exact sequence relating the cohomology groups of the three complexes. In this case, this gives:

$$\begin{aligned}
0 \longrightarrow H^0(\mathcal{U}, \mathcal{F}_1) \longrightarrow H^0(\mathcal{U}, \mathcal{F}_2) \longrightarrow H^0(\mathcal{U}, \mathcal{F}_3) \xrightarrow{\delta} H^1(\mathcal{U}, \mathcal{F}_1)& \\
\longrightarrow H^1(\mathcal{U}, \mathcal{F}_2) \longrightarrow H^1(\mathcal{U}, \mathcal{F}_3) \xrightarrow{\delta} H^2(\mathcal{U}, \mathcal{F}_1) \longrightarrow \cdots .&
\end{aligned}$$

Moreover, we may pass to the limit over refinements, getting:

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{F}_1) \longrightarrow H^0(X, \mathcal{F}_2) \longrightarrow H^0(X, \mathcal{F}_3) \xrightarrow{\delta} H^1(X, \mathcal{F}_1)& \\
\longrightarrow H^1(X, \mathcal{F}_2) \longrightarrow H^1(X, \mathcal{F}_3) \xrightarrow{\delta} H^2(X, \mathcal{F}_1) \longrightarrow \cdots .&
\end{aligned}$$

In almost all applications, we are only interested in the cohomology of *sheaves* and unfortunately short exact sequences of sheaves are seldom exact as sequences of presheaves. Still, in reasonable cases the long exact cohomology sequence continues to hold. The problem can be analyzed as follows: let

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

be a short exact sequence of *sheaves*. If we define a subpresheaf $\mathcal{F}_3^* \subset \mathcal{F}_3$ by

$$\mathcal{F}_3^*(U) = \text{Image}\left[\mathcal{F}_2(U) \longrightarrow \mathcal{F}_3(U)\right],$$

then

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3^* \longrightarrow 0$$

is an exact sequence of presheaves, hence we get a long exact sequence:

$$\cdots \longrightarrow H^i(X, \mathcal{F}_1) \longrightarrow H^i(X, \mathcal{F}_2) \longrightarrow H^i(X, \mathcal{F}_3^*) \xrightarrow{\delta} H^{i+1}(X, \mathcal{F}_1) \longrightarrow \cdots$$

Now $\mathcal{F}_3$ is the sheafification of $\mathcal{F}_3^*$ so a long exact sequence for the cohomology of the sheaves $\mathcal{F}_i$ follows if we can prove the more general assertion:

$$(*) \qquad \begin{array}{l} \text{for all presheaves } \mathcal{F}, \text{ the canonical maps} \\ \qquad H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathrm{sh}(\mathcal{F})) \\ \text{are isomorphisms.} \end{array}$$

Breaking up $\mathcal{F} \to \mathrm{sh}(\mathcal{F})$ into a diagram of presheaves:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{F} \arrow[rr] \arrow[dr] & & \mathrm{sh}\,\mathcal{F} \arrow[r] & \mathcal{C} \arrow[r] & 0 \\ & & & \mathcal{F}' \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

($\mathcal{K}$ = kernel, $\mathcal{C}$ = cokernel, $\mathcal{F}'$ = image) and applying twice the long exact sequence for presheaves, $(*)$ follows from:

$$(**) \qquad \text{If } \mathcal{F} \text{ is a presheaf such that } \mathrm{sh}(\mathcal{F}) = (0), \text{ then } H^i(X, \mathcal{F}) = (0).$$

The standard case where $(**)$ and hence $(*)$ is satisfied is for paracompact Hausdorff spaces[2] X: we will use this fact once in (3.11) below and §VIII.3 in comparing classical and algebraic De Rham cohomology for complex varieties. Schemes however are far from Hausdorff so we need to take a different tack. In fact, suppose X is a scheme (separated as usual) and

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of *quasi-coherent* sheaves. Then in the above notations:

$$\mathcal{F}_3^*(U) \xrightarrow{\approx} \mathcal{F}_3(U), \quad \text{all } \textit{affine } U$$

so

$$\mathcal{K}(U) = \mathcal{C}(U) = (0), \quad \text{all affine } U.$$

[2]The proof is as follows: We may compute $H^i(X, \mathcal{F})$ by locally finite coverings $\mathcal{U}$ so let $\mathcal{U}$ be one and let $s \in C^i(\mathcal{U}, \mathcal{F})$. A paracompact space is normal so one easily constructs a covering $\mathcal{V}$ with the same index set I such that $\overline{V}_\alpha \subset U_\alpha$, $\forall \alpha \in I$. Now for all $x \in X$, the local finiteness of $\mathcal{U}$ shows that $\exists$ neighborhood N_x of x such that

a) $x \in U_{\alpha_0} \cap \cdots \cap U_{\alpha_i} \Longrightarrow N_x \subset U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}$ and $\mathrm{res}_{N_x}\, s(\alpha_0, \ldots, \alpha_i) = 0$. Shrinking N_x, we can also assume that N_x meets only a finite set of U_α's hence there is a smaller neighborhood $M_x \subset N_x$ of x such that:
b) $M_x \subset$ some V_α and if $M_x \cap V_\beta \neq \emptyset$, then $M_x \subset V_\beta$. Let $\mathcal{W} = \{M_x\}_{x \in X}$. Then $\mathcal{W}$ refines $\mathcal{V}$ and it follows immediately that $\mathrm{ref}_{\mathcal{V},\mathcal{W}}(s) \equiv 0$ as a cochain.

Now if $\mathcal{U}$ is any affine open covering of X, then X separated implies $U_{\alpha_0} \cap \cdots \cap U_{\alpha_i}$ affine for all $\alpha_0, \ldots, \alpha_i$, hence $C^i(\mathcal{U}, \mathcal{K}) = C^i(\mathcal{U}, \mathcal{C}) = (0)$, hence $H^i(\mathcal{U}, \mathcal{K}) = H^i(\mathcal{U}, \mathcal{C}) = (0)$. Since affine coverings are cofinal among all coverings, $H^i(X, \mathcal{K}) = H^i(X, \mathcal{C}) = (0)$, hence $H^i(X, \mathcal{F}_3^*) \xrightarrow{\approx} H^i(X, \mathcal{F}_3)$ and we get a long exact sequence for the cohomology of the $\mathcal{F}_i$'s for much more elementary reasons!

What are the functorial properties of cohomology groups? Here are three important kinds:

a) If $f\colon X \to Y$ is a continuous map of topological spaces, $\mathcal{F}$ (resp. $\mathcal{G}$) a presheaf on X (resp. Y), and $\alpha\colon \mathcal{G} \to \mathcal{F}$ a homomorphism covering f, (i.e., a set of homomorphisms:
$$\alpha(U)\colon \mathcal{G}(U) \longrightarrow \mathcal{F}(f^{-1}(U)), \qquad \text{all open } U \subset Y$$
commuting with restriction), then we get canonical maps:
$$(f, \alpha)^*\colon H^i(Y, \mathcal{G}) \longrightarrow H^i(X, \mathcal{F}), \qquad \text{all } i.$$

b) If we have two short exact sequences of presheaves and a commutative diagram:
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \\ & & {\scriptstyle\alpha_1}\downarrow & & {\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\alpha_3}\downarrow & & \\ 0 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{G}_3 & \longrightarrow & 0, \end{array}$$
then the δ's in the long exact cohomology sequences give a commutative diagram:
$$\begin{array}{ccc} H^i(X, \mathcal{F}_3) & \xrightarrow{\delta} & H^{i+1}(X, \mathcal{F}_1) \\ {\scriptstyle\alpha_3}\downarrow & & \downarrow{\scriptstyle\alpha_1} \\ H^i(X, \mathcal{G}_3) & \xrightarrow{\delta} & H^{i+1}(X, \mathcal{G}_1), \end{array}$$
(i.e., the $H^i(X, \mathcal{F})$'s together are a "cohomological δ-functor").

c) If $\mathcal{F}$ and $\mathcal{G}$ are two presheaves of abelian groups, define a presheaf $\mathcal{F} \otimes \mathcal{G}$ by $(\mathcal{F} \otimes \mathcal{G})(U) = \mathcal{F}(U) \otimes \mathcal{G}(U)$. Then there is a bilinear map:
$$H^i(X, \mathcal{F}) \times H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes \mathcal{G})$$
called cup product, and written $\cup$.

To construct the map in (a), take the obvious map of cocycles and check that it commutes with δ; (b) is a straightforward computation; as for (c), define $\cup$ on couples by:
$$(s \cup t)(\alpha_0, \ldots, \alpha_{i+j}) = \operatorname{res} s(\alpha_0, \ldots, \alpha_i) \otimes \operatorname{res} t(\alpha_i, \ldots, \alpha_{i+j})$$
and check that $\delta(s \cup t) = \delta s \cup t + (-1)^i s \cup \delta t$. It is not hard to check that $\cup$ is associative and has a certain skew-commutativity property:

c$'$) If $s_i \in H^{k_i}(X, \mathcal{F}_i)$, $i = 1, 2, 3$, then in the group $H^{k_1+k_2+k_3}(X, \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3)$ we have
$$(s_1 \cup s_2) \cup s_3 = s_1 \cup (s_2 \cup s_3).$$

c$''$) If $\operatorname{Symm}^2 \mathcal{F}$ is the quotient presheaf of $\mathcal{F} \otimes \mathcal{F}$ by the subsheaf of elements $a \otimes b - b \otimes a$, and $s_i \in H^{k_i}(X, \mathcal{F})$, $i = 1, 2$, then in the group $H^{k_1+k_2}(X, \operatorname{Symm}^2 \mathcal{F})$ we have
$$s_1 \cup s_2 = (-1)^{k_1 k_2} s_2 \cup s_1.$$

The proofs are left to the reader.

The cohomology exact sequence leads to the method of computing cohomology by *acyclic resolutions*: suppose a sheaf $\mathcal{F}$ is given and we construct a long exact sequence of sheaves
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{G}_2 \longrightarrow \cdots,$$
such that:

a) $H^i(X, \mathcal{G}_k) = (0)$, $i \geq 1$, $k \geq 0$.
b) If $\mathcal{K}_k = \mathrm{Ker}(\mathcal{G}_{k+1} \to \mathcal{G}_{k+2})$ and $\mathcal{C}_k =$ cokernel as presheaf $(\mathcal{G}_{k-1} \to \mathcal{G}_k)$ so that $\mathcal{K}_k = \mathrm{sh}(\mathcal{C}_k)$, then assume

$$H^i(X, \mathcal{C}_k) \xrightarrow{\approx} H^i(X, \mathcal{K}_k), \quad i \geq 0, \ k \geq 0.$$

Then $H^i(X, \mathcal{F})$ is isomorphic to the i-th cohomology group of the complex:

$$0 \longrightarrow \mathcal{G}_0(X) \longrightarrow \mathcal{G}_1(X) \longrightarrow \mathcal{G}_2(X) \longrightarrow \cdots .$$

To see this, use induction on i. We may split off the first part of our resolution like this:

i) $0 \to \mathcal{F} \to \mathcal{G}_0 \to \mathcal{C}_0 \to 0$, exact as *presheaves.*
ii) $0 \to \mathcal{K}_0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to \cdots$, exact as *sheaves.*

So by the cohomology sequence of (i) and induction applied to the resolution (ii):

a)

$$\begin{aligned} H^0(X, \mathcal{F}) &\cong \mathrm{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{C}_0)\right] \\ &\cong \mathrm{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{K}_0)\right] \\ &\cong \mathrm{Ker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{G}_1)\right] \end{aligned}$$

b)

$$\begin{aligned} H^1(X, \mathcal{F}) &\cong \mathrm{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{C}_0)\right] \\ &\cong \mathrm{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow H^0(\mathcal{K}_0)\right] \\ &\cong \mathrm{Coker}\left[H^0(\mathcal{G}_0) \longrightarrow \mathrm{Ker}\left[H^0(\mathcal{G}_1) \longrightarrow H^0(\mathcal{G}_2)\right]\right] \\ &\cong H^1(\text{the complex } H^0(\mathcal{G}_\bullet)). \end{aligned}$$

c)

$$\begin{aligned} H^i(X, \mathcal{F}) &\cong H^{i-1}(X, \mathcal{C}_0) \\ &\cong H^{i-1}(X, \mathcal{K}_0) \\ &\cong H^i(\text{the complex } H^0(\mathcal{G}_\bullet)), \quad i \geq 2. \end{aligned}$$

If $\mathcal{F}$ is a sheaf, we have seen that $H^0(X, \mathcal{F})$ is just $\Gamma(X, \mathcal{F})$ or $\mathcal{F}(X)$. $H^1(X, \mathcal{F})$ also has a simple interpretation in terms of "twisted structures" over X. Define

A principal $\mathcal{F}$-sheaf

$=$ a sheaf of sets $\mathcal{G}$, plus an action of $\mathcal{F}$ on $\mathcal{G}$
(i.e., $\mathcal{F}(U)$ acts on $\mathcal{G}(U)$ commuting with restriction)
such that $\exists$ a covering $\{U_\alpha\}$ of X where:
res_{U_α} ($\mathcal{G}$, as sheaf with $\mathcal{F}$-action)
$\cong \mathrm{res}_{U_\alpha}$ ($\mathcal{F}$, with $\mathcal{F}$-action on itself by translation).

Then if $\mathcal{F}$ is a sheaf:

$(*)$ $H^1(X, \mathcal{F}) \cong \{$set of principal $\mathcal{F}$-sheaves, modulo isomorphism$\}$.

$$H^1(\mathcal{U}, \mathcal{F}) \cong \left\{ \begin{array}{l} \text{subset of those principal } \mathcal{F}\text{-sheaves which are trivial} \\ \text{on the open sets } U_\alpha \text{ of the covering } \mathcal{U} \end{array} \right\}.$$

In fact:

a) Given $\mathcal{G}$, let $\phi_\alpha\colon \mathcal{G}|_{U_\alpha} \xrightarrow{\approx} \mathcal{F}|_{U_\alpha}$ be an $\mathcal{F}$-isomorphism. Then on $U_\alpha \cap U_\beta$, $\phi_\alpha \circ \phi_\beta^{-1}\colon \mathcal{F}|_{U_\alpha \cap U_\beta} \longrightarrow \mathcal{F}|_{U_\alpha \cap U_\beta}$ is an $\mathcal{F}$-automorphism. If it carries the 0-section to $s(\alpha,\beta) \in \mathcal{F}(U_\alpha \cap U_\beta)$, it will be the map $x \mapsto x + s(\alpha,\beta)$. One checks that s is a 1-cocycle, hence it defines a cohomology class in $H^1(\mathcal{U},\mathcal{F})$, and by refinement in $H^1(X,\mathcal{F})$.

b) Conversely, given $\sigma \in H^1(X,\mathcal{F})$, represent σ by a 1-cocycle $s(\alpha,\beta)$ for a covering $\{U_\alpha\}$. Define a sheaf $\mathcal{G}_\sigma$ by

$$\mathcal{G}_\sigma(V) = \left\{ \begin{array}{l} \text{collections of elements } t_\alpha \in \mathcal{F}(V \cap U_\alpha) \text{ such that} \\ \operatorname{res} t_\alpha + s(\alpha,\beta) = \operatorname{res} t_\beta \text{ in } \mathcal{F}(V \cap U_\alpha \cap U_\beta) \end{array} \right\}.$$

Intuitively, $\mathcal{G}_\sigma$ is obtained by "glueing" the sheaves $\mathcal{F}|_{U_\alpha}$ together by translation by $s(\alpha,\beta)$ on $U_\alpha \cap U_\beta$.

We leave it to the reader to check that $\mathcal{G}_\sigma$ is independent of the choice of s and that the constructions (a) and (b) are inverse to each other. The same ideas exactly allow you to prove:

If $\mathcal{O}_X$ is a sheaf of *rings* on X and $\mathcal{O}_X^* =$ subsheaf of units in $\mathcal{O}_X$, then

$$H^1(X,\mathcal{O}_X^*) \cong \left\{ \begin{array}{l} \text{set of sheaves of } \mathcal{O}_X\text{-modules, locally isomorphic} \\ \text{to } \mathcal{O}_X \text{ itself, modulo isomorphism} \end{array} \right\}$$

(cf. §III.6)

and

If X is locally connected and $(\mathbb{Z}/n\mathbb{Z})_X =$ sheafification of the constant presheaf $\mathbb{Z}/n\mathbb{Z}$, then

$$H^1(X,(\mathbb{Z}/n\mathbb{Z})_X) \cong \left\{ \begin{array}{l} \text{set of covering spaces } \pi\colon Y \to X \text{ with } \mathbb{Z}/n\mathbb{Z} \\ \text{acting on } Y \text{, permuting freely and transitively} \\ \text{the points of each set } \pi^{-1}(x),\ x \in X \end{array} \right\}.$$

2. The case of schemes: Serre's theorem

From now on, we assume that X is a *scheme*[3] and that $\mathcal{F}$ is a *quasi-coherent sheaf*. The main result is this:

THEOREM 2.1 (Serre). *Let $\mathcal{U}$ and $\mathcal{V}$ be two affine open coverings of X, with $\mathcal{V}$ refining $\mathcal{U}$. Then*

$$\operatorname{res}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U},\mathcal{F}) \longrightarrow H^i(\mathcal{V},\mathcal{F})$$

is an isomorphism.

The proof consists in two steps. The first is a general criterion for res to be an isomorphism. The second is an explicit computation for modules and distinguished affine coverings. The general criterion is this:

PROPOSITION 2.2. *Let X be any topological space, $\mathcal{F}$ a sheaf of abelian groups on X, and $\mathcal{U}$ and $\mathcal{V}$ two open coverings of X. Suppose $\mathcal{V}$ refines $\mathcal{U}$. For every finite subset $S_0 = \{\alpha_0,\dots,\alpha_p\} \subset S$, let*

$$U_{S_0} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$$

and let $\mathcal{V}|_{U_{S_0}}$ denote the covering of U_{S_0} induced by $\mathcal{V}$. Assume:

$$H^i(\mathcal{V}|_{U_{S_0}},\mathcal{F}|_{U_{S_0}}) = (0), \quad \textit{all } S_0,\ i > 0.$$

[3]Our approach works only because all our schemes are separated. In the general case, Čech cohomology is not good and either derived functors (via Grothendieck) or a Cech complex@Čech!modifiedmodified Čech complex (via Lubkin or Verdier) must be used.

Then $\mathrm{ref}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U},\mathcal{F}) \to H^i(\mathcal{V},\mathcal{F})$ *is an isomorphism for all* i.

Proof. The technique is to compare the two Čech cohomologies via a big double complexes:

$$C^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p \in S} \prod_{\beta_0,\ldots,\beta_q \in T} \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}).$$

By ignoring either the α's or the β's and taking δ in the β's or α's, we get two coboundary maps:

$$\delta_1\colon C^{p,q} \longrightarrow C^{p+1,q}$$

$$\delta_1 s(\alpha_0,\ldots,\alpha_{p+1},\beta_0,\ldots,\beta_q) = \sum_{j=0}^{p+1} (-1)^j s(\alpha_0,\ldots,\widehat{\alpha}_j,\ldots,\alpha_{p+1},\beta_0,\ldots,\beta_q)$$

and

$$\delta_2\colon C^{p,q} \longrightarrow C^{p,q+1}$$

$$\delta_2 s(\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\beta_{q+1}) = \sum_{j=0}^{q+1} (-1)^j s(\alpha_0,\ldots,\alpha_p,\beta_0,\ldots,\widehat{\beta}_j,\ldots,\beta_{q+1}).$$

One checks immediately that these satisfy $\delta_1^2 = \delta_2^2 = 0$ and $\delta_1\delta_2 = \delta_2\delta_1$. As in §1, we get a "total complex" by setting:

$$C^{(n)} = \sum_{\substack{p+q=n \\ p,q \geq 0}} C^{p,q}$$

and with $d = \delta_1 + (-1)^p\delta_2\colon C^{(n)} \to C^{(n+1)}$ as differential. Here is the picture:

$$\begin{tikzcd}
{} & & & \\
C^{0,2} \arrow[u] \arrow[r] & {} & & \\
C^{0,1} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{1,1} \arrow[u] \arrow[r] & {} & \\
C^{0,0} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{1,0} \arrow[u, "\delta_2"] \arrow[r, "\delta_1"] & C^{2,0} \arrow[u] \arrow[r] & {}
\end{tikzcd}$$

where

$$\begin{aligned}
C^{0,2} &= \prod_{\substack{\alpha\in S\\ \beta_0,\beta_1,\beta_2\in T}} \mathcal{F}(U_\alpha \cap V_{\beta_0} \cap V_{\beta_1} \cap V_{\beta_2})\\
C^{0,1} &= \prod_{\substack{\alpha\in S\\ \beta_0,\beta_1\in T}} \mathcal{F}(U_\alpha \cap V_{\beta_0} \cap V_{\beta_1})\\
C^{1,1} &= \prod_{\substack{\alpha_0,\alpha_1\in S\\ \beta_0,\beta_1\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_{\beta_0} \cap V_{\beta_1})\\
C^{0,0} &= \prod_{\substack{\alpha\in S\\ \beta\in T}} \mathcal{F}(U_\alpha \cap V_{\beta})\\
C^{1,0} &= \prod_{\substack{\alpha_0,\alpha_1\in S\\ \beta\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_{\beta})\\
C^{2,0} &= \prod_{\substack{\alpha_0,\alpha_1,\alpha_2\in S\\ \beta\in T}} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap V_{\beta}).
\end{aligned}$$

We need to observe four things about this situation:

(A) The columns of this double complex are just products of the Čech complexes for the coverings $\mathcal{V}|_{U_{S_0}}$ for various $S_0 \subset S$: in fact the p-th column $C^{p,0} \to C^{p,1} \to \cdots$ is the product of these complexes for all S_0 with $\#S_0 = p+1$. By assumption these complexes have no cohomology beyond the first place, hence

the δ_2-cohomology of the columns

$$\text{Ker}\left[\delta_2 \colon C^{p,q} \longrightarrow C^{p,q+1}\right] / \text{Image}\left[\delta_2 \colon C^{p,q-1} \longrightarrow C^{p,q}\right]$$

is (0) *for all* $p \geq 0$*, all* $q > 0$.

(B) The rows of this double complex are similarly products of the Čech complexes for the coverings $\mathcal{U}|_{V_{T_0}}$ for various $T_0 \subset T$. Now $V_{T_0} \subset$ some $V_\beta \subset$ some U_α, hence the covering $\mathcal{U}|_{V_{T_0}}$ of V_{T_0} includes among its open sets the whole space V_{T_0}. For such silly coverings, Čech cohomology always vanishes —

Lemma 2.3. *X a topological space, $\mathcal{F}$ a sheaf, and $\mathcal{U}$ an open covering of X such that $X \in \mathcal{U}$. Then $H^i(\mathcal{U}, \mathcal{F}) = (0)$, $i > 0$.*

Proof of Lemma 2.3. Let $X = U_\zeta \in \mathcal{U}$. For all $s \in Z^i(\mathcal{U}, \mathcal{F})$, define an $(i-1)$-cochain by:

$$t(\alpha_0, \ldots, \alpha_{i-1}) = s(\zeta, \alpha_0, \ldots, \alpha_{i-1})$$

[OK since $U_\zeta \cap U_{\alpha_0} \cap \cdots \cap U_{\alpha_{i-1}} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_{i-1}}$!] An easy calculation shows that $s = \delta t$. ☐

Hence

the δ_1-cohomology of the rows is (0) *at the* (p,q)*-th spot, for all* $p > 0$*,* $q > 0$.

(C) Next there is a big diagram-chase —

Lemma 2.4 (The easy lemma of the double complex). *Let $\{C^{p,q}, \delta_1, \delta_2\}_{p,q\geq 0}$ be any double complex (meaning $\delta_1^2 = \delta_2^2 = 0$ and $\delta_1\delta_2 = \delta_2\delta_1$). Assume that the δ_2-cohomology:*

$$H^{p,q}_{\delta_2} = \text{Ker}\left[\delta_2 \colon C^{p,q} \longrightarrow C^{p,q+1}\right] / \text{Image}\left[\delta_2 \colon C^{p,q-1} \longrightarrow C^{p,q}\right]$$

is (0) *for all* $p \geq 0$, $q > 0$. *Then there is an isomorphism:*

$\left(\delta_1\text{-}\textit{cohomology of } H^{p,0}_{\delta_2}\right) = ((d = \delta_1 + (-1)^p\delta_2)\text{-}\textit{cohomology of total complex})$

i.e.,

$$\frac{\{x \in C^{p,0} \mid \delta_1 x = \delta_2 x = 0\}}{\{\delta_1 x \mid x \in C^{p-1,0} \textit{ with } \delta_2 x = 0\}} \cong \frac{\left\{x \in \sum_{i+j=p} C^{i,j} \mid dx = 0\right\}}{\left\{dx \mid x \in \sum_{i+j=p-1} C^{i,j}\right\}}.$$

Proof of Lemma 2.4. We give the proof in detail for $p = 2$ in such a way that it is clear how to set up the proof in general. Start with $x = (x_{2,0}, x_{1,1}, x_{0,2}) \in \sum_{i+j=2} C^{i,j}$ such that $dx = 0$, i.e.,

$$\delta_1 x_{2,0} = 0;\ \delta_1 x_{1,1} + \delta_2 x_{2,0} = 0;\ \delta_1 x_{0,2} - \delta_2 x_{1,1} = 0;\ \delta_2 x_{0,2} = 0.$$

$$\begin{tikzcd}
0 & & & \\
x_{0,2} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm y & & \\
& x_{1,1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm z & \\
& & x_{2,0} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & 0
\end{tikzcd}$$

Now $\delta_2 x_{0,2} = 0 \implies x_{0,2} = \delta_2 x_{0,1}$ for some $x_{0,1}$. Alter x by the coboundary $d(0, -x_{0,1})$: we find

$$x \sim x' = (x'_{2,0}, x'_{1,1}, 0) \qquad (\sim \text{ means cohomologous}).$$

But then $dx' = 0 \implies \delta_2 x'_{1,1} = 0 \implies x'_{1,1} = \delta_2 x_{1,0}$ for some $x_{1,0}$. Alter x' by the coboundary $d(x_{1,0}, 0)$: we find

$$x' \sim x'' = (x''_{2,0}, 0, 0)$$

and $dx'' = 0 \implies \delta_1 x''_{2,0}$ and $\delta_2 x''_{2,0}$ are 0. Thus $x''_{2,0}$ defines an element of $H^2_{\delta_1}\left(\text{the complex } H^{p,0}_{\delta_2}\right)$. This argument (generalized in the obvious way) shows that the map:

$$\Phi\colon \left(\delta_1\text{-cohomology of } H^{p,0}_{\delta_2}\right) \longrightarrow (d\text{-cohomology of total complex})$$

is *surjective*. Now say $x_{2,0} \in C^{2,0}$ satisfies $\delta_1 x_{2,0} = \delta_2 x_{2,0} = 0$. Say $(x_{2,0}, 0, 0) = dx$, $x = (x_{1,0}, x_{0,1}) \in \sum_{i+j=1} C^{i,j}$, i.e.,

$$x_{2,0} = \delta_1 x_{1,0};\ -\delta_2 x_{1,0} + \delta_1 x_{0,1} = 0; \delta_2 x_{0,1} = 0.$$

$$\begin{tikzcd}
0 & & \\
x_{0,1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & \pm y & \\
& x_{1,0} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & x_{2,0}
\end{tikzcd}$$

Then $\delta_2 x_{0,1} = 0 \implies x_{0,1} = \delta_2 x_{0,0}$, for some $x_{0,0}$. Alter x by the coboundary $-dx_{0,0}$: we find

$$x \sim x' = (x'_{0,0}, 0)$$

and $dx' = (x_{2,0}, 0, 0)$. Then $\delta_2 x'_{1,0} = 0$ and $\delta_1 x'_{1,0} = x_{2,0}$, i.e., $x_{2,0}$ goes to 0 in the δ_1-cohomology of $H^{p,0}_{\delta_2}$. This gives injectivity of Φ. $\square$

If we combine (A), (B) and (C), applied both to the rows and columns of our double complex, we find isomorphisms:

$$\begin{aligned} H^n_d(\text{total complex } C^{(\cdot)}) &\cong H^n_{\delta_1}\left(\text{the complex } \operatorname{Ker}\delta_2 \text{ in } C^{\cdot,0}\right) \\ &\cong H^n_{\delta_2}\left(\text{the complex } \operatorname{Ker}\delta_1 \text{ in } C^{0,\cdot}\right). \end{aligned}$$

But

$$\begin{aligned} \operatorname{Ker}\left(\delta_2 \colon C^{n,0} \longrightarrow C^{n,1}\right) &\cong \prod_{\alpha_0,\dots,\alpha_n\in S} \mathcal{F}(U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}) = C^n(\mathcal{U},\mathcal{F}) \\ \operatorname{Ker}\left(\delta_1 \colon C^{0,n} \longrightarrow C^{1,n}\right) &\cong \prod_{\beta_0,\dots,\beta_n\in T} \mathcal{F}(V_{\beta_0}\cap\cdots\cap V_{\beta_n}) = C^n(\mathcal{V},\mathcal{F}), \end{aligned}$$

so in fact

$$\begin{aligned} H^n_d\left(\text{total complex } C^{(\cdot)}\right) &\cong H^n(\mathcal{U},\mathcal{F}) \\ &\cong H^n(\mathcal{V},\mathcal{F}). \end{aligned}$$

It remains to check:

(D) The above isomorphism is the refinement map, i.e., if $s(\alpha_0,\dots,\alpha_n)$ is an n-cocycle for $\mathcal{U}$, then $s \in C^{n,0}$ and $\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s \in C^{0,n}$ are cohomologous in the total complex. In fact, define $t \in C^{(n-1)}$ by setting its $(l, n-1-l)$-th component equal to:

$$t_l(\alpha_0,\dots,\alpha_l,\beta_0,\dots,\beta_{n-1-l}) = (-1)^l \operatorname{res} s(\alpha_0,\dots,\alpha_l,\sigma\beta_0,\dots,\sigma\beta_{n-1-l}).$$

Thus a straightforward calculation shows that $dt = (\operatorname{ref}^{\sigma}_{\mathcal{U},\mathcal{V}} s) - s$.

This completes the proof of Proposition 2.2. $\square$

Now return to the proof of Theorem 2.1 for quasi-coherent sheaves on schemes! The second step in its proof is the following explicit calculation:

PROPOSITION 2.5. *Let* $\operatorname{Spec} R$ *be an affine scheme,* $\mathcal{U} = \{\operatorname{Spec} R_{f_i}\}_{i\in I}$ *a finite distinguished affine covering and* $\widetilde{M}$ *a quasi-coherent sheaf on* X*. Then* $H^i(\mathcal{U},\widetilde{M}) = (0)$*, all* $i > 0$*.*

PROOF. Since $\widetilde{M}(\operatorname{Spec} R_f) \cong M_f$ and $\bigcap_{i\in I_0} \operatorname{Spec} R_{f_i} = \operatorname{Spec} R_{(\prod_{i\in I_0} f_i)}$, the complex of Čech cochains reduces to:

$$\prod_{i\in I} M_{f_i} \longrightarrow \prod_{i_0,i_1\in I} M_{(f_{i_0}\cdot f_{i_1})} \longrightarrow \prod_{i_0,i_1,i_2\in I} M_{(f_{i_0}\cdot f_{i_1}\cdot f_{i_2})} \longrightarrow \cdots.$$

Using the fact that the covering is finite, we can write a k-cochain:

$$m(i_0,\dots,i_k) = \frac{m_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^N}, \quad m_{i_0,\dots,i_k} \in M$$

with fixed denominator. Then

$$\begin{aligned} (\delta m)(i_0,\dots,i_{k+1}) &= \frac{m_{i_1,\dots,i_{k+1}}}{(f_{i_1}\cdots f_{i_k})^N} - \frac{m_{i_0,i_2,\dots,i_{k+1}}}{(f_{i_0}f_{i_2}\cdots f_{i_{k+1}})^N} + \cdots + (-1)^{k+1}\frac{m_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^N} \\ &= \frac{f_{i_0}^N m_{i_1,\dots,i_{k+1}} - f_{i_1}^N m_{i_0,i_2,\dots,i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^N m_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_{k+1}})^N}. \end{aligned}$$

If $\delta m = 0$, then this expression is 0 in $M_{(f_{i_0}\cdots f_{i_{k+1}})}$, hence

$$(f_{i_0}\cdots f_{i_{k+1}})^{N'}\left[f_{i_0}^N m_{i_1,\dots,i_{k+1}} - f_{i_1}^N m_{i_0,i_2,\dots,i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^N m_{i_0,\dots,i_k}\right] = 0$$

in M if N' is sufficiently large. But rewriting the original cochain m with N replaced by $N+N'$, we have

$$m(i_0,\dots,i_k) = \frac{m'_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^{N+N'}}, \quad m'_{i_0,\dots,i_k} = (f_{i_0}\cdots f_{i_k})^{N'} m_{i_0,\dots,i_k}$$

so that

$$(*)\quad f_{i_0}^{N+N'} m'_{i_1,\dots,i_{k+1}} - f_{i_1}^{N+N'} m'_{i_0,i_2,\dots,i_{k+1}} + \cdots + (-1)^{k+1} f_{i_{k+1}}^{N+N'} m'_{i_0,\dots,i_k} = 0 \quad \text{in } M.$$

Now since

$$\operatorname{Spec} R = \bigcup \operatorname{Spec} R_{f_i} = \bigcup \operatorname{Spec} R_{(f_i^{N+N'})},$$

it follows that $1 \in (\dots, f_i^{N+N'}, \dots)$, i.e., we can write

$$1 = \sum_{i\in I} g_i \cdot f_i^{N+N'}$$

for some $g_i \in R$. Now define a $(k-1)$-cochain n by the formula:

$$n(i_0,\dots,i_{k-1}) = \frac{n_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_{k-1}})^{N+N'}}$$

$$n_{i_0,\dots,i_{k-1}} = \sum_{l\in I} g_l \cdot m'_{l,i_0,\dots,i_{k=1}}.$$

Then $m = \delta n$! In fact

$$\begin{aligned}
(\delta n)(i_0,\dots,i_k) &= \sum_{j=0}^{k} (-1)^j \cdot \frac{n_{i_0},\dots,\widehat{i_j},\dots,i_k}{(f_{i_0}\cdots \widehat{f_{i_j}}\cdots f_{i_k})^{N+N'}} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{j=0}^{k} (-1)^j f_{i_j}^{N+N'} \cdot \sum_{l\in I} g_l m'_{l,i_0,\dots,\widehat{i_j},\dots,i_k} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l \sum_{j=0}^{k} (-1)^j f_{i_j}^{N+N'} m'_{l,i_0,\dots,\widehat{i_j},\dots,i_k} \\
&= \frac{1}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l f_i^{N+N'} m'_{i_0,\dots,i_k} \quad \text{(by (*))} \\
&= \frac{m'_{i_0,\dots,i_k}}{(f_{i_0}\cdots f_{i_k})^{N+N'}} \sum_{l\in I} g_l f_i^{N+N'} \\
&= m(i_0,\dots,i_k).
\end{aligned}$$

□

COROLLARY 2.6. *Let X be an affine scheme, $\mathcal{U}$ any affine covering of X and $\widetilde{M}$ a quasi-coherent sheaf on X. Then $H^i(\mathcal{U},\widetilde{M}) = (0)$, $i > 0$.*

PROOF. Since the distinguished affines form a basis for the topology of X, and X is quasi-compact, we can find a finite distinguished affine covering $\mathcal{V}$ of X refining $\mathcal{U}$. Consider the map

$$\operatorname{ref}_{\mathcal{U},\mathcal{V}}\colon H^i(\mathcal{U},\widetilde{M}) \longrightarrow H^i(\mathcal{V},\widetilde{M}).$$

By Proposition 2.5, $H^i(\mathcal{V},\widetilde{M}) = (0)$ all $i > 0$, *and* $H^i(\mathcal{V}|_{U_{S_0}}, \widetilde{M}|_{U_{S_0}}) = (0)$ for all $i > 0$ and for all finite intersections $U_{S_0} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ (since each $V_\beta \cap U_{S_0}$ is a distinguished affine in U_{S_0} too). Therefore by Proposition 2.2, $\operatorname{ref}_{\mathcal{U},\mathcal{V}}$ is an isomorphism, hence $H^i(\mathcal{U},\widetilde{M}) = (0)$ for all $i > 0$. □

Theorem 2.1 now follows immediately from Proposition 2.2 and Corollary 2.6, in view of the fact that since X is separated, each U_{S_0} as a finite intersection of affines, is also affine as are the open sets $V_\beta \cap U_{S_0}$ that cover it.

Theorem 2.1 implies:

Corollary 2.7. *For all schemes X, quasi-coherent $\mathcal{F}$ and affine covering $\mathcal{U}$, the natural map:*

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$$

is an isomorphism.

The "easy lemma of the double complex" (Lemma 2.4) has lots of other applications in homological algebra. We sketch one that we can use later on.

a) Let R be any commutative ring, let $M^{(1)}$, $M^{(2)}$ be R-modules, choose free resolutions $F_\bullet^{(1)} \to M^{(1)}$ and $F_\bullet^{(2)} \to M^{(2)}$, i.e., exact sequences

$$\longrightarrow F_n^{(1)} \longrightarrow F_{n-1}^{(1)} \longrightarrow \cdots \longrightarrow F_1^{(1)} \longrightarrow F_0^{(1)} \longrightarrow M^{(1)} \longrightarrow 0$$
$$\longrightarrow F_n^{(2)} \longrightarrow F_{n-1}^{(2)} \longrightarrow \cdots \longrightarrow F_1^{(2)} \longrightarrow F_0^{(2)} \longrightarrow M^{(2)} \longrightarrow 0$$

where all $F_j^{(i)}$ are free R-modules. Look at the double complex $C_{i,j} = F_i^{(1)} \otimes_R F_j^{(2)}$, $0 \leq i, j$ with boundary maps

$$d^{(1)}\colon C_{i,j} \longrightarrow C_{i-1,j}$$
$$d^{(2)}\colon C_{i,j} \longrightarrow C_{i,j-1}$$

induced by the d's in the two resolutions. Then Lemma 2.4 shows that

$$\begin{aligned} H_n(\text{total complex } C_{\bullet,\bullet}) &\cong H_n(\text{complex } F_\bullet^{(1)} \otimes_R M^{(2)}) \\ &\cong H_n(\text{complex } M^{(1)} \otimes_R F_\bullet^{(2)}). \end{aligned}$$

Note that the arrows here are reversed compared to the situation in the text. For complexes in which d decreases the index, we take homology H_n instead of cohomology H^n. It is not hard to check that the above R-modules are *independent* of the resolutions $F_\bullet^{(1)}$, $F_\bullet^{(2)}$. They are called $\operatorname{Tor}_n^R(M^{(1)}, M^{(2)})$. The construction could be globalized: if X is a scheme, $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ are quasi-coherent sheaves, then there are canonical quasi-coherent sheaves $\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})$ such that for all affine open $U \subset X$, if

$$U = \operatorname{Spec} R$$
$$\mathcal{F}^{(i)} = \widetilde{M^{(i)}},$$

then

$$\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F}^{(1)}, \mathcal{F}^{(2)})|_U = \operatorname{Tor}_n^R(M^{(1)}, M^{(2)})^{\sim}.$$

I want to conclude this section with the classical explanation of the "meaning" of $H^1(X, \mathcal{O}_X)$, via so-called "Cousin data". Let me digress to give a little history: in the 19th century Mittag-Leffler proved that for any discrete set of points $\alpha_i \in \mathbb{C}$ and any positive integer n_i, there is a meromorphic function $f(z)$ with poles of order n_i at α_i and no others. Cousin generalized this to meromorphic functions $f(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ in the following form: say $\{U_i\}$ is an open covering of $\mathbb{C}^n$ and f_i is a meromorphic function on U_i such that $f_i - f_j$ is holomorphic on $U_i \cap U_j$. Then there exists a meromorphic function f such that $f - f_i$ is holomorphic on U_i. We can easily pose an algebraic analog of this —

a) Let X be a reduced and irreducible scheme.
b) Let $\mathbf{R}(X)$ = function field of X.

c) Cousin data consists in an open covering $\{U_\alpha\}_{\alpha\in S}$ of X plus $f_\alpha \in \mathbf{R}(X)$ for each α such that

$$f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X), \quad \text{all } \alpha, \beta.$$

d) The Cousin problem for this data is to find $f \in \mathbf{R}(X)$ such that

$$f - f_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X), \quad \text{all } \alpha,$$

i.e., f and f_α have the same "polar part" in U_α.

For all Cousin data $\{f_\alpha\}$, let $g_{\alpha\beta} = f_\alpha - f_\beta \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$. Then $\{g_{\alpha\beta}\}$ is a 1-cocycle in $\mathcal{O}_X$ for the covering $\{U_\alpha\}$ and by refinement, it defines an element of $H^1(X, \mathcal{O}_X)$, which we call $\mathrm{ob}(\{f_\alpha\})$(= the "obstruction").

Proposition 2.8. $\mathrm{ob}(\{f_\alpha\}) = 0$ *iff the Cousin problem has a solution.*

Proof. If $\mathrm{ob}(\{f_\alpha\}) = 0$, then there is a finer covering $\{V_\alpha\}_{\alpha\in T}$ and $h_\alpha \in \Gamma(V_\alpha, \mathcal{O}_X)$ such that if $\sigma\colon T \to S$ is a refinement map, then

$$h_\alpha - h_\beta = \operatorname{res} g_{\sigma\alpha,\sigma\beta} = \operatorname{res}(f_{\sigma\alpha} - f_{\sigma\beta})$$

(equality here being in the ring $\Gamma(V_\alpha \cap V_\beta, \mathcal{O}_X)$). But then in $\mathbf{R}(X)$,

$$h_\alpha - f_{\sigma\alpha} = h_\beta - f_{\sigma\beta},$$

i.e., $f_{\sigma\alpha} - h_\alpha = F$ is independent of α. Then F has the same polar part as $f_{\sigma\alpha}$ in V_α. And for any $x \in U_\alpha$, take β so that $x \in V_\beta$ too; then since $f_\alpha - f_{\sigma\beta} \in \mathcal{O}_{x,X}$, it follows that $F - f_\alpha = (F - f_{\sigma\beta}) + (f_{\sigma\beta} - f_\alpha) \in \mathcal{O}_{x,X}$, i.e., F has the same polar part as f_α throughout U_α, so F solves the Cousin problem. Conversely, if such F exists, let $h_\alpha = f_\alpha - F$; then $h_\alpha - h_\beta = g_{\alpha\beta}$ and $h_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X)$, i.e., $\{g_{\alpha\beta}\} = \delta(\{h_\alpha\})$ is a 1-coboundary. $\square$

3. Higher direct images and Leray's spectral sequence

One of the main tools that is used over and over again in computing cohomology is the higher direct image sheaf and the Leray spectral sequence. Let $f\colon X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups on X. For all $i \geq 0$, consider the presheaf on Y:

a) $U \longmapsto H^i(f^{-1}(U), \mathcal{F}), \quad \forall U \subset Y$ open
b) if $U_1 \subset U_2$, then

$$\operatorname{res}\colon H^i(f^{-1}(U_2), \mathcal{F}) \longrightarrow H^i(f^{-1}(U_1), \mathcal{F})$$

is the canonical map.

Definition 3.1. $R^i f_*(\mathcal{F}) =$ the sheafification of this presheaf, i.e., the universal sheaf which receives homomorphisms:

$$H^i(f^{-1}(U), \mathcal{F}) \longrightarrow R^i f_* \mathcal{F}(U), \quad \text{all } U.$$

Proposition 3.2. *If X and Y are schemes, $f\colon X \to Y$ is quasi-compact and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $R^i f_*(\mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-module. Moreover, if U is affine or if $i = 0$, then*

$$H^i(f^{-1}(U), \mathcal{F}) \longrightarrow R^i f_*(\mathcal{F})(U)$$

is an isomorphism.

PROOF. In fact, by the sheaf axiom for $\mathcal{F}$, it follows immediately that the presheaf $U \mapsto H^0(f^{-1}(U), \mathcal{F}) = \mathcal{F}(f^{-1}(U))$ is a sheaf on Y. Therefore $H^0(f^{-1}(U), \mathcal{F}) \to R^0 f_* \mathcal{F}(U)$ is an isomorphism for all U. The rest of the proposition falls into the set-up of (I.5.9). As stated there, it suffices to verify that if U is affine, $R = \Gamma(U, \mathcal{O}_X)$ and $g \in R$, then we get an isomorphism:

$$H^i(f^{-1}(U), \mathcal{F}) \otimes_R R_g \xrightarrow{\approx} H^i(f^{-1}(U_g), \mathcal{F}).$$

But since f is quasi-compact, we may cover $f^{-1}(U)$ by a finite set of affines $\{V_1, \ldots, V_N\} = \mathcal{V}$. Then $f^{-1}(U_g)$ is covered by

$$\left\{(V_1)_{f^*(g)}, \ldots, (V_N)_{f^*(g)}\right\} = \mathcal{V}|_{f^{-1}(U_g)}$$

which is again an affine covering. Therefore

$$\begin{aligned} H^i(f^{-1}(U), \mathcal{F}) &= H^i(C^\cdot(\mathcal{V}, \mathcal{F})) \\ H^i(f^{-1}(U_g), \mathcal{F}) &= H^i(C^\cdot(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F})). \end{aligned}$$

The cochain complexes are:

$$\begin{aligned} C^i(\mathcal{V}, \mathcal{F}) &= \prod_{1 \le \alpha_0, \ldots, \alpha_i \le N} \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \\ C^i(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F}) &= \prod_{1 \le \alpha_0, \ldots, \alpha_i \le N} \mathcal{F}((V_{\alpha_0})_{f^*g} \cap \cdots \cap (V_{\alpha_i})_{f^*g}). \end{aligned}$$

Since if $S = \Gamma(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}, \mathcal{O}_X)$:

$$\begin{aligned} \mathcal{F}((V_{\alpha_0})_{f^*g} \cap \cdots \cap (V_{\alpha_i})_{f^*g}) &\cong \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \otimes_S S_{f^*g} \\ &\cong \mathcal{F}(V_{\alpha_0} \cap \cdots \cap V_{\alpha_i}) \otimes_R R_g, \end{aligned}$$

it follows that

$$C^i(\mathcal{V}|_{f^{-1}(U_g)}, \mathcal{F}) \cong C^i(\mathcal{V}, \mathcal{F}) \otimes_R R_g$$

(since $\otimes$ commutes with finite products). But now localizing commutes with kernels and cokernels, so for any complex $A^\cdot$ of R-modules, $H^i(A^\cdot) \otimes_R R_g \cong H^i(A^\cdot \otimes_R R_g)$. Thus

$$H^i(f^{-1}(U_g), \mathcal{F}) \cong H^i(f^{-1}(U), \mathcal{F}) \otimes_R R_g$$

as required. □

COROLLARY 3.3. *If $f\colon X \to Y$ is an affine morphism (cf. Proposition-Definition* I.7.3*) and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module, then*

$$R^i f_* \mathcal{F} = 0, \qquad \forall i > 0.$$

A natural question to ask now is whether the cohomology of $\mathcal{F}$ on X can be reconstructed by taking the cohomology on Y of the higher direct images $R^i f_* \mathcal{F}$. The answer is: almost. The relationship between them is a *spectral sequence*. These are the biggest monsters that occur in homological algebra and have a tendency to strike terror into the heart of all eager students. I want to try to debunk their reputation of being so difficult[4].

DEFINITION 3.4. A spectral sequence $E_2^{pq} \Longrightarrow E^n$ consists in two pieces of data[5]:

---

[4](Added in publication) Fancier notions of "derived categories and derived functors" have since become indispensable not only in algebraic geometry but also in analysis, mathematical physics, etc. Among accessible references are: Hartshorne [**55**], Kashiwara-Schapira [**65**], [**66**] and Gelfand-Manin [**40**].

[5]Sometimes one also has a spectral sequence that "begins" with an E_1^{pq}. Then the first differential is

$$d_1^{pq}\colon E_1^{pq} \longrightarrow E_1^{p+1,q}$$

and if you set $E_2^{pq} = (\operatorname{Ker} d_1^{p,q})/(\operatorname{Image} d_1^{p-1,q})$, you get a spectral sequence as above.

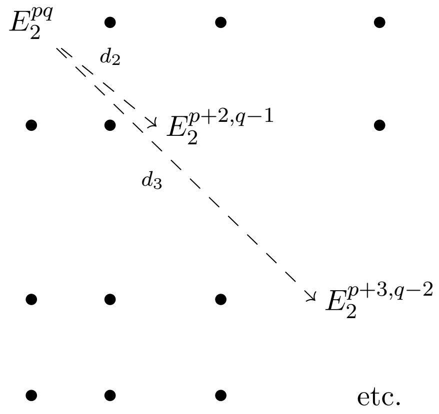

FIGURE VII.2

(A) A doubly infinite collection of abelian groups E_2^{pq}, ($p, q \in \mathbb{Z}$, $p, q \geq 0$) called the initial terms plus filtrations on each E_2^{pq}, which we write like this:

$$E_2^{pq} = Z_2^{pq} \supset Z_3^{pq} \supset Z_4^{pq} \supset \cdots \supset B_4^{pq} \supset B_3^{pq} \supset B_2^{pq} = (0),$$

also, let

$$Z_\infty^{pq} = \bigcap_r Z_r^{pq}$$
$$B_\infty^{pq} = \bigcup_r B_r^{pq},$$

plus a set of homomorphisms d_r^{pq} that allow us to determine inductively Z_{r+1}^{pq}, B_{r+1}^{pq} from the previous ones Z_r^{pq}, B_r^{pq}:

$$d_r^{pq} \colon Z_r^{pq} \longrightarrow E_2^{p+r,q-r+1}/B_r^{p+r,q-r+1}$$

(cf. Figure VII.2).

The d's should have the properties

i) $B_r^{pq} \subset \mathrm{Ker}(d_r^{pq})$, $Z_r^{p+r,q-r+1} \supset \mathrm{Image}(d_r^{pq})$ so that d_r^{pq} induces a map

$$Z_r^{pq}/B_r^{pq} \longrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}.$$

This sub-quotient of E_2^{pq} is called E_r^{pq}.

ii) $d^2 = 0$; more precisely, the composite

$$Z_r^{pq}/B_r^{pq} \longrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1} \longrightarrow Z_r^{p+2r,q-2r+2}/B_r^{p+2r,q-2r+2}$$

is 0.

iii) $Z_{r+1}^{pq} = \mathrm{Ker}(d_r^{pq})$; $B_{r+1}^{p+r,q-r+1} = \mathrm{Image}(d_r^{pq})$. This implies that E_{r+1}^{pq} is the cohomology of the complex formed by the E_r^{pq}'s and the d_r's!

(B) The so-called "abutment": a simply infinite collection of abelian groups E^n plus a filtration on each E^n whose successive quotients are precisely the groups $E_\infty^{p,n-p} = Z_\infty^{p,n-p}/B_\infty^{p,n-p}$:

$$E^n = \underbrace{F^0(E^n) \supset F^1}_{\cong E_\infty^{0,n}}\underbrace{(E^n) \supset \cdots}_{\cong E_\infty^{1,n-1}}\underbrace{\cdots\cdots \supset F^n}_{\cdots}\underbrace{(E^n) \supset F^{n+1}(E^n)}_{\cong E_\infty^{n,0}} = (0)$$

To illustrate what is going on here, look at the terms of lowest total degree. One sees easily that one gets the following exact sequences:

a) $E_2^{00} \cong E^0$.

b) $0 \longrightarrow E_2^{1,0} \longrightarrow E^1 \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow E^2.$

c) For all n, one gets "edge homomorphisms"

$$E_2^{n,0} \longrightarrow E_\infty^{n,0} \longrightarrow E^n$$

and

$$E^n \longrightarrow E_\infty^{0,n} \longrightarrow E_2^{0,n} :$$

i.e.,

$$\begin{tikzcd} & & E^n \arrow[dll] \\ E_2^{0,n} \arrow[dr, dashed, no head] & & \\ & E_2^{n,0} \arrow[uur] & \end{tikzcd}$$

THEOREM 3.5. [6] *Given any quasi-compact morphism $f : X \to Y$ and quasi-coherent sheaf $\mathcal{F}$ on X, there is a canonical spectral sequence, called* Leray's spectral sequence*, with initial terms*

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F})$$

and abutment $E^n = H^n(X, \mathcal{F})$.

PROOF. Choose open affine coverings $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ of Y and $\mathcal{V} = \{V_\beta\}_{\beta \in T}$ of X and consider the double complex introduced in §2 for the two coverings $f^{-1}(\mathcal{U})$ and $\mathcal{V}$ of X:

$$C^{pq} = \prod_{\alpha_0, \dots, \alpha_p \in S} \prod_{\beta_0, \dots, \beta_q \in T} \mathcal{F}(f^{-1}U_{\alpha_0} \cap \cdots \cap f^{-1}U_{\alpha_p} \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}).$$

Note that all the open sets here are affine because of Proposition II.4.5.

Now the q-th row of our double complex is the product over all $\beta_0, \dots, \beta_q \in T$ of the Čech complex $C^\bullet(f^{-1}(\mathcal{U}) \cap V_{\beta_0} \cap \cdots \cap V_{\beta_q}, \mathcal{F})$, i.e., the Čech complex for an affine open covering of an affine $V_{\beta_0} \cap \cdots \cap V_{\beta_q}$. Therefore all the rows are exact except at their first terms where their cohomology is $\prod_{\beta_0, \dots, \beta_q} \mathcal{F}(V_{\beta_0} \cap \cdots \cap V_{\beta_q})$, i.e., $C^q(\mathcal{V}, \mathcal{F})$. Hence by the easy lemma of the double complex (Lemma 2.4),

1)
$$\begin{aligned} H^n(\text{total complex}) &\cong H^n(C^\bullet(\mathcal{V}, \mathcal{F})) \\ &\cong H^n(X, \mathcal{F}). \end{aligned}$$

But on the other hand, the p-th column of our double complex is the product over all $\alpha_0, \dots, \alpha_p \in S$ of the Čech complex $C^\bullet(\mathcal{V} \cap f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}), \mathcal{F})$. The cohomology of this complex at the q-th spot is $H^q(f^{-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}), \mathcal{F})$ which is also the same as $R^q f_* \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. Therefore:

2) [vertical δ_2-cohomology of p-th column at (p,q)] $\cong \prod_{\alpha_0, \dots, \alpha_p \in S} R^q f_* \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$. But now the horizontal maps $\delta_1 : C^{p,q} \to C^{p+1,q}$ induce maps from the [δ_2-cohomology at (p,q)-th spot] $\to$ [δ_2-cohomology at $(p+1,q)$-th spot] and we see easily that

3) [q-th row of vertical cohomology groups] $\underset{\text{as complex}}{\cong}$ Čech complex $C^\bullet(\mathcal{U}, R^q f_* \mathcal{F})$. Therefore finally:

4) [horizontal δ_1-cohomology at (p,q) of vertical δ_2-cohomology group] $\cong H^p(Y, R^q f_* \mathcal{F})$!

Theorem 3.5 is now reduced to:

[6] Theorem 3.5 also holds for continuous maps of paracompact Hausdorff spaces and arbitrary sheaves $\mathcal{F}$, but we will not use this.

LEMMA 3.6 (The hard lemma of the double complex). *Let $(C^{p,q}, \delta_1, \delta_2)$ be any double complex. Make no assumption on the δ_2-cohomology, but consider instead its δ_1-cohomology:*

$$E_2^{p,q} = H^p_{\delta_1}(H^q_{\delta_2}(C^{\cdot,\cdot})).$$

Then there is a spectral sequence starting at $E_2^{p,q}$ and abutting at the cohomology of the total complex. Alternatively, one can "start" this spectral sequence at

$$E_1^{p,q} = H^q_{\delta_2}(C^{p,\cdot}) = (\textit{cohomology in vertical direction})$$

with d_1 being the maps induced by δ_1 on δ_2-cohomology[7]*. Also, since the rows and columns of a double complex play symmetric roles, one gets as a consequence a second spectral sequence with*

$$E_2^{p,q} = H^p_{\delta_2}(H^q_{\delta_1}(C^{\cdot,\cdot}))$$

or

$$E_1^{p,q} = H^q_{\delta_1}(C^{\cdot,p}) = (\textit{cohomology in horizontal direction}),$$

abutting also to the cohomology of the total complex.

A hard-nosed detailed proof of this is not very long but quite unreadable. I think the reader will find it easier if I sketch the idea of the proof far enough so that he/she can work out for himself/herself as many details as he/she wants. To begin with, we may describe E_2^{pq} rather more explicitly as:

$$E_2^{pq} = \frac{\{x \in C^{p,q} \mid \delta_2 x = 0 \text{ and } \delta_1 x = \delta_2 y, \text{ some } y \in C^{p+1,q-1}\}}{\delta_2(C^{p,q-1}) + \delta_1\{x \in C^{p-1,q} \mid \delta_2 x = 0\}}.$$

The idea is — how hard is it to "extend" the δ_2-cocycle x to a whole d-cocycle in the total complex: more precisely, to a set of elements

$$\begin{array}{ll} x \in C^{p,q} & \delta_2 x = 0 \\ y_1 \in C^{p+1,q-1} & \delta_2 y_1 = \delta_1 x \\ y_2 \in C^{p+2,q-2} & \delta_2 y_2 = \delta_1 y_1 \\ \text{etc.} & \text{etc.} \end{array}$$

so that $d(x \pm y_1 \pm y_2 \pm \cdots) = 0$ (the signs being mechanically chosen here taking into account that $d = \delta_1 + (-1)^p \delta_2$). See Figure VII.3.

Define Z_∞^{pq} to be the subgroup of E_2^{pq} for which such a sequence of y_i's exist; define Z_3^{pq} to be the set of x's such that such y_1 and y_2 exist; define Z_4^{pq} to be the set of x's such that such y_1, y_2 and y_3 exist; etc.

On the other hand, a δ_2-cocycle x may be a d-coboundary in various ways — let

$$B_3^{pq} = \text{image in } E_2^{pq} \text{ of } \left\{ x \in C^{p,q} \;\middle|\; \begin{array}{l} w_1 \in C^{p-1,q},\ w_2 \in C^{p-2,q-1} \\ \delta_1 w_1 = x,\ \delta_2 w_1 = \delta_1 w_2,\ \delta_2 w_2 = 0 \end{array} \right\}$$

$$B_4^{pq} = \text{image in } E_2^{pq} \text{ of } \left\{ x \in C^{p,q} \;\middle|\; \begin{array}{l} w_1, w_2 \text{ as above},\ w_3 \in C^{p-3,q-2} \\ \delta_1 w_1 = x,\ \delta_2 w_1 = \delta_1 w_2,\ \delta_2 w_2 = \delta_1 w_3,\ \delta_2 w_3 = 0 \end{array} \right\}$$

etc.

(cf. Figure VII.4)

[7] More precisely, to construct the spectral sequence, one doesn't need both gradings on $\bigoplus C^{p,q}$ and both differentials; it is enough to have one grading (the grading by total degree), one filtration ($F_k = \bigoplus_{p \geq k} C^{p,q}$) and the total differential: for details cf. MacLane [**77**, Chapter 11, §§3 and 6].

$$\begin{tikzcd}
0 \\
x \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & z_1 \\
& y_1 \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & z_2 \\
& & y_2 \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & {} \\
& & & \ddots & \ddots & & d_r x! \arrow[dd, dashed, bend right] \\
& & & & y_{r-1} \arrow[r, mapsto, "\delta_1"] & z_{r-1} \\
& & & & & y_{r-1} \arrow[u, mapsto, "\delta_2"] \arrow[r, mapsto, "\delta_1"] & ?
\end{tikzcd}$$

FIGURE VII.3

$$\begin{tikzcd}
0 \\
w_r \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & u_{r-1} \\
& w_{r-1} \arrow[u, mapsto, "\delta_2"'] & \ddots \\
& & \ddots & \ddots \\
& & & \ddots & u_2 \\
& & & & w_2 \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & u_1 \\
& & & & & w_1 \arrow[u, mapsto, "\delta_2"'] \arrow[r, mapsto, "\delta_1"'] & x
\end{tikzcd}$$

FIGURE VII.4

As for $d_r^{pq}\colon Z_r^{pq} \to E_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$, suppose $x \in C^{p,q}$ defines an element of Z_r^{pq}, i.e., $\exists y_1 \in C^{p+1,q-1}, \dots, y_{r-1} \in C^{p+r-1,q-r+1}$ such that $\delta_2 y_{i+1} = \delta_1 y_i$, $i < r-1$; $\delta_2 y_1 = \delta_1 x$. Define

$$d_r^{pq}(x) = \delta_1 y_{r-1}.$$

This is an element of $C^{p+r,q-r+1}$ killed by δ_1 and δ_2, hence it defines an element of $E_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}$. At this point there are quite a few points to verify — that d_r is well-defined so long as the image is taken modulo B_r and that d_r has the three properties of the definition. These are all mechanical and we omit them.

$$
\begin{array}{cccc|cccc}
0 & & & & & & & \\
 & 0 & & & F^k & & & \\
 & & \ddots & & & & & \\
 & & & 0 & & & & \\
 & & & & x_{k,n-k} & & & \\
 & & & & & \ddots & & \\
 & & & & & & x_{n-1,1} & \\
 & & & & & & & x_{n,0}
\end{array}
$$

FIGURE VII.5

Finally, define the filtration on the cohomology of the total complex:

$$F^k(E^n) = \text{those elements of } \left[\frac{\operatorname{Ker} d \text{ in } \sum_{p+q=n} C^{p,q}}{d\left(\sum_{p+q=n-1} C^{p,q}\right)}\right]$$

which can be represented by a d-cocycle

with components $x_{pq} \in C^{p,q}$, $x_{pq} = 0$ if $p < k$

(cf. Figure VII.5). The whole point of these definitions, which is now reasonable I hope, is the isomorphism:

$$F^p E^n / F^{p+1} E^n \cong Z_\infty^{p,n-p} / B_\infty^{p,n-p}.$$

The details are again omitted. □

An important remark is that the edge homomorphisms in the Leray spectral sequence:

a) $H^n(Y, f_*\mathcal{F}) \cong E_2^{n,0} \to E^n \cong H^n(X, \mathcal{F})$

b) $H^n(X, \mathcal{F}) \cong E^n \to E_2^{0,n} \cong H^0(Y, R^n f_*\mathcal{F})$

are just the maps induced by the functorial properties of cohomology (i.e., the set of maps $f_*\mathcal{F}(U) \to \mathcal{F}(f^{-1}(U))$ means that there is a map of sheaves "$f_*\mathcal{F} \to \mathcal{F}$ with respect to f" in an obvious sense and this gives (a); and the maps $H^n(X, \mathcal{F}) \to H^n(f^{-1}U, \mathcal{F}) \to R^n f_*\mathcal{F}(U)$ for all U give (b)). This comes out if $\mathcal{V}$ is a refinement of $f^{-1}(\mathcal{U})$ by the calculation used in the proof of Theorem 3.5.

PROPOSITION 3.7. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $f\colon X \to Y$ is an affine morphism (cf. Proposition-Definition I.7.3), then*

$$H^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(Y, f_*\mathcal{F}), \qquad \forall p.$$

PROOF. Leray's spectral sequence (Theorem 3.5) and Corollary 3.3. □

COROLLARY 3.8. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. If $i\colon X \to Y$ is a closed immersion of schemes (cf. Definition* 3.1*), then*

$$H^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(Y, i_*\mathcal{F}), \qquad \forall p.$$

REMARK. If X is identified with its image $i(X)$ in Y, $i_*\mathcal{F}$ is nothing but the quasi-coherent $\mathcal{O}_Y$-module obtained as the extension of the $\mathcal{O}_X$-module $\mathcal{F}$ by (0) outside X.

A second important application of the hard lemma (Lemma 3.6) is to hypercohomology and in particular to De Rham cohomology (cf. §VIII.3 below). Let $\mathcal{F}^\cdot$ be any complex of sheaves on a topological space X. Then if $\mathcal{U}$ is an open covering, $\mathbb{H}^n(\mathcal{U}, \mathcal{F}^\cdot)$ is by definition the cohomology of the total complex of the double complex $C^q(\mathcal{U}, \mathcal{F}^p)$, hence we get two spectral sequences abutting to it. The first is gotten by taking vertical cohomology (with respect to the superscript q):

$$E_1^{pq} = H^q(\mathcal{U}, \mathcal{F}^p) \Longrightarrow E^n = \mathbb{H}^n(\mathcal{U}, \mathcal{F}^\cdot)$$

(with d_1^{pq} the map induced on cohomology by $d\colon \mathcal{F}^p \to \mathcal{F}^{p+1}$).

Passing to the limit over finer coverings, we get:

$$E_1^{pq} = H^q(X, \mathcal{F}^p) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\cdot). \tag{3.9}$$

The second is gotten by taking horizontal cohomology (with respect to p) and *then* vertical cohomology. To express this conveniently, define *presheaves* $\mathcal{H}^p_{\mathrm{pre}}(\mathcal{F}^\cdot)$ by

$$\mathcal{H}^p_{\mathrm{pre}}(\mathcal{F}^\cdot)(U) = \frac{\mathrm{Ker}(\mathcal{F}^p(U) \to \mathcal{F}^{p+1}(U))}{\mathrm{Image}(\mathcal{F}^{p-1}(U) \to \mathcal{F}^p(U))}.$$

The sheafification of these presheaves are just:

$$\mathcal{H}^p(\mathcal{F}^\cdot) = \frac{\mathrm{Ker}(\mathcal{F}^p \to \mathcal{F}^{p+1})}{\mathrm{Image}(\mathcal{F}^{p-1} \to \mathcal{F}^p)}$$

but $\mathcal{H}^p_{\mathrm{pre}}$ will not generally be a sheaf already. The horizontal cohomology of the double complex $C^q(\mathcal{U}, \mathcal{F}^p)$ is just $C^q(\mathcal{U}, \mathcal{H}^p_{\mathrm{pre}})$ and the vertical cohomology of this is $H^q(\mathcal{U}, \mathcal{H}^p_{\mathrm{pre}})$, hence we get the second spectral sequence:

$$E_2^{pq} = H^p(\mathcal{U}, \mathcal{H}^q_{\mathrm{pre}}(\mathcal{F}^\cdot)) \Longrightarrow E^n = \mathbb{H}^n(\mathcal{U}, \mathcal{F}^\cdot).$$

Passing to the limit over $\mathcal{U}$, this gives:

$$E_2^{pq} = H^p(X, \mathcal{H}^q_{\mathrm{pre}}(\mathcal{F}^\cdot)) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\cdot). \tag{3.10}$$

In good cases, e.g., X paracompact Hausdorff (cf. §1), the cohomology of a presheaf is the cohomology of its sheafification, so we get finally:

$$E_2^{pq} = H^p(X, \mathcal{H}^q(\mathcal{F}^\cdot)) \Longrightarrow E^n = \mathbb{H}^n(X, \mathcal{F}^\cdot). \tag{3.11}$$

4. Computing cohomology (1): Push $\mathcal{F}$ into a huge acyclic sheaf

Although the apparatus of cohomology of quasi-coherent sheaves may seem at first acquaintace rather formidable, it should always be remembered that it is really only fancy linear algebra. In many specific cases, it is no great problem to compute it. To stress the flexibility of the tools available for computing cohomology, we present in a fugal style four calculations each using a different method.

A standard approach for cohomology is via a resolution of the type:

$$0 \longrightarrow \mathcal{F} \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots\cdots$$

where the I_k's are injective, or "flasque" or "mou" or at least are acyclic. (See Godement [**42**] or Swan [**110**].) Sheaves of this type tend to be huge monsters, but there has been quite a bit of work done on injectives in the category of sheaves of $\mathcal{O}_X$-modules on a noetherian X (see Hartshorne [**55**, p. 120]). We use the method as follows:

LEMMA 4.1. *If $U \subset X$ is affine and $i\colon U \to X$ the inclusion map, then for all quasi-coherent $\mathcal{F}$ on U, $i_*\mathcal{F}$ is acyclic, i.e., $H^p(X, i_*\mathcal{F}) = (0)$, all $p \geq 1$.*

PROOF. In fact, for $V \subset X$ affine, $i^{-1}(V) = U \cap V$ is affine, so the presheaf $V \mapsto H^p(i^{-1}V, \mathcal{F})$ is (0) on affines ($p \geq 1$). Thus $R^p i_*\mathcal{F} = (0)$ if $p \geq 1$. Then Leray's spectal sequence (Theorem 3.5) degenerates since

$$E_2^{pq} = H^p(X, R^q i_*\mathcal{F}) = (0), \quad q \geq 1.$$

Thus $E_2^{pq} \cong E_\infty^{pq} \cong E^{p+q}$, and the edge homomorphism

$$H^p(X, i_*\mathcal{F}) \longrightarrow H^p(U, \mathcal{F})$$

is an isomorphism. Since $H^p(U, \mathcal{F}) = (0)$, $p \geq 1$, the lemma is proven. $\square$

If $\mathcal{F}$ is quasi-coherent on X, and $i\colon U \to X$ is the inclusion of an affine, there is a canonical map:

$$\phi\colon \mathcal{F} \longrightarrow i_*(\mathcal{F}|_U)$$

via

$$\mathcal{F}(V) \xrightarrow{\text{res}} \mathcal{F}(U \cap V) \cong i_*(\mathcal{F}|_U)(V), \quad \forall \text{open } V,$$

which is an isomorphism on U. We can apply this to prove:

PROPOSITION 4.2. *Let X be a noetherian scheme and $\mathcal{F}$ a quasi-coherent sheaf on X. Let $n = \dim(\operatorname{Supp}\mathcal{F})$, i.e., n is the maximum length of chains:*

$$Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_r \subset \operatorname{Supp}(\mathcal{F}), \qquad Z_i \textit{ closed irreducible.}$$

Then $H^i(X, \mathcal{F}) = (0)$ if $i > n$.

PROOF. Use induction on n. If $n = 0$, then $\operatorname{Supp}\mathcal{F}$ is a finite set of closed points $\{x_1, \ldots, x_N\}$. For all i, let $U_i \subset X$ be an affine neighborhood of x_i such that $x_j \notin U_i$, all $j \neq i$; let $\{U_\beta\}_{\beta \in T}$ be an affine covering of $X \setminus \{x_1, \ldots, x_N\}$. Then $\{U_1, \ldots, U_N\} \cup \{U_\beta\}$ is an affine covering of X such that for any two distinct open sets U_α, $U_{\alpha'}$ in it, $U_\alpha \cap U_{\alpha'} \cap \operatorname{Supp}\mathcal{F} = \emptyset$. Thus $C^i(\mathcal{U}, \mathcal{F}) = (0)$, $i \geq 1$, and hence $H^i(X, \mathcal{F}) = (0)$, $i \geq 1$.

In general, decompose $\operatorname{Supp}\mathcal{F}$ into irreducible sets:

$$\operatorname{Supp}\mathcal{F} = S_1 \cup \cdots \cup S_N.$$

Let $U_i \subset X$ be an affine open set such that

$$\begin{aligned} &U_i \cap S_i \neq \emptyset \\ &U_i \cap S_j = \emptyset, \quad \text{all } j \neq i. \end{aligned}$$

Let $i_k\colon U_k \to X$ be the inclusion map, and let

$$\mathcal{F}_k = i_{k,*}(\mathcal{F}|_{U_k}).$$

As above we have a canonical map:

$$\mathcal{F} \xrightarrow{\phi} \bigoplus_{k=1}^N \mathcal{F}_k$$

given by:

$$\mathcal{F}(V) \xrightarrow{\text{res}} \bigoplus_{k=1}^N \mathcal{F}(U_k \cap V) \cong \left[\bigoplus_{k=1}^N i_{k,*}(\mathcal{F}|_{U_k})\right](V).$$

Concerning ϕ, we have the following facts:

a) If $i \neq j$, $U_i \cap U_j \cap \operatorname{Supp} \mathcal{F} = \emptyset$, hence $\mathcal{F}(U_i \cap U_j) = (0)$. Therefore if $V \subset U_{k_0}$,

$$\bigoplus_{k=1}^{N} \mathcal{F}(U_k \cap V) = \mathcal{F}(U_{k_0} \cap V) = \mathcal{F}(V).$$

Therefore ϕ is an *isomorphism* of sheaves on each of the open sets U_k.

b) If $V \cap S_k = \emptyset$, then $V \cap U_k \cap \operatorname{Supp} \mathcal{F} = \emptyset$ so $\mathcal{F}_k(V) = \mathcal{F}(U_k \cap V) = (0)$. Thus $\operatorname{Supp} \mathcal{F}_k \subset S_k$.

c) Each $\mathcal{F}_k$ is quasi-coherent by Proposition 3.2, hence $\mathcal{K}_1 = \operatorname{Ker} \phi$ and $\mathcal{K}_2 = \operatorname{Coker} \phi$ are quasi-coherent.

Putting all this together, if $i = 1, 2$

$$\begin{aligned} \operatorname{Supp} \mathcal{K}_i &\subset (S_1 \cup \cdots \cup S_N) \setminus (\text{open set where } \phi \text{ is an isomorphism}) \\ &\subset \bigcup_{k=1}^{N} (S_k \setminus S_k \cap U_k). \end{aligned}$$

Therefore $\dim \operatorname{Supp} \mathcal{K}_i < n$, and we can apply induction. If we set $\mathcal{K}_3 = \mathcal{F}/\mathcal{K}_1$, we get two short exact sequences:

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{K}_3 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K}_3 \longrightarrow \bigoplus_{k=1}^{N} \mathcal{F}_k \longrightarrow \mathcal{K}_2 \longrightarrow 0,$$

hence if $p > n$:

$$(*) \qquad \begin{array}{ccccc} H^p(X, \mathcal{K}_1) & \longrightarrow & H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{K}_3) \\ \| \text{ by induction} & & & & \\ (0) & & & & (0) \\ \| \text{ by induction} & & & & \| \text{ by Lemma 4.1} \\ H^{p-1}(X, \mathcal{K}_2) & \longrightarrow & H^p(X, \mathcal{K}_3) & \longrightarrow & \bigoplus_{k=1}^{N} H^p(X, \mathcal{F}_k). \end{array}$$

This proves that $H^p(X, \mathcal{F}) = (0)$ if $p > n$. $\square$

5. Computing cohomology (2): Directly via the Čech complex

We illustrate this approach by calculating $H^i(\mathbb{P}^n_R, \mathcal{O}(m))$ for any ring R. We need some more definitions first:

a) Let R be a ring, $f_1, \ldots, f_n \in R$. Let M be an R-module. Introduce formal symbols $\omega_1, \ldots, \omega_n$ such that

$$\omega_i \wedge \omega_j = -\omega_j \wedge \omega_i, \quad \omega_i \wedge \omega_i = 0.$$

Define an R-module:

$$K^p(f_1, \ldots, f_n; M) = \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} M \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

Define

$$d \colon K^p(f_1, \ldots, f_n; M) \longrightarrow K^{p+1}(f_1, \ldots, f_n; M)$$

by

$$dm = \left(\sum_{i=1}^{n} f_i \omega_i \right) \wedge m.$$

Note that $d^2 = 0$. This gives us the *Koszul complex* $K^\bullet(f_1, \ldots, f_n; M)$:

$$\begin{array}{ccccccc} 0 \longrightarrow K^0((f);M) & \longrightarrow & K^1((f);M) & \longrightarrow \cdots \longrightarrow & K^n((f);M) & \longrightarrow 0 \\ \| & & \| & & \| & \\ M & & \bigoplus_{i=1}^n M \cdot \omega_i & & M \cdot \omega_1 \wedge \cdots \wedge \omega_n & \end{array}$$

b) Now say R is a graded ring, $f_i \in R_{d_i}$ is homogeneous and M is a graded module. Then we assign ω_i the degree $-d_i$, so that $\sum f_i\omega_i$ is homogeneous of degree 0. Then $K^\bullet(f_1, \ldots, f_n; M)$ is a complex of graded modules with degree preserving maps d. We let $K^\bullet(f_1, \ldots, f_n; M)^0$ denote the degree 0 subcomplex, i.e.,

$$K^p(f_1, \ldots, f_n; M)^0 = \bigoplus_{i_1 < \cdots < i_p} M_{d_{i_1} + \cdots + d_{i_p}} \cdot \omega_{i_1} \wedge \cdots \wedge \omega_{i_p}.$$

c) Next compare the Koszul complexes $K^\bullet(f_1^\nu, \ldots, f_n^\nu; M)$ for various $\nu \geq 1$. If we write

$$K^p(f_1^\nu, \ldots, f_n^\nu; M) = \bigoplus_{i_1 < \cdots < i_p} M \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}$$

and set

$$\omega_i^{(\nu)} = f_i^{(\nu')} \cdot \omega_i^{(\nu+\nu')}$$

then we get a natural homomorphism

$$K^p(f_1^\nu, \ldots, f_n^\nu; M) \longrightarrow K^p(f_1^{\nu+\nu'}, \ldots, f_n^{\nu+\nu'}; M)$$

which commutes with d.

The point of all this is:

Proposition 5.1. *If R is a graded ring, $f_i \in R_{d_i}$, $1 \leq i \leq n$, M a graded R-module, $U_i = (\operatorname{Proj} R)_{f_i}$, $\mathcal{U} = \{U_1, \ldots, U_n\}$, then there is a natural isomorphism:*

$$C_{\text{alt}}^{p-1}(\mathcal{U}, \widetilde{M}) \cong \varinjlim_{\nu} K^p(f_1^\nu, \ldots, f_n^\nu; M)^0, \quad p \geq 1,$$

under which the Čech coboundary δ and the Koszul d correspond.

Proof. We have

$$\begin{aligned} C_{\text{alt}}^{p-1}(\mathcal{U}, \widetilde{M}) &= \bigoplus_{i_1 < \ldots < i_p} \widetilde{M}(U_{i_1} \cap \cdots \cap U_{i_p}) \\ &= \bigoplus_{i_1 < \cdots < i_p} \left(M_{f_{i_1} \cdots f_{i_p}}\right)^0 \end{aligned}$$

and

$$\begin{aligned} \varinjlim_{\nu} K^p((f^\nu); M)^0 &= \bigoplus_{i_1 < \ldots < i_p} \varinjlim_{\nu} \left[M \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}\right]^0 \\ &= \bigoplus_{i_1 < \cdots < i_p} \varinjlim_{\nu} M_{\nu(d_{i_1} + \cdots + d_{i_p})} \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)}. \end{aligned}$$

Define a homomorphism:

$$M_{\nu(d_{i_1} + \cdots + d_{i_p})} \cdot \omega_{i_1}^{(\nu)} \wedge \cdots \wedge \omega_{i_p}^{(\nu)} \longrightarrow (M_{f_{i_1} \cdots f_{i_p}})^0$$

by taking $\omega_j^{(\nu)}$ to $1/f_j^\nu$. This clearly commutes with the limit operation in ν and since for any ring S, any S-module N and $g \in S$, $N_g \cong$ direct limit of the system

$$N \xrightarrow{\text{multiplication by } g} N \xrightarrow{\text{multiplication by } g} N \xrightarrow{\text{multiplication by } g} \cdots,$$

it follows that

$$\varinjlim_{\nu} M_{\nu(d_{i_1}+\cdots d_{i_p})}\omega_{i_1}^{(\nu)}\wedge\cdots\wedge\omega_{i_p}^{(\nu)} \cong (M_{f_{i_1}\cdots f_{i_p}})^0.$$

We leave it to the reader to check that δ and d correspond. □

The complex $\{K^p\}$ goes down to $p=0$ while $\{C^{p-1}\}$ only goes down to $p=1$. We can extend $\{C^{p-1}\}$ one further step so that it matches up with $\{K^p\}$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \xrightarrow{\varepsilon} & C^0_{\text{alt}}(\mathcal{U},\widetilde{M}) & \longrightarrow & C^1_{\text{alt}}(\mathcal{U},\widetilde{M}) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \varinjlim_\nu (K^0)^0 & \longrightarrow & \varinjlim_\nu (K^1)^0 & \longrightarrow & \varinjlim_\nu (K^2)^0 & \longrightarrow & \cdots \end{array}$$

where ε is the composite of the canonical maps:

$$M_0 \longrightarrow \Gamma(\operatorname{Proj} R, \widetilde{M}) \longrightarrow C^0_{\text{alt}}(\mathcal{U},\widetilde{M}).$$

What we need next is a criterion for a Koszul complex to be exact:

Proposition 5.2 (Koszul). *Let R be a ring, $f_1,\ldots,f_n\in R$ and M an R-module. If f_s is a non-zero-divisor in $M/(f_1,\ldots,f_{s-1})\cdot M$ for $1\le s\le t$, then the complex $K^{\bullet}(f_1,\ldots,f_n;M)$ is exact at $K^s(f_1,\ldots,f_n;M)$ for $0\le s\le t-1$.*

Proof. To see how simple this is, it's better to take the first non-trivial case and check it, rather than getting confused in a general inductive proof. Take $t=3$ and check that

$$\bigoplus_i M\omega_i \xrightarrow{d} \bigoplus_{i_1<i_2} M\omega_{i_1}\wedge\omega_{i_2} \xrightarrow{d} \bigoplus_{i_1<i_2<i_3} M\omega_{i_1}\wedge\omega_{i_2}\wedge\omega_{i_3}$$

is exact. Write an element η of the middle module as

$$\eta = m\omega_1\wedge\omega_2 + \omega_1\wedge\left(\sum_{i=3}^n n_i\omega_i\right) + \sum_{2\le i<j} p_{ij}\omega_i\wedge\omega_j.$$

Assume $d\eta=0$. Looking at the coefficient of $\omega_1\wedge\omega_2\wedge\omega_3$, it follows

$$f_3m = f_2n_3 - f_1p_{23} \in (f_1,f_2)M.$$

Therefore by hypothesis $m = f_1q_1+f_2q_2$, $q_i\in M$. Replace η by $\eta - d(q_1\omega_2 - q_2\omega_1)$ and the coefficient m becomes 0. Assuming we have any η with $m=0$, look at the coefficient of $\omega_1\wedge\omega_2\wedge\omega_i$, $i\ge 3$. It follows

$$f_2n_i = f_1p_{23} \in f_1M.$$

Therefore $n_i = f_1q_i$, $q_i\in M$. Replace η by $\eta - d(\sum_{i=3}^n q_i\omega_i)$ and now all the coefficients m, n_i are 0. Assuming we have any η with $m=n_i=0$, look at the coefficient of $\omega_1\wedge\omega_i\wedge\omega_j$, $2\le i<j$. It follows that

$$f_1p_{ij}=0$$

whence by hypothesis $p_{ij}=0$, hence $\eta=0$. This idea works for any t. □

Combining the two propositions, we get:

Proposition 5.3. *Let R be a graded ring generated by homogeneous elements $f_i\in R_{d_i}$, $1\le i\le n$. Let M be a graded R-module. Fix an integer t and assume[8] that, for every ν, and every s, $1\le s\le t$, f_s^ν is a non-zero-divisor in $M/(f_1^\nu,\ldots,f_{s-1}^\nu)\cdot M$. Then*

a) *If $t\ge 1$, $M_d\to\Gamma(\operatorname{Proj} R,\widetilde{M}(d))$ is injective for all d.*
b) *If $t\ge 2$, $M_d\to\Gamma(\operatorname{Proj} R,\widetilde{M}(d))$ is an isomorphism for all d.*

[8] A closer analysis shows that if this condition holds for $\nu=1$, it automatically holds for larger ν.

c) *If* $t \geq 3$, $H^i(\operatorname{Proj} R, \widetilde{M}(d)) = (0)$, $1 \leq i \leq t-2$, *for all* d.

This follows by combining Propositions 5.1 and 5.2, taking note that we must augment the Čech complex $C^\bullet(\mathcal{U}, \widetilde{M}(d))$ by "$0 \to M_d \to$" to get the $\varinjlim$ of Koszul complexes (and also using the fact that a direct limit of exact sequences is exact).

COROLLARY 5.4. *Let S be a ring. Then*

a)
$$\begin{bmatrix} S\text{-module of homogeneous polynomials} \\ f(X_0, \dots, X_l) \text{ of degree } d \end{bmatrix} \longrightarrow \Gamma(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d))$$
is an isomorphism for all $d \in \mathbb{Z}$,

b) $H^i(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d)) = (0)$, *for all* $d \in \mathbb{Z}$, $1 \leq i \leq l-1$.

PROOF. Apply Proposition 5.3 to $R = S[X_0, \dots, X_l]$, $n = l+1$, $f_i = X_{i-1}$, $1 \leq i \leq l+1$ and $M = R$. Then in fact multiplication by X_j^ν is injective in the ring of truncated polynomials:
$$R/(X_0^\nu, \dots, X_{j-1}^\nu) \cdot R$$
so Proposition 5.3 applies with $t = n$. □

On the other hand, for any quasi-coherent $\mathcal{F}$ on $\mathbb{P}^l_S$, using the affine covering $U_i = (\mathbb{P}^l_S)_{X_i}$, $0 \leq i \leq l$, we get non-zero alternating cochains $C^i_{\text{alt}}(\mathcal{U}, \mathcal{F})$ only for $0 \leq i \leq l$. Therefore:

(5.5) $$H^i(\mathbb{P}^l_S, \mathcal{F}) = (0), \quad i > l, \text{ all quasi-coherent } \mathcal{F}.$$

If we look more closely, we can describe the groups $H^l(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d))$ too. Look first at the general situation R, $(f_1, \dots, f_n)$, M:
$$\begin{aligned} H^{n-1}(\mathcal{U}, \widetilde{M}) &= H^{n-1}(C^\bullet_{\text{alt}}(\mathcal{U}, \widetilde{M})) \\ &= C^{n-1}_{\text{alt}}(\mathcal{U}, \widetilde{M})/\delta(C^{n-2}_{\text{alt}}) \\ &= \widetilde{M}(U_1 \cap \dots \cap U_n) \Big/ \sum_{i=1}^n \operatorname{res} \widetilde{M}(U_1 \cap \cdots \widehat{U}_i, \dots, \cap U_n) \\ &= \Big(M_{(\prod f_j)}\Big)^0 \Big/ \sum_{i=1}^n \Big(M_{(\prod_{j \neq i} f_j)}\Big)^0 . \end{aligned}$$
Thus in the special case:

(5.6) $$\begin{aligned} H^l(\mathbb{P}^l_S, \mathcal{O}_{\mathbb{P}^l}(d)) &\cong \text{elements of degree } d \text{ in the } S[X_0, \dots, X_l]\text{-module} \\ & \qquad S[X_0, \dots, X_l]_{(\prod X_j)} \Big/ \sum_{i=0}^l S[X_0, \dots, X_l]_{(\prod_{j \neq i} X_j)} \\ &\cong S\text{-module of rational functions} \\ & \qquad \sum_{\substack{\alpha_i \leq -1 \\ \sum \alpha_i = d}} c_{\alpha_0, \dots, \alpha_l} X_0^{\alpha_0} \cdots X_l^{\alpha_l}. \end{aligned}$$

In particular $H^l(\mathcal{O}_{\mathbb{P}^l}(d)) = (0)$ if $d > -l-1$. It is natural to ask to what extent this is a canonical description of H^l — for instance, if you change coordinates, how do you change the description of an element of H^l by a rational function. The theory of this goes back to Macaulay and his "inverse systems", cf. Hartshorne [**58**, Chapter III].

Koszul complexes have many applications to the local theory too. For instance in Chapter V, we presented smooth morphisms locally as:

$$\begin{array}{c} X = \operatorname{Spec} R[X_1, \ldots, X_{n+r}]/(f_1, \ldots, f_r) \\ \Big\downarrow{\scriptstyle f} \\ Y = \operatorname{Spec} R \end{array}$$

and in Proposition V.3.19, we described the syzygies among the equations f_i locally. We can strengthen Proposition V.3.19 as follows: let $x \in X$, $y = f(x)$ so that

$$\mathcal{O}_{x,X} = \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}/(f_1, \ldots, f_r)$$

for some prime ideal $\mathfrak{p}$. Then I claim:

$$K^{\bullet}((f), \mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}) \longrightarrow \mathcal{O}_{x,X} \longrightarrow 0$$

is a resolution of $\mathcal{O}_{x,X}$ as module over $\mathcal{O}_{y,Y}[X_1, \ldots, X_{n+r}]_{\mathfrak{p}}$. This follows from the general fact:

PROPOSITION 5.7. *Let R be a regular local ring, M its maximal ideal and let $f_1, \ldots, f_r \in M$ be independent in M/M^2. Then*

$$0 \longrightarrow K^0((f), R) \longrightarrow \cdots \longrightarrow K^r((f), R) \longrightarrow R/(f_1, \ldots, f_r) \longrightarrow 0$$

is exact.

PROOF. Use Proposition 5.2. □

Proposition 5.7 may also be applied to prove that if R is regular, $f_1, \ldots, f_n \in M$ are independent in M/M^2, then:

$$\operatorname{Tor}_i^R(R/(f_1, \ldots, f_k), R/(f_{k+1}, \ldots, f_n)) = (0), \quad i > 0.$$

(cf. discussion of Serre's theory of intersection multiplicity, §V.1.)

6. Computing cohomology (3): Generate $\mathcal{F}$ by "known" sheaves

There are usually no projective objects in categories of sheaves, but it is nontheless quite useful to examine resolutions of the type:

$$\cdots \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where, for instance, the $\mathcal{E}_i$ are locally free sheaves of $\mathcal{O}_X$-modules (on *affine* schemes, such $\mathcal{E}_i$ are projective in the category of quasi-coherent sheaves).

Let S be a noetherian ring. We proved in Theorem III.4.3 due to Serre that for every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^l_S$ there is an integer n_0 such that $\mathcal{F}(n_0)$ is generated by global sections. This means that for some m_0, equivalently,

a) there is a surjection

$$\mathcal{O}_{\mathbb{P}^l_S}^{m_0} \longrightarrow \mathcal{F}(n_0) \longrightarrow 0$$

or

b) there is a surjection

$$\mathcal{O}_{\mathbb{P}^l_S}(-n_0)^{m_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Iterating, we get a resolution of $\mathcal{F}$ by "known" sheaves:

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_1)^{m_1} \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_0)^{m_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

We are now in a position to prove Serre's Main Theorem in his classic paper [**99**]:

THEOREM 6.1 (Fundamental theorem of F.A.C.). *Let S be a noetherian ring, and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^l_S$. Then*

1) *$H^i(\mathbb{P}^l_S, \mathcal{F}(n))$ is a finitely generated S-module for all $i \geq 0$, $n \in \mathbb{Z}$.*
2) *$\exists n_0$ such that $H^i(\mathbb{P}^l_S, \mathcal{F}(n)) = (0)$ if $i \geq 1$, $n \geq n_0$.*
3) *Every $\mathcal{F}$ is of the form $\widetilde{M}$ for some finitely generated graded $S[X_0, \ldots, X_l]$-module M; and if $\mathcal{F} = \widetilde{M}$ where M is finitely generated, then $\exists n_1$ such that $M_n \to H^0(\mathbb{P}^l_S, \mathcal{F})$ is an isomorphism if $n \geq n_1$.*

PROOF. We prove (1) and (2) by descending induction on i. If $i > l$, then as we have seen $H^i(\mathcal{F}(n)) = (0)$, all n (cf. Proposition 4.2). Suppose we know (1) and (2) for all $\mathcal{F}$ and $i > i_0 \geq 1$. Given $\mathcal{F}$, put it in an exact sequence as before:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(-n_1)^{n_2} \longrightarrow \mathcal{F} \longrightarrow 0.$$

For every $n \in \mathbb{Z}$, this gives us:

$$0 \longrightarrow \mathcal{G}(n) \longrightarrow \mathcal{O}_{\mathbb{P}^l_S}(n-n_1)^{n_2} \longrightarrow \mathcal{F}(n) \longrightarrow 0,$$

hence

$$H^{i_0}(\mathcal{O}_{\mathbb{P}^l_S}(n-n_1))^{n_2} \longrightarrow H^{i_0}(\mathcal{F}(n)) \longrightarrow H^{i_0+1}(\mathcal{G}(n)).$$

By induction $H^{i_0+1}(\mathcal{G}(n))$ is finitely generated for all n and (0) for $n \gg 0$ and by §5, $H^{i_0}(\mathcal{O}_{\mathbb{P}^l_S}(n-n_1))$ is finitely generated for all n and (0) for $n \gg 0$: therefore the same holds for $\mathcal{F}(n)$.

The first half of (3) has been proven in Proposition III.4.4. Suppose $\mathcal{F} = \widetilde{M}$. Let $R = S[X_0, \ldots, X_l]$ and let

$$\bigoplus_\beta R(-n_\beta) \longrightarrow \bigoplus_\alpha R(-m_\alpha) \longrightarrow M \longrightarrow 0$$

be a presentation of M by twists of the free rank one module R. Taking $\widetilde{\ }$, this gives a presentation of $\mathcal{F}$:

$$\bigoplus_\beta \mathcal{O}_{\mathbb{P}^l_S}(-n_\beta) \longrightarrow \bigoplus_\alpha \mathcal{O}_{\mathbb{P}^l_S}(-m_\alpha) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{6.2}$$

Twisting by n and taking sections, we get a diagram:

$$\begin{tikzcd} \bigoplus_\beta R_{n-n_\beta} \arrow[r] \arrow[d, "\approx"] & R_{n-m_\alpha} \arrow[r] \arrow[d, "\approx"] & M_n \arrow[r] \arrow[d] & 0 \\ \bigoplus_\beta \Gamma(\mathcal{O}_{\mathbb{P}^l_S}(n-n_\beta)) \arrow[r] & \bigoplus_\alpha \Gamma(\mathcal{O}_{\mathbb{P}^l_S}(n-m_\alpha)) \arrow[r] & H^0(\mathcal{F}(n)) \arrow[r] & 0 \end{tikzcd} \tag{6.3}$$

with top row exact, but the bottom row need not be so. But break up (6.2) into short exact sequences

$$0 \longrightarrow \mathcal{G} \longrightarrow \bigoplus_\alpha \mathcal{O}_{\mathbb{P}^l_S}(-m_\alpha) \longrightarrow \mathcal{F} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{H} \longrightarrow \bigoplus_\beta \mathcal{O}_{\mathbb{P}^l_S}(-n_\beta) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Choose n_1 so that

$$H^1(\mathcal{G}(n)) = H^1(\mathcal{H}(n)) = (0), \quad n \geq n_1.$$

Then if $n \geq n_1$

$$0 \longrightarrow H^0(\mathcal{G}(n)) \longrightarrow \bigoplus_\alpha H^0(\mathcal{O}_{\mathbb{P}^l_S}(n-m_\alpha)) \longrightarrow H^0(\mathcal{F}(n)) \longrightarrow 0$$

$$0 \longrightarrow H^0(\mathcal{H}(n)) \longrightarrow \bigoplus_\alpha H^0(\mathcal{O}_{\mathbb{P}^l_S}(n-n_\beta)) \longrightarrow H^0(\mathcal{G}(n)) \longrightarrow 0$$

are exact, hence so is the bottom row of (6.3). This proves (3). $\square$

COROLLARY 6.4. *Let $f\colon X \to Y$ be a projective morphism (cf. Definition* II.5.8*) with Y a noetherian scheme. Let $\mathcal{L}$ be a relatively ample invertible sheaf on X. Then for all coherent $\mathcal{F}$ on X:*

1) *$R^i f_*(\mathcal{F})$ is coherent on Y.*
2) *$\exists n_0$ such that $R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) = (0)$ if $i \geq 1$, $n \geq n_0$.*
3) *$\exists n_1$ such that all the natural map*

$$f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

is surjective if $n \geq n_1$.

PROOF. Since Y can be covered by a *finite* set of affines, to prove all of these it suffices to prove them over some fixed affine $U = \operatorname{Spec} R \subset Y$. Then choose $n \geq 1$ and $(s_0, \ldots, s_k) \in \Gamma(f^{-1}(U), \mathcal{L}^{\otimes n})$ defining a closed immersion $i\colon f^{-1}(U) \hookrightarrow \mathbb{P}^k_R$. Let $X' \subset \mathbb{P}^k_R$ be the image of i, and let $\mathcal{F}'$, $\mathcal{L}'$ be coherent sheaves on $\mathbb{P}^k_R$, (0) outside X' and isomorphic on X' to $\mathcal{F}|_{f^{-1}(U)}$ and $\mathcal{L}|_{f^{-1}(U)}$. By construction $\mathcal{O}_{X'}(1) \cong (\mathcal{L}')^{\otimes n}$. Then applying Serre's theorem (Theorem 6.1):

1) $R^i f_*(\mathcal{F})|_U \cong (H^i(X', \mathcal{F}'))^{\sim}$ is coherent.
2) For any fixed m,

$$\begin{aligned} R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes(m+\nu n)})|_U &\cong (H^i(X', \mathcal{F}' \otimes (\mathcal{L}')^{\otimes(m+\nu n)}))^{\sim} \\ &\cong (H^i(X', (\mathcal{F}' \otimes (\mathcal{L}')^{\otimes m})(\nu)))^{\sim} \\ &= (0), \qquad \text{if } \nu \geq \nu_0. \end{aligned}$$

Apply this for $m = 0, 1, \ldots, n-1$ to get (2) of Corollary 6.4.

3) For any fixed m,

$$\begin{aligned} f^* f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes(m+\nu n)})|_{f^{-1}(U)} &\cong H^0(X', \mathcal{F}' \otimes (\mathcal{L}')^{\otimes(m+\nu n)}) \otimes_R \mathcal{O}_{X'} \\ &\cong H^0(X', \mathcal{F}' \otimes (\mathcal{L}')^{\otimes m}(\nu)) \otimes_R \mathcal{O}_{X'} \end{aligned}$$

and this maps *onto* $\mathcal{F}' \otimes (\mathcal{L}')^{\otimes m}$ if $\nu \geq \nu_1$.

Apply this for $m = 0, 1, \ldots, n-1$ to get (3) of Corollary 6.4. □

Combining this with Chow's lemma (Theorem II.6.3) and the Leray spectral sequence (Theorem 3.5), we get:

THEOREM 6.5 (Grothendieck's coherency theorem). *Let $f\colon X \to Y$ be a proper morphism with Y a noetherian scheme. If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then $R^i f_*(\mathcal{F})$ is a coherent $\mathcal{O}_Y$-module for all i.*

PROOF. The result being local on Y, we need to prove that if $Y = \operatorname{Spec} S$, then $H^i(X, \mathcal{F})$ is a finitely generated S-module. Since X is also a noetherian scheme, its closed subsets satisfy the descending chain condition and we may make a "noetherian induction", i.e., assume the theorem holds for all coherent $\mathcal{G}$ with $\operatorname{Supp} \mathcal{G} \subsetneqq \operatorname{Supp} \mathcal{F}$. Also, if $\mathcal{I} \subset \mathcal{O}_X$ is the ideal of functions f such that multiplication by f is 0 in $\mathcal{F}$, we may replace X by the closed subscheme X', $\mathcal{O}_{X'} = \mathcal{O}_X/\mathcal{I}$. This has the effect that $\operatorname{Supp} \mathcal{F} = X$. Now apply Chow's lemma to construct

$$\begin{array}{ccc} & & X' \\ & \overset{\pi}{\swarrow} & \\ X & & \\ {\scriptstyle f}\big\downarrow & & \\ Y & & \end{array} \qquad \text{with } \pi \text{ and } f \circ \pi \text{ projective}$$

where $\operatorname{res}\pi\colon \pi^{-1}(U_0) \to U_0$ is an isomorphism for an open dense $U_0 \subset X$. Now consider the canonical map of sheaves $\alpha\colon \mathcal{F} \to \pi_*(\pi^*\mathcal{F})$ defined by the collection of maps:

$$\alpha(U)\colon \mathcal{F}(U) \longrightarrow \pi^*\mathcal{F}(\pi^{-1}U) = \pi_*(\pi^*)(U).$$

$\mathcal{F}$ coherent implies $\pi^*\mathcal{F}$ coherent and since π is projective, $\pi_*(\pi^*\mathcal{F})$ is coherent by Corollary 6.4. Look at the kernel, cokernel, and image:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F} \arrow[rr, "\alpha"] \arrow[dr] & & \pi_*(\pi^*\mathcal{F}) \arrow[r] & \mathcal{K}_2 \arrow[r] & 0. \\
 & & & \mathcal{F}/\mathcal{K}_1 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Since α is an isomorphism on U_0, $\operatorname{Supp}\mathcal{K}_i \subset X \setminus U_0 \subsetneqq X$. Thus $H^j(\mathcal{K}_i)$ are finitely generated S-modules by induction. But now using the long exact sequences:

$$\begin{tikzcd}
H^{i-1}(\mathcal{K}_2) \arrow[r] & H^i(\mathcal{F}/\mathcal{K}_1) \arrow[r] & H^i(\pi_*\pi^*\mathcal{F}) \\
\text{finitely generated} \arrow[u, dashed] \arrow[d, dashed] & & \\
H^i(\mathcal{K}_1) \arrow[r] & H^i(\mathcal{F}) \arrow[r] & H^i(\mathcal{F}/\mathcal{K}_1)
\end{tikzcd}$$

it follows readily that if $H^i(\pi_*\pi^*\mathcal{F})$ is finitely generated, so is $H^i(\mathcal{F})$. But now consider the Leray spectral sequence:

$$H^p(R^q\pi_*(\pi^*\mathcal{F})) = E_2^{pq} \Longrightarrow E^n = \underbrace{H^n(\pi^*\mathcal{F})}_{\substack{\text{finitely generated } S\text{-module} \\ \text{because } X' \text{ is projective} \\ \text{over } \operatorname{Spec} S.}}$$

If $q \geq 1$, then $R^q\pi_*(\pi^*\mathcal{F})|_{U_0} = (0)$; and since π is projective, $R^q\pi_*(\pi^*\mathcal{F})$ is coherent by Corollary 6.4. Therefore by noetherian induction, $H^p(R^q\pi_*(\pi^*\mathcal{F}))$ is finitely generated if $q \geq 1$. In other words, we have a spectral sequence of S-modules with E^n (all n) and E_2^{pq} ($q \geq 1$) finitely generated. It is a simple lemma that in such a case E_2^{p0} must be finitely generated too. $\square$

7. Computing cohomology (4): Push $\mathcal{F}$ into a coherent acyclic one

This is a variant on Method (1) taking advantage of what we have learned already — that at least on $\mathbb{P}^l_S$ there are plenty of coherent acyclic sheaves obtained by twists. It is the closest in spirit to the original Italian methods out of which cohomology grew. For simplicity we work only on $\mathbb{P}^n_k$ (and its closed subschemes) for k an infinite field for the rest of §7.

Let $\mathcal{F}$ be coherent on $\mathbb{P}^n_k$. Then if $F(X_0, \ldots, X_n)$ is a homogeneous polynomial of degree d, multiplication by F defines a homomorphism:

$$\mathcal{F} \xrightarrow{F} \mathcal{F}(d).$$

If d is sufficiently large, $H^i(\mathbb{P}^n_k, \mathcal{F}(d)) = (0)$, $i > 0$, and the cohomology of $\mathcal{F}$ can be deduced from the kernel $\mathcal{K}_1$ and cokernel $\mathcal{K}_2$ of F as follows:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \mathcal{K}_1 \arrow[r] & \mathcal{F} \arrow[rr] \arrow[dr] & & \mathcal{F}(d) \arrow[r] & \mathcal{K}_2 \arrow[r] & 0 \\
 & & & \mathcal{F}/\mathcal{K}_1 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$\begin{array}{lll} (7.1) & \to H^i(\mathcal{K}_1) \to H^i(\mathcal{F}) \to H^i(\mathcal{F}/\mathcal{K}_1) \longrightarrow & H^{i+1}(\mathcal{K}_1) \longrightarrow \\ & & \quad \approx\uparrow \\ & & H^{i-1}(\mathcal{K}_2),\ \text{if } i \geq 2 \\ & & \text{or} \\ & & H^0(\mathcal{K}_2)/H^0(\mathcal{F}(d)),\ \text{if } i = 1. \end{array}$$

This reduces properties of the cohomology of $\mathcal{F}$ to those of $\mathcal{K}_1$ and $\mathcal{K}_2$ which have, in general, lower dimensional support. In fact, one can easily arrange that F is injective, hence $\mathcal{K}_1 = (0)$ too. In terms of $\mathrm{Ass}(\mathcal{F})$, defined in §II.3, we can give the following criterion:

PROPOSITION 7.2. *Given a coherent $\mathcal{F}$ on $\mathbb{P}^n_k$, let $\mathrm{Ass}(\mathcal{F}) = \{x_1, \ldots, x_t\}$. Then $F\colon \mathcal{F} \to \mathcal{F}(d)$ is injective if and only if $F(x_a) \neq 0$, $1 \leq a \leq t$ (more precisely, if $x_a \notin V(X_{n_a})$, then the function $F/X^d_{n_a}$ is not 0 at x_a).*

PROOF. Let $U_a = \mathbb{P}^n_k \setminus V(X_{n_a})$. If $(F/X^d_{n_a})(x_a) = 0$, then $F/X^d_{n_a} = 0$ on $\overline{\{x_a\}} \cap U_a$. But $\exists s \in \mathcal{F}(U_a)$ with $\mathrm{Supp}(s) = \overline{\{x_a\}} \cap U_a$, so $(F/X^d_{n_a})^N \cdot s = 0$ if $N \gg 0$. Choose N_a so that $(F/X^d_{n_a})^{N_a} \cdot s \neq 0$ but $(F/X^d_{n_a})^{N_a+1} \cdot s = 0$. Then

$$F \cdot \left(\frac{F}{X^d_{n_a}}\right)^{N_a} \cdot s = 0 \quad \text{in } \mathcal{F}(d)(U_a)$$

so F is not injective. Conversely if $F(x_a) \neq 0$ for all a and $s \in \mathcal{F}(U)$ is not 0, then for some a, $s_{x_a} \in \mathcal{F}_{x_a}$ is not 0. But $F/X^d_{n_a}$ is a unit in $\mathcal{O}_{x_a}$, so $(F/X^d_{n_a}) \cdot s_{x_a} \neq 0$, so $F \cdot s_{x_a} \neq 0$. □

Assuming then that F is *injective*, we get

$$(7.1^*) \qquad \begin{array}{l} H^i(\mathcal{F}) \xrightarrow{\ \approx\ } H^{i-1}(\mathcal{K}_2) \text{ if } i \geq 2 \\[2ex] H^1(\mathcal{F}) \xrightarrow{\ \approx\ } H^0(\mathcal{K}_2)/\,\mathrm{Image}\, H^0(\mathcal{F}(d)). \end{array}$$

It is at this point that we make contact with the Italian methods. Let $X \subset \mathbb{P}^n_k$ be a projective variety, i.e., a reduced and irreducible closed subscheme. Let D be a Cartier divisor on X and $\mathcal{O}_X(D)$ the invertible sheaf of functions "with poles on D" (cf. §III.6). Then $\mathcal{O}_X(D)$, extended by (0) outside X, is a coherent sheaf on $\mathbb{P}^n_k$ of $\mathcal{O}_{\mathbb{P}^n}$-modules (cf. Remark after Corollary 3.8) and its cohomology may be computed by (7.1*).

In fact, we may do even better and describe its cohomology by induction *using only sheaves of the same type $\mathcal{O}_X(D)$*! First, some notation —

DEFINITION 7.3. If X is an irreducible reduced scheme, $Y \subset X$ an irreducible reduced subscheme and D is a Cartier divisor on X, then if $Y \not\subseteq \mathrm{Supp}\, D$, define $\mathrm{Tr}_Y\, D$ to be the Cartier divisor on Y whose local equations at $y \in Y$ are just the restrictions to Y of its local equations at $y \in X$. Note that:

$$\mathcal{O}_Y(\mathrm{Tr}_Y\, D) \cong \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_Y.$$

Now take a homogeneous polynomial F endowed with the following properties:

a) $X \not\subseteq V(F)$ and the effective Cartier divisor $H = \mathrm{Tr}_X(V(F))$ is reduced and irreducible,
b) no component D_j of $\mathrm{Supp}\, D$ is contained in $V(F)$.

It can be shown that such an F exists (in fact, in the affine space of all F's, any F outside a proper union of subvarieties will have these properties). Take a second F' with the property

c) $H \not\subseteq V(F')$

and let $H' = \mathrm{Tr}_X(V(F'))$. Start with the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

and tensor with $\mathcal{O}_X(D+H')$. We find

$$0 \longrightarrow \mathcal{O}_X(D+H'-H) \longrightarrow \mathcal{O}_X(D+H') \longrightarrow \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H') \longrightarrow 0.$$

But the first sheaf is just $\mathcal{O}_X(D)$ via:

$$\mathcal{O}_X(D) \xrightarrow[\approx]{\text{multiply } F/F'} \mathcal{O}_X(D+H'-H)$$

and the second sheaf is just $\mathcal{O}_X(D)(d)$ and the whole sequence is the same exact sequence as before:

$$\tag{7.4} \begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X(D) \arrow[r, "\substack{\text{multiplication}\\ \text{by } F}"] \arrow[d, "\approx", "\substack{\text{multiplication}\\ \text{by } F/F'}"'] & \mathcal{O}_X(D)(d) \arrow[r] \arrow[d, "\approx"', "\substack{\text{multiplication}\\ \text{by } F/F'}"] & \mathcal{K}_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_X(D+H'-H) \arrow[r, "\substack{\text{natural}\\ \text{inclusion}}"'] & \mathcal{O}_X(D+H') \arrow[r] & \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H') \arrow[r] & 0 \end{tikzcd}$$

Thus $\mathcal{K}_2 \approx \mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H')$. This inductive precedure allowed the Italian School to discuss the cohomology in another language without leaving the circle of ideas of linear systems. For instance

$$\begin{aligned} H^1(\mathcal{O}_X(D)) &\cong \mathrm{Coker}\left[H^0(\mathcal{O}_X(D+H')) \longrightarrow H^0(\mathcal{O}_H(\mathrm{Tr}_H D + \mathrm{Tr}_H H'))\right] \\ &\cong \begin{bmatrix} \text{space of linear conditions that must be imposed} \\ \text{on an } f \in \mathbf{R}(H) \text{ with poles on } \mathrm{Tr}_H D + \mathrm{Tr}_H H' \text{ before} \\ \text{it can be extended to an } f \in \mathbf{R}(X) \text{ with poles in } D + H' \end{bmatrix}. \end{aligned}$$

Classically one dealt with the projective space $|D+H'|_X$ of divisors $V(s)$, $s \in H^0(\mathcal{O}_X(D+H'))$, (which is just the set of 1-dimensional subspaces of $H^0(\mathcal{O}_X(D+H'))$), and provided $\dim X \geq 2$, we can look instead at:

$$\begin{aligned} \left\{ \begin{array}{l} \text{subset of } |D+H'|_X \text{ of divisors} \\ E \text{ with } H \not\subseteq \mathrm{Supp}\, E \end{array} \right\} &\xrightarrow{\mathrm{Tr}_H} |\,\mathrm{Tr}_H D + \mathrm{Tr}_H H'|_H \\ E &\longmapsto \mathrm{Tr}_H E. \end{aligned}$$

Then

$$\begin{aligned} \dim H^1(\mathcal{O}_X(D)) = &\text{codimension of Image of } \mathrm{Tr}_H, \text{ called} \\ &\text{the “deficiency”of } \mathrm{Tr}_H |D+H'|_X. \end{aligned}$$

We go on now to discuss another application of method (4) — to the Hilbert polynomial. First of all, suppose X is any scheme proper over k and $\mathcal{F}$ is a coherent sheaf on X. Then one defines:

$$\tag{7.5} \begin{aligned} \chi(\mathcal{F}) &= \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, \mathcal{F}) \\ &= \text{the Euler characteristic of } \mathcal{F}, \end{aligned}$$

which makes sense because the H^i are finite-dimensional by Grothendieck's coherency theorem (Theorem 6.5). The importance of this particular combination of the $\dim H^i$'s is that if

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of coherent sheaves, then it follows from the associated long exact cohomology sequence by a simple calculation that:

$$\chi(\mathcal{F}_2) = \chi(\mathcal{F}_1) + \chi(\mathcal{F}_3). \tag{7.6}$$

This makes χ particularly easy to compute. In particular, we get:

THEOREM 7.7. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Then there exists a polynomial $P(t)$ with $\deg P = \dim \operatorname{Supp} \mathcal{F}$ such that*

$$\chi(\mathcal{F}(\nu)) = P(\nu), \quad \textit{all } \nu \in \mathbb{Z}.$$

In particular, by Theorem 6.1, there exists an ν_0 such that

$$\dim H^0(\mathcal{F}(\nu)) = P(\nu), \quad \textit{if } \nu \in \mathbb{Z},\ \nu \geq \nu_0.$$

$P(t)$ is called the Hilbert polynomial of $\mathcal{F}$.

PROOF. This is a geometric form of Part I [**87**, (6.21)] and the proof is parallel: Let $L(X)$ be a linear form such that $L(x_a) \neq 0$ for any of the associated points x_a of $\mathcal{F}$. Then as above we get an exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{L} \mathcal{F}(1) \longrightarrow \mathcal{G} \longrightarrow 0$$

for some coherent $\mathcal{G}$, with

$$\operatorname{Supp} \mathcal{G} = \operatorname{Supp} \mathcal{F} \cap V(L)$$

hence

$$\dim \operatorname{Supp} \mathcal{G} = \dim \operatorname{Supp} \mathcal{F} - 1.$$

Tensoring by $\mathcal{O}_{\mathbb{P}^n}(l)$ we get exact sequences:

$$0 \longrightarrow \mathcal{F}(l) \longrightarrow \mathcal{F}(l+1) \longrightarrow \mathcal{G}(l) \longrightarrow 0 \tag{7.8}$$

for every $l \in \mathbb{Z}$, hence

$$\chi(\mathcal{F}(l+1)) = \chi(\mathcal{F}(l)) + \chi(\mathcal{G}(l)).$$

Now we prove the theorem by induction: if $\dim \operatorname{Supp} \mathcal{F} = 0$, $\operatorname{Supp} \mathcal{F}$ is a finite set, so $\operatorname{Supp} \mathcal{G} = \emptyset$ and $\mathcal{F}(l) \xrightarrow{\approx} \mathcal{F}(l+1)$ for all l by (7.8). Therefore $\chi(\mathcal{F}(l)) = \chi(\mathcal{F}) =$ constant, a polynomial of degree 0! In general, if $s = \dim \operatorname{Supp} \mathcal{F}$, then by induction $\chi(\mathcal{G}(l)) = Q(l)$, Q a polynomial of degree $s-1$. Then

$$\chi(\mathcal{F}(l+1)) - \chi(\mathcal{F}(l)) = Q(l)$$

hence as in Part I [**87**, (6.21)], $\chi(\mathcal{F}(l)) = P(l)$ for some polynomial P of degree s. □

This leads to the following point of view. Given $\mathcal{F}$, one often would like to compute $\dim_k \Gamma(\mathcal{F})$: for $\mathcal{F} = \mathcal{O}_X(D)$, this is the typical problem of the *additive theory of rational functions on X*. But because of the formula (7.6), it is often easier to compute either $\chi(\mathcal{F})$ directly, or $\dim_k \Gamma(\mathcal{F}(\nu))$ for $\nu \gg 0$, hence the Hilbert polynomial, hence $\chi(\mathcal{F})$ again. The Italians called $\chi(\mathcal{F})$ the *virtual dimension* of $\Gamma(\mathcal{F})$ and viewed it as $\dim \Gamma(\mathcal{F})$ (the main term) followed by an alternating sum of "error terms" $\dim H^i(\mathcal{F})$, $i \geq 1$. Thus one of the main reasons for computing the higher cohomology groups is to find how far $\dim \Gamma(\mathcal{F})$ has diverged from $\chi(\mathcal{F})$.

Recall that in Part I [**87**, (6.28)], we defined the arithmetic genus $p_a(X)$ of a projective variety $X \subset \mathbb{P}^n_k$ with a given projective embedding to be

$$p_a(X) = (-1)^r (P(0) - 1)$$
$$\text{where } P(x) = \text{Hilbert polynomial of } X,\ r = \dim X.$$

It now follows:

COROLLARY 7.9 (Zariski-Muhly).

$$p_a(X) = \dim H^r(\mathcal{O}_X) - \dim H^{r-1}(\mathcal{O}_X) + \cdots + (-1)^{r-1} \dim H^1(\mathcal{O}_X)$$

hence $p_a(X)$ is independent of the projective embedding of X.

PROOF. By Theorem 7.7, $P(0) = \chi(\mathcal{O}_X)$ so the formula follows using $\dim H^0(\mathcal{O}_X) = 1$ (Corollary II.6.10). □

I'd like to give one somewhat deeper result analyzing the "point" vis-a-vis tensoring with $\mathcal{O}(\nu)$ at which the higher cohomology vanishes; and which shows how the vanishing of higher cohomology groups alone can imply the existence of sections:

THEOREM 7.10 (Generalized lemma of Castelnuovo and syzygy theorem of Hilbert). *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. Then the following are equivalent:*

i) $H^i(\mathcal{F}(-i)) = (0)$, $1 \le i \le n$,
ii) $H^i(\mathcal{F}(m)) = (0)$, *if* $m + i \ge 0$, $i \ge 1$,
iii) *there exists a "Spencer resolution":*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n+1)^{r_{n-1}} \longrightarrow \cdots \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

If these hold, then the canonical map

$$H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l)) \longrightarrow H^0(\mathcal{F}(l))$$

is surjective, $l \ge 0$.

PROOF. We use induction on n: for $n = 0$, $\mathbb{P}^n_k = \operatorname{Spec} k$, $\mathcal{F} = \widetilde{k^n}$ and the result is clear. So we may suppose we know the result on $\mathbb{P}^{n-1}_k$. The implication (ii) $\Longrightarrow$ (i) is obvious and (iii) $\Longrightarrow$ (ii) follows easily from what we know of the cohomology of $\mathcal{O}_{\mathbb{P}^n}(l)$, by splitting the resolution up into a set of short exact sequences:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \to \mathcal{O}_{\mathbb{P}^n}(-n+1)^{r_{n-1}} \to \mathcal{F}_{n-1} \to 0$$
$$- - - - - - - - - - - - - - - -$$
$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \longrightarrow \mathcal{F}_1 \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

So assume (i). Choose a linear form $L(X)$ such that $L(x_a) \neq 0$ for any associated points x_a of $\mathcal{F}$, getting sequences

$$0 \longrightarrow \mathcal{F}(l-1) \xrightarrow{\otimes L} \mathcal{F}(l) \longrightarrow \mathcal{G}(l) \longrightarrow 0, \quad \text{all } l \in \mathbb{Z}$$

where $\mathcal{G}$ is a coherent sheaf on the hyperplane $H = V(L)$. In fact $\mathcal{G}$ is not only supported on H but is annihilated by the local equations L/X_j of H: hence $\mathcal{G}$ is a sheaf of $\mathcal{O}_H$-modules. Since $H \cong \mathbb{P}^{n-1}_k$, we are in a position to apply our induction hypothesis. The cohomology sequences give:

$$\longrightarrow H^i(\mathcal{F}(-i)) \longrightarrow H^i(\mathcal{G}(-i)) \longrightarrow H^{i+1}(\mathcal{F}(-i-1)) \longrightarrow .$$

Applying this for $i \ge 1$, we find that $\mathcal{G}$ satisfies (i) also; applying it for $i = 0$, we find that $H^0(\mathcal{F}) \to H^0(\mathcal{G})$ is surjective. Therefore by the theorem for $\mathcal{G}$,

$$H^0(\mathcal{G}) \otimes H^0(\mathcal{O}_H(l)) \longrightarrow H^0(\mathcal{G}(l))$$

is surjective. Consider the maps:

$$\begin{array}{ccc} H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l)) & \xrightarrow{\gamma} & H^0(\mathcal{F}(l)) \\ \alpha\Big\downarrow & & \beta\Big\downarrow \\ H^0(\mathcal{G}) \otimes H^0(\mathcal{O}_H(l)) & \longrightarrow & H^0(\mathcal{G}(l)). \end{array}$$

We prove next that γ is surjective for all $l \geq 0$. By Proposition III.1.8, $H^0(\mathcal{O}_H(l))$ is the space of homogeneous polynomials of degree l in the homogeneous coordinates on H: therefore each is obtained by restricting to H a polynomial $P(X_0, \ldots, X_n)$ of degree l and $H^0(\mathcal{O}_{\mathbb{P}^n}(l)) \to H^0(\mathcal{O}_H(l))$ is surjective. Therefore α is surjective. It follows that if $s \in H^0(\mathcal{F}(l))$, then $\beta(s) = \sum \overline{u}_q \otimes \overline{v}_q$, $\overline{u}_q \in H^0(\mathcal{G})$, $\overline{v}_q \in H^0(\mathcal{O}_H(l))$; hence lifgint $\overline{u}_q$ to $u_q \in H^0(\mathcal{F})$, $\overline{v}_q$ to $v_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l))$, $s - \sum u_q \otimes v_q$ lies in $\operatorname{Ker} \beta$. But $\operatorname{Ker} \beta =$ Image of $H^0(\mathcal{F}(l-1))$ under the map $\otimes L \colon \mathcal{F}(l-1) \to \mathcal{F}(l)$ and by induction on l, anything in $H^0(\mathcal{F}(l-1))$ is in $H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l-1))$. Thus

$$s - \sum u_q \otimes v_q = \left(\sum u'_q \otimes v'_q\right) \otimes L, \quad u'_q \in H^0(\mathcal{F}),\ v'_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l-1)).$$

Thus

$$s = \sum u_q \otimes v_q + \sum u'_q \otimes (v'_q \otimes L), \quad \text{where } v'_q \otimes L \in H^0(\mathcal{O}_{\mathbb{P}^n}(l))$$

as required.

Next, note that this implies that $\mathcal{F}$ is generated by $H^0(\mathcal{F})$. In fact, if $x \in \mathbb{P}^n_k$, $x \notin V(X_j)$, and $s \in \mathcal{F}_x$, then $X_j^l \cdot s \in \mathcal{F}(l)_x$. For $l \gg 0$, $\mathcal{F}(l)$ is generated by $H^0(\mathcal{F}(l))$. So

$$\begin{array}{c} X_j^l \cdot s \in H^0(\mathcal{F}(l)) \cdot (\mathcal{O}_{\mathbb{P}^n})_x \\ \| \\ \big(H^0(\mathcal{F}) \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(l))\big) \cdot (\mathcal{O}_{\mathbb{P}^n})_x \end{array}$$

i.e.,

$$X_j^l \cdot s = \sum u_q \otimes v_q \cdot a_q, \quad u_q \in H^0(\mathcal{F}),\ v_q \in H^0(\mathcal{O}_{\mathbb{P}^n}(l)),\ a_q \in (\mathcal{O}_{\mathbb{P}^n})_x$$

hence

$$s = \sum u_q \otimes \underbrace{\frac{v_q}{X_j^l} \cdot a_q}_{\in (\mathcal{O}_{\mathbb{P}^n})_x}.$$

We can now begin to construct a Spencer resolution: let $s_1, \ldots, s_{r_0}$ be a basis of $H^0(\mathcal{F})$ and define

$$\mathcal{O}_{\mathbb{P}^n}^{r_0} \longrightarrow \mathcal{F} \longrightarrow 0$$

by

$$(a_1, \ldots, a_{r_0}) \longmapsto \sum_{q=0}^{r_0} a_q s_q.$$

If $\mathcal{F}_1$ is the kernel, then from the cohomology sequence it follows immediately that $\mathcal{F}_1(1)$ satisfies Condition (i) of the theorem. Hence $\mathcal{F}_1(1)$ is also generated by its sections and choosing a basis $t_1, \ldots, t_{r_1}$ of $H^0(\mathcal{F}_1(1))$, we get the next step:

$$\mathcal{O}_{\mathbb{P}^n}^{r_1} \longrightarrow \mathcal{F}_1(1) \longrightarrow 0$$

$$(a_1, \ldots, a_{r_1}) \longmapsto \sum_{q=1}^{r_1} a_q t_q$$

hence

$$\begin{tikzcd}[column sep=small]
\mathcal{O}_{\mathbb{P}^n}(-1)^{r_1} \arrow[rr] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n}^{r_0} \arrow[r] & \mathcal{F} \arrow[r] & 0. \\
& \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Continuing in this way, we derive the whole Spencer resolution. It remains to check that after the last step:

$$0 \longrightarrow \mathcal{F}_{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-n)^{r_n} \longrightarrow \mathcal{F}_n \longrightarrow 0,$$

the sheaf $\mathcal{F}_{n+1}$ is actually (0)! To prove this, we compute $H^i(\mathcal{F}_{n+1}(l))$ for $0 \le i \le n$, $0 \le l \le n$, using all the cohomology sequences $(*)_m$ associated to:

$$0 \longrightarrow \mathcal{F}_{m+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-m)^{r_m} \longrightarrow \mathcal{F}_m \longrightarrow 0.$$

We get:

a) $H^0(\mathcal{F}_{n+1}(l)) \underset{\text{by } (*)_n}{=} (0)$ (using injectivity of $H^0(\mathcal{F}^{r_n}) \to H^0(\mathcal{F}_n(n))$ when $l = n$)

b)

$$H^1(\mathcal{F}_{n+1}) \begin{cases} \underset{\text{by } (*)_n}{=} (0) & \text{if } l = n \quad \text{(using surjectivity of } \\ & \qquad\qquad H^0(\mathcal{O}^{r_n}) \to H^0(\mathcal{F}_n(n))) \\ \underset{\text{by } (*)_n}{\cong} H^0(\mathcal{F}_n(l)) \underset{\text{by } (*)_{n-1}}{=} (0) & \text{if } l < n \quad \text{(using injectivity of} \\ & \qquad\qquad H^0(\mathcal{O}^{r_{n-1}}) \to H^0(\mathcal{F}_{n-1}(n-1)) \\ & \qquad\qquad \text{when } l = n-1) \end{cases}$$

. . . .

x)

$$\begin{aligned} H^n(\mathcal{F}_{n+1}(l)) &\underset{\text{by } (*)_n}{\cong} H^{n-1}(\mathcal{F}_n(l)) \underset{\text{by } (*)_{n-1}}{\cong} \cdots \\ \cdots &\underset{\text{by } (*)_2}{\cong} H^1(\mathcal{F}_2(l)) \end{aligned}$$

$$\begin{cases} \underset{\text{by } (*)_1}{=} (0) & \text{if } l \ge 1 \quad \text{(using surjectivity of} \\ & \qquad\qquad H^0(\mathcal{O}(l-1)^{r_1}) \to H^0(\mathcal{F}_1(l))) \\ \underset{\text{by } (*)_1}{\cong} H^0(\mathcal{F}_1) \underset{\text{by } (*)_0}{=} (0) & \text{if } l = 1 \quad \text{(using injectivity of} \\ & \qquad\qquad H^0(\mathcal{O}^{r_0}) \to H^0(\mathcal{F})). \end{cases}$$

So all these groups are (0). Thus $\chi(\mathcal{F}_{n+1}(l)) = 0$, for $n+1$ distinct values $l = 0, \ldots, n$. Since $\chi(\mathcal{F}_{n+1}(l))$ is a polynomial of degree at most n, it must be identically 0. But then for $l \gg 0$, $\dim H^0(\mathcal{F}_{n+1}(l)) = \chi(\mathcal{F}_{n+1}(l)) = 0$, hence $H^0(\mathcal{F}_{n+1}(l)) = (0)$ and since these sections generate $\mathcal{F}_{n+1}(l)$, $\mathcal{F}_{n+1}(l) = (0)$ too. $\square$

8. Serre's criterion for ampleness

This section gives a cohomological criterion equivalent to ampleness for an invertible sheaf introduced in §III.5. We apply it later to questions of positivity of intersections, formulated in terms of the Euler characteristic.

THEOREM 8.1. *Let X be a scheme over a noetherian ring A, embedded as a closed subscheme in a projective space over A, with canonical sheaf $\mathcal{O}_X(1)$. Let $\mathcal{F}$ be coherent on X. Then for all $i \geq 0$, $H^i(X, \mathcal{F})$ is a finite A-module, and there exists an integer n_0 such that for $n \geq n_0$ we have*

$$H^i(X, \mathcal{F}(n)) = 0 \quad \textit{for all } i \geq 1.$$

PROOF. We have already seen in Corollary 3.8 that under a closed immersion $X \hookrightarrow \mathbb{P}^r_A$ the cohomology of $\mathcal{F}$ over X is the same as the cohomology of $\mathcal{F}$ viewed as a sheaf over projective space. Consequently we may assume without loss of generality that $X = \mathbb{P}^r_A$, which we denote by $\mathbb{P}$.

The explicit computation of cohomology $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ in Corollary 5.4 and (5.6) shows that the theorem is true when $\mathcal{F} = \mathcal{O}_{\mathbb{P}}(n)$ for all integers n. Now let $\mathcal{F}$ be an arbitrary coherent sheaf on $\mathbb{P}$. We can represent $\mathcal{F}$ in a short exact sequence (cf. §6)

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}$ is a finite direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$ for appropriate integers d, and $\mathcal{G}$ is defined to be the kernel of $\mathcal{E} \to \mathcal{F}$. We use the cohomology sequence, and write the cohomology groups without $\mathbb{P}$ for simplicity:

$$\longrightarrow H^i(\mathcal{E}) \longrightarrow H^i(\mathcal{F}) \longrightarrow H^{i+1}(\mathcal{G}) \longrightarrow$$

We apply descending induction. For $i > r$ we have $H^i(\mathcal{F}) = 0$ because $\mathbb{P}$ can be covered by $r+1$ open affine subsets, and the Čech complex is 0 with respect to this covering in dimension $\geq r+1$ (cf. (5.5)). If, by induction, $H^{i+1}(\mathcal{G})$ is finite over A, then the finiteness of $H^i(\mathcal{E})$ implies that $H^i(\mathcal{F})$ is finite.

Furthermore, twisting by n, that is, taking tensor products with $\mathcal{O}_{\mathbb{P}}(n)$, is an exact functor, so the short exact sequence tensored with $\mathcal{O}_{\mathbb{P}}(n)$ remains exact. This gives rise to the cohomology exact sequence:

$$\to H^i(\mathcal{E}(n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{G}(n)) \to$$

Again by induction, $H^{i+1}(\mathcal{G}(n)) = 0$ for n sufficiently large, and $H^i(\mathcal{E}(n)) = 0$ because of the special nature of $\mathcal{E}$ as a direct sum of sheaves $\mathcal{O}_{\mathbb{P}}(d)$. This implies that $H^i(\mathcal{F}(n)) = 0$ for n sufficiently large, and concludes the proof of the theorem. □

THEOREM 8.2 (Serre's criterion). *Let X be a scheme, proper over a noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if the following condition holds: For any coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that for all $n \geq n_0$ we have*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \textit{for all } i \geq 1.$$

PROOF. Suppose that $\mathcal{L}$ is ample, so $\mathcal{L}^d$ is very ample for some d. We have seen (cf. Theorem III.5.4 and §II.6) that X is projective over A. We apply Theorem 8.1 to the tensor products

$$\mathcal{F}, \mathcal{F} \otimes \mathcal{L}, \ldots, \mathcal{F} \otimes \mathcal{L}^{d-1}$$

and the very ample sheaf $\mathcal{L}^d = \mathcal{O}_X(1)$ to conclude the proof that the cohomology groups vanish for $i \geq 1$.

Conversely, assume the condition on the cohomology groups. We want to prove that $\mathcal{L}$ is ample. It suffices to prove that for any coherent sheaf $\mathcal{F}$ the tensor product $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections for n sufficiently large. (cf. Definition III.5.1) By Definition III.2.1 it will suffice to prove that for every closed point P, the fibre $\mathcal{F} \otimes \mathcal{L}^n \otimes \Bbbk(P)$ is generated by global sections. Let $\mathcal{I}_P$ be the ideal sheaf defining the closed point P as a closed subscheme. We have an exact sequence

$$0 \longrightarrow \mathcal{I}_P\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \Bbbk(P) \longrightarrow 0.$$

Since $\mathcal{L}^n$ is locally free, tensoring with $\mathcal{L}^n$ preserves exactness, and yields the exact sequence

$$0 \longrightarrow \mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \Bbbk(P) \otimes \mathcal{L}^n \longrightarrow 0$$

whence the cohomology exact sequence

$$H^0(\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(\mathcal{F} \otimes \Bbbk(P) \otimes \mathcal{L}^n) \longrightarrow 0$$

because $H^1(\mathcal{I}_P\mathcal{F} \otimes \mathcal{L}^n) = 0$ by hypothesis. This proves that the fibre at P of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections, and concludes the proof of the theorem. □

9. Functorial properties of ampleness

This section gives a number of conditions relating ampleness on a scheme with ampleness on certain subschemes.

PROPOSITION 9.1. *Let X be a scheme of finite type over a noetherian ring and $\mathcal{L}$ an invertible sheaf, ample on X. For every closed subscheme Y, the restriction $\mathcal{L}|_Y = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_Y$ is ample on Y.*

PROOF. Taking a power of $\mathcal{L}$ we may assume without loss of generality that $\mathcal{L}$ is very ample (cf. Theorem III.5.4), so $\mathcal{O}_X(1)$ in a projective embedding of X. Then $\mathcal{O}_X|_Y = \mathcal{O}_Y(1)$ in that same embedding. Thus the proposition is immediate. □

Let X be a scheme. For each open subset U we let $\mathcal{N}il(U)$ be the ideal of nilpotnet elements in $\mathcal{O}_X(U)$. Then $\mathcal{N}il$ is a sheaf of ideals, and the quotient sheaf $\mathcal{O}_X/\mathcal{N}il$ defines a closed subscheme called the reduced scheme X_{red}. Its sheaf of rings has no nilpotent elements. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then we let

$$\mathcal{F}_{\text{red}} = \mathcal{F}/\mathcal{N}\mathcal{F} \quad \text{where } \mathcal{N} = \mathcal{N}il\,.$$

Alternatively, we can say that $\mathcal{F}_{\text{red}}$ is the restriction of $\mathcal{F}$ to X_{red}.

PROPOSITION 9.2. *Let X be a scheme, proper over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}_{\text{red}}$ is ample on X_{red}.*

PROOF. By Proposition 9.1, it suffices to prove one side of the equivalence, namely: if $\mathcal{L}_{\text{red}}$ is ample then $\mathcal{L}$ is ample. Since X is noetherian, there exists an integer r such that if $\mathcal{N} = \mathcal{N}il$ is the sheaf of nilpotent elements, then $\mathcal{N}^r = 0$. Hence we get a finite filtration

$$\mathcal{F} \supset \mathcal{N}\mathcal{F} \supset \mathcal{N}^2\mathcal{F} \supset \cdots \supset \mathcal{N}^r\mathcal{F} = 0.$$

For each $i = 1, \ldots, r-1$ we have the exact sequence

$$0 \longrightarrow \mathcal{N}^i\mathcal{F} \longrightarrow \mathcal{N}^{i-1}\mathcal{F} \longrightarrow \mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F} \longrightarrow 0$$

whence the exact cohomology sequence

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, \mathcal{N}^{i-1}\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, (\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}) \otimes \mathcal{L}^n).$$

For each i, $\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}$ is a coherent $\mathcal{O}_X/\mathcal{N}$-module, and thus is a sheaf on X_{red}. By hypothesis, and Theorem 8.2, we know that

$$H^p(X, (\mathcal{N}^{i-1}\mathcal{F}/\mathcal{N}^i\mathcal{F}) \otimes \mathcal{L}^n) = 0$$

for all n sufficiently large and all $p \geq 1$. But $\mathcal{N}^i\mathcal{F} = 0$ for $i \geq r$. We use descending induction on i. We have

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \text{for all } p > 0, i \geq r,$$

and n sufficiently large. Hence inductively,

$$H^p(X, \mathcal{N}^i\mathcal{F} \otimes \mathcal{L}^n) = 0 \quad \text{for all } p > 0$$

implies that $H^p(X, \mathcal{N}^{i-1}\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $p > 0$ and n sufficiently large. This concludes the proof. □

PROPOSITION 9.3. *Let X be a proper scheme over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on X. Then $\mathcal{L}$ is ample if and only if $\mathcal{L}|_{X_i}$ is ample on each irreducible component X_i of X.*

PROOF. Since an irreducible component is a closed subscheme of X, Proposition 9.1 shows that it suffices here to prove one implication. So assume that $\mathcal{L}|_{X_i}$ is ample for all i. Let $\mathcal{I}_i$ be the coherent sheaf of ideals defining X_i, and say $i = 1, \ldots, r$. We use induction on r. We consider the exact sequence

$$0 \longrightarrow \mathcal{I}_1\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{I}_1\mathcal{F} \longrightarrow 0,$$

giving rise to the exact cohomology sequence

$$H^p(X, \mathcal{I}_1\mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^p(X, (\mathcal{F}/\mathcal{I}_1\mathcal{F}) \otimes \mathcal{L}^n).$$

Since $\mathcal{L}|_{X_1}$ is ample by hypothesis, it follows that

$$H^p(X, (\mathcal{F}/\mathcal{I}_1\mathcal{F}) \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$. Furthermore, $\mathcal{I}_1\mathcal{F}$ is a sheaf with support in $X_2 \cup \cdots \cup X_r$, so by induction we have

$$H^p(X, \mathcal{I}_1\mathcal{F} \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$. The exact sequence then gives

$$H^p(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$$

for all $p > 0$ and $n \geq n_0$, thus concluding the proof. □

PROPOSITION 9.4. *Let $f\colon X \to Y$ be a finite (cf. Definition* II.6.6*) surjective morphism of proper schemes over a noetherian ring. Let $\mathcal{L}$ be an invertible sheaf on Y. Then $\mathcal{L}$ is ample if and only if $f^*\mathcal{L}$ is ample on X.*

PROOF. First note that f is affine (cf. Proposition-Definition I.7.3 and Definition II.6.6). Let $\mathcal{F}$ be a coherent sheaf on X, so $f_*\mathcal{F}$ is coherent on Y. For $p \geq 0$ we get:

$$\begin{aligned} H^p(Y, f_*(\mathcal{F}) \otimes \mathcal{L}^n) &= H^p(Y, f_*(\mathcal{F} \otimes (f^*\mathcal{L})^n)) \\ &= H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) \end{aligned}$$

by the *projection formula*[9] and by Proposition 3.7.

[9]Let $f\colon X \to Y$ be a morphism, $\mathcal{F}$ an $\mathcal{O}_X$-module and $\mathcal{L}$ an $\mathcal{O}_Y$-module. The identity homomorphism $f^*\mathcal{L} \to f^*\mathcal{L}$ induces an $\mathcal{O}_Y$-homomorphism $\mathcal{L} \to f_*f^*\mathcal{L}$. Tensoring this with $f_*\mathcal{F}$ over $\mathcal{O}_Y$ and composing the result with a canonical homomorphism, one gets a canonical homomorphism

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L} \longrightarrow f_*\mathcal{F} \otimes_{\mathcal{O}_Y} f_*f^*\mathcal{L} \longrightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}).$$

This can be easily shown to be an isomophism if $\mathcal{L}$ is a locally free $\mathcal{O}_Y$-module of finite rank, giving rise to the "*projection formula*"

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L} \xrightarrow{\sim} f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}).$$

If $\mathcal{L}$ is ample, then the left hand side is 0 for $n \geq n_0$ and $p > 0$, so this proves that $f^*\mathcal{L}$ is ample on X.

Conversely, assume $f^*\mathcal{L}$ ample on X. We show that for any coherent $\mathcal{O}_Y$-module $\mathcal{G}$, one has

$$H^p(Y, \mathcal{G} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0$$

by noetherian induction on $\operatorname{Supp}(\mathcal{G})$.

By Propositions 9.2 and 9.3, we may assume X and Y to be integral. We follow Hartshorne [**56**, §4, Lemma 4.5, pp. 25–27] and first prove:

LEMMA 9.5. *Let $f\colon X \to Y$ be a finite surjective morphism of degree m of noetherian integral schemes X and Y. Then for every coherent $\mathcal{O}_Y$-module $\mathcal{G}$ on Y, there exist a coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-homomorphism $\xi\colon f_*\mathcal{F} \to \mathcal{G}^{\oplus m}$ that is a generic isomorphism (i.e., ξ is an isomorphism in a neighborhood of the generic point of Y).*

PROOF OF LEMMA 9.5. By assumption, the function field $\mathbf{R}(X)$ is an algebraic extension of $\mathbf{R}(Y)$ of degree m. Let $U = \operatorname{Spec} A \subset X$ be an affine open set. Since $\mathbf{R}(X)$ is the quotient field of A, we can choose $s_1, \dots, s_m \in A$ such that $\{s_1, \dots, s_m\}$ is a basis of $\mathbf{R}(X)$ as a vector space over $\mathbf{R}(Y)$. The $\mathcal{O}_X$-submodule $\mathcal{H} = \sum_{i=1}^m \mathcal{O}_X s_i$ of the constant $\mathcal{O}_X$-module $\mathbf{R}(X)$ is coherent. Since $s_1, \dots, s_m \in H^0(X, \mathcal{H}) = H^0(Y, f_*\mathcal{H})$, we have an $\mathcal{O}_Y$-homomorphism

$$\eta\colon \mathcal{O}_Y^{\oplus m} = \sum_{i=1}^m \mathcal{O}_Y e_i \longrightarrow f_*\mathcal{H}, \qquad e_i \longmapsto s_i \quad (i = 1, \dots, m),$$

which is a generic isomorphism by the choice of $s_1, \dots, s_m$. If a coherent $\mathcal{O}_Y$-module $\mathcal{G}$ is given, η induces an $\mathcal{O}_Y$-homomorphism

$$\xi\colon \mathcal{H}' = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{H}, \mathcal{G}) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y^{\oplus m}, \mathcal{G}) = \mathcal{G}^{\oplus m},$$

which is a generic isomorphism. Since $\mathcal{H}'$ is an $f_*\mathcal{O}_X$-module through the first factor of $\mathcal{H}om$ and f is finite, we have $\mathcal{H}' = f_*\mathcal{F}$ for a coherent $\mathcal{O}_X$-module $\mathcal{F}$. $\square$

To continue the proof of Proposition 9.4, let $\mathcal{G}$ be a coherent $\mathcal{O}_Y$-module $\mathcal{G}$. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module as in Lemma 9.5, and let $\mathcal{K}$ and $\mathcal{C}$ be the kernel and cokernel of the $\mathcal{O}_Y$-homomorphism $\xi\colon f_*(\mathcal{F}) \to \mathcal{G}^{\oplus m}$. We have exact sequences

$$0 \longrightarrow \mathcal{K} \longrightarrow f_*\mathcal{F} \longrightarrow \operatorname{Image}(\xi) \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Image}(\xi) \longrightarrow \mathcal{G}^{\oplus m} \longrightarrow \mathcal{C} \longrightarrow 0.$$

$\mathcal{K}$ and $\mathcal{C}$ are coherent $\mathcal{O}_Y$-modules, and $\operatorname{Supp}(\mathcal{K}) \subsetneqq Y$ and $\operatorname{Supp}(\mathcal{C}) \subsetneqq Y$, since ξ is a generic isomorphism. Hence by the induction hypothesis, we have

$$H^p(Y, \mathcal{K} \otimes \mathcal{L}^n) = H^p(Y, \mathcal{C} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0.$$

By the cohomology long exact sequence, we have

$$\begin{array}{c} H^p(Y, (f_*\mathcal{F}) \otimes \mathcal{L}^n) \xrightarrow{\sim} H^p(Y, \operatorname{Image}(\xi) \otimes \mathcal{L}^n) \xrightarrow{\sim} H^p(Y, \mathcal{G} \otimes \mathcal{L}^n)^{\oplus m} \\ \| \\ H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) \end{array}$$

for all $p > 0$ and $n \gg 0$, the equality on the left hand side being again by the projection formula. $H^p(X, \mathcal{F} \otimes (f^*\mathcal{L})^n) = 0$ for all $p > 0$ and $n \gg 0$, since $f^*\mathcal{L}$ is assumed to be ample. Hence

$$H^p(Y, \mathcal{G} \otimes \mathcal{L}^n) = 0, \qquad \forall p > 0 \text{ and } n \gg 0.$$

$\square$

PROPOSITION 9.6. *Let X be a proper scheme over a noetherian ring A. Let $\mathcal{L}$ be an invertible sheaf on X, and assume that $\mathcal{L}$ is generated by its global sections. Suppose that for every closed integral curve C in X the restriction $\mathcal{L}|_C$ is ample. Then $\mathcal{L}$ is ample on X.*

For the proof we need the following result given in Proposition VIII.1.7:

Let C' be a geometrically irreducible curve, proper and smooth over a field k.
An invertible sheaf $\mathcal{L}'$ on C' is ample if and only if $\deg \mathcal{L}' > 0$.

PROOF. By Propositions 9.2 and 9.3 we may assume without loss of generality that X is integral. Since $\mathcal{L}$ is generated by global sections, a finite number of these define a morphism

$$\varphi\colon X \longrightarrow \mathbb{P}^n_A$$

such that $\mathcal{L} = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$. Then φ is a finite morphism. For otherwise, by Corollary V.6.5 some fiber of φ contains a closed integral curve C. Let $\varphi(C) = P$, a closed point of $\mathbb{P}^n_A$. Let $f\colon C' \to C$ be a morphism obtained as follows: C' is the normalization of C in a composite field $\overline{\Bbbk(P)}\mathbf{R}(C)$ obtained as a quotient of

$$\overline{\Bbbk(P)} \otimes_{\Bbbk(P)} \mathbf{R}(C),$$

where $\overline{\Bbbk(P)}$ is the algebraic closure of $\Bbbk(P)$. (C' is regular by Proposition V.5.11, hence is proper and smooth over $\overline{\Bbbk(P)}$.) Since $\mathcal{L}|_C$ is ample, so is $\mathcal{L}' = f^*\mathcal{L}$ by Proposition 9.4. But then $\deg \mathcal{L}' > 0$ by Proposition VIII.1.7, while $\mathcal{L}' = f^*\mathcal{L} = f^*\varphi^*\mathcal{O}_{\mathbb{P}}(1)$. This contradicts the fact that $\varphi(C) = P$ is a point. Hence φ is finite. Propositions 9.2, 9.3 and 9.4 now conclude the proof. □

10. The Euler characteristic

Throughout this section, we let A be a local artinian ring. We let

$$X \longrightarrow \operatorname{Spec}(A)$$

be a projective morphism. We let $\mathcal{F}$ be a coherent sheaf on X.

By Theorem 8.1, the cohomology groups $H^i(X,\mathcal{F})$ are finite A-modules, and since A is artinian, they have finite length. By (5.5) and Corollary 3.8, we also have $H^i(X,\mathcal{F}) = 0$ for i sufficiently large. We define the *Euler characteristic*

$$\chi_A(X,\mathcal{F}) = \chi_A(\mathcal{F}) = \sum_{i=0}^{\infty}(-1)^i \operatorname{length} H^i(X,\mathcal{F}).$$

This is a generalization of what we introduced in (7.5) in the case $A = k$ a field. As a generalization of Theorem 7.7, we have:

PROPOSITION 10.1. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be a short exact sequence of coherent sheaves on X. Then

$$\chi_A(\mathcal{F}) = \chi_A(\mathcal{F}') + \chi_A(\mathcal{F}'').$$

PROOF. This is immediate from the exact cohomology sequence

$$\longrightarrow H^p(X,\mathcal{F}') \longrightarrow H^p(X,\mathcal{F}) \longrightarrow H^p(X,\mathcal{F}'') \longrightarrow$$

which has 0's for $p < 0$ and p sufficiently large. cf. Lang [**75**, Chapter IV]. □

We now compute this Euler characteristic in an important special case.

PROPOSITION 10.2. *Suppose $\mathbb{P} = \mathbb{P}^r_A$. Then*

$$\chi_A(\mathcal{O}_{\mathbb{P}}(n)) = \binom{n+r}{r} = \frac{(n+r)(n+r-1)\cdots(n+1)}{r!} \quad \textit{for all } n \in \mathbb{Z}.$$

PROOF. For $n > 0$, we can apply Corollary 5.4 to conclude that

$$\chi_A(\mathcal{O}_{\mathbb{P}}(n)) = \operatorname{length} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)),$$

which is the number of monomials in $T_0, \ldots, T_r$ of degree n, and is therefore equal to the binomial coefficient as stated. If $n \leq -r-1$, then similarly by (5.6), we have

$$\chi_k(\mathcal{O}_{\mathbb{P}}(n)) = (-1)^r \operatorname{length} H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)).$$

From the explicit determination of the cohomology in (5.6) if we put $n = -r-d$ then the length of $H^r(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n))$ over A is equal to the number of r-tuples $(q_0, \ldots, q_r)$ of integers $q_j > 0$ such that $\sum q_j = r + d$, which is equal to the number of r-tuples $(q'_0, \ldots, q'_r)$ of integers ≥ 0 such that $\sum q'_j = d - 1$. This is equal to

$$\binom{d-1+r}{r} = (-1)^r \binom{n+r}{r}.$$

Finally, let $-r \leq n \leq 0$. Then $H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)) = 0$ for all $i > 0$ once more by Corollary 5.4 and (5.6). Also the binomial coefficient is 0. This proves the proposition. □

Starting with the explicit case of projective space as in Proposition 10.2, we can now derive a general result, which is a generalization of Theorem 7.7 in the case $A = k$ a field.

THEOREM 10.3. *Let A be a local artinian ring. Let X be a projective scheme over $Y = \operatorname{Spec}(A)$. Let $\mathcal{L}$ be an invertible sheaf on X, very ample over Y, and let $\mathcal{F}$ be a coherent sheaf on X. Put*

$$\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^n \quad \textit{for } n \in \mathbb{Z}.$$

i) *There exists a unique polynomial $P(T) \in \mathbb{Q}[T]$ such that*

$$\chi_A(\mathcal{F}(n)) = P(n) \quad \textit{for all } n \in \mathbb{Z}.$$

ii) *For n sufficiently large, $\chi_A(\mathcal{F}(n)) = \operatorname{length} H^0(X, \mathcal{F}(n))$.*

iii) *The leading coefficient of $P(T)$ is ≥ 0.*

PROOF. By Theorem 8.1 we know that

$$H^i(\mathcal{F}(n)) = 0 \quad \text{for } i \geq 1 \text{ and } n \text{ large.}$$

Hence $\chi_A(\mathcal{F}(n))$ is the length of $H^0(\mathcal{F}(n))$ as asserted in (ii). In particular, $\chi_A(\mathcal{F}(n))$ is ≥ 0 for n large, so the leading coefficient of $P(T)$ is ≥ 0 if such polynomial exists. Its uniqueness is obvious.

To show the existence, we reduce to the case of Proposition 10.2 by Jordan-Hölder techniques. Suppose we have an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

Taking the tensor product with $\mathcal{L}$ preserves exactness. It follows immediately that if (i) is true for $\mathcal{F}'$ and $\mathcal{F}''$, then (i) is true for $\mathcal{F}$. Let $\mathfrak{m}$ be the maximal ideal of A. Then there is a finite filtration

$$\mathcal{F} \supset \mathfrak{m}\mathcal{F} \supset \mathfrak{m}^2\mathcal{F} \supset \cdots \supset \mathfrak{m}^s\mathcal{F} = 0.$$

By the above remark, we are reduced to proving (i) when $\mathfrak{m}\mathcal{F} = 0$, because $\mathfrak{m}$ annihilates each factor sheaf $\mathfrak{m}^j\mathcal{F}/\mathfrak{m}^{j+1}\mathcal{F}$.

Suppose now that $\mathfrak{m}\mathcal{F} = 0$. Then $\mathcal{F}$ can be viewed as a sheaf on the fibre X_y, where y is the closed point of $Y = \operatorname{Spec}(A)$. The restriction of $\mathcal{L}$ to X_y is ample by Proposition 9.1, and the cohomology groups of a sheaf on a closed subscheme are the same as those of that same sheaf viewed on the whole scheme. The twisting operation also commutes with passing to a closed subscheme. This reduces the proof that $\chi_A(\mathcal{F}(n))$ is a polynomial to the case when A is a field k. Thus we are done by Theorem 7.7. $\square$

When $A = k$ is a field, the length is merely the dimension over k. For any coherent sheaf $\mathcal{F}$ on X we have by definition

$$\chi(\mathcal{F}) = \sum_{i=0}^{d} (-1)^i \dim_k H^i(X, \mathcal{F}),$$

where $d = \dim X$. By Theorem 7.7, we know that

$$P(n) = \chi(\mathcal{F}(n))$$

is a polynomial of degree e where $e = \dim \operatorname{Supp} \mathcal{F}$.

Remark. (Added in publication) As a part of the results on cohomology and base change, Grothendieck showed in EGA [**1**, Chapter III, Theorem 7.9.4] the following:

> Let $f\colon X \to Y$ be a proper morphism of noetherian schemes, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module flat over Y. Then
>
> $$Y \ni y \longmapsto \chi(X_y, \mathcal{F}_y) := \sum_{i=0}^{\infty} (-1)^i \dim_{\Bbbk(y)} H^i(X_y, \mathcal{F}_y)$$
>
> is locally constant, where
>
> $$X_y := X \times_Y \operatorname{Spec}(\Bbbk(y)) \qquad \text{a scheme over } \Bbbk(y)$$
> $$\mathcal{F}_y := \mathcal{F} \otimes_{\mathcal{O}_{Y,y}} \Bbbk(y) \qquad \text{an } \mathcal{O}_{X_y}\text{-module.}$$

For the proof, see also Mumford [**85**, Chapter II, §5].

11. Intersection numbers

Throughout this section we let X be a proper scheme over a field k. We let $\chi = \chi_k$.

Theorem 11.1 (Snapper). *Let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be invertible sheaves on X and let $\mathcal{F}$ be a coherent sheaf. Let $d = \dim \operatorname{Supp}(\mathcal{F})$. Then there exists a polynomial P with rational coefficients, in r variables, such that for all integers $n_1, \dots, n_r$ we have*

$$P(n_1, \dots, n_r) = \chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}).$$

This polynomial P has total degree $\leq d$.

Proof. Suppose first that $\mathcal{L}_1, \dots, \mathcal{L}_r$ are very ample. Then the assertion follows by induction on r and Theorem 7.7 (generalized in Theorem 10.3). Suppose X projective. Then there exists a very ample invertible sheaf $\mathcal{L}_0$ such that $\mathcal{L}_0\mathcal{L}_1, \dots, \mathcal{L}_0\mathcal{L}_r$ are very ample (take any very ample sheaf, raise it to a sufficiently high power and use Theorem III.5.10). Let

$$Q(n_0, n_1, \dots, n_r) = \chi\left(\mathcal{L}_0^{n_0} \otimes (\mathcal{L}_0\mathcal{L}_1)^{n_1} \otimes \cdots \otimes (\mathcal{L}_0\mathcal{L}_r)^{n_r} \otimes \mathcal{F}\right).$$

Then

$$P(n_1, \dots, n_r) = Q(-n_1 - \cdots - n_r, n_1, \dots, n_r)$$

and the theorem follows.

If X is not projective, the proof is more complicated. We follow Kleiman [**70**]. The proof proceeds by induction on $d = \dim \mathrm{Supp}(\mathcal{F})$. Since the assertion is trivial if $d = -1$, i.e., $\mathcal{F} = (0)$, we assume $d \geq 0$.

Replacing X by the closed subscheme $\mathrm{Spec}_X(\mathcal{O}_X/\mathrm{Ann}(\mathcal{F}))$ defined by the annihilator ideal $\mathrm{Ann}(\mathcal{F})$, we may assume $\mathrm{Supp}(\mathcal{F}) = X$. The induction hypothesis then means that the assertion is true for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\mathrm{Supp}(\mathcal{F}) \subsetneqq X$, i.e., for torsion $\mathcal{O}_X$-modules $\mathcal{F}$.

Let $\mathbf{K}$ be the abelian category of coherent $\mathcal{O}_X$-modules, and let $\mathbf{K}' \subset \mathrm{Ob}(\mathbf{K})$ consist of those $\mathcal{F}$'s for which the assertion holds. $\mathbf{K}'$ is obviously exact in the sense of Definition II.6.11. By dévissage (Theorem II.6.12), it suffices to show that $\mathcal{O}_Y \in \mathbf{K}'$ for any closed integral subscheme Y of X. In view of the induction hypothesis, we may assume $Y = X$, that is, X itself is integral. Then by Proposition III.6.2, there exists a Cartier divisor D on X such that $\mathcal{L}_1 = \mathcal{O}_X(D)$ and that the intersections $\mathcal{I} = \mathcal{O}_X(-D) \cap \mathcal{O}_X$ as well as $\mathcal{J} = \mathcal{O}_X(D) \cap \mathcal{O}_X$ taken inside the function field $\mathbf{R}(X)$ are coherent $\mathcal{O}_X$-ideals not equal to $\mathcal{O}_X$. Obviously, we have $\mathcal{J} = \mathcal{I} \otimes \mathcal{O}_X(D) = \mathcal{I} \otimes \mathcal{L}_1$. Tensoring the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0$ (resp. $0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{J} \to 0$) with $\mathcal{L}_1^{n_1}$ (resp. $\mathcal{L}_1^{n_1-1}$), we have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I} \otimes \mathcal{L}_1^{n_1} & \longrightarrow & \mathcal{L}_1^{n_1} & \longrightarrow & \mathcal{L}_1^{n_1} \otimes (\mathcal{O}_X/\mathcal{I}) & \longrightarrow & 0 \\
 & & \| & & & & & & \\
0 & \longrightarrow & \mathcal{J} \otimes \mathcal{L}_1^{n_1-1} & \longrightarrow & \mathcal{L}_1^{n_1-1} & \longrightarrow & \mathcal{L}_1^{n_1-1} \otimes (\mathcal{O}_X/\mathcal{J}) & \longrightarrow & 0.
\end{array}$$

Thus tensoring both sequences with $\mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}$ and taking the Euler characteristic, we have

$$\begin{aligned}
&\chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) - \chi(\mathcal{L}_1^{n_1-1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r}) \\
&\quad = \chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes (\mathcal{O}_X/\mathcal{I})) - \chi(\mathcal{L}_1^{n_1-1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes (\mathcal{O}_X/\mathcal{J})).
\end{aligned}$$

The right hand side is a polynomial with rational coefficients in $n_1, \ldots, n_r$ of total degree $< d$ since $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{J}$ are torsion $\mathcal{O}_X$-modules. Hence we are done, since $\chi(\mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r})$ is a polynomial in $n_2, \ldots, n_r$ of total degree $\leq d$ by induction on r. □

We recall here the following result on integral valued polynomials.

Lemma 11.2. *Let $P(x_1, \ldots, x_r) \in \mathbb{Q}[x_1, \ldots, x_r] = \mathbb{Q}[x]$ be a polynomial with rational coefficients, and integral valued on $\mathbb{Z}^r$. Then P admits an expression*

$$P(x_1, \ldots, x_r) = \sum a(i_1, \ldots, i_r) \binom{x_1 + i_1}{i_1} \cdots \binom{x_r + i_r}{i_r}$$

where $a(i_1, \ldots, i_r) \in \mathbb{Z}$, the sum is taken for $i_1, \ldots, i_r \geq 0$,

$$\binom{x+i}{i} = \frac{(x+i)(x+i-1)\cdots(x+1)}{i!} \quad \text{if } i > 0,$$

and the binomial coefficients is 1 if $i = 0$, and 0 if $i < 0$.

Proof. This is proved first for one variable by induction, and then for several variables by induction again. We leave this to the reader. □

Lemma 11.3. *The coefficient of $n_1 \cdots n_r$ in $\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F})$ is an integer.*

Proof. Immediate from Lemma 11.2. □

Let us define the *intersection symbol*:

$$(\mathcal{L}_1.\mathcal{L}_2 \ldots \mathcal{L}_r.\mathcal{F}) = \text{coefficient of } n_1 \cdots n_r \text{ in the polynomial } \chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}).$$

Lemma 11.4.

(i) *The function* $(\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F})$ *is multilinear in* $\mathcal{L}_1, \dots, \mathcal{L}_r$.
(ii) *If* c *is the coefficient of* n^r *in* $\chi(\mathcal{L}^n \otimes \mathcal{F})$, *then*

$$(\mathcal{L}.\mathcal{L} \dots \mathcal{L}.\mathcal{F}) = r!c \quad (\mathcal{L} \textit{ is repeated } r \textit{ times.})$$

PROOF. Let $\mathcal{L}$, $\mathcal{M}$ be invertible sheaves. Then

$$\chi(\mathcal{L}^n \otimes \mathcal{M}^m \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{F}) = ann_2 \cdots n_r + bmn_2 \cdots n_r + \cdots$$

with rational coefficients a, b. Putting $n = 0$ and $m = 0$ shows that

$$a = (\mathcal{L}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}) \qquad \text{and} \qquad b = (\mathcal{M}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}).$$

Let $m = n = n_1$. It follows that

$$((\mathcal{L} \otimes \mathcal{M}).\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}) = (\mathcal{L}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}) + (\mathcal{M}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}).$$

Similarly, $(\mathcal{L}^{-1}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}) = -(\mathcal{L}.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F})$. This proves the first assertion.

As to the second, let $P(n) = \chi(\mathcal{L}^n.\mathcal{F})$ and

$$Q(n_1, \dots, n_r) = \chi(\mathcal{L}_1^{n_1} \dots \mathcal{L}_r^{n_r}.\mathcal{F}).$$

Let ∂ be the derivative, and $\partial_1, \dots, \partial_r$ be the partial derivatives. Then the second assertion follows from the relation

$$\partial_1 \cdots \partial_r Q(0, \dots, 0) = \partial^r P(0).$$

□

The next lemma gives the additivity as a function of $\mathcal{F}$, in the sense of the Grothendieck group.

LEMMA 11.5. *Let*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

be an exact sequence of coherent sheaves. Then

$$(\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}) = (\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}') + (\mathcal{L}_1 \dots \mathcal{L}_r.\mathcal{F}'').$$

PROOF. Immediate since the Euler characteristic satisfies the same type of relation. □

REMARK. (Added in publication) Let X be a proper scheme over an algebraically closed field k. For a connected noetherian scheme T over k, consider the scheme $\mathcal{X} := X \times_{\mathrm{Spec}(k)} T$ over T and a coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ that is flat over T. For any closed point t of T, we have a family of coherent $\mathcal{O}_X$-modules $\mathcal{F}_t$ on $\mathcal{X}_t = X$. By what we remarked at the end of §10,

$$\chi(X, \mathcal{F}_t) \qquad \text{is independent of } t.$$

Consequently for invertible sheaves $\mathcal{L}_1, \dots, \mathcal{L}_r$ on X, we have

$$(\mathcal{L}_1.\mathcal{L}_2 \dots \mathcal{L}_r.\mathcal{F}_t) \qquad \text{is independent of } t.$$

12. The criterion of Nakai-Moishezon

Let X be a proper scheme over a field k.

Let Y be a closed subscheme of X. Then Y is defined by a coherent sheaf of ideals $\mathcal{I}_Y$, and

$$\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}_Y$$

is its structure sheaf. Let $D_1, \ldots, D_r$ be divisors on X, by which we always mean Cartier divisors, so they correspond to invertible sheaves $\mathcal{L}_1 = \mathcal{O}_X(D_1), \ldots, \mathcal{L}_r = \mathcal{O}_X(D_r)$. *Suppose Y has dimension r.* We define the *intersection number*

$$(D_1.D_2 \ldots D_r.Y) = \text{coefficient of } n_1 \cdots n_r \text{ in the polynomial}$$
$$\chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{O}_Y).$$
$$(D^r.Y) = (D \ldots D.Y), \quad \text{where } D \text{ is repeated } r \text{ times.}$$

LEMMA 12.1.

(i) *The intersection number $(D_1.D_2 \ldots D_r.Y)$ is an integer, and the function*

$$(D_1, \ldots, D_r) \longmapsto (D_1 \ldots D_r.Y)$$

is multilinear symmetric.

(ii) *If a is the coefficient of n^r in $\chi(\mathcal{L}^n \otimes \mathcal{O}_Y)$, and $\mathcal{L} = \mathcal{O}_X(D)$, then $(D^r.Y) = r!a$.*

PROOF. This is merely a repetition of Lemma 11.4 in the present context and notation. $\square$

REMARK. (Added in publication) Let X be a proper scheme over an algebraically closed field k. For a connected noetherian scheme T over k, consider the scheme $\mathcal{X} := X \times_{\mathrm{Spec}(k)} T$ over T and its closed subscheme $\mathcal{Y}$ that is flat over T with r-dimensional fibres. We thus have a family of r-dimensional closed subschemes $\mathcal{Y}_t$ of $\mathcal{X}_t = X$ parametrized by closed points t of T. By what we remarked at the end of §11, we see that

$$(D_1.D_2 \ldots D_r.\mathcal{Y}_t) \qquad \text{is independent of } t.$$

REMARK. Suppose that Y is zero dimensional, so Y consists of a finite number of closed points. Then the higher cohomology groups are 0, and

$$(Y) = \chi(\mathcal{O}_Y) = \dim H^0(Y, \mathcal{O}_Y) > 0,$$

because $H^0(Y, \mathcal{O}_Y)$ is the vector space of global sections, and is not 0 since Y is affine. One can reduce the general intersection symbol to this case by means of the next lemma.

LEMMA 12.2. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be invertible sheaves on X such that $\mathcal{L}_1$ is very ample. Let D_1 be a divisor corresponding to $\mathcal{L}_1$ such that D_1 does not contain any associated point of $\mathcal{O}_Y$. Let Y' be the scheme intersection of Y and D_1. Then*

$$(D_1 \ldots D_r.Y) = (D_2 \ldots D_r.Y').$$

In particular, if $D_1, \ldots, D_r$ are ample, then

$$(D_1 \ldots D_r.Y) > 0.$$

PROOF. If $\mathcal{I}_Y$ is the sheaf of ideals defining Y, and $\mathcal{I}_1$ is the sheaf of ideals defining D_1, the $(\mathcal{I}_Y, \mathcal{I}_1)$ defines $Y \cap D_1$. By §III.6 we know that $\mathcal{I}_1$ is locally principal. The assumption in the lemma implies that we have an exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{I}_1 \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y \cap D_1} \longrightarrow 0.$$

Indeed, let $\mathrm{Spec}(A)$ be an open affine subset of X containing a generic point of Y, and such that D_1 is represented in A by the local equation $f = 0$, while Y is defined by the ideal I. Then the above sequence translates to

$$0 \longrightarrow fA \otimes A/I \longrightarrow A/I \longrightarrow A/(I, f) \longrightarrow 0$$

which is exact on the left by our assumption on f.

But $\mathcal{I}_1 = \mathcal{L}_1^{-1}$ by the definitions. Tensoring the sequence $(*)$ with

$$\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r},$$

taking the Euler characteristic, and using the additivity of the Euler characteristic, we get

$$\begin{aligned}(D_1 \ldots D_r.Y)n_1 \cdots n_r - (D_1 \ldots D_r.Y)(n_1 - 1)n_2 \cdots n_r &+ \text{lower terms} \\ = (D_2 \ldots D_r.Y')n_2 \cdots n_r &+ \text{lower terms.}\end{aligned}$$

This proves the lemma. $\square$

The intersection number $(D^r.Y)$ was taken with respect to the scheme X and it is sometimes necessary to include X in the notation, so we write

$$(D_1 \ldots D_r.Y)_X \quad \text{or} \quad (\mathcal{L}_1 \ldots \mathcal{L}_r.Y)_X.$$

On the other hand, let Z be a closed subscheme of X. Then we may induce the sheaves to Z to get $\mathcal{L}_1|_Z, \ldots, \mathcal{L}_r|_Z$.

LEMMA 12.3. *Let $Y \subset Z \subset X$ be inclusions of closed subschemes. Suppose Y has dimension r as before. Then*

$$(\mathcal{L}_1 \ldots \mathcal{L}_r.Y)_X = (\mathcal{L}_1|_Z \ldots \mathcal{L}_r|_Z.Y)_Z.$$

PROOF. In the tensor products

$$\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_r^{n_r} \otimes \mathcal{O}_Y$$

we may tensor with $\mathcal{O}_Z$ each one of the factors without changing this tensor product. The cohomology of a sheaf supported by a closed subscheme is the same as the cohomology of the sheaf in the scheme itself (cf. Corollary 3.8), so the assertion of the lemma is now clear. $\square$

THEOREM 12.4 (Criterion of Nakai-Moishezon). *Let X be a proper scheme over a field k. Then a divisor D is ample on X if and only if $(D^r.Y) > 0$ for all integral closed subschemes Y of dimension r, for all $r \leq \dim X$.*

PROOF. Suppose D is ample. Replacing D by a positive multiple, we may assume without loss of generality that D is very ample. Let $\mathcal{L} = \mathcal{O}(D)$, and let $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}}(1)$ for a projective embedding $f \colon X \to \mathbb{P}$ over k. Abbreviate $\mathcal{H} = \mathcal{O}_{\mathbb{P}}(1)$. Then the Euler characteristic

$$\chi_k(\mathcal{L}^{n_1} \otimes \cdots \otimes \mathcal{L}^{n_r} \otimes \mathcal{O}_Y)$$

is the same as the Euler characteristic

$$\chi_k(\mathcal{H}^{n_1} \otimes \cdots \otimes \mathcal{H}^{n_r} \otimes \mathcal{O}_Y)$$

where $\mathcal{O}_Y$ is now viewed as a sheaf on $\mathbb{P}$. This reduces the positivity to the case of projective space, and D is a hyperplane, which is true by Lemma 12.2.

The converse is more difficult and is the essence of the Nakai-Moishezon theorem. We assume that $(D^r.Y) > 0$ for all integral closed subschemes Y of X of dimension $r \leq \dim X$ and we want to prove that D is ample. By Propositions 9.2 and 9.3, we may assume that X is integral (reduced and irreducible), so X is a variety.

For the rest of the proof we let $\mathcal{L} = \mathcal{O}(D)$.

By Lemma 12.3 and induction we may assume that $\mathcal{L}|_Z$ is ample for every closed subscheme Z of X, $Z \neq X$.

Lemma 12.5. *For n large, $H^0(X, \mathcal{L}^n) \neq 0$.*

Proof of Lemma 12.5. First we remark that $\chi(\mathcal{L}^n) \to \infty$ as $n \to \infty$, for by Lemma 12.1 (ii),

$$\chi(\mathcal{L}^n) = an^d + \text{lower terms}$$

where $d = \dim X$, and $r!a = (D^d.X) > 0$ by assumption.

Next, we prove that $H^i(\mathcal{L}^n) \approx H^i(\mathcal{L}^{n-1})$ for $i \geq 2$ and $n \geq n_0$. Since X is integral, we can identify $\mathcal{L}$ as a subsheaf of the sheaf of rational functions on X. We let

$$\mathcal{I} = \mathcal{L}^{-1} \cap \mathcal{O}_X.$$

Then $\mathcal{I}$ is a coherent sheaf of ideals of $\mathcal{O}_X$, defining a closed subscheme $Y \neq X$. Furthermore $\mathcal{I} \otimes \mathcal{L}$ is also a coherent sheaf of ideals, defining a closed subscheme $Z \neq X$. We have two exact sequences

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{L} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

We tensor the first with $\mathcal{L}^n$ and the second with $\mathcal{L}^{n-1}$. By induction, $H^i(\mathcal{L}^n|_Y) = H^i(\mathcal{L}^{n-1}|_Z) = 0$ for $i \geq 1$ and $n \geq n_0$. Then the exact cohomology sequence gives isomorphisms for $i \geq 2$ and $n \geq n_0$:

$$H^i(\mathcal{I} \otimes \mathcal{L}^n) \approx H^i(\mathcal{L}^n) \quad \text{and} \quad H^i(\mathcal{I} \otimes \mathcal{L} \otimes \mathcal{L}^{n-1}) \approx H^i(\mathcal{L}^{n-1}).$$

This proves that $H^i(\mathcal{L}^n) \approx H^i(\mathcal{L}^{n-1})$ for $i \geq 2$. But then

$$\dim H^0(\mathcal{L}^n) \geq \chi(\mathcal{L}^n) \to \infty,$$

thus proving the lemma. □

A global section of $\mathcal{L}^n$ then implies the existence of an effective divisor $E \sim nD$, and since the intersection number depends only on the linear equivalence class (namely, on the isomorphism class of the invertible sheaves), the hypothesis of the theorem implies that $(E^r.Y) > 0$ for all closed subschemes Y of X. It will suffice to prove that E is ample. This reduces the proof of the theorem to the case when D is effective, which we now assume.

Lemma 12.6. *Assume D effective. Then for sufficiently large n, $\mathcal{L}^n$ is generated by its global sections.*

Proof of Lemma 12.6. We have $\mathcal{L} = \mathcal{O}(D)$ where D is effective, so we have an exact sequence

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Tensoring with $\mathcal{L}^n$ yields the exact sequence

$$0 \longrightarrow \mathcal{L}^{n-1} \longrightarrow \mathcal{L}^n \longrightarrow \mathcal{L}^n|_D \longrightarrow 0.$$

By induction, $\mathcal{L}^n|_D$ is ample on D, so $H^1(\mathcal{L}^n|_D) = 0$ for n large. The cohomology sequence

$$H^0(\mathcal{L}^n) \longrightarrow H^0(\mathcal{L}^n|_D) \longrightarrow H^1(\mathcal{L}^{n-1}) \longrightarrow H^1(\mathcal{L}^n) \longrightarrow H^1(\mathcal{L}^n|_D)$$

shows that $H^1(\mathcal{L}^{n-1}) \to H^1(\mathcal{L}^n)$ is surjective for n large. Since the vector spaces $H^1(\mathcal{L}^n)$ are finite dimensional, there exists n_0 such that

$$H^1(\mathcal{L}^{n-1}) \longrightarrow H^1(\mathcal{L}^n) \quad \text{is an isomorphism for } n \geq n_0.$$

Now the first part of the cohomology exact sequence shows that

$$H^0(\mathcal{L}^n) \longrightarrow H^0(\mathcal{L}^n|_D) \quad \text{is surjective for } n \geq n_0.$$

Since $\mathcal{L}^n|_D$ is ample on D, it is generated by global sections. By Nakayama, it follows that $\mathcal{L}^n$ is generated by global sections. This proves Lemma 12.6 □

We return to the proof of Theorem 12.4 proper. If $\dim X = 1$, then $(D) > 0$, X is a curve, and every effective non-zero divisor on a curve is ample (cf. Proposition VIII.1.7 below).

Suppose $\dim X \geq 2$. For every integral curve (subscheme of dimension one) C on X, we know by induction that $\mathcal{L}^n|_C$ is ample on C. We can apply Proposition 9.6 to conclude the proof. □

Exercise—Modifications needed

Bezout's Theorem via the Spencer resolution.

(1) If C is any abelian category, define

$$K^0(C) = \left(\begin{array}{l} \text{free abelian group on elements } [X]\text{, one for each} \\ \text{isomorphism class of objects in } C\text{, modulo relations} \\ [X_2] = [X_1] + [X_3] \text{ for each short exact sequence:} \\ 0 \to X_1 \to X_2 \to X_3 \to 0 \\ \text{in } C. \end{array} \right)$$

If X is any noetherian scheme, define

$K_0(X) = K^0(\text{Category of coherent sheaves of } \mathcal{O}_X\text{-modules on } X)$

$K^0(X) = K^0(\text{Category of locally free finite rank sheaves of } \mathcal{O}_X\text{-modules})$.

Prove:

a) $\exists$ a natural map $K^0(X) \to K_0(X)$.

b) $K^0(X)$ is a contravariant functor in X, i.e., $\forall$ morphism $f\colon X \to Y$, we get $f^*\colon K^0(Y) \to K^0(X)$ with the usual properties.

c) $K^0(X)$ is a commutative ring via

$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2]$$

and $K_0(X)$ is a $K^0(X)$-module via

$$[\mathcal{E}] \cdot [\mathcal{F}] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}].$$

d) $K_0(X)$ is a covariant functor for *proper* morphisms $f\colon X \to Y$ via

$$f_*([\mathcal{F}]) = \sum_{n=0}^{\infty} (-1)^n [R^n f_* \mathcal{F}].$$

(2) Return to the case where k is an infinite field.

a) Using the Spencer resolution, show that

$$K^0(\mathbb{P}^n_k) \longrightarrow K_0(\mathbb{P}^n_k)$$

is surjective and that they are both generated by the sheaves $[\mathcal{O}_{\mathbb{P}^n}(l)]$, $l \in \mathbb{Z}$.
Hint: On any scheme X, if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

is exact, $\mathcal{E}_i$ locally free and finitely generated, then $\mathcal{F}$ is locally free and finitely generated, and *locally* on X, the sequence *splits*, i.e., $\mathcal{E}_1 \cong \mathcal{F} \oplus \mathcal{E}_0$.

b) Consider the Koszul complex $K^\cdot(X_0, \dots, X_n; k[X_0, \dots, X_n])$. Take $\widetilde{}$ and hence show that $[\mathcal{O}_{\mathbb{P}^n}(l)] \in K^0(\mathbb{P}^n_k)$ satisfy

$$(*) \qquad \sum_{\nu=0}^{n+1} (-1)^\nu \binom{n+1}{\nu} [\mathcal{O}_{\mathbb{P}^n}(\nu + \nu_0)] = 0, \quad \forall \nu_0 \in \mathbb{Z},$$

hence $K^0(\mathbb{P}^n_k)$ is generated by $[\mathcal{O}_{\mathbb{P}^n}(\nu)]$ for any set of ν's of the form $\nu_0 \leq \nu \leq \nu_0 + n$. Show that $[\mathcal{O}_{L^\nu}]$, $0 \leq \nu \leq n$, $L^\nu =$ a fixed linear space of dimension ν, generate $K_0(\mathbb{P}^n_k)$.

c) Let

$$S_n = \left(\begin{array}{l} \text{group of rational polynomials } P(t) \text{ of degree } \leq n \\ \text{taking integer values at integers} \end{array} \right)$$
$$= \left(\begin{array}{l} \text{free abelian group on the polynomials} \\ P_\nu(t) = \binom{t}{\nu}, \quad 0 \leq \nu \leq n \end{array} \right).$$

Prove that

$$[\mathcal{F}] \longmapsto \text{Hilbert polynomial of } \mathcal{F}$$

defines

$$K_0(\mathbb{P}^n_k) \longrightarrow S_n.$$

d) Combining (a), (b) and (c), show that $K^0(\mathbb{P}^n_k) \xrightarrow{\sim} K_0(\mathbb{P}^n_k)$.

e) Using the result of Part I [**87**, §6C] show that if $Z \subset \mathbb{P}^n_k$ is any subvariety, of dimension r and

$$\begin{aligned} g_\nu &= p_a(Z \cdot H_1 \cdots H_{r-\nu}) \\ &= \text{arithmetic genus of the } \nu\text{-dimensional} \\ &\qquad \text{linear section of } Z, \quad (1 \leq \nu \leq r) \end{aligned}$$

$d = \deg Z$, then in $K_0(\mathbb{P}^n_k)$:

$$\begin{aligned} [\mathcal{O}_Z] = d \cdot [\mathcal{O}_{L^n}] &+ (1 - d - g_1)[\mathcal{O}_{L^{r-1}}] + (g_1 + g_2)[\mathcal{O}_{L^{r-2}}] + \\ &\cdots + (-1)^n (g_{r-1} + g_r)[\mathcal{O}_{L^0}]. \end{aligned}$$

(3) Because of (2), (d), $K_0(\mathbb{P}^n_k)$ inherits a ring structure. Using the sheaves $\mathcal{T}or_i$ defined in §2 as one of the applications of the "easy lemma of the double complex" (Lemma 2.4), show that this ring structure is given by

$$(*) \qquad [\mathcal{F}_1] \cdot [\mathcal{F}_2] = \sum_{i=0}^{n} (-1)^i [\mathcal{T}or_i(\mathcal{F}_1, \mathcal{F}_2)].$$

In particular, check that $\mathcal{T}or_i = (0)$ if $i > n$. (In fact, on any regular scheme X, it can be shown that $\mathcal{T}or_i = (0)$, $i > \dim X$; and that $(*)$ defines a ring structure in $K_0(X)$). Next apply this with $\mathcal{F}_1 = \mathcal{O}_{X_1}$, $\mathcal{F}_2 = \mathcal{O}_{X_2}$, X_1, X_2 subvarieties of $\mathbb{P}^n_k$ intersecting properly and transversely at generic points of the components $W_1, \dots, W_\nu$ of $X_1 \cap X_2$ (cf. Part I [**87**, §5B]). Show by Ex. 2, §5D2???, that if $i \geq 1$,

$$\dim \operatorname{Supp}(\mathcal{T}or_i(\mathcal{O}_{X_1}, \mathcal{O}_{X_2})) < \dim X_1 \cap X_2.$$

Combining this with the results of (2), show Bezout's Theorem:

$$(\deg X_1) \cdot (\deg X_2) = \sum_{i=1}^{\nu} \deg W_i.$$

Hint: Show that $[\mathcal{O}_{L^r}] \cdot [\mathcal{O}_{L^s}] = [\mathcal{O}_{L^{r+s-n}}]$. Show next that if $i \geq 1$

$$[\mathcal{T}or_i(\mathcal{O}_{X_1}, \mathcal{O}_{X_2})] = \text{combination of } [\mathcal{O}_{L^t}] \text{ for } t < \dim(X_1 \cap X_2).$$

CHAPTER VIII

Applications of cohomology

In this chapter, we hope to demonstrate the usefulness of the formidable tool that we developed in Chapter VII. We will deal with several topics that are tied together by certain common themes, although not in a linear sequence. We will start with possibly the most famous theorem in all algebraic geometry: the Riemann-Roch theorem for curves. This has always been the principal non-trivial result of an introduction to algebraic geometry and we would not dare to omit it. Besides being the key to the higher theory of curves, it also brings in differentials in an essential way — foreshadowing the central role played by the cohomology of differentials on all varieties. This theme, that of De Rham cohomology is discussed in §3. In order to be able to prove strong result there, we must first discuss in §2 Serre's cohomological approach to Chow's theorem, comparing analytic and algebraic coherent cohomologies. In §4 we discuss the application, following Kodaira, Spencer and Grothendieck, of the cohomology of Θ, the sheaf of vector fields, to deformation of varieties. Finally, in §§2, 3 and 4, we build up the tools to be able at the end to give Grothendieck's results on the partial computation of π_1 of a curve in characteristic p.

1. The Riemann-Roch theorem

As we discussed in §VII.7, cohomology, disguised in classical language, grew out of the attempt to develop formulas for the dimension of:

$$H^0(\mathcal{O}_X(D)) = \left\{ \begin{array}{l} \text{space of 0 and non-zero rational functions } f \text{ on } X \\ \text{with poles at most } D\text{, i.e., } (f)+D \geq 0 \end{array} \right\}.$$

(See also the remark in §III.6.)

Put another way, the general problem is to describe the filtration of the function field $\mathbf{R}(X)$ given by the size of the poles. This one may call the fundamental problem of the *additive* theory of functions on X (as opposed to the *multiplicative* theory dealing with the group $\mathbf{R}(X)^*$, and leading to $\mathrm{Pic}(X)$). Results on $\dim H^0(\mathcal{O}_X(D))$ lead in turn to results on the projective embeddings of X and other rational maps of X to $\mathbb{P}^n$, hence to many results on the geometry and classification of varieties X.

The first and still the most complete result of this type is the *Riemann-Roch theorem for curves.* This may be stated as follows:

THEOREM 1.1 (Riemann-Roth theorem). *Let k be a field and let X be a curve, smooth and proper over k such that X is* geometrically irreducible *(also said to be absolutely irreducible, i.e., $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \overline{k}$ is irreducible with $\overline{k}$ = algebraic closure of k). If $\sum n_i P_i$ ($P_i \in X$, closed points) is a divisor on X, define*

$$\deg(\sum n_i P_i) = \sum n_i [\Bbbk(P_i) : k].$$

Then for any divisor D on X:

1) $\dim_k H^0(\mathcal{O}_X(D)) - \dim_k H^1(\mathcal{O}_X(D)) = \deg D - g + 1$, *where $g = \dim_k H^1(\mathcal{O}_X)$ is the genus of X, and*

2) *(weak form)* $\dim_k H^1(\mathcal{O}_X(D)) = \dim_k H^0(\Omega^1_{X/k}(-D))$.

The first part follows quickly from our general theory like this:

PROOF OF 1). Note first that $H^0(\mathcal{O}_X)$ consists only in constants in k. In fact $H^0(\mathcal{O}_X)$ is a finite-dimensional k-algebra (cf. Proposition II.6.9), without nilpotents because X is reduced and without non-trivial idempotents because X is connected. Therefore $H^0(\mathcal{O}_X)$ is a field L, finite over k. By the theory of §IV.2, X smooth over $k \implies \mathbf{R}(X)$ separable over $k \implies L$ separable over k; and $X \times_k \overline{k}$ irreducible $\implies k$ separable algebraically closed in $\mathbf{R}(X) \implies L$ purely inseparable over k. Thus $L = k$, and (1) can be rephrased:

$$\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X).$$

Therefore Part (1) of Theorem 1.1 follows from:

LEMMA 1.2. *If P is a closed point on X and $\mathcal{L}$ is an invertible sheaf, then*

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + [\Bbbk(P) : k].$$

PROOF OF LEMMA 1.2. Use the exact sequence:

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \Bbbk(P) \longrightarrow 0$$

and the fact that $\mathcal{L}$ invertible $\implies \mathcal{L} \otimes_{\mathcal{O}_X} \Bbbk(P) \cong \underline{\Bbbk(P)}$ (where: $\underline{\Bbbk(P)} =$ sheaf (0) outside P, with stalk $\Bbbk(P)$ at P). Thus

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + \chi(\underline{\Bbbk(P)})$$

and since $H^0(\underline{\Bbbk(P)}) = \Bbbk(P)$, $H^1(\underline{\Bbbk(P)}) = (0)$, the result follows. □

□

To explain the rather mysterious second part, consider the first case $k = \mathbb{C}$, $D = \sum_{i=1}^d P_i$ with the P_i distinct, so that $\deg D = d$. Let $z_i \in \mathcal{O}_{P_i,X}$ vanish to first order at P_i, so that z_i is a local analytic coordinate in a small (classical) neighborhood of P_i. Then if $f \in H^0(\mathcal{O}_X(D))$, we can expand f near each P_i as:

$$f = \frac{a_i}{z_i} + \text{function regular at } P_i,$$

and we can map

$$H^0(\mathcal{O}_X(D)) \longrightarrow \mathbb{C}^d$$

$$f \longmapsto (a_1, \ldots, a_d)$$

by assigning the coefficients of their poles to each f. Since only constants have no poles, this shows right away that

$$\dim H^0(\mathcal{O}_X(D)) \leq d + 1.$$

Suppose on the other hand we start with $a_1, \ldots, a_d \in \mathbb{C}$ and seek to construct f. From elementary complex variable theory we find obstructions to the existence of this f! Namely, regarding X as a compact Riemann surface (= compact 1-dimensional complex manifold), we use the fact that if ω is a meromorphic differential on X, then the sum of the residues of ω at all its poles is zero (an immediate consequence of Cauchy's theorem). Now $\Omega^1_{X/\mathbb{C}}$ is the sheaf of *algebraic* differential forms on X and for any Zariski-open $U \subset X$ and $\omega \in \Omega^1_{X/\mathbb{C}}(U)$, ω defines a *holomorphic* differential form on U. (In fact if locally near $x \in U$,

$$\omega = \sum a_j db_j, \quad a_j, b_j \in \mathcal{O}_{x,X}$$

then a_j, b_j are holomorphic functions near x too and $\sum a_j db_j$ defines a holomorphic differential form: we will discuss this rather fine point more carefully in §3 below.) So if $\omega \in \Gamma(\Omega^1_{X/\mathbb{C}})$, then write ω near P_i as:

$$\omega = (b_i(\omega) + \text{function zero at } P_i) \cdot dz_i, \quad b_i(\omega) \in \mathbb{C}.$$

If f exists with poles a_i/z_i at P_i, then $f\omega$ is a meromorphic differential such that:

$$f\omega = a_i \cdot b_i(\omega) \cdot \frac{dz_i}{z_i} + (\text{differential regular at } P_i),$$

hence

$$\operatorname{res}_{P_i}(f\omega) = a_i \cdot b_i(\omega)$$

hence

$$0 = \sum_{i=1}^{d} \operatorname{res}_{P_i}(f\omega) = \sum_{i=1}^{d} a_i \cdot b_i(\omega).$$

This is a linear condition on $(a_1, \ldots, a_d)$ that must be satisfied if f is to exist. Now Assertion (2) of Theorem 1.1 in its most transparent form is just the converse: if $\sum a_i \cdot b_i(\omega) = 0$ for every $\omega \in \Gamma(\Omega^1_{X/\mathbb{C}})$, then f with polar parts a_i/z_i exists. How does this imply (2) as stated? Consider the pairing:

$$\mathbb{C}^d \times H^0(\Omega^1_{X/\mathbb{C}}) \longrightarrow \mathbb{C}$$

$$((a_i), \omega) \longmapsto \sum a_i \cdot b_i(\omega).$$

Clearly the null-space of this pairing on the $H^0(\Omega^1_{X/\mathbb{C}})$-side is the space of ω's zero at each P_i, i.e., $H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i))$. We have claimed that the null-space on the $\mathbb{C}^d$-side is Image $H^0(\mathcal{O}_X(\sum P_i))$. Thus we have a *non-degenerate* pairing:

$$\Big(\mathbb{C}^d / \operatorname{Image} H^0(\mathcal{O}_X(\sum P_i))\Big) \times \Big(H^0(\Omega^1_{X/\mathbb{C}}) / H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i))\Big) \longrightarrow \mathbb{C}.$$

Taking dimensions,

$$(*) \qquad d - \dim H^0(\mathcal{O}_X(\sum P_i)) + 1 = \dim H^0(\Omega^1_{X/\mathbb{C}}) - \dim H^0(\Omega_{X/\mathbb{C}}(-\sum P_i)).$$

Now it turns out that if $\sum_{i=1}^d P_i$ is a large enough positive divisor, $H^1(\mathcal{O}_X(\sum P_i)) = (0)$ and $H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i)) = (0)$ and this equation reads:

$$d - \chi(\mathcal{O}_X(\sum P_i)) + 1 = \dim H^0(\Omega^1_{X/\mathbb{C}}),$$

and since by Part (1) of Theorem 1.1, $\chi(\mathcal{O}_X(\sum P_i)) = d - g + 1$, it follows that $g = \dim H^0(\Omega^1_{X/\mathbb{C}})$. Putting this back in $(*)$, and using Part (1) of Theorem 1.1 again we get

$$\begin{aligned} g - \dim H^0(\Omega^1_{X/\mathbb{C}}(-\sum P_i)) &= d + 1 - \chi(\mathcal{O}_X(\sum P_i)) - \dim H^1(\mathcal{O}_X(\sum P_i)) \\ &= g - \dim H^1(\mathcal{O}_X(\sum P_i)) \end{aligned}$$

hence Part (2) of Theorem 1.1.

A more careful study of the above residue pairing leads quite directly to a *proof* of Assertion (2) of Theorem 1.1 when $k = \mathbb{C}$. Let us first generalize the residue pairing: if D_1 and D_2 are any two divisors on X such that $D_2 - D_1$ is positive (D_1, D_2 themselves arbitrary), then we get a pairing:

$$\left(\bigoplus_x \mathcal{O}_X(D_2)_x / \mathcal{O}_X(D_1)_x\right) \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}$$

as follows: given $(\overline{f}_x)$ representing a member of the left hand side ($f_x \in \mathcal{O}_X(D_2)_x$) and $\omega \in H^0(\Omega^1_{X/\mathbb{C}}(-D_1))$, pair these to $\sum_x \mathrm{res}_x(f_x \cdot \omega)$. Here $f_x \cdot \omega$ may have a pole of order >1 at x, but res_x still makes good sense: expand

$$f_x\omega = \left(\sum_{n=-N}^{+\infty} c_n t^n\right) dt$$

where t has a simple zero at x, and set $\mathrm{res}_x = c_{-1}$. Since

$$c_{-1} = \frac{1}{2\pi i}\oint f_x\omega$$

(taken around a small loop around x), c_{-1} is independent of the choice of t. Note that if $f'_x \in f_x + \mathcal{O}_X(D_1)_x$, then $f'_x \cdot \omega - f_x \cdot \omega \in \Omega^1_x$, hence $\mathrm{res}_x(f'_x\omega) = \mathrm{res}_x(f_x\omega)$. If $D_2 = \sum P_i$, $D_1 = 0$, we get the special case considered already. By the fact that the sum of the residues of any $\omega \in \Omega^1_{\mathbf{R}(X)/\mathbb{C}}$ is 0, the pairing factors as follows:

$$\text{(residue pairing)} \qquad \frac{\bigoplus_x \mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x}{\mathrm{Image}\, H^0(\mathcal{O}_X(D_2))} \times \frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D_2))} \longrightarrow \mathbb{C}.$$

It is trivial that this is non-degenerate on the right: i.e., if $\omega \in H^0(\Omega^1_{X/\mathbb{C}}(-D_1))\backslash H^0(\Omega^1_{X/\mathbb{C}}(-D_2))$, then for some $(\overline{f}_x)$, $\mathrm{res}_x(f_x\omega) \neq 0$. But in fact:

THEOREM 1.3 (Riemann-Roch theorem (continued)). (2)-strong form: *For every D_1, D_2 with $D_2 - D_1$ positive, the residue pairing is non-degenerate on both sides.*

PROOF OF THEOREM 1.3. First, note that the left hand side can be interpreted via H^1's: namely the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \mathcal{O}_X(D_2) \longrightarrow \bigoplus_x \mathcal{O}_x(D_2)/\mathcal{O}_x(D_1) \longrightarrow 0,$$

where $\mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x$ is the skyscraper sheaf at x with stalk $\mathcal{O}_x(D_2)/\mathcal{O}_x(D_1)$, induces an isomorphism

$$\frac{\bigoplus_x \mathcal{O}_x(D_2)/\mathcal{O}_x(D_1)}{\mathrm{Image}\, H^0(\mathcal{O}_X(D_2))} \cong \mathrm{Ker}\left[H^1(\mathcal{O}_X(D_1)) \longrightarrow H^1(\mathcal{O}_X(D_2))\right].$$

Now let D_2 increase. Whenever $D_2 < D'_2$ (i.e., $D'_2 - D_2$ a positive divisor), it follows that there are natural maps:

$$\frac{\bigoplus_x \mathcal{O}_X(D_2)_x/\mathcal{O}_X(D_1)_x}{\mathrm{Image}\, H^0(\mathcal{O}_X(D_2))} \underset{\text{injective}}{\hookrightarrow} \frac{\bigoplus_x \mathcal{O}_X(D'_2)_x/\mathcal{O}_X(D_1)_x}{\mathrm{Image}\, H^0(\mathcal{O}_X(D'_2))}$$

and

$$\frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D_2))} \underset{\text{surjective}}{\twoheadrightarrow} \frac{H^0(\Omega^1_{X/\mathbb{C}}(-D_1))}{H^0(\Omega^1_{X/\mathbb{C}}(-D'_2))}$$

compatible with the pairing. Passing to the limit, we get a pairing:

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbf{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbf{R}(X)\ \text{(embedded diagonally)}} \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}.$$

It follows immediately that if this is non-degenerate on the left, so is the original pairing. Note here that the left hand side can be interpreted as an H^1: namely the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \underset{\substack{\text{constant}\\ \text{sheaf}}}{\mathbf{R}(X)} \longrightarrow \bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbf{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow 0,$$

where $\mathbf{R}(X)/\mathcal{O}_X(D_1)_x$ is the skyscraper sheaf at x with stalk $\mathbf{R}(X)/\mathcal{O}_X(D_1)_x$, induces an isomorphism:

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbf{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbf{R}(X)} \cong H^1(\mathcal{O}_X(D_1)).$$

Thus we are now trying to show that we have via residue a perfect pairing:

$$H^1(\mathcal{O}_X(D_1)) \times H^0(\Omega^1_{X/\mathbb{C}}(-D_1)) \longrightarrow \mathbb{C}.$$

This pairing is known as "Serre duality". To continue, suppose

$$l\colon \bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbf{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow \mathbb{C}$$

is *any* linear function. Then $l = \sum l_x$, where

$$l_x\colon \mathbf{R}(X)/\mathcal{O}_X(D_1)_x \longrightarrow \mathbb{C}$$

is a linear function. Now if t_x has a simple zero at x, and $n_x =$ order of x in the divisor D_1, then let

$$c_\nu = l_x(t_x^{-\nu}), \quad \text{all } \nu \in \mathbb{Z}.$$

Note that $c_\nu = 0$ if $\nu \le -n_x$. Then we can write l_x formally:

$$l_x(f) = \operatorname{res}_x(f\cdot\omega_x)$$

where

$$\omega_x = \sum_{\nu=-n_x+1}^{+\infty} c_\nu t_x^\nu \cdot \frac{dt_x}{t_x}$$

is a *formal* differential at x; in fact

$$\omega_x \in \Omega^1_X(-D_1)_x.$$

This suggests defining, for the purposes of the proof only, *pseudo-section* of $\Omega^1_X(-D_1)$ to be a collection $(\omega_x)_{x\in X,\ \text{closed}}$, where $\omega_x \in \Omega^1_X(-D_1)_x$ are formal differentials and where

$$\sum_{\substack{x\in X\\ \text{closed}}} \operatorname{res}_x(f\cdot\omega_x) = 0, \quad \text{all } f\in\mathbf{R}(X).$$

If we let $\widetilde{H}^0(\Omega^1_X(-D_1))$ be the vector space of such pseudo-sections, then we see that

$$\frac{\bigoplus_{\substack{x\in X\\ \text{closed}}} \mathbf{R}(X)/\mathcal{O}_X(D_1)_x}{\mathbf{R}(X)} \times \widetilde{H}^0(\Omega^1_X(-D_1)) \longrightarrow \mathbb{C}$$

is indeed a perfect pairing, and we must merely check that all pseudo-sections are true sections to establish the assertion. Now let D_1 tend to $-\infty$ as a divisor. If $D_1' < D_1$, we get a diagram:

$$\begin{array}{ccc} H^0(\Omega^1_X(-D_1')) & \subset & \widetilde{H}^0(\Omega^1_X(-D_1')) \\ \cup & & \cup \\ H^0(\Omega^1_X(-D_1)) & \subset & \widetilde{H}^0(\Omega^1_X(-D_1)) \end{array}$$

and clearly:

$$H^0(\Omega^1_X(-D_1')) \cap \widetilde{H}^0(\Omega^1_X(-D_1)) = H^0(\Omega^1_X(-D_1)).$$

Passing to the limit, we get:

$$\Omega^1_{\mathbf{R}(X)/\mathbb{C}} \subset \widetilde{\Omega}^1_{\mathbf{R}(X)/\mathbb{C}}$$

where

$$\widetilde{\Omega}^1_{\mathbf{R}(X)/\mathbb{C}} = \left\{ \begin{array}{l} \text{set of meromorphic pseudo-differentials, i.e.,} \\ \text{collections of } \omega_{x,X} \in \Omega_{x,X} \otimes_{\mathcal{O}_x} \mathbf{R}(X) \text{ such that} \\ \sum_x \operatorname{res}_x(f \cdot \omega_x) = 0, \text{ all } f \in \mathbf{R}(X) \end{array} \right\}.$$

It suffices to prove that $\Omega^1 = \widetilde{\Omega}^1$. But it turns out that if D_1' is sufficiently negative, then $-D_1'$ is very positive and

$$H^1(\Omega^1_X(-D_1')) = H^0(\mathcal{O}_X(D_1')) = (0).$$

Thus

$$\begin{aligned} \dim H^0(\Omega^1_X(-D_1')) &= \deg \Omega^1_X - \deg D_1' - g + 1 \\ \dim \widetilde{H}^0(\Omega^1_X(-D_1')) &= \dim H^1(\mathcal{O}_X(D_1')) \\ &= -\deg D_1' + g - 1 \end{aligned}$$

hence

$$\dim_{\mathbb{C}} \left(\widetilde{H}^0(\Omega^1_X(-D_1'))/H^0(\Omega^1_X(-D_1')) \right) = 2g - 2 - \deg \Omega^1_X \quad \text{(independent of } D_1'\text{)}.$$

Thus $\dim_{\mathbb{C}} \left(\widetilde{\Omega}^1_{\mathbf{R}(X)/\mathbb{C}} / \Omega^1_{\mathbf{R}(X)/\mathbb{C}} \right) < +\infty$. But $\widetilde{\Omega}^1_{\mathbf{R}(X)/\mathbb{C}}$ is an $\mathbf{R}(X)$-vector space! So if $\widetilde{\Omega}^1 \supsetneqq \Omega^1$, then $\dim_{\mathbb{C}} \widetilde{\Omega}^1/\Omega^1 = +\infty$. Therefore $\widetilde{\Omega}^1 = \Omega^1$ as required. □

All this uses the assumption $k = \mathbb{C}$ only in two ways: first in order to know that if we define the residue of a formal meromorphic differential via:

$$\operatorname{res} \left(\sum_{n=-N}^{+\infty} c_n t^n dt \right) = c_{-1},$$

then the residue remains unchanged if we take a new local coordinate $t' = a_1 t + a_2 t^2 + \cdots$, $(a_1 \neq 0)$. Secondly, if $\omega \in \Omega^1_{\mathbf{R}(X)/k}$, then we need the deep fact:

$$\sum_{\substack{x \in X \\ \text{closed}}} \operatorname{res}_x \omega = 0.$$

Given these facts, our proof works over any algebraically closed ground field k (and with a little more work, over any k at all). For a long time, only rather roundabout proofs of these facts were known in characteristic p (when characteristic $= 0$, there are simple algebraic proofs or one can reduce to the case $k = \mathbb{C}$). Around the time this manuscript was being written, Tate [**111**] discovered a very elementary and beautiful proof of these facts: we reproduce his proofs in an appendix to this section. Note that his "dualizing sheaf" is exactly the same as our "pseudo-differentials".

We finish the section with a few applications.

COROLLARY 1.4. *If X is a geometrically irreducible curve, proper and smooth over a field k, then:*

a) *For all $f \in \mathbf{R}(X)$, $\deg(f) = 0$; hence if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, then $\deg D_1 = \deg D_2$. This means we can assign a degree to an invertible sheaf $\mathcal{L}$ by requiring:*

$$\deg \mathcal{L} = \deg D \quad \text{if } \mathcal{L} \cong \mathcal{O}_X(D).$$

b) *If $\deg D < 0$, then $H^0(\mathcal{O}_X(D)) = (0)$.*

Proof. Multiplication by f is an isomorphism $\mathcal{O}_X \xrightarrow{\approx} \mathcal{O}_X((f))$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_X((f)))$, so by Riemann-Roch (Theorem 1.1), $\deg(f) = 0$. Secondly, if $f \in H^0(\mathcal{O}_X(D))$, $f \neq 0$, then $D + (f) \geq 0$, so

$$\deg D = \deg(D + (f)) \geq 0.$$

□

Corollary 1.5. *If X is a geometrically irreducible curve, proper and smooth over a field k of genus g ($g \underset{def}{=} \dim H^1(\mathcal{O}_X)$), then:*

a) $\dim_k H^0(\Omega^1_{X/k}) = g$, $\dim_k H^1(\Omega^1_{X/k}) = 1$,
b) *If K is a divisor such that $\Omega^1_{X/k} \cong \mathcal{O}_X(K)$ — a so-called* canonical divisor *— then*

$$\deg K = 2g - 2.$$

Proof. Apply Riemann-Roch (Theorem 1.1) with $D = K$. □

Corollary 1.6. *If X is a geometrically irreducible curve, proper and smooth over a field k of genus g, then $\deg D > 2g - 2$ implies:*

a) $H^1(\mathcal{O}_X(D)) = (0)$
b) $\dim H^0(\mathcal{O}_X(D)) = \deg D - g + 1$.

Proof. If $\Omega^1_{X/k} \cong \mathcal{O}_X(K)$, then $\deg(K - D) < 0$, hence $H^0(\Omega^1_{X/k}(-D)) = (0)$. Thus by Riemann-Roch (Theorem 1.1), $H^1(\mathcal{O}_X(D)) = (0)$ and $\dim H^0(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X(D)) = \deg D - g + 1$. □

Proposition 1.7. $\boxed{\textit{Added}}$ *Let X be a geometrically irreducible curve proper and smooth over a field k. An invertible sheaf $\mathcal{L}$ on X is ample if and only if $\deg \mathcal{L} > 0$.*

Proof. We use Serre's cohomological criterion (Theorem VII.8.2). Note that the cohomology groups H^p for $p > 1$ of coherent $\mathcal{O}_X$-modules vanish since $\dim X = 1$ (cf. Proposition VII.4.2). Thus we need to show that

$$\text{for any coherent } \mathcal{O}_X\text{-module } \mathcal{F} \text{ one has} \quad H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \qquad n \gg 0$$

if and only if $\deg \mathcal{L} > 0$.

Let $r = \operatorname{rk} \mathcal{F}$, i.e., the dimension of the $\mathbf{R}(X)$-vector space $\mathcal{F}_\eta$ ($\eta =$ generic point). Then we claim that $\mathcal{F}$ has a filtration

$$(0) \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{F}$$

by coherent $\mathcal{O}_X$-submodules such that

$$\begin{aligned} \mathcal{F}_0 \quad &= \text{torsion } \mathcal{O}_X\text{-module} \\ \mathcal{F}_j/\mathcal{F}_{j-1} \quad &= \text{invertible } \mathcal{O}_X\text{-module for } j = 1, \ldots, r. \end{aligned}$$

Indeed, $\mathcal{O}_{x,X}$ for closed points x are discrete valuation rings since X is a regular curve. Thus for the submodule $(\mathcal{F}_x)_{\text{tor}}$ of torsion elements in the finitely generated $\mathcal{O}_{x,X}$-module $\mathcal{F}_x$, the quotient $\mathcal{F}_x/(\mathcal{F}_x)_{\text{tor}}$ is a free $\mathcal{O}_{x,X}$-module. $\mathcal{F}_0$ is the $\mathcal{O}_X$-submodule of $\mathcal{F}$ with $(\mathcal{F}_0)_x = (\mathcal{F}_x)_{\text{tor}}$ for all closed points x and $\mathcal{F}/\mathcal{F}_0$ is locally free of rank r. X is projective by Proposition V.5.11. Thus if we choose a very ample sheaf on X, then a sufficient twist of $\mathcal{F}/\mathcal{F}_0$ by it has a section. Untwising the result, we get an invertible subsheaf $\mathcal{M} \subset \mathcal{F}/\mathcal{F}_0$. Let $\mathcal{F}_1 \subset \mathcal{F}$ be the $\mathcal{O}_X$-submodule containing $\mathcal{F}_0$ such that $\mathcal{F}_1/\mathcal{F}_0 \supset \mathcal{M}$ and that $(\mathcal{F}_1/\mathcal{F}_0)/\mathcal{M}$ is the $\mathcal{O}_X$-submodule of torsions of $(\mathcal{F}/\mathcal{F}_0)/\mathcal{M}$. Obviously, $\mathcal{F}_1/\mathcal{F}_0$ is an invertible submodule of $\mathcal{F}/\mathcal{F}_0$ with $\mathcal{F}/\mathcal{F}_1$ locally free of rank $r - 1$. The above claim thus follows by induction.

Since $H^1(X, \mathcal{F}_0 \otimes \mathcal{L}^n) = 0$ for any n again by Proposition VII.4.2, the proposition follows if we show that

$$\text{for } \mathcal{F} \text{ invertible one has } \quad H^1(X, \mathcal{F} \otimes \mathcal{L}^n) = 0, \qquad n \gg 0$$

if and only if $\deg \mathcal{L} > 0$. But this is immediate, since the cohomology group vanishes if $\deg(\mathcal{F} \otimes \mathcal{L}^n) = \deg \mathcal{F} + n \deg \mathcal{L} > 2g - 2$ by Corollary 1.6, (a). $\square$

Remark. $\boxed{\text{Added}}$ Using the filtration appearing in the proof above, we can generalize Theorem 1.1 (Riemann-Roch), (1) for a locally free sheaf $\mathcal{E}$ of rank r as:

$$\dim_k H^0(X, \mathcal{E}) - \dim_k H^1(X, \mathcal{E}) = \deg(\bigwedge^r \mathcal{E}) + r(1 - g).$$

Remark. Let X be a curve proper and smooth over an algebraically closed field k, and $\mathcal{L}$ an invertible sheaf on X. We can show:

- If $\deg \mathcal{L} \geq 2g$, then $\mathcal{L}$ is generated by global sections.
- If $\deg \mathcal{L} \geq 2g + 1$, then $\mathcal{L}$ is very ample (over k).

Corollary 1.8. *If X is a geometrically irreducible curve smooth and proper over a field k of genus 0, and X has at least one k-rational point x (e.g., if k is algebraically closed; or k a finite field, cf. Proposition* IV.3.5*), then $X \cong \mathbb{P}^1_k$.*

Proof. Apply Riemann-Roch (Theorem 1.1) to $\mathcal{O}_X(x)$. It follows that

$$\dim_k H^0(\mathcal{O}_X(x)) \geq 2,$$

hence $\exists f \in H^0(\mathcal{O}_X(x))$ which is not a constant. This f defines a morphism

$$f' \colon X \longrightarrow \mathbb{P}^1_k$$

such that $(f')^{-1}(\infty) = \{x\}$, with reduced structure. Then f' must be finite; and thus $\mathcal{O}_{x,X}$ is a finite $\mathcal{O}_{\infty,\mathbb{P}^1}$-module such that

$$\begin{array}{ccc}
\mathcal{O}_{\infty,\mathbb{P}^1}/\mathfrak{m}_{\infty,\mathbb{P}^1} & \longrightarrow & \mathcal{O}_{x,X}/(\mathfrak{m}_{\infty,\mathbb{P}^1} \cdot \mathcal{O}_{x,X}) \\
\| & & \| \\
k & & \Bbbk(x) \\
& & \| \\
& & k
\end{array}$$

is an isomorphism. Thus $\mathcal{O}_{x,X} \cong \mathcal{O}_{\infty,\mathbb{P}^1}$, hence f' is birational, hence by Zariski's Main Theorem (§V.6), f' is an isomorphism. $\square$

Corollary 1.9. *If X is a geometrically irreducible curve smooth and proper over a field k of genus 1, then $\Omega^1_{X/k} \cong \mathcal{O}_X$. Moreover the map*

$$X(k) = \left\{ \begin{array}{l} \textit{set of } k\textit{-rational} \\ \textit{points } x \in X \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{invertible sheaves } \mathcal{L} \\ \textit{of degree 1 on } X \end{array} \right\}$$

$$x \longmapsto \mathcal{O}_X(x)$$

is an isomorphism, hence if $x_0 \in X(k)$ is a base point, $X(k)$ is a group via $x + y = z$ if and only if

$$\mathcal{O}_X(x) \otimes \mathcal{O}_X(y) \cong \mathcal{O}_X(z) \otimes \mathcal{O}_X(x_0).$$

PROOF. Since $H^0(\Omega^1_{X/k}) \neq (0)$, $\Omega^1_{X/k} \cong \mathcal{O}_X(D)$ for some non-negative divisor D. But then

$$\deg D = 2g - 2 = 0,$$

so $D = 0$, i.e., $\Omega^1_{X/k} \cong \mathcal{O}_X$. Next, if $\mathcal{L}$ is an invertible sheaf of degree 1, then by Corollary 1.6, $H^1(\mathcal{L}) = (0)$, hence by Riemann-Roch (Theorem 1.1), $\dim_k H^0(\mathcal{L}) = 1$. This means there is a unique non-negative divisor D such that $\mathcal{L} \cong \mathcal{O}_X(D)$. Since $\deg D = 1$, $D = x$ where $x \in X(k)$. Finally, the invertible sheaves of degree 0 form a group under $\otimes$, hence so do the sheaves of degree 1 if we multiply them by:

$$(\mathcal{L}, \mathcal{M}) \longmapsto \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{O}_X(-x_0).$$

This proves that $X(k)$ is a group. $\square$

In fact, it can be shown that X is a group scheme (in fact an abelian variety) over k (cf. §VI.1) with origin x_0: especially there is a morphism

$$\mu \colon X \times_{\operatorname{Spec} k} X \longrightarrow X$$

inducing the above addition on $X(k)$: see Mumford [**85**, Chapter I, p. 36], and compare Part I [**87**, §7D].

COROLLARY 1.10. *If $\Theta_X = \mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \cong \mathcal{O}_X(-K)$ is the tangent sheaf to X, then its cohomology is:*

	$g = 0$	$g = 1$	$g > 1$
$\dim H^0(\Theta_X)$	3	1	0
$\dim H^1(\Theta_X)$	0	1	$3g - 3$

In fact, the three sections of Θ when $X = \mathbb{P}^1_k$ come from the infinitesimal section of the 3-dimensional group scheme $\mathrm{PGL}_{2,k}$ acting on $\mathbb{P}^1_k$; the one section of Θ when $g = 1$ comes from the infinitesimal action of X on itself, and the absence of sections when $g > 1$ is reflected in the fact that the group of automorphisms of such curves is finite. Thus three way division of curves, according as $g = 0$, $g = 1$, $g > 1$ is the algebraic side of the analytic division of Riemann surfaces according as whether they are a) the Gauss sphere, b) the plane modulo a discrete translation group or c) the unit disc modulo a freely acting Fuchsian group; and of the differential geometric division of compact surfaces according as they admit a metric with constant curvature K, with $K > 0$, $K = 0$, or $K < 0$.

For further study of curves, an excellent reference is Serre [**103**, Chapters 2–5]. Classical references on curves are: Hensel-Landsberg [**59**], Coolidge [**32**], Severi [**107**] and Weyl [**116**][1].

What happens in higher dimensions?[2]

The necessity of the close analysis of all higher cohomolgy groups becomes much more apparent as the dimension increases. Part (1) of the curve Riemann-Roch theorem (Theorem 1.1) was generalized by Hirzebruch [**62**], and by Grothendieck (cf. [**25**])[3] to a formula for computing $\chi(\mathcal{O}_X(D))$ — for any smooth, projective variety X and divisor D — by a "universal polynomial" in terms of D and the Chern classes of X; this polynomial can be taken in a suitable cohomology ring of X, or else in the so-called Chow ring — a ring formed by cycles $\sum n_i Z_i$ (Z_i subvarieties of X) modulo "rational equivalence" with product given by intersection. For this theory, see Chevalley Seminar [**30**] and Samuel [**96**].

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[1](Added in publication) See also Iwasawa [**63**].

[2](Added in publication) There have been considerable developments on Kodaira dimension, Minimal model program, etc. See §IX.1

[3](Added in publication) See also SGA6 [**9**] for further developments.

Part (2) of the curve Riemann-Roch theorem (Theorem 1.1) was generalized by Serre and Grothendieck (see Serre [**99**], Altman-Kleiman [**13**] and Hartshorne [**55**]) to show, if X is a smooth complete varietie of dimension n, that

a) a canonical isomorphism $\epsilon\colon H^n(X, \Omega^n_{X/k}) \xrightarrow{\approx} k$, and
b) that — plus cup product induces a non-degenerate pairing

$$H^i(X, \mathcal{O}_X(D)) \times H^{n-i}(X, \Omega^n_X(-D)) \to k$$

for all divisors D and all i.

Together however, these results do not give any formula in dim ≥ 2 involving H^0's alone. Thus geometric applications of Riemann-Roch requires a good deal more ingenuity (cf. for instance Shafarevitch et al. [**108**]).

Three striking examples of cases where the higher cohomology groups can be dealt with so that a geometric conclusion is deduced from a cohomological hypothesis are:

Theorem 1.11 (Criterion of Nakai-Moishezon). *Let k be a field, X a scheme proper over k, and $\mathcal{L}$ an invertible sheaf on X. Then*

$$\begin{bmatrix} \mathcal{L} \textit{ is ample, i.e., } n \geq 1 \textit{ and} \\ \textit{a closed immersion} \\ \phi\colon X \to \mathbb{P}^N \textit{ such that} \\ \phi^*(\mathcal{O}_{\mathbb{P}^N}(1)) \cong \mathcal{L}^n \end{bmatrix} \iff \begin{bmatrix} \forall \textit{ reduced and irreducible} \\ \textit{subvarieities } Y \subset X \textit{ of} \\ \textit{positive dimension,} \\ \chi(\mathcal{L}^n \otimes \mathcal{O}_Y) \to \infty \textit{ as } n \to \infty \end{bmatrix}.$$

(This is another form of Theorem VII.12.4. See also Kleiman [**69**]).

Theorem 1.12 (Criterion of Kodaira). *Let X be a compact complex analytic manifold and $\mathcal{L}$ an invertible analytic sheaf on X. Then*

$$\begin{bmatrix} X \textit{ is a projective} \\ \textit{variety and } \mathcal{L} \textit{ is} \\ \textit{an ample sheaf on it} \end{bmatrix} \iff \begin{bmatrix} \mathcal{L} \textit{ can be defined by transition functions } \{f_{\alpha\beta}\} \\ \textit{for a covering } \{U_\alpha\} \textit{ of } X\textit{, where} \\ |f_{\alpha\beta}|^2 = g_\alpha/g_\beta,\ g_\alpha \textit{ positive real } C^\infty \textit{ on } U_\alpha \textit{ and} \\ (\partial^2 \log g_\alpha/\partial z_i \partial \bar{z}_j)(P) \\ \textit{positive definite Hermitian form at all } P \in U_\alpha \end{bmatrix}.$$

(For a proof, cf. Gunning-Rossi [**54**].)

Theorem 1.13 (Vanishing theorem of Kodaira-Akizuki-Nakano). *Let X be an n-dimensional complex projective variety, $\mathcal{L}$ an ample invertible sheaf on X. Then*

$$H^p(X, \Omega^q_X \otimes \mathcal{L}) = (0), \quad \textit{if } p+q > n.$$

(For a proof, cf. Akizuki-Nakano [**11**].)

Appendix: Residues of differentials on curves by John Tate

(Added in publication)

We reproduce here, in our notation, the very elementary and beautiful proof of Tate [**111**].

Here is the key to Tate's proof: Let V be a vector space over a field k. A k-linear endomorphism $\theta \in \mathrm{End}_k(V)$ is said to be *finite potent* if $\theta^n V$ is finite dimensional for a positive integer n. For such a θ, the *trace*

$$\mathrm{Tr}_V(\theta) \in k$$

is defined and has the following properties:

(T1) If $\dim V < \infty$, then $\mathrm{Tr}_V(\theta)$ is the ordinary trace.

(T2) If $W \subset V$ is a subspace with $\theta W \subset W$, then

$$\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta) + \mathrm{Tr}_{V/W}(\theta).$$

(T3) $\mathrm{Tr}_V(\theta) = 0$ if θ is nilpotent.

(T4) Suppose $F \subset \mathrm{End}_k(V)$ is a *finite potent* k-subspace, i.e., there exists a positive integer n such that $\theta_1 \circ \theta_2 \circ \cdots \circ \theta_n V$ is finite dimensional for any $\theta_1, \ldots, \theta_n \in F$. Then

$$\mathrm{Tr}_V \colon F \to k \quad \text{is } k\text{-linear}.$$

(It does not seem to be known if

$$\mathrm{Tr}_V(\theta + \theta') = \mathrm{Tr}_V(\theta) + \mathrm{Tr}_V(\theta')$$

holds in general under the condition θ, θ' and $\theta + \theta'$ are finite potent.)

(T5) Let $\phi\colon V' \to V$ and $\psi\colon V \to V'$ be k-linear maps with $\phi \circ \psi\colon V \to V$ finite potent. Then $\psi \circ \phi\colon V' \to V'$ is finite potent and

$$\mathrm{Tr}_V(\phi \circ \psi) = \mathrm{Tr}_{V'}(\psi \circ \phi).$$

(T1), (T2) and (T3) characterize $\mathrm{Tr}_V(\theta)$: Indeed, by assumption, $W = \theta^n V$ is finite dimensional for some n. Then $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_W(\theta)$.

For the proof of (T4), we may assume F to be finite dimensional and compute the trace on the finite dimensional subspace $W = F^n V$.

As for (T5), ϕ and ψ induce isomorphisms between the subspaces $W = (\phi \circ \psi)^n V$ and $W' = (\psi \circ \phi)^n V'$ for $n \gg 0$, under which $(\psi \circ \phi)|_{W'}$ and $(\phi \circ \psi)|_W$ correspond.

Definition 1. Let A and B be k-subspaces of V.

- A is said to be "not much bigger than" B (denoted $A \prec B$) if $\dim(A + B)/B < \infty$.
- A is said to be "about the same size as" B (denoted $A \sim B$) if $A \prec B$ and $A \succ B$.

Proposition 2. *Let A be a k-subspace of V.*

(1) $E = \{\theta \in \mathrm{End}_k(V) \mid \theta A \prec A\}$ *is a k-subalgebra of* $\mathrm{End}_k(V)$.

(2) *The subspaces*

$$\begin{aligned} E_1 &= \{\theta \in \mathrm{End}_k(V) \mid \theta V \prec A\} \\ E_2 &= \{\theta \in \mathrm{End}_k(V) \mid \theta A \prec (0)\} \\ E_0 &= E_1 \cap E_2 = \{\theta \in \mathrm{End}_k(V) \mid \theta V \prec A, \theta A \prec (0)\} \end{aligned}$$

are two-sided ideals of E with $E = E_1 + E_2$, and E_0 is finite potent so that there is a k-linear map $\mathrm{Tr}_V\colon E_0 \to k$. Moreover, E, E_1, E_2 and E_0 depend only on the $\sim$-equivalence class of A.

(3) *Let $\phi, \psi \in \mathrm{End}_k(V)$. If either* (i) *$\phi \in E_0$ and $\psi \in E$, or* (ii) *$\phi \in E_1$ and $\psi \in E_2$, then*

$$[\phi, \psi] := \phi \circ \psi - \psi \circ \phi \in E_0$$

with $\mathrm{Tr}_V([\phi, \psi]) = 0$.

Proof. (1) is obvious. As for (2), express V as a direct sum $V = A \oplus A'$, and denote by $\varepsilon\colon V \twoheadrightarrow A$, $\varepsilon'\colon V \twoheadrightarrow A'$ the projections. Then $\mathrm{id}_V = \varepsilon + \varepsilon'$ with $\varepsilon \in E_1$ and $\varepsilon \in E_2$, so that $\theta = \theta\varepsilon + \theta\varepsilon'$ for all $\theta \in E$. Obviously, $\theta_1 \circ \theta_2 V$ is finite dimensional for any $\theta_1, \theta_2 \in E_0$. (3) follows easily from (T5). □

Theorem 3 (Abstract residue). *Let K be a commutative k-algebra (with 1), V a k-vector space which is also a K-module, and $A \subset V$ a k-subspace such that $fA \prec A$ for all $f \in K$. (Hence K acts on V through $K \to E \subset \mathrm{End}_k(V)$ with the image in E of each $f \in K$ denoted* *by the same letter f, where E, E_1, E_2 and E_0 are defined with respect to the present A as in Proposition 2.) Then there exists a unique k-linear map*

$$\mathrm{Res}_A^V\colon \Omega^1_{K/k} \longrightarrow k$$

such that for any pair $f, g \in K$, we have

$$\mathrm{Res}_A^V(fdg) = \mathrm{Tr}_V([f_1, g_1])$$

for $f_1, g_1 \in E$ such that

i) $f \equiv f_1 \pmod{E_2}$, $g \equiv g_1 \pmod{E_2}$

ii) *either $f_1 \in E_1$ or $g_1 \in E_1$.*

The k-linear map is called the residue *and satisfies the following properties:*

(R1) $\mathrm{Res}_A^V = \mathrm{Res}_A^{V'}$ *if $V \supset V' \supset A$ and $KV' = V'$. Moreover,* $\mathrm{Res}_A^V = \mathrm{Res}_{A'}^V$ *if $A \sim A'$.*

(R2) *(Continuity in f and g) We have*

$$fA + fgA + fg^2A \subset A \Longrightarrow \mathrm{Res}_A^V(fdg) = 0.$$

Thus $\mathrm{Res}_A^V(fdg) = 0$ if $fA \subset A$ and $gA \subset A$. In particular, $\mathrm{Res}_A^V = 0$ if $A \subset V$ is a K-submodule.

(R3) *For $g \in K$ and an integer n, we have*

$$\mathrm{Res}_A^V(g^n dg) = 0 \quad \text{if} \quad \begin{cases} n \geq 0 \\ \text{or} \\ n \leq -2 \text{ and } g \text{ invertible in } K. \end{cases}$$

In particular, $\mathrm{Res}_A^V(dg) = 0$.

(R4) *If $g \in K$ is invertible and $h \in K$ with $hA \subset A$, then*

$$\mathrm{Res}_A^V(hg^{-1}dg) = \mathrm{Tr}_{A/(A\cap gA)}(h) - \mathrm{Tr}_{gA/(A\cap gA)}(h).$$

In particular, if $g \in K$ is invertible and $gA \subset A$, then

$$\mathrm{Res}_A^V(g^{-1}dg) = \dim_k(A/gA).$$

(R5) *Suppose $B \subset V$ is another subspace such that $fB \prec B$ for all $f \in K$. Then $f(A+B) \prec A+B$ and $f(A\cap B) \prec A\cap B$ hold for all $f \in K$, and*

$$\mathrm{Res}_A^V + \mathrm{Res}_B^V = \mathrm{Res}_{A+B}^V + \mathrm{Res}_{A\cap B}^V.$$

(R6) *Suppose K' is a commutative K-algebra that is a free K-module of finite rank r. For a K-basis $\{x_1, \ldots, x_r\}$ of K', let*

$$V' = K' \otimes_K V \supset A' = \sum_{i=1}^r x_i \otimes A.$$

Then $f'A' \prec A'$ holds for any $f' \in K'$, and the $\sim$-equivalence class of A' depends only on that of A and not on the choice of $\{x_1, \ldots, x_r\}$. Moreover,

$$\mathrm{Res}_{A'}^{V'}(f'dg) = \mathrm{Res}_A^V((\mathrm{Tr}_{K'/K} f')dg), \qquad \forall f' \in K',\ \forall g \in K.$$

Proof of the existence of residue. By assumption, we have $f, g \in E = E_1 + E_2$. Thus f_1 and g_1 satisfying (i) and (ii) can be chosen. Then $[f_1, g_1] \in E_1$ by (ii), and $[f_1, g_1] \equiv [f, g] \pmod{E_2}$ by (i) with $[f, g] = 0$ by the commutativity of K. Thus $[f_1, g_1] \in E_1 \cap E_2 = E_0$ and $\mathrm{Tr}_V([f_1, g_1])$ is defined. By Proposition 2, (3), it is unaltered if f_1 or g_1 is changed by an element of E_2 as long as the other is in E_1. Moreover by (T4), $\mathrm{Tr}_V([f_1, g_1])$ is a k-bilinear function of f and g. Thus there is a k-linear map

$$\beta\colon K \otimes_k K \longrightarrow k \qquad \text{such that } \beta(f \otimes g) = \mathrm{Tr}_V([f_1, g_1]).$$

We now show that

$$\beta(f \otimes gh) = \beta(fg \otimes h) + \beta(fh \otimes g), \qquad \forall f, g, h \in K,$$

hence r factors through the canonical surjective homomorphism

$$c\colon K \otimes_k K \longrightarrow \Omega^1_{K/k}, \qquad c(f \otimes g) = f dg.$$

Indeed, for $f, g, h \in K$, choose suitable $f_1, g_1, h_1 \in E_1$ and let $(fg)_1 = f_1 g_1$, $(fh)_1 = h_1 f_1$ and $(gh)_1 = g_1 h_1$. Then we obviously have

$$[f_1, g_1 h_1] = [f_1 g_1, h_1] + [h_1 f_1, g_1].$$

$\square$

We use the following lemma in proving the rest of Theorem 3:

Lemma 4. *For $f, g \in K$, define subspaces $B, C \subset V$ by*

$$B = A + gA$$
$$C = B \cap f^{-1}(A) \cap (fg)^{-1}(A) = \{v \in B \mid fv \in A \text{ and } fgv \in A\}.$$

Then B/C is finite dimensional and

$$\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_{B/C}([\varepsilon f, g]),$$

where $\varepsilon\colon V \twoheadrightarrow A$ is a k-linear projection.

Proof. B/C is finite dimensional, since $B/\{v \in B \mid fv \in A\}$ and $B/\{v \in B \mid fgv \in A\}$ are mapped injectively into the finite dimensional space $(A+fA+fgA+fg^2A)/A$. Moreover, $\varepsilon f \in E_1$ and $\varepsilon f \equiv f \pmod{E_2}$, hence $\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_V([\varepsilon f, g])$. On the other hand, $[\varepsilon f, g] = \varepsilon f g - g \varepsilon f$ maps V into B, and C into 0, since $fg = gf$. Thus the assertion follows by (T2), since $\mathrm{Tr}_V = \mathrm{Tr}_{V/B} + \mathrm{Tr}_{B/C} + \mathrm{Tr}_C$. $\square$

Proof of Theorem 3 continued. (R1) follows easily from Lemma 4, since $B, C \subset V'$.

As for (R2), we have $B = C$ in Lemma 4.

To prove (R3), choose $g_1 \in E_1$ such that $g_1 \equiv g \pmod{E_2}$. If $n \geq 0$, we have $\mathrm{Res}^V_A(g^n dg) = \mathrm{Tr}_V([g_1^n, g_1]) = 0$ since g_1^n and g_1 commute. If g is invertible, then $g^{-2-n} dg = -(g^{-1})^n d(g^{-1})$, whose residue is 0 if $n \geq 0$ by what we have just seen.

For the proof of (R4), let $f = hg^{-1}$ and apply Lemma 4. We have $[\varepsilon f, g] = \varepsilon h - \varepsilon_1 h$, where $\varepsilon_1 = g \varepsilon g^{-1}$ is a projection of V onto gA. Since both A and gA are stable under h, we have

$$\mathrm{Res}^V_A(f dg) = \mathrm{Tr}_{(A+gA)/(A\cap gA)}(\varepsilon h) - \mathrm{Tr}_{(A+gA)/(A\cap gA)}(g\varepsilon g^{-1} h)$$

and we are done by computing the traces through $A + gA \supset A \supset A \cap gA$ and $A + gA \supset gA \supset A \cap gA$, respectively.

To prove (R5), choose projections $\varepsilon_A\colon V \twoheadrightarrow A$, $\varepsilon_B\colon V \twoheadrightarrow B$, $\varepsilon_{A+B}\colon V \twoheadrightarrow A + B$, $\varepsilon_{A\cap B}\colon V \twoheadrightarrow A \cap B$ such that

$$\varepsilon_A + \varepsilon_B = \varepsilon_{A+B} + \varepsilon_{A\cap B}.$$

Then $[\varepsilon_A f, g]$ and $[\varepsilon_{A+B} f, g]$ belong to

$$F = \{\theta \in E \mid \theta V \prec A + B, \theta(A + B) \prec A, \theta A \prec (0)\},$$

which is finite potent, since $\theta_1 \circ \theta_2 \circ \theta_3 V$ is finite dimensional for any $\theta_1, \theta_2, \theta_3 \in F$. Since $\varepsilon_A f \in E_1$, $\varepsilon_A f \equiv f \pmod{E_2}$, $\varepsilon_{A+B} f \in E_1$ and $\varepsilon_{A+B} f \equiv f \pmod{E_2}$, one has

$$\begin{aligned}
\mathrm{Res}^V_A(f dg) - \mathrm{Res}^V_{A+B}(f dg) &= \mathrm{Tr}_V([\varepsilon_A f, g]) - \mathrm{Tr}_V([\varepsilon_{A+B} f, g]) \\
&= \mathrm{Tr}_V([(\varepsilon_A - \varepsilon_{A+B}) f, g]) \\
&= \mathrm{Tr}_V([(\varepsilon_{A\cap B} - \varepsilon_B) f, g]),
\end{aligned}$$

which, by a similar argument using

$$F' := \{\theta \in \mathrm{End}_k(V) \mid \theta V \prec B, \theta B \prec A \cap B, \theta(A \cap B) \prec 0\},$$

equals $\mathrm{Res}^V_{A\cap B}(fdg) - \mathrm{Res}^V_B(fdg)$.

As for (R6), a k-endomorphism φ of V' can be expressed as an $r \times r$ matrix (φ_{ij}) of endomorphisms of V by the rule

$$\varphi(\sum_j x_j \otimes v_j) = \sum_{ij} x_i \otimes \varphi_{ij} v_j, \qquad \text{for } v_j \in V.$$

If $F \subset \mathrm{End}_k(V)$ is a finite potent subspace, then φ's such that $\varphi_{ij} \in F$ for all i, j form a finite potent subspace $F' \subset \mathrm{End}_k(V')$. We see that $\mathrm{Tr}_{V'}(\varphi) = \sum_i \mathrm{Tr}_V(\varphi_{ii})$ for all $\varphi \in F'$ by decomposing the matrix (φ_{ij}) into the sum of a diagonal matrix, a nilpotent triangular matrix having zeros on and below the diagonal, and another nilpotent triangular matrix having zeros on and above the diagonal. For $f' \in K'$, write $f'x_j = \sum_i x_i f_{ij}$ with $f_{ij} \in K$. Let $\varepsilon \colon V \twoheadrightarrow A$ be a k-linear projection and put $\varepsilon'(\sum_i x_i \otimes v_i) = \sum_i x_i \otimes \varepsilon v_i$. Then $\varepsilon' \colon V' \twoheadrightarrow A'$ is a projection, and

$$[f'\varepsilon', g]_{ij} = [f_{ij}\varepsilon, g].$$

We are done since $\mathrm{Tr}_{K'/K} f = \sum_i f_{ii}$. $\square$

We are now ready to deal with residues of differentials on curves.

Let X be a regular irreducible curve proper over a field k, and denote by X_0 the set of closed points of X. For each $x \in X_0$ let

$$A_x = \widehat{\mathcal{O}}_{x,X} = \mathfrak{m}_{x,X}\text{-adic completion of } \mathcal{O}_{x,X}$$
$$K_x = \text{quotient field of } A_x.$$

Define

$$\mathrm{Res}_x \colon \Omega^1_{\mathbf{R}(X)/k} \longrightarrow k$$

by

$$\mathrm{Res}_x(fdg) = \mathrm{Res}^{K_x}_{A_x}(fdg), \qquad f, g \in \mathbf{R}(X),$$

which makes sense since $\Bbbk(x) = A_x/\mathfrak{m}_{x,X}A_x$ is a finite dimensional k-vector space so that $A_x \sim \mathfrak{m}^n_{x,X}A_x$ for any $n \in \mathbb{Z}$ and that for any non-zero $f \in K_x$ we have $fA_x \prec A_x$ since $fA_x = \mathfrak{m}^n_{x,X}A_x$ for some n.

THEOREM 5.

i) *Suppose $x \in X_0$ is k-rational so that $A_x = k[[t]]$ and $K_x = k((t))$. For*

$$f = \sum_{\nu \gg \infty} a_\nu t^\nu, \quad g = \sum_{\mu \gg \infty} b_\mu t^\mu \in K_x,$$

we have

$$\mathrm{Res}_x(fdg) = \textit{coefficient of } t^{-1} \textit{ in } f(t)g'(t)$$
$$= \sum_{\nu+\mu=0} \mu a_\nu b_\mu.$$

ii) *For any subset $S \subset X_0$, let $\mathcal{O}(S) = \bigcap_{x\in S} \mathcal{O}_{x,X} \subset \mathbf{R}(X)$. Then*

$$\sum_{x\in S} \mathrm{Res}_x(\omega) = \mathrm{Res}^{\mathbf{R}(X)}_{\mathcal{O}(S)}(\omega), \qquad \forall \omega \in \Omega^1_{\mathbf{R}(X)/k}.$$

In particular

$$\sum_{x\in X_0} \mathrm{Res}_x(\omega) = 0, \qquad \forall \omega \in \Omega^1_{\mathbf{R}(X)/k}.$$

iii) *Let $\varphi\colon X' \to X$ be a finite surjective morphism of irreducible regular curves proper over k. Then*

$$\sum_{x' \in \varphi^{-1}(x)} \mathrm{Res}_{x'}(f'dg) = \mathrm{Res}_x((\mathrm{Tr}_{\mathbf{R}(X')/\mathbf{R}(X)}\, f')dg)$$

if $f' \in \mathbf{R}(X')$, $g \in \mathbf{R}(X)$ and $x \in X_0$, while

$$\mathrm{Res}_{x'}(f'dg) = \mathrm{Res}_x((\mathrm{Tr}_{K'_{x'}/K_x}\, f')dg)$$

if $x' \in X'_0$ with $\varphi(x') = x$, $f' \in K'_{x'}$ and $g \in K_x$. ($K'_{x'}$ is the quotient field of the $\mathfrak{m}_{x',X'}$-adic completion $A'_{x'}$ of $\mathcal{O}_{x',X'}$.)

Proof. (i) By the continuity (R2), we may assume that only finitely many of the a_ν and b_μ are non-zero. Indeed, express f and g as

$$f = \phi_1(t) + \phi_2(t)$$
$$g = \psi_1(t) + \psi_2(t)$$

in such a way that $\phi_1(t)$ and $\psi_1(t)$ are Laurent polynomials and that $\phi_2(t), \psi_2(t) \in t^n A_x$ for large enough n so that

$$\phi_1(t)\psi_2'(t) + \phi_2(t)\psi_1'(t) + \phi_2(t)\psi_2'(t) \in A_x.$$

Then $fdg = f(t)g'(t)dt$, and only the term in t^{-1} can give non-zero residue by (R3). By (R4) we have

$$\mathrm{Res}^{V_x}_{A_x}(t^{-1}dt) = \dim_k \Bbbk(x) = 1.$$

(Note that in positive characteristics it is not immediately obvious that the coefficient in question is independent of the choice of the uniformizing parameter t).

For (ii), let

$$A_S = \prod_{x \in S} A_x$$
$$V_S = {\prod}'_{x \in S} K_x$$
$$= \{f = (f_x) \mid f_x \in K_x,\ \forall x \in S \text{ and } f_x \in A_x \text{ for all but a finite number of } x\}.$$

Embedding $\mathbf{R}(X)$ diagonally into V_S, we see that $\mathbf{R}(X) \cap A_S = \mathcal{O}(S)$. By (R5) we have

$$\mathrm{Res}^{V_S}_{A_S} + \mathrm{Res}^{V_S}_{\mathbf{R}(X)} = \mathrm{Res}^{V_S}_{\mathcal{O}(S)} + \mathrm{Res}^{V_S}_{(\mathbf{R}(X)+A_S)}.$$

$\mathrm{Res}^{V_S}_{\mathbf{R}(X)} = 0$ by (R2), since $\mathbf{R}(X)$ is an $\mathbf{R}(X)$-module. We now show $V_S/(\mathbf{R}(X) + A_S)$ to be finite dimensional, hence $\mathrm{Res}^{V_S}_{(\mathbf{R}(X)+A_S)} = 0$ by (R1). It suffices to prove the finite dimensionality when $S = X_0$ because of the projection $V_{X_0} \twoheadrightarrow V_S$. Regarding $\mathbf{R}(X)$ as a constant sheaf on X, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \underset{\substack{\text{constant}\\\text{sheaf}}}{\mathbf{R}(X)} \longrightarrow \mathbf{R}(X)/\mathcal{O}_X = \bigoplus_{x \in X_0} K_x/A_x \longrightarrow 0,$$

where K_x/A_x is the skyscraper sheaf at x with stalk K_x/A_x. The associated cohomology long exact sequence induces an isomorphism

$$V_{X_0}/(\mathbf{R}(X) + A_{X_0}) \xrightarrow{\sim} H^1(X, \mathcal{O}_X),$$

the right hand side of which is finite dimensional since X is proper over k. To complete the proof of (ii), it remains to show

$$\mathrm{Res}^{V_S}_{A_S}(\omega) = \sum_{x \in S} \mathrm{Res}_x(\omega), \qquad \forall \omega = fdg.$$

Let $S' \subset S$ be a finite subset containing all poles of f and g. We write

$$V_S = V_{S\setminus S'} \times \prod_{x\in S'} K_x$$
$$A_S = A_{S\setminus S'} \times \prod_{x\in S'} A_x.$$

By (R5) and (R1),

$$\operatorname{Res}^{V_S}_{A_S}(f dg) = \operatorname{Res}^{V_{S\setminus S'}}_{A_{S\setminus S'}}(f dg) + \sum_{x\in S'} \operatorname{Res}_x(f dg).$$

$\operatorname{Res}^{V_{S\setminus S'}}_{A_{S\setminus S'}}(f dg) = 0$ and $\operatorname{Res}_x(f dg) = 0$ for $x \in S \setminus S'$ by the choice of S'. The last assertion in (ii) follows, since

$$\mathcal{O}(X_0) = \bigcap_{x\in X_0} \mathcal{O}_{x,X} = H^0(X, \mathcal{O}_X)$$

is finite dimensional over k so that $\mathcal{O}(X_0) \sim (0)$ and $\operatorname{Res}^{V_{X_0}}_{\mathcal{O}(X_0)} = 0$ by (R1).

To prove (iii), regard the function field $\mathbf{R}(X')$ of X' as a finite algebraic extension of $\mathbf{R}(X)$. Then (iii) follows from (R6), since the integral closure of $\mathcal{O}_x$ (resp. A_x) in $\mathbf{R}(X')$ (resp. $K'_{x'}$) is a finite module over $\mathcal{O}_x$ (resp. A_x). □

Recall that X_0 is the set of closed points of an irreducible regular curve X proper over k. Each $x \in X_0$ determines a prime divisor on X, which we denote by $[x]$. Thus a divisor D on X is of the form

$$D = \sum_{x\in X_0} n_x[x], \qquad \text{with } n_x = 0 \text{ for all but a finite number of } x.$$

We denote $\operatorname{ord}_x D = n_x$.

Let

$$V = V_{X_0} = {\prod}'_{x\in X_0} K_x$$
$$A = A_{X_0} = \prod_{x\in X_0} A_x.$$

For a divisor D on X, let

$$V(D) = \{f = (f_x) \in V \mid \operatorname{ord}_x f_x \geq -\operatorname{ord}_x D,\ \forall x \in X_0\}.$$

Then by an argument similar to that in the proof of Theorem 5, (ii), we get

$$H^1(X, \mathcal{O}_X(D)) \cong V/(\mathbf{R}(X) + V(D)).$$

Let

$$\begin{aligned} J_{\mathbf{R}(X)/k} &= \{\lambda \in \operatorname{Hom}_k(V, k) \mid \lambda(\mathbf{R}(X) + V(D)) = 0,\ \exists D \text{ divisor}\} \\ &= \varinjlim_{D} \operatorname{Hom}_k(H^1(X, \mathcal{O}_X(D)), k), \end{aligned}$$

which is nothing but the space of meromorphic "pseudo-differentials" appearing in §1.

$J_{\mathbf{R}(X)/k}$ is a vector space over $\mathbf{R}(X)$ by the action

$$(g\lambda)(f) = \lambda(gf), \qquad \forall g \in \mathbf{R}(X),\ \forall f = (f_x) \in V,$$

since obviously $(g\lambda)(\mathbf{R}(X)) = 0$, while $(g\lambda)(V((g) + D)) = 0$.

As in §1, let us assume X to be smooth and proper over k and geometrically irreducible. Then $\mathbf{R}(X)$ is a regular transcendental extension of transcendence degree one so that the module $\Omega^1_{\mathbf{R}(X)/k}$ is a one-dimensional vector space over $\mathbf{R}(X)$. Moreover, for any $x \in X_0$, the stalk

$\Omega^1_{X/k,x}$ is a free $\mathcal{O}_{x,X}$-submodule of $\Omega^1_{\mathbf{R}(X)/k}$ of rank one. If t_x is a local parameter at x so that $\mathfrak{m}_{x,X} = t_x\mathcal{O}_{x,X}$, then each $\omega \in \Omega^1_{\mathbf{R}(X)/k}$ can be expressed as

$$\omega = h dt_x, \qquad \text{for some } h \in \mathbf{R}(X).$$

Let us denote $\mathrm{ord}_x(\omega) = \mathrm{ord}_x(h)$, which is independent of the choice of the local parameter t_x. We then denote

$$(\omega) = \sum_{x \in X_0} \mathrm{ord}_x(\omega)[x],$$

which is easily seen to be a divisor on X.

For any divisor D, one has

$$H^0(X, \Omega^1_{X/k}(-D)) = \{\omega \in \Omega^1_{\mathbf{R}(X)/k} \mid (\omega) \geq D\}$$

and

$$\Omega^1_{\mathbf{R}(X)/k} = \varinjlim_D H^0(X, \Omega^1_{X/k}(-D)).$$

The abstract residue gives rise to an $\mathbf{R}(X)$-linear map

$$\sigma\colon \Omega^1_{\mathbf{R}(X)/k} \longrightarrow J_{\mathbf{R}(X)/k}$$

defined by

$$\sigma(\omega)(f) = \sum_{x \in X_0} \mathrm{Res}_x(f_x\omega), \qquad \forall \omega \in \Omega^1_{\mathbf{R}(X)/k},\ \forall f = (f_x) \in V.$$

This makes sense, since $\sigma(\omega)(\mathbf{R}(X)) = 0$ by Theorem 5, (ii), while $\sigma(\omega)(V(D)) = 0$ for $D = (\omega)$ by (R2).

For any divisor D, we see easily that σ induces a k-linear map

$$\sigma_D\colon H^0(X, \Omega^1_{X/k}(-D)) \longrightarrow \mathrm{Hom}_k(V/(\mathbf{R}(X) + V(D)), k) = \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D)), k).$$

Theorem 6 (Serre duality). *As in the Riemann-Roch theorem (Theorem* 1.1*), let X be a curve, smooth, proper and geometrically irreducible over a field k. Then*

$$\sigma\colon \Omega^1_{\mathbf{R}(X)/k} \longrightarrow J_{\mathbf{R}(X)/k}$$

is an isomorphism, which induces an isomorphism

$$\sigma_D\colon H^0(X, \Omega^1_{X/k}(-D)) \xrightarrow{\sim} \mathrm{Hom}_k(V/(\mathbf{R}(X) + V(D)), k) = \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D)), k),$$

for any divisor D. Consequently, (2)*-strong form of the Riemann-Roch theorem (Theorem* 1.3*) holds, giving rise to a non-degenerate bilinear pairing*

$$H^0(X, \Omega^1_{X/k}(-D)) \times H^1(X, \mathcal{O}_X(D)) \longrightarrow k.$$

In particular, Part (2) *of Theorem* 1.1 *holds.*

To show that σ and σ_D are isomorphisms, we follow Serre [**103**, Chapter II, §§6 and 8].

Lemma 7.

$$\dim_{\mathbf{R}(X)} J_{\mathbf{R}(X)/k} \leq 1.$$

Proof. Suppose $\lambda, \lambda' \in J_{\mathbf{R}(X)/k}$ were $\mathbf{R}(X)$-linearly independent. Hence we have an injective homomorphism

$$\mathbf{R}(X) \oplus \mathbf{R}(X) \ni (g, h) \longmapsto g\lambda + h\lambda' \in J_{\mathbf{R}(X)/k}.$$

There certainly exists D such that $\lambda(V(D)) = 0$ and $\lambda'(V(D)) = 0$. Fix $x \in X_0$ and let $P = [x]$. For a positive integer n and $g, h \in \mathbf{R}(X)$ with $(g) + nP \geq 0$ and $(h) + nP \geq 0$, we have $(g\lambda + h\lambda')(V(D - nP)) = 0$. Thus we have an injective homomorphism

$$H^0(X, \mathcal{O}_X(nP)) \oplus H^0(X, \mathcal{O}_X(nP)) \ni (g, h) \longmapsto g\lambda + h\lambda' \in \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D - nP)), k).$$

Hence we have

$$(*) \qquad \dim H^1(X, \mathcal{O}_X(D - nP)) \geq 2 \dim_k H^0(X, \mathcal{O}_X(nP)).$$

The right hand side of $(*)$ is greater than or equal to $2(n \deg P - g + 1)$ by Theorem 1.1, (1). On the other hand, again by Theorem 1.1, (1), the left hand side of $(*)$ is equal to

$$\begin{aligned} &-\deg(D - nP) + g - 1 + \dim_k H^0(X, \mathcal{O}_X(D - nP)) \\ &= \ n \deg P + (g - 1 - \deg D) + \dim_k H^0(X, \mathcal{O}_X(D - nP)). \end{aligned}$$

However, one has $\deg(D - nP) < 0$ for $n \gg 0$, hence $H^0(X, \mathcal{O}_X(D - nP)) = 0$ by Corollary 1.4, (b). Thus $(*)$ obviously leads to a contradiction for $n \gg 0$. $\square$

To continue the proof, we need to consider the base extension $X' = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ with respect to a finite extension k' of k. By assumption, X' is proper and smooth over k' and we obviously have a canonical commutative diagram

$$\begin{tikzcd} \Omega^1_{\mathbf{R}(X')/k'} \arrow[r, "\sigma'"] & J_{\mathbf{R}(X')/k'} \\ k' \otimes_k \Omega^1_{\mathbf{R}(X)/k} \arrow[u, hook] \arrow[r, "k' \otimes \sigma"] & k' \otimes_k J_{\mathbf{R}(X)/k}. \arrow[u, hook] \end{tikzcd}$$

The base extension $D' = D \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')$ of a divisor D on X induces

$$\begin{tikzcd} H^0(X', \Omega^1_{X'/k'}(-D')) \arrow[r, "\sigma'_{D'}"] \arrow[d, equal] & \mathrm{Hom}_{k'}(H^1(X', \mathcal{O}_{X'}(D')), k') \arrow[d, equal] \\ k' \otimes_k H^0(X, \Omega^1_{X/k}(-D)) \arrow[r, "k' \otimes \sigma_D"] & k' \otimes_k \mathrm{Hom}_k(H^1(X, \mathcal{O}_X(D)), k). \end{tikzcd}$$

Lemma 8. *Under the* $\mathbf{R}(X)$*-linear map*

$$\sigma \colon \Omega^1_{\mathbf{R}(X)/k} \longrightarrow J_{\mathbf{R}(X)/k},$$

$\omega \in \Omega^1_{\mathbf{R}(X)/k}$ *belongs to* $H^0(X, \Omega^1_{X/k}(-D))$ *if* $\sigma(\omega)(V(D)) = 0$.

Proof. Otherwise, there exists $y \in X_0$ such that $\mathrm{ord}_y(\omega) < \mathrm{ord}_y D$. Replacing k by a finite extension k', we may assume y to be k-rational so that $\Bbbk(y) = k$. Let $n = \mathrm{ord}_y(\omega) + 1$, hence $n \leq \mathrm{ord}_y D$. Define $f = (f_x) \in V$ by

$$\begin{cases} f_x = 0 & \text{if } x \neq y \\ f_y = 1/t_y^n & (t_y \text{ being a local parameter at } y). \end{cases}$$

Obviously, $\mathrm{Res}_x(f_x\omega) = 0$ for $x \neq y$, while

$$\mathrm{ord}_y(f_y\omega) = \mathrm{ord}_y((1/t_y^n)\omega) = -n + \mathrm{ord}_y(\omega) = -1,$$

hence $\sigma(\omega)(f) = \mathrm{Res}_y(f_y\omega) = 1 \neq 0$ by (R4). Since $n \leq \mathrm{ord}_y(D)$, one has $f \in V(D)$, a contradiction to the assumption $\sigma(\omega)(V(D)) = 0$. $\square$

Remark. (Added in publication) (Chai) Without resorting to any base extension of k, we can prove the lemma as follows: Pick a k-morphism α from an open neighborhood U of y to $\mathbb{A}^1 = \operatorname{Spec} k[s]$ which is étale at y. Let $z = \alpha(y)$ and $L = k(s)$. Write $\omega = h\alpha^* ds$, $h \in \mathbf{R}(X)$, and $\operatorname{ord}_y(h) = n$. We need to show that there exists an element $f_y \in K_y$ with $\operatorname{ord}_y(f_y) = -n-1$ such that $\operatorname{Res}_y(f_y h ds) \neq 0$. We know that $\operatorname{Tr}_{K_y/L_z}(A_y) = A_z$ because α is étale at y, so that by Theorem 5 (iii) it suffices to exhibit an element $g_z \in L_z$ such that $\operatorname{ord}_z(g_z) = -1$ and $\operatorname{Res}_z(g_z ds) \neq 0$. Let $q(s)$ be a monic irreducible polynomial in $k[s]$ corresponding to the closed point $z \in \mathbb{A}^1_k$, and let $d = \deg(q(s))$. Then everything follows from the formula

$$\operatorname{Res}_z\left(\left(\sum_{1\le i\le N} \frac{a_{-i}(s)}{q(s)^i}\right) ds\right) = b_{d-1}, \qquad a_{-1}(s) = \sum_{0\le j\le d-1} b_j s^j$$

for local residues at z of rational differentials, where each $a_{-i}(s)$ is a polynomial in $k[s]$ of degree at most $d-1$. This formula can be proved either by direct computation using Tate's definition, or using the residue theorem and computing

$$-\operatorname{Res}_\infty\left(\left(\sum_{1\le i\le N} \frac{a_{-i}(s)}{q(s)^i}\right) ds\right) = b_{d-1},$$

because z and ∞ are the only poles.

Proof of Theorem 6. σ is injective, for if $\sigma(\omega) = 0$, then $\omega \in H^0(X, \Omega^1_{X/k}(-D))$ for all D by Lemma 8, hence $\omega = 0$.

σ is surjective, since σ is a non-zero $\mathbf{R}(X)$-linear map with $\dim_{\mathbf{R}(X)} \le 1$ by Lemma 7. Moreover, σ_D is surjective, for if $\lambda \in J_{\mathbf{R}(X)/k}$ satisfies $\lambda(V(D)) = 0$, then there exists $\omega \in \Omega^1_{\mathbf{R}(X)/k}$ with $\sigma(\omega) = \lambda$. We see that $\omega \in H^0(X, \Omega^1_{X/k}(-D))$ by Lemma 8. □

Remark. (Added in publication) (Chai)

(1) The classical style of treating algebraic curves via valuations and adèles (as in the book by Chevelley, following a 1938 paper by Weil written in German) amounts to, in modern language, considering one-dimensional irreducible regular scheme X of finite type proper over a field k. Let $K := \mathbf{R}(X)$ be the function field of X. We may and do assume that k is algebraically closed in K. Then X can be recovered from K by considering discrete valuations on K that are trivial on k.

In general, the scheme X may not be smooth over k. There are two potential problems. First, the field K may not be separable over k, i.e., $K/k(x)$ is not a finite separable extension for any $x \in K$; equivalently, $\operatorname{Spec} K$ may not be smooth over $\operatorname{Spec} k$. An example: $k = F(u, v)$, with u, v transcendental over a field $F \supset \mathbb{F}$, and K is the fraction field of $k[x, y]/(x^p - uy^p - v)$.

Even when K/k is a regular extension (i.e., K/k is separable and k is algebraically closed in K), the morphism $X \to \operatorname{Spec} k$ may still be non-smooth. An example: $k \supset \mathbb{F}_p$, $a \notin k^p$, p odd, and K is the fraction field of $k[x, y]/(y^2 - x^p + a)$. Then X is the algebraic curve containing $\operatorname{Spec} k[x, y]/(y^2 - x^p + a)$ as an affine open set, with $\operatorname{Spec} k[x, y]/(y^2 - x^p + a)$ regular but not smooth over k. Note that the genus of K is $(p-1)/2$, while the genus of $K \cdot k^{1/p}$ over $k^{1/p}$ is 0. (Whenever X is not smooth over k, the phenomenon "genus change under constant field extension" occurs—see Artin [**15**, Chap. 15].)

(2) Tate's definition of residue gives a K-linear map $\sigma\colon \Omega^1_{K/k} \to J_{K/k}$. This map is an isomorphism if K/k is separable (hence regular) as shown in Theorem 5, while it is identically zero if K is not separable over k. This last assertion follows from Theorem 5, (iii). For any $x \in K$ transcendental over k, K is inseparable over $k(x)$, hence $\operatorname{Tr}_{K/k(x)}$ is identically zero.

Sheafification of the map σ gives an $\mathcal{O}_X$-linear map

$$\sigma_X : \Omega^1_{X/k} \longrightarrow J_{X/k},$$

where $J_{X/k}$ is an invertible $\mathcal{O}_X$-module, while $\Omega^1_{X/k}$ is an invertible $\mathcal{O}_X$-module if and only if X is smooth over k.

We have seen that the map σ_X is identically zero when K/k is not separable. When X is not smooth over k, the map σ_X is never injective (since $\Omega^1_{X/k}$ has torsion), and it may not be surjective either. Consider the case where K is the fraction field of $k[x,y]/(y^2 - x^p + a)$, with $a \notin k^p$ and p odd. Let P_0 be the closed point of X corresponding to the principal ideal $yk[x,y]/(y^2 - x^p + a)$ of $k[x,y]/(y^2 - x^p + a)$. Then the image of σ_X is $J_{X/k}(-P_0)$ as can be checked by an easy computation using Theorem 5, (iii).

[Here is the proof: Let $\mathcal{O}'$ be the completed local ring at P_0, and let $\mathcal{O}$ be the completion of the localization of $k[x]$ at the principal ideal generated by $x^p - a$. The maximal ideals of these two discrete valuation rings are generated by y and $x^p - a$, respectively. Then the sheaf of continuous differentials $\Omega^1_{\mathcal{O}'/k}$ is generated by dx and dy, with the relation $ydy = 0$. Hence $\sigma_{X,P_0}(dy) = 0$, and the image of σ_{x,P_0} is determined by Theorem 5, (iii). An easy computation shows that $\mathrm{Tr}_{\mathcal{O}'/\mathcal{O}}(y^{-1}\mathcal{O}') = \mathcal{O}$, while $\mathrm{Tr}_{\mathcal{O}'/\mathcal{O}}(y^{-2}\mathcal{O}') = (x^p - a)^{-1}\mathcal{O}$.]

2. Comparison of algebraic with analytic cohomology

In almost all of this section, we work only with complex projective space and its non-singular subvarieties. We abbreviate $\mathbb{P}^n_{\mathbb{C}}$ to $\mathbb{P}^n$ and recall that the set of closed points of $\mathbb{P}^n$ has two topologies: the Zariski topology and the much finer classical (or ordinary) topology. By ($\mathbb{P}^n$ in the classical topology) we mean the set of closed points of $\mathbb{P}^n$ in the classical topology and by ($\mathbb{P}^n$ in the Zariski topology) we mean the scheme $\mathbb{P}^n_{\mathbb{C}}$ as usual. Note that there is a continuous map

$$\epsilon \colon (\mathbb{P}^n \text{ in classical topology}) \longrightarrow (\mathbb{P}^n \text{ in the Zariski topology}).$$

We shall consider sheaves on the space on the left. The following class is very important.

DEFINITION 2.1. The holomorphic or analytic *structure sheaf* $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$ on ($\mathbb{P}^n$ in the classical topology) is the sheaf:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(U) = \text{ring of analytic functions } f \colon U \to \mathbb{C}.$$

If $U \subset \mathbb{P}^n$ is an open set, then a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules on U is called a *coherent analytic sheaf* if the following conditions are satisfied:

- $\mathcal{F}$ is locally of finite type: for all $x \in U$, there exists a (classical) open neighborhood $U_x \subset U$ of x and a surjective homomorphism
$$\mathcal{O}^m_{\mathbb{P}^n,\mathrm{an}}|_{U_x} \twoheadrightarrow \mathcal{F}|_{U_x}$$
of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules on U_x,
- for any open set $V \subset U$ and any homomorphism
$$h \colon \mathcal{O}^l_{\mathbb{P}^n,\mathrm{an}}|_V \to \mathcal{F}|_V$$
of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}|_V$-modules, $\mathrm{Ker}(h)$ is locally of finite type.

For basic results on coherent analytic sheaves, we refer to Gunning-Rossi [**54**]. Among the standard results given there are:

(2.2) If $\phi \colon \mathcal{F} \to \mathcal{G}$ is an $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-module homomorphism of coherent analytic sheaves on some U, then $\mathrm{Ker}\,\phi$, $\mathrm{Image}\,\phi$ and $\mathrm{Coker}\,\phi$ are coherent; thus the coherent analytic sheaves on U form an abelian category.

(2.3) If $U \subset \mathbb{P}^n_{\mathbb{C}}$ is a polycylinder in some affine piece and $\mathcal{F}$ is coherent on U, then $H^i(U, \mathcal{F}|_U) = (0)$, $i > 0$, and $\mathcal{F}|_U$ is generated as $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-module by $H^0(U, \mathcal{F}|_U)$.

(2.4) If $X \subset U$ is a closed analytic subset, then the sheaf $\mathcal{I}_X$ of analytic functions vanishing on X is coherent. If $\mathcal{F}$ is coherent, then $\{x \in \mathbb{P}^n \mid \mathcal{F}_x \neq (0)\}$ is a closed analytic subset.

Now if $\mathcal{F}$ is an algebraic coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$, one can define canonically an associated analytic coherent sheaf $\mathcal{F}_{\mathrm{an}}$ as follows: for all classical open U, let:

$$\mathcal{F}_{\mathrm{an}}(U) = \text{ submodule of } \prod_{x \in U} (\mathcal{O}_{\mathbb{P}^n,\mathrm{an}})_x \otimes_{\mathcal{O}_{\mathbb{P}^n,x}} \mathcal{F}_x$$

consisting of families $\{s_x\}$ satisfying the following condition:
for all $x \in U$, classical neighborhood U_1 of x and a Zariski neighborhood U_2 of x, $f_i \in \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(U_1)$ and $t_i \in \mathcal{F}(U_2)$ such that
$s_x = \sum f_i \otimes t_i, \quad x \in U_1 \cap U_2$.

This looks a bit cumbersome but, in fact, it is the natural way to define $f^*\mathcal{F}$ for any morphism f of ringed spaces, and sheaf of modules $\mathcal{F}$ on the image space. In the present situation, one has $\mathcal{F}_{\mathrm{an}} = \epsilon^*\mathcal{F}$. An elementary calculation gives the stalks of $\mathcal{F}_{\mathrm{an}}$:

$$(\mathcal{F}_{\mathrm{an}})_x = (\mathcal{O}_{\mathbb{P}^n,\mathrm{an}})_x \otimes_{\mathcal{O}_{\mathbb{P}^n,x}} \mathcal{F}_x.$$

Also, $\mathcal{F} \longmapsto \mathcal{F}_{\mathrm{an}}$ is obviously a functor, i.e., any $\mathcal{O}_{\mathbb{P}^n}$-homomorphism $\phi\colon \mathcal{F}_1 \to \mathcal{F}_2$ induces $\phi_{\mathrm{an}}\colon \mathcal{F}_{1,\mathrm{an}} \to \mathcal{F}_{2,\mathrm{an}}$. We now invoke the basic fact:

Lemma 2.5 (Serre). *$\mathbb{C}\{X_1, \ldots, X_n\}$, the ring of convergent power series, is flat as a module over $\mathbb{C}[X_1, \ldots, X_n]$.*

Proof. In fact, the completion $\widehat{\mathcal{O}}$ of a noetherian local ring $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-module (Atiyah-MacDonald [**20**, (10.14) and Exercise 7, Chapter 10]), hence $\mathbb{C}[[X_1, \ldots, X_n]]$ is faithfully flat over $\mathbb{C}\{X_1, \ldots, X_n\}$ and over $\mathbb{C}[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$. Hence $\forall M \to N \to P$ over $\mathbb{C}[X_1, \ldots, X_n]$,

$$\begin{aligned} M \to N \to P \text{ exact} &\Longrightarrow M \otimes \mathbb{C}[[X]] \to N \otimes \mathbb{C}[[X]] \to P \otimes \mathbb{C}[[X]] \text{ exact} \\ &\Longrightarrow M \otimes \mathbb{C}\{X\} \to N \otimes \mathbb{C}\{X\} \to P \otimes \mathbb{C}\{X\} \text{ exact.} \end{aligned}$$

□

Corollary 2.6. *$\mathcal{F} \longmapsto \mathcal{F}_{\mathrm{an}}$ is an exact functor from the category of all $\mathcal{O}_{\mathbb{P}^n}$-modules to the category of all $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules.*

Proof. If $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact, then by Proposition IV.4.3

$$\mathcal{F}_{\mathrm{an},x} \longrightarrow \mathcal{G}_{\mathrm{an},x} \longrightarrow \mathcal{H}_{\mathrm{an},x}$$

is exact for all x, hence $\mathcal{F}_{\mathrm{an}} \to \mathcal{G}_{\mathrm{an}} \to \mathcal{H}_{\mathrm{an}}$ is exact. □

Corollary 2.7. *If $\mathcal{F}$ is a coherent algebraic sheaf, then $\mathcal{F}_{\mathrm{an}}$ is a coherent analytic sheaf.*

The proof is left to the reader.

Note that covering the identity map

$$\epsilon\colon (\mathbb{P}^n \text{ in the classical topology}) \longrightarrow (\mathbb{P}^n \text{ in the Zariski topology})$$

there is a map $\epsilon^*\colon \mathcal{F} \to \mathcal{F}_{\mathrm{an}}$ of sheaves. This induces a canonical map on cohomology $H^i(\mathcal{F}) \to H^i(\mathcal{F}_{\mathrm{an}})$.

Our goal is now the following fundamental theorem:

Theorem 2.8 (Serre). (Fundamental "GAGA"[4] comparison theorem)

[4]Short for "géométrie analytique et géométrie algébrique"

i) *For every coherent algebraic $\mathcal{F}$, and every i,*

$$H^i(\mathbb{P}^n \textit{ in the Zariski topology}, \mathcal{F}) \cong H^i(\mathbb{P}^n \textit{ in the classical topology}, \mathcal{F}_{\mathrm{an}}).$$

ii) *The categories of coherent algebraic and coherent analytic sheaves are equivalent, i.e., every coherent analytic $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\mathrm{an}}$, some $\mathcal{F}$, and*

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}}(\mathcal{F}_{\mathrm{an}}, \mathcal{G}_{\mathrm{an}}).$$

Remark. (Added in publication) The statement of Serre's GAGA theorem (Theorem 2.8) holds also when the underlying variety is replaced by a scheme proper over $\mathbb{C}$, by an argument using Chow's lemma and noetherian induction similar to that in the proof of Grothendieck's coherency theorem (Theorem VII.6.5).

We will omit the details of the first and most fundamental step in the proof (for these we refer the reader to Gunning-Rossi [**54**, Chapter VIII A]). This is the finiteness assertion: given a coherent analytic $\mathcal{F}$, then $\dim_{\mathbb{C}} H^i(\mathbb{P}^n, \mathcal{F}) < +\infty$, for all i. The proof goes as follows:

a) For all $C > 1$, $0 \le i \le n$, let

$$U_{i,C} = \left\{ x \in \mathbb{P}^n \;\middle|\; x \notin V(X_i) \text{ and } \left|\frac{X_j}{X_i}(x)\right| < C,\ 0 \le j \le n \right\}.$$

b) Then $\bigcup_{i=0}^n U_{i,C} = \mathbb{P}^n$ so we have an open covering $\mathcal{U}_C = \{U_{0,C}, \cdots, U_{n,C}\}$ of $\mathbb{P}^n$.

c) Note that each intersection $U_{i_1,C} \cap \cdots \cap U_{i_k,C}$ can be mapped bihilomorphically onto a closed analytic subset Z of a high-dimensional polycylinder D by means of the set of functions X_j/X_{i_l}, $0 \le j \le n$, $1 \le l \le k$. Therefore, every coherent analytic $\mathcal{F}$ on $U_{i_1,C} \cap \cdots U_{i_k,C}$ corresponds to a sheaf $\mathcal{F}'$ on Z and, extending it to $D \setminus Z$ by (0) outside Z, a coherent analytic $\mathcal{F}'$ on D. Then

$$H^i(U_{i_1,C} \cap \cdots U_{i_k,C}, \mathcal{F}) \cong H^i(Z, \mathcal{F}') \cong H^i(D, \mathcal{F}') \cong (0),\ i > 0.$$

d) Therefore by Proposition VII.2.2 it follows that

$$H^i(\mathbb{P}^n, \mathcal{F}) \cong H^i(\mathcal{U}_C, \mathcal{F})$$

and that the refinement maps (for $C > C' > 1$):

$$\mathrm{ref}^i_{C,C'}: C^i(\mathcal{U}_C, \mathcal{F}) \longrightarrow C^i(\mathcal{U}_{C'}, \mathcal{F})$$

induce an isomorphism on cohomology.

e) The key step is to show that the space of sections:

$$\mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C})$$

is a *topological* vector space, in fact, a Fréchet space in a natural way; and that all restriction maps such as

$$\mathcal{F}(U_{i_2,C} \cap \cdots U_{i_k,C}) \longrightarrow \mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C})$$

are continuous, and that restriction to a relatively compact open subset, as in

$$\mathcal{F}(U_{i_1,C} \cap \cdots U_{i_k,C}) \longrightarrow \mathcal{F}(U_{i_1,C'} \cap \cdots U_{i_k,C'}) \qquad (C > C')$$

is *compact*. This last is a generalization of Montel's theorem that

$$\mathrm{res}\colon \left\{ \begin{array}{l} \text{holomorphic functions on} \\ \text{disc } |z| < C \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{holomorphic functions on} \\ \text{disc } |z| < C' \end{array} \right\}$$

is compact. It follows that $C^i(\mathcal{U}_C, \mathcal{F})$ is a complex of Fréchet spaces and continuous maps, and that $\mathrm{ref}^i_{C,C'}$ is compact.

f) By step (d),

$$\begin{aligned} Z^i(\mathcal{U}_C, \mathcal{F}) \oplus C^{i-1}(\mathcal{U}_{C'}) &\xrightarrow{\text{ref}+\delta} Z^i(\mathcal{U}_{C'}, \mathcal{F}) \\ (a, b) &\longmapsto \operatorname{ref}^i_{C,C'} a + \delta b \end{aligned}$$

is surjective. A standard fact in the theory of Fréchet spaces is that if

$$\alpha, \beta \colon V_1 \longrightarrow V_2$$

are two continuous maps of Fréchet spaces, with α surjective and β compact, then $\alpha+\beta$ has closed image of finite codimension. Apply this with $\alpha = \text{ref}+\delta$, $\beta = -\text{ref}$ and we find that $\operatorname{Coker}(\delta) = H^i(\mathcal{U}_{C'}, \mathcal{F})$ is finite-dimensional.

The second step in the proof is the vanishing theorem — if $\mathcal{F}$ is coherent analytic, then for $i > 0$, $m \gg 0$, $H^i(\mathbb{P}^n, \mathcal{F}(m)) = (0)$. (Here $\mathcal{F}(m) \underset{\text{def}}{=} \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n,\text{an}}} \mathcal{O}_{\mathbb{P}^n,\text{an}}(m)$ as usual.) We prove this by induction on n, the complex dimension of the ambient projective space, since it is obvious for $n = 0$. As in §VII.7, we use the $\otimes L \colon \mathcal{F}(m) \to \mathcal{F}(m+1)$, where $L = \sum c_i X_i$ is a linear form. This induces exact sequences:

(2.9)
$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{G}_L(m) \arrow[r] & \mathcal{F}(m) \arrow[rr] \arrow[dr] & & \mathcal{F}(m+1) \arrow[r] & \mathcal{H}_L(m) \arrow[r] & 0 \\ & & & \mathcal{F}'_L(m) \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

where both $\mathcal{G}_L$ and $\mathcal{H}_L$ are annihilated by L/X_i on $\mathbb{P}^n \setminus V(X_i)$. Therefore they are coherent analytic sheaves on $V(L) \cong \mathbb{P}^{n-1}$, and the induction assumption applies to them, i.e., $\exists m_0(L)$ such that

$$\left.\begin{aligned} H^i(\mathbb{P}^n, \mathcal{G}_L(m)) &\cong (0) \\ H^i(\mathbb{P}^n, \mathcal{H}_L(m)) &\cong (0) \end{aligned}\right. \quad \text{if } m \geq m_0(L),\ 1 \leq i \leq n.$$

The cohomology sequence of (2.9) then gives us:

$$\otimes L \colon H^i(\mathbb{P}^n, \mathcal{F}(m)) \xrightarrow{\sim} H^i(\mathbb{P}^n, \mathcal{F}(m+1)), \quad m \geq m_0(L).$$

In particular, $\dim H^i(\mathbb{P}^n, \mathcal{F}(m)) = N_i$, independent of m for $m \geq m_0(L)$. Now fix one linear form L and consider the maps:

$$\otimes F \colon H^i(\mathcal{F}(m_0(L))) \longrightarrow H^i(\mathcal{F}(m_0(L) + d))$$

for all homogeneous F of degree d. If R_d is the vector space of such F's, then choosing fixed bases of the above cohomology groups, we have a linear map:

$$\begin{aligned} R_d &\longrightarrow \text{vector space of } (N_i \times N_i)\text{-matrices} \\ F &\longmapsto \text{matrix for } \otimes F. \end{aligned}$$

Let I_d be the kernel. It is clear that $I = \sum_{d=1}^{\infty} I_d$ is an ideal in $R = \bigoplus_{d=0}^{\infty} R_d$ and that $\dim R_d/I_d \leq N_i^2$. Thus the degree of the Hilbert polynomial of R/I is 0, hence the subscheme $V(I) \subset \mathbb{P}^n$ with structure sheaf $\mathcal{O}_{\mathbb{P}^n}/\widetilde{I}$ is 0-dimensional. If $V(I) = \{x_1, \ldots, x_t\}$, it follows that the only associated prime ideals of I can be either

$$\mathfrak{m}_{x_i} = \text{ideal of forms } F \text{ with } F(x_i) = 0,\ 1 \leq i \leq t$$

or

$$(X_0, \ldots, X_n).$$

By the primary decomposition theorem, it follows that

$$I \supset (X_0, \ldots, X_n)^{d_0} \cap \bigcap_{i=1}^{t} \mathfrak{m}_{x_i}^{d_i}$$

for some $d_0, \ldots, d_t$. Now fix linear forms L_i such that $L_i(x_i) = 0$. Let

$$F = L^{\max(d_0, m_0(L_i) - m_0(L))} \cdot \prod_{i=1}^{t} L_i^{d_i}.$$

On the one hand, we see that $F \in I$, hence $\otimes F$ on $H^i(\mathcal{F}(m_0(L)))$ is 0. But on the other hand, if $m_1 = \max(d_0, m_0(L_i) - m_0(L))$, then $\otimes F$ factors:

$$\begin{aligned} H^i(\mathcal{F}(m_0(L))) \underset{\approx}{\xrightarrow{\otimes L}} \cdots \underset{\approx}{\xrightarrow{\otimes L}} & H^i(\mathcal{F}(m_0(L) + m_1)) \\ & \underset{\approx}{\xrightarrow{\otimes L_1}} \cdots \underset{\approx}{\xrightarrow{\otimes L_t}} H^i(\mathcal{F}(m_0(L) + m_1 + \sum d_i)) \end{aligned}$$

which is an isomorphism. It follows that $H^i(\mathcal{F}(m_0(L))) = (0)$ as required.

The third step is to show that if $\mathcal{F}$ is coherent analytic, then $\mathcal{F}(\nu)$ is generated by its sections if $\nu \gg 0$. For each $x \in \mathbb{P}^n$, let $\mathfrak{m}_x$ = sheaf of functions zero at x, and consider the exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{m}_x \cdot \mathcal{F}(\nu) & \longrightarrow & \mathcal{F}(\nu) & \longrightarrow & \mathcal{F}(\nu)/\mathfrak{m}_x \cdot \mathcal{F}(\nu) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & (\mathfrak{m}_x \cdot \mathcal{F})(\nu). & & & & & & \end{array}$$

There exists ν_x such that if $\nu \geq \nu_x$, then $H^1(\mathfrak{m}_x \cdot \mathcal{F}(\nu)) = (0)$, hence $H^0(\mathcal{F}(\nu)) \to H^0(\mathcal{F}(\nu)/\mathfrak{m}_x \cdot \mathcal{F}(\nu))$ is onto. Let $\mathcal{G}$ be the cokernel:

$$H^0(\mathcal{F}(\nu_x)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}} \longrightarrow \mathcal{F}(\nu_x) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then $\mathcal{G}$ is coherent analytic and

$$\mathcal{G}_x/\mathfrak{m}_x \cdot \mathcal{G}_x \cong \mathcal{F}(\nu)_x / \big(\mathfrak{m}_x \cdot \mathcal{F}(\nu)_x + \mathrm{Image}\, H^0(\mathcal{F}(\nu)\big) = (0).$$

Therefore by Nakayama's lemma, $\mathcal{G}_x = (0)$ and by coherency, $\exists$ a neighborhood U_x of x in which $\mathcal{G} \equiv (0)$. It follows that $\mathcal{F}(\nu_x)$ is generated by $H^0(\mathcal{F}(\nu_x))$ in U_x and hence $\mathcal{F}(\nu)$ is generated by $H^0(\mathcal{F}(\nu))$ in U_x for $\nu \geq \nu_x$ too! By compactness $\mathbb{P}^n$ is covered by finitely many of these U_x's, say $U_{x_1}, \ldots, U_{x_t}$. Then if $\nu \geq \max(\nu_{x_i})$, $\mathcal{F}(\nu)$ is generated everywhere by $H^0(\mathcal{F}(\nu))$.

The fourth step is to show that

$$H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)) = \begin{array}{l} \text{vector space of homogeneous forms of} \\ \text{degree } m \text{ in } X_0, \ldots, X_n \end{array}$$

just as in the algebraic case. We do this by induction first on n, since it is clear for $n = 0$; and then by a second induction on m, since it is also clear for $m = 0$, i.e., by the maximum principle the only global analytic functions on the compact space $\mathbb{P}^n$ are constants. The induction step uses the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-1) \xrightarrow{\otimes X_n} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1},\mathrm{an}}(m) \longrightarrow 0$$

which gives:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0,\dots,X_n\\ \text{of degree } m-1\end{array}\right\} & \xrightarrow{X_n} & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0,\dots,X_n\\ \text{of degree } m\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{Polynomials in}\\ X_0,\dots,X_{n-1}\\ \text{of degree } m\end{array}\right\} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-1)) & \xrightarrow{\otimes X_n} & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{n-1},\mathrm{an}}(m)). & \end{array}$$

"Chasing" this diagram shows the required assertion for $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m)$.

The fifth step is that every coherent analytic $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\mathrm{an}}$, some coherent algebraic $\mathcal{F}$. By the third step there is a surjection:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}^{n_0} \longrightarrow \mathcal{F}'(m_0) \longrightarrow 0$$

for suitable n_0 and m_0, hence a surjection:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Applying the same reasoning to the kernel, we get a presentation:

$$\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_1)^{n_1} \xrightarrow{\phi'} \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Now ϕ' is given by an $(n_0 \times n_1)$-matrix of sections ϕ'_{ij} of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m_1 - m_0)$, hence by an $(n_0 \times n_1)$-matrix F_{ij} of polynomials of degree $m_1 - m_0$. Thus the F_{ij} defines ϕ, with cokernel $\mathcal{F}$:

$$\mathcal{O}_{\mathbb{P}^n}(-m_1)^{n_1} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^n}(-m_0)^{n_0} \longrightarrow \mathcal{F} \longrightarrow 0.$$

By exactness of the functor $\mathcal{G} \mapsto \mathcal{G}_{\mathrm{an}}$, it follows that $\mathcal{F}' \cong \mathcal{F}_{\mathrm{an}}$. Using the same set-up, we can also conclude that $H^0(\mathcal{F}(m)) \cong H^0(\mathcal{F}_{\mathrm{an}}(m))$ for $m \gg 0$. In fact, twist enough so that the H^1 of the kernel and image of both ϕ and ϕ' are all (0): then the usual sequences show that the two rows below are exact:

$$\begin{array}{ccccccc} H^0(\mathcal{O}_{\mathbb{P}^n}(m-m_1)^{n_1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n}(m-m_0)^{n_0}) & \longrightarrow & H^0(\mathcal{F}(m)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-m_1)^{n_1}) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(m-m_0)^{n_0}) & \longrightarrow & H^0(\mathcal{F}_{\mathrm{an}}(m)) & \longrightarrow & 0. \end{array}$$

Thus $H^0(\mathcal{F}(m)) \to H^0(\mathcal{F}_{\mathrm{an}}(m))$ is an isomorphism.

The sixth step is to compare the cohomologies of $\mathcal{F}(m)$ and $\mathcal{F}_{\mathrm{an}}(m)$ for all m. We know that for $m \gg 0$, all their cohomology groups are isomorphic and we may assume by induction on n that we know the result for sheaves on $\mathbb{P}^{n-1}$. We use a second induction on m, i.e., assuming the result for $H^i(\mathcal{F}(m+1))$, all i, deduce it for $H^i(\mathcal{F}(m))$, all i. Use the diagram (2.9) above for any linear form L. We get

$$\begin{array}{ccccccccc} H^{i-1}(\mathcal{F}(m+1)) & \longrightarrow & H^{i-1}(\mathcal{H}_L(m)) & \longrightarrow & H^i(\mathcal{F}'_L(m)) & \longrightarrow & H^i(\mathcal{F}(m+1)) & \longrightarrow & H^i(\mathcal{H}_L(m)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{i-1}(\mathcal{F}_{\mathrm{an}}(m+1)) & \to & H^{i-1}(\mathcal{H}_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}_{\mathrm{an}}(m+1)) & \to & H^i(\mathcal{H}_{L,\mathrm{an}}(m)) \end{array}$$

and

$$\begin{array}{ccccccccc} H^{i-1}(\mathcal{F}'_L(m)) & \longrightarrow & H^i(\mathcal{G}_L(m)) & \longrightarrow & H^i(\mathcal{F}(m)) & \longrightarrow & H^i(\mathcal{F}'_L(m)) & \longrightarrow & H^{i+1}(\mathcal{G}_L(m)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{i-1}(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{G}_{L,\mathrm{an}}(m)) & \to & H^i(\mathcal{F}_{\mathrm{an}}(m)) & \to & H^i(\mathcal{F}'_{L,\mathrm{an}}(m)) & \to & H^{i+1}(\mathcal{G}_{L,\mathrm{an}}(m)). \end{array}$$

By the 5-lemma, the result for $H^i(\mathcal{F}(m+1))$ and $H^{i-1}(\mathcal{F}(m+1))$ implies it for $H^i(\mathcal{F}'_L(m))$. And the result for $H^i(\mathcal{F}'_L(m))$ and $H^{i-1}(\mathcal{F}'_L(m))$ implies it for $H^i(\mathcal{F}(m))$.

The seventh step is to compare $\mathrm{Hom}(\mathcal{F},\mathcal{G})$ and $\mathrm{Hom}(\mathcal{F}_{\mathrm{an}},\mathcal{G}_{\mathrm{an}})$. Presenting $\mathcal{F}$ as before, we get:

$$\begin{aligned}\mathrm{Hom}(\mathcal{F},\mathcal{G}) &\cong \mathrm{Ker}\left[\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-m_0)^{n_0},\mathcal{G}) \xrightarrow{\circ\phi} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-m_1)^{n_1},\mathcal{G})\right]\\ &\cong \mathrm{Ker}\left[H^0(\mathcal{G}(m_0)^{n_0}) \xrightarrow{\otimes F_{ij}} H^0(\mathcal{G}(m_1)^{n_1})\right]\\ &\cong \mathrm{Ker}\left[H^0(\mathcal{G}_{\mathrm{an}}(m_0)^{n_0}) \xrightarrow{\otimes F_{ij}} H^0(\mathcal{G}_{\mathrm{an}}(m_1)^{n_1})\right]\\ &\cong \mathrm{Ker}\left[\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_0)^{n_0},\mathcal{G}_{\mathrm{an}}) \xrightarrow{\circ\phi} \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}(-m_1)^{n_1},\mathcal{G}_{\mathrm{an}})\right]\\ &\cong \mathrm{Hom}(\mathcal{F}_{\mathrm{an}},\mathcal{G}_{\mathrm{an}}).\end{aligned}$$

□

COROLLARY 2.10. *A new proof of Chow's theorem (*Part I [**87**, (4.6)]*): If $X \subset \mathbb{P}^n$ is a closed analytic subset, then X is a closed algebraic subset.*

PROOF. If $X \subset \mathbb{P}^n$ is a closed analytic subset, then $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$ is a coherent analytic sheaf, so $\mathcal{I}_X = \mathcal{J}_{\mathrm{an}}$ for some coherent algebraic $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^n}$. So $X = \mathrm{Supp}\,\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}/\mathcal{I}_X = \mathrm{Supp}\,\mathcal{O}_{\mathbb{P}^n}/\mathcal{J}$ is a closed algebraic subset. □

COROLLARY 2.11. *If X_1 and X_2 are two complete verieties over $\mathbb{C}$, then every holomorphic map $f\colon X_1 \to X_2$ is algebraic, i.e., a morphism.*

PROOF. Apply Chow's lemma (Theorem II.6.3) to find proper birational $\pi_1\colon X_1' \to X_1$ and $\pi_2\colon X_2' \to X_2$ with X_i' projective. Let $\Gamma \subset X_1 \times X_2$ be the graph of f. Then $(\pi_1 \times \pi_2)^{-1}\Gamma \subset X_1' \times X_2'$ is a closed analytic subset of projective space, hence is algebraic by Chow's theorem. Since $\pi_1 \times \pi_2$ is proper, $\Gamma = (\pi_1 \times \pi_2)[(\pi_1 \times \pi_2)^{-1}\Gamma]$ is also a closed algebraic set. In order to see that it is the graph of a morphism, we must check that $p_1\colon \Gamma \to X_1$ is an isomorphism. This follows from:

LEMMA 2.12. *Let $f\colon X \to Y$ be a bijective morphism of varieties. If f is an analytic isomorphism, then f is an algebraic isomorphism.*

PROOF OF LEMMA 2.12. Note that f is certainly birational since $\#f^{-1}(y) = 1$ for all $y \in Y$. Let $x \in X$, $y = f(x)$. We must show that $f^*\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. The local rings of analytic functions on X and Y at x and y and the formal completions of these rings are related by:

$$\begin{array}{ccc}\mathcal{O}_{Y,y} & \xrightarrow{\quad f^* \quad} & \mathcal{O}_{X,x}\\ \downarrow & & \downarrow\\ (\mathcal{O}_{Y,\mathrm{an}})_y & \xrightarrow{\quad f^*_{\mathrm{an}} \quad} & (\mathcal{O}_{X,\mathrm{an}})_x\\ \downarrow & & \downarrow\\ \widehat{\mathcal{O}}_{Y,y} & \xrightarrow{\quad f^* \quad} & \widehat{\mathcal{O}}_{X,x}.\end{array}$$

Now f^*_{an} is an isomorphism by assumption. If $a \in \mathcal{O}_{X,x}$, write $a = b/c$, $b, c \in \mathcal{O}_{Y,y}$ using the fact that f is birational. Then

$$\begin{aligned} f^*_{\mathrm{an}} \text{ isomorphism} &\Longrightarrow \exists a' \in (\mathcal{O}_{Y,\mathrm{an}})_y \text{ with } b = c\cdot a'\\ &\Longrightarrow b \in \mathcal{O}_{Y,y} \cap c\cdot\widehat{\mathcal{O}}_{Y,y}.\end{aligned}$$

But for any ideal $\mathfrak{a} \subset \mathcal{O}_{Y,y}$, $\mathfrak{a} = \mathcal{O}_{Y,y} \cap \mathfrak{a}\cdot\widehat{\mathcal{O}}_{Y,y}$, so $b \in c\cdot\mathcal{O}_{Y,y}$, i.e., $a \in \mathcal{O}_{Y,y}$. □

□

Corollary 2.13 (Projective case of Riemann's Existence Theorem). *Let X be a complex projective variety*[5]. *Let $\widetilde{Y}$ be a compact topological space and*

$$\widetilde{\pi}\colon \widetilde{Y} \longrightarrow (X \textit{ in the classical topology})$$

a covering map (since $\widetilde{Y}$ is compact, this amounts merely to requiring that $\widetilde{\pi}$ is a local homeomorphism). Then there is a unique scheme Y and étale proper morphism $\pi\colon Y \to X$ such that there exists a homeomorphism ρ:

$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "\approx", "\rho"'] \arrow[dr] & & (Y \textit{ in the classical topology}) \arrow[dl] \\
& (X \textit{ in the classical topology}). &
\end{tikzcd}$$

Proof. Given $\widetilde{Y}$, note first that since $\widetilde{\pi}$ is a local homeomorphism we can put a unique analytic structure on it making $\widetilde{\pi}$ into a local analytic isomorphism. Let $\mathcal{B} = \widetilde{\pi}_*(\mathcal{O}_{\widetilde{Y}})$: this is a sheaf of $\mathcal{O}_{X,\mathrm{an}}$-algebras. Now every $x \in X$ has a neighborhood U such that $\widetilde{\pi}^{-1}(U) \cong$ disjoint union of l copies of U; hence $\mathcal{B}|_U \cong \bigoplus_{i=1}^{l} \mathcal{O}_{X,\mathrm{an}}$ as a sheaf of algebras. In particular, $\mathcal{B}$ is a coherent analytic sheaf of $\mathcal{O}_{X,\mathrm{an}}$-modules. Recall that we can identify sheaves of $\mathcal{O}_{X,\mathrm{an}}$-modules with sheaves of $\mathcal{O}_{\mathbb{P}^n,\mathrm{an}}$-modules, (0) outside X and killed by multiplication by $\mathcal{I}_X$. Therefore by the fundamental GAGA Theorem 2.8, $\mathcal{B} \cong \mathfrak{B}_{\mathrm{an}}$ for some algebraic coherent sheaf of $\mathcal{O}_X$-modules $\mathfrak{B}$. Multiplication in $\mathcal{B}$ defines an $\mathcal{O}_{X,\mathrm{an}}$-module homomorphism

$$\mu\colon \mathcal{B} \otimes_{\mathcal{O}_{X,\mathrm{an}}} \mathcal{B} \longrightarrow \mathcal{B},$$

hence by the GAGA Theorem 2.8 again this is induced by some $\mathcal{O}_X$-module homomorphism:

$$\nu\colon \mathfrak{B} \otimes_{\mathcal{O}_X} \mathfrak{B} \longrightarrow \mathfrak{B}.$$

The associative law for μ implies it for ν and so this makes $\mathfrak{B}$ into a sheaf of $\mathcal{O}_X$-algebras. The unit in $\mathcal{B}$ similarly gives a unit in $\mathfrak{B}$. We now define $Y = \mathrm{Spec}_X(\mathfrak{B})$, with $\pi\colon Y \to X$ the canonical map (proper since $\mathfrak{B}$ is coherent by Proposition II.6.5). How are Y and $\widetilde{Y}$ related? We have

i) a continuous map $\zeta\colon \widetilde{Y} \to$ (closed points of X)
ii) a map backwards covering ζ:

$$\zeta^*\colon \mathfrak{B} \longrightarrow (\text{sheaf of continuous } \mathbb{C}\text{-valued functions on } \widetilde{Y})$$

such that $\forall x \in \widetilde{Y}$ and $\forall f \in \mathfrak{B}_{\zeta(x)}$,

$$\zeta^* f(x) = f(\zeta(x)),$$

defined as the composite

$$\mathfrak{B}(U) \longrightarrow \mathcal{B}(U) \longrightarrow \mathcal{O}_{\widetilde{Y},\mathrm{an}}(\widetilde{\pi}^{-1}U) \longrightarrow [\text{continuous functions on } \widetilde{\pi}^{-1}U].$$

These induce a continuous map:

$$\eta\colon \widetilde{Y} \longrightarrow (\text{closed points of } Y)$$

[5]The theorem is true in fact for any variety X and finite-sheeted covering $\pi^*\colon Y^* \to X$, but this is harder, cf. SGA4 [**7**, Theorem 4.3, Exposé 11], where Artin deduces the general case from Grauert-Remmert [**43**]; or SGA1 [**4**, Exposé XII, Théorème 5.1, p. 332], where Grothendieck deduces it from Hironaka's resolution theorems [**61**].

by

$$\eta(x) = \text{ point corresponding to maximal ideal } \frac{\{f \in \mathfrak{B}_{\zeta(x)} \mid \zeta^* f(x) = 0\}}{\mathfrak{m}_{X,\zeta(x)} \cdot \mathfrak{B}_{\zeta(x)}}$$
via the correspondence
$$\pi^{-1}(\zeta(x)) \leftrightarrows \text{maximal ideals in } \mathfrak{B}_{\zeta(x)}/\mathfrak{m}_{\zeta(x)} \cdot \mathfrak{B}_{\zeta(x)}.$$

Now η has the property

2.14. $\forall f \in \mathcal{O}_Y(V)$, *the composite map*

$$\eta^{-1}(V) \xrightarrow{\eta} (\textit{closed points of } V) \xrightarrow{f} \mathbb{C}$$

is a continuous function on $\eta^{-1}(V)$ *(in the classical topology).*

But a basis for open sets in the classical topology on Y is given by finite intersections of the sets:

$$\begin{aligned} &V \text{ Zariski open, } f \in \mathcal{O}_Y(V), \text{ let} \\ &W_{f,\epsilon} = \{x \in V \mid x \text{ closed and } |f(x)| < \epsilon\}. \end{aligned}$$

Because of (2.14), $\eta^{-1}(W_{f,\epsilon})$ is open in $\widetilde{Y}$, i.e., η is a continuous map from $\widetilde{Y}$ to (Y in the classical topology). Now in fact η is bijective too. In fact, if $U \subset X$ is a classical open so that $\pi^{-1}(U) =$ (disjoint union of n copies of U) and $\mathcal{B}|_U = \bigoplus_{i=1}^{l} \mathcal{O}_{X,\mathrm{an}}|_U$, then for all $x \in U$,

$$\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \cong \bigoplus_{i=1}^{l} \mathbb{C}$$

and the correspondence between points of $\pi^{-1}(x)$ and maximal ideals of $\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x$ given by $y \mapsto \{f \mid f(y) = 0\}$ is bijective. On the other hand, since $\mathcal{B}_x \cong \mathfrak{B}_x \otimes_{\mathcal{O}_{x,x}} (\mathcal{O}_{X,\mathrm{an}})_x$, it follows that $\mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \cong \mathfrak{B}_x/\mathfrak{m}_x \cdot \mathfrak{B}_x$. Thus η is a continuous bijective map from a compact space $\widetilde{Y}$ to (Y in the classical topology). Thus η is a homeomorphism. Finally $\mathcal{B}_x$ is a free $(\mathcal{O}_{X,\mathrm{an}})_x$-module, hence it follows that $\mathfrak{B}_x$ is a free $\mathcal{O}_{X,x}$-module: Hence $\pi\colon Y \to X$ is a flat morphism. And the scheme-theoretic fibre is:

$$\begin{aligned} \pi^{-1}(x) &= \operatorname{Spec} \mathfrak{B}_x/\mathfrak{m}_x \cdot \mathfrak{B}_x \\ &\cong \operatorname{Spec} \mathcal{B}_x/\mathfrak{m}_x \cdot \mathcal{B}_x \\ &\cong \operatorname{Spec} \bigoplus_{i=1}^{l} \mathbb{C} = l \text{ reduced points.} \end{aligned}$$

Thus π is étale.

As for the uniqueness of Y, it is a consequence of the stronger result: say

$$\begin{tikzcd} Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

are two étale proper morphisms. Then any map continuous in the classical topology:

$$(Y_1 \text{ in the classical topology}) \xrightarrow{f} (Y_2 \text{ in the classical topology})$$

with $\pi_2 \circ f = \pi_1$ is a morphism. To see this, note that π_i are local analytic isomorphisms, hence f is analytic, hence by Corollary 2.11, f is a morphism. □

This Corollary 2.13 implies profound connections between topology and field theory. To explain these, we must first define the *algebraic fundamental group* $\pi_1^{\mathrm{alg}}(X)$ for any normal noetherian scheme X[6]. We have seen in §V.6 that morphisms:

$$\text{(a)}\qquad \begin{array}{l} \pi\colon Y \longrightarrow X \\ Y \text{ normal irreducible} \\ \pi \text{ proper, surjective, } \pi^{-1}(x) \text{ finite for all } x, \\ \pi \text{ generically smooth} \end{array}$$

are uniquely determined by the function field extension $\mathbf{R}(Y) \supset \mathbf{R}(X)$, which is necessarily separable; and that conversely, given any finite separable $K \supset \mathbf{R}(X)$, we obtain such a π by setting $Y =$ the normalization of X in K. In particular, suppose we start with a morphism:

$$\text{(b)}\qquad \begin{array}{l} \pi\colon Y \longrightarrow X \\ Y \text{ connected} \\ \pi \text{ proper and étale.} \end{array}$$

Then Y is smooth over a normal X, hence is normal by Proposition V.5.5. Being connected, Y is also irreducible. Thus $Y =$ normalization of X in $\mathbf{R}(Y)$. Now choose a specific separable algebraic closure $\overline{\mathbf{R}(X)}$ of $\mathbf{R}(X)$ and let

$$\begin{aligned} G &= \mathrm{Gal}(\overline{\mathbf{R}(X)}/\mathbf{R}(X)), \text{ the Galois group} \\ &\cong \varprojlim_K \mathrm{Aut}(K/\mathbf{R}(X)) \\ &\qquad \text{where } \mathbf{R}(X) \subset K \subset \overline{\mathbf{R}(X)},\ K \text{ normal over } \mathbf{R}(X) \text{ with } [K:\mathbf{R}(X)] < +\infty. \end{aligned}$$

As usual, G, being an inverse limit of finite groups, has a natural structure of compact, totally disconnected topological group. One checks easily[7] that there is an intermediate field:

$$\mathbf{R}(X) \subset \widetilde{\mathbf{R}(X)} \subset \overline{\mathbf{R}(X)}$$

such that for all $K \subset \overline{\mathbf{R}(X)}$, finite over $\mathbf{R}(X)$:

$$\left.\begin{array}{l} \text{the normalization } Y_K \text{ of } X \\ \text{in } K \text{ is étale over } X \end{array}\right\} \Longleftrightarrow K \subset \widetilde{\mathbf{R}(X)}.$$

---

[6]Normality is not necessary and noetherian can be weakened. For a discussion of the results below in more general case, see SGA1 [**4**, Exposés V and XII].

[7]This follows from two simple facts:

a) $K_1 \subset K_2$, Y_{K_2} étale over $X \implies Y_{K_1}$ étale over X,
b) Y_{K_1} and Y_{K_2} étale over $X \implies Y_{K_1\cdot K_2}$ étale over X.

To prove (a), note that we have a diagram

$$Y_{K_2} \longrightarrow Y_{K_1} \longrightarrow X.$$

Now Y_{K_2} étale over $X \implies \Omega_{Y_{K_2}/X} = (0) \implies \Omega_{Y_{K_2}/Y_{K_1}} = (0) \implies Y_{K_2}$ étale over Y_{K_1} by Criterion 4.1 for smoothness in §V.4. In particular, Y_{K_2} is flat over Y_{K_1}, hence if $y_2 \in Y_{K_2}$ has images y_1 and x in Y_{K_1} and X, then $\mathcal{O}_{y_2}$ is flat over $\mathcal{O}_{y_1}$, hence $\mathfrak{m}_x \cdot \mathcal{O}_{y_2} \cap \mathcal{O}_{y_1} = \mathfrak{m}_x \cdot \mathcal{O}_{y_1}$. Thus

$$\mathcal{O}_{y_1}/\mathfrak{m}_x \cdot \mathcal{O}_{y_1} \subset \mathcal{O}_{y_2}/\mathfrak{m}_x \cdot \mathcal{O}_{y_2} \cong \text{product of separable field extensions of } \Bbbk(x)$$

hence $\mathcal{O}_{y_1}/\mathfrak{m}_x \cdot \mathcal{O}_{y_1}$ is also a product of separable field extensions of $\Bbbk(x)$. This shows $\Omega_{Y_{K_1}/X} \otimes \Bbbk(y_1) = (0)$, hence by Nakayama's lemma, $\Omega_{Y_{K_1}/X} = (0)$ near y_1, hence by Criterion 4.1 for smoothness in §V.4, Y_{K_1} is étale over X at y_1.

To prove (b), note that $Y_{K_1} \times_X Y_{K_2}$ will be étale over X, hence normal. We get a morphism

$$Y_{K_1\cdot K_2} \xrightarrow{\phi} Y_{K_1} \times_X Y_{K_2}$$

and if $Z =$ component of $Y_{K_1} \times_X Y_{K_2}$ containing Image ϕ, then $\phi\colon Y_{K_1\cdot K_2} \to Z$ is birational. Since Z is normal and the fibres of ϕ are finite, ϕ is an isomorphism by Zariski's Main Theorem in §V.6.

Note that because of its defining property, $\widetilde{\mathbf{R}(X)}$ is invariant under all automorphisms of $\overline{\mathbf{R}(X)}$, i.e., it is normal over $\mathbf{R}(X)$ and its Galois group over $\mathbf{R}(X)$ is a quotient G/N of G. By Galois theory, the closed subgroups of finite index in G are in one-to-one correncepondence with the subfields $K \subset \overline{\mathbf{R}(X)}$ finite over $\mathbf{R}(X)$. So the closed subgroups of finite index in G/N are in one-to-one correspondence with the subfields $K \subset \widetilde{\mathbf{R}(X)}$ finite over $\mathbf{R}(X)$, hence with the set of schemes Y_K étale over X. It is therefore reasonable to call G/N the *algebraic fundamental group* of X, or $\pi_1^{\mathrm{alg}}(X)$:

$$\pi_1^{\mathrm{alg}}(X) = \mathrm{Gal}(\widetilde{\mathbf{R}(X)}/\mathbf{R}(X)). \tag{2.15}$$

Next in the complex projective case again choose a universal covering space Ω of X in the classical topology. Then the topological fundamental group is:

$$\pi_1^{\mathrm{top}}(X) = \text{group of homeomorphisms of } \Omega \text{ over } X,$$

and its subgroups of finite index are in one-to-one correspondence with the compact covering spaces $\widetilde{Y}$ dominated by Ω:

$$\Omega \longrightarrow \widetilde{Y} \xrightarrow{\tilde{\pi}} X,$$

which give, by algebraization (Corollary 2.13), connected normal complete varieties Y, étale over X. This must simply force a connection between the two groups and, in fact, it implies this:

THEOREM 2.16. *Let X be a normal subvariety of $\mathbb{P}^n_{\mathbb{C}}$ and let*

$$\begin{aligned}\widehat{\pi}_1^{\mathrm{top}}(X) &= \varprojlim_H \pi_1^{\mathrm{top}}(X)/H, \ \text{ over all } H \subset \pi_1^{\mathrm{top}}(X) \text{ of finite index}\\ &= \text{“pro-finite completion” of } \pi_1^{\mathrm{top}}(X).\end{aligned}$$

Then $\widehat{\pi}_1^{\mathrm{top}}(X)$ and $\pi_1^{\mathrm{alg}}(X)$ are isomorphic as topological groups, the isomorphism being canonical up to an inner automorphism.

PROOF. Choose a sequence $\{H_\nu\}$ of normal subgroups of π_1^{top} of finite index, with $H_{\nu+1} \subset H_\nu$, such that for any H of finite index, $H_\nu \subset H$ for some ν. Let $\pi_1^{\mathrm{top}}/H_\nu = G_\nu$ and let H_ν define $\widetilde{Y}_\nu \to X$. Then

$$\widehat{\pi}_1^{\mathrm{top}} \cong \varprojlim_\nu G_\nu$$

$$\text{and} \qquad G_\nu \cong \text{group of homeomorphisms of } \widetilde{Y}_\nu \text{ over } X.$$

Algebraize $\widetilde{Y}_\nu$ to a scheme Y_ν étale over X by Corollary 2.13. Then the map $\widetilde{Y}_{\nu+1} \to \widetilde{Y}_\nu$ comes from a morphism $Y_{\nu+1} \to Y_\nu$ and we get a tower of function field extensions:

$$\cdots \longleftarrow \mathbf{R}(Y_{\nu+1}) \longleftarrow \mathbf{R}(Y_\nu) \longleftarrow \cdots \longleftarrow \mathbf{R}(X).$$

Note that

$$\mathrm{Aut}_{\mathbf{R}(X)}(\mathbf{R}(Y_\nu)) \cong \mathrm{Aut}_X(Y_\nu) \cong \mathrm{Aut}_X(\widetilde{Y}_\nu) \cong G_\nu$$

and since $\#G_\nu =$ degree of the covering $(\widetilde{Y}_\nu \to X) = [\mathbf{R}(Y_\nu) : \mathbf{R}(X)]$, this shows that $\mathbf{R}(Y_\nu)$ is a normal extension of $\mathbf{R}(X)$. The fact that $Y_\nu \cong Y_{\mathbf{R}(Y_\nu)}$ is étale over X shows that $\mathbf{R}(Y_\nu)$ is isomorphic to a subfield of $\widetilde{\mathbf{R}(X)}$. Now choose an $\mathbf{R}(X)$-isomorphism:

$$\phi\colon \bigcup_{\nu=1}^{\infty} \mathbf{R}(Y_\nu) \longrightarrow \widetilde{\mathbf{R}(X)}.$$

It is easy to see that ϕ is surjective by going backwards from an étale $Y_K \to X$ to a topological covering $\widetilde{Y}_K \to X$ and dominating this by Ω. So we get the sought for isomorphism:

$$\begin{aligned}\widehat{\pi}_1^{\text{top}} &\cong \varprojlim_\nu G_\nu \\ &\cong \varprojlim_\nu \operatorname{Gal}(\mathbf{R}(Y_\nu)/\mathbf{R}(X)) \\ &\cong \operatorname{Gal}(\widetilde{\mathbf{R}(X)}/\mathbf{R}(X)) \\ &\cong \pi_1^{\text{alg}}.\end{aligned}$$

The only choice here is of ϕ and varying ϕ changes the above isomorphism by an inner automorphism. □

As a final topic I would like to discuss Grothendieck's *formal analog* of Serre's fundamental theorem. His result is this:

Let $R =$ noetherian ring, complete in the topology
defined by the powers of an ideal I.
Let $X \longrightarrow \operatorname{Spec} R$ be a proper morphism.

Consider the schemes:

$$X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec} R/I^{n+1}$$

i.e.,

$$\begin{array}{ccccccccc} X_0 & \subset & \cdots & \subset & X_n & \subset & \cdots\cdots & \subset & X \\ \downarrow & & & & \downarrow & & & & \downarrow \\ \operatorname{Spec} R/I & \subset & \cdots & \subset & \operatorname{Spec} R/I^{n+1} & \subset & \cdots\cdots & \subset & \operatorname{Spec} R \end{array}$$

Define: a *formal coherent sheaf* $\mathcal{F}$ on X is a set of coherent sheaves $\mathcal{F}_n$ on X_n plus isomorphisms:

$$\mathcal{F}_{n-1} \cong \mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}}.$$

Note that every coherent $\mathcal{F}$ on X induces a formal $\mathcal{F}_{\text{for}}$ by letting

$$\mathcal{F}_{\text{for},n} \cong \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}.$$

Then:

THEOREM 2.17 (Grothendieck). (Fundamental "GFGA"[8] comparison theorem)

i) *For every coherent algebraic $\mathcal{F}$ on X and every i,*

$$H^i(X, \mathcal{F}) \cong \varprojlim_n H^i(X_n, \mathcal{F}_n)$$

where $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$.

ii) *The categories of formal and algebraic coherent sheaves are equivalent, i.e., every formal $\mathcal{F}'$ is isomorphic to $\mathcal{F}_{\text{for}}$, some $\mathcal{F}$, and*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \text{Formal } \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_{\text{for}}, \mathcal{G}_{\text{for}}).$$

The result for H^0 is essentially due to Zariski, whose famous [**118**] proving this and applying it to prove the connectedness theorem (see (V.6.3) Fundamental theorem of "holomorphic functions") started this whole development. A complete proof of Theorem 2.17 can be found in EGA [**1**, Chapter 3, §§4 and 5][9].

[8]Short for "géométrie formelle et géométrie algébrique".

[9](Added in publication) Illusie's account in FAG [**3**, Chapter 8] "provides an introduction, explaining the proofs of the key theorems, discussing typical applications, and updating when necessary."

Here we will prove only the special case:

R complete local, $I =$ maximal ideal, $k = R/I$
X projective over $\operatorname{Spec} R$

(which suffices for most applications)[10]. If X is projective over $\operatorname{Spec} R$, we can embed X in $\mathbb{P}^m_R$ for some m, and extend all sheaves from X to $\mathbb{P}^m_R$ by (0): thus it suffices to prove Theorem 2.17 for $X = \mathbb{P}^m_R$.

Before beginning the proof, we need elementary results on the category of coherent formal sheaves. For details, we refer the reader to EGA [**1**, Chapter 0, §7 and Chapter 1, §10]; however none of these facts are very difficult and the reader should be able to supply proofs.

2.18. *If A is a noetherian ring, complete in its I-adic topology and $U_n = \operatorname{Spec} A/I^{n+1}$, then there is an equivalence of categories between:*

a) *sets of coherent sheaves $\mathcal{F}_n$ on U_n plus isomorphism*

$$\mathcal{F}_{n-1} \cong \mathcal{F}_n \otimes_{\mathcal{O}_{U_n}} \mathcal{O}_{U_{n-1}}$$

b) *finitely generated A-modules M*

given by:

$$M = \varprojlim_n \Gamma(U_n, \mathcal{F}_n)$$
$$\mathcal{F}_n = \widetilde{M/I^{n+1}M}.$$

In particular, Category (a) is abelian: [but kernel is *not* the usual sheaf-theoretic kernel because $M_1 \subset M_2$ does *not* imply $M_1/I^{n+1}M_1 \subset M_2/I^{n+1}M_2$!].

2.19. *Given A as above, and $f \in A$, then*

$$A_f = \varprojlim_n A_f/I^n \cdot A_f = \varprojlim_n (A/I^n A)_f$$

is flat over A.

Corollary 2.20. *The category of coherent formal sheaves $\{\mathcal{F}_n\}$ on a scheme X, proper over $\operatorname{Spec} R$ (R as above) is abelian with*

$$\operatorname{Coker}[\{\mathcal{F}_n\} \longrightarrow \{\mathcal{G}_n\}] = \{\operatorname{Coker}(\mathcal{F}_n \longrightarrow \mathcal{G}_n)\}_{n=0,1,\ldots}$$

but

$$\operatorname{Ker}[\{\mathcal{F}_n\} \longrightarrow \{\mathcal{G}_n\}] = \{\mathcal{H}_n\}$$

where for each affine $U \subset X$:

$$\mathcal{H}_n(U) = \mathcal{H}(U)/I^{n+1}\mathcal{H}(U)$$
$$\mathcal{H}(U) = \operatorname{Ker}\left[\varprojlim_n \mathcal{F}_n(U) \longrightarrow \varprojlim_n \mathcal{G}_n(U)\right].$$

Proof of Corollary 2.20. Applying (2.18) with $A = \varprojlim_n \mathcal{O}_{X_n}(U)$ we construct kernels of $\{\mathcal{F}_n|_U\} \to \{\mathcal{G}_n|_U\}$ for each affine U as described above. Use (2.19) to check that on each distinguished open $U_f \subset U$, the restriction of the kernel on U is the kernel on U_f. $\square$

[10](Added in publication) See the remark at the end of this section.

2.21. *If A is any ring and*

$$0 \longrightarrow K_n \longrightarrow L_n \longrightarrow M_n \longrightarrow 0$$

are exact sequences of A-modules for each $n \geq 0$ fitting into an inverse system

$$\begin{array}{ccccccccc} 0 & \to & K_{n+1} & \to & L_{n+1} & \to & M_{n+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_n & \longrightarrow & L_n & \longrightarrow & M_n & \longrightarrow & 0 \end{array}$$

and if for each n, the decreasing set of submodules $\mathrm{Image}(K_{n+k} \to K_n)$ *of K_n is stationary for k large enough, then*

$$0 \longrightarrow \varprojlim_n K_n \longrightarrow \varprojlim_n L_n \longrightarrow \varprojlim_n M_n \longrightarrow 0$$

is exact.

Proof of Theorem 2.17. We now begin the proof of GFGA. To start off, say $\mathcal{F} = \{\mathcal{F}_n\}$ is a coherent formal sheaf on $\mathbb{P}^m_R$. Introduce

$$\begin{aligned} \operatorname{gr} R = \bigoplus_{n=0}^{\infty} I^n/I^{n+1} : &\quad \text{a finitely generated graded } k\text{-algebra} \\ S = \operatorname{Spec}(\operatorname{gr} R) : &\quad \text{an affine scheme of finite type over } k \\ \operatorname{gr} \mathcal{F} = \bigoplus_{n=0}^{\infty} I^n \cdot \mathcal{F}_n : &\quad \text{a quasi-coherent sheaf on } \mathbb{P}^m_k. \end{aligned}$$

Note that $\operatorname{gr}\mathcal{F}$ is in fact a sheaf of $(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}) \otimes \mathcal{O}_{\mathbb{P}^m_k}$-modules and since

$$\left(I^n/I^{n+1}\right) \otimes_k \mathcal{F}_0 \longrightarrow I^n \cdot \mathcal{F}_n$$

is surjective, $\operatorname{gr}\mathcal{F}$ is finitely generated as a sheaf of $(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}) \otimes \mathcal{O}_{\mathbb{P}^m_k}$-modules. In other words, we can form a *coherent* sheaf $\widetilde{\operatorname{gr}\mathcal{F}}$ on

$$\operatorname{Spec}_S\left(\left(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}\right) \otimes \mathcal{O}_{\mathbb{P}^m_k}\right) = \mathbb{P}^m_S.$$

Moreover,

$$H^q(\mathbb{P}^m_S, \widetilde{\operatorname{gr}\mathcal{F}}) = \bigoplus_{n=0}^{\infty} H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n).$$

This same holds after twisting $\mathcal{F}$ by the standard invertible sheaf $\mathcal{O}(l)$, hence:

$$H^q(\mathbb{P}^m_S, \widetilde{\operatorname{gr}\mathcal{F}(l)}) = \bigoplus_{n=0}^{\infty} H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n(l)).$$

But since $\operatorname{gr} R$ is a noetherian ring, the left hand side is (0) if $l \geq l_0$ (for some l_0) and $q \geq 1$. Thus:

$$H^q(\mathbb{P}^m_k, I^n \cdot \mathcal{F}_n(l)) = (0), \text{ if } q \geq 1,\ n \geq 0,\ l \geq l_0.$$

Now look at the exact sequences:

$$0 \longrightarrow I^n \cdot \mathcal{F}_n(l) \longrightarrow \mathcal{F}_n(l) \longrightarrow \mathcal{F}_{n-1}(l) \longrightarrow 0.$$

It follows from the cohomology sequences by induction on n that:

$$\begin{aligned} &H^q(\mathbb{P}^m_R, \mathcal{F}_n(l)) = (0), \text{ if } q \geq 1, \\ \text{and } &H^0(\mathbb{P}^m_R, \mathcal{F}_n(l)) \longrightarrow H^0(\mathbb{P}^m_R, \mathcal{F}_{n-1}(l)) \text{ surjective for all } n \geq 0,\ l \geq l_0. \end{aligned}$$

The next step (like the third step of the GAGA Theorem 2.8) is that for some l_1, $\{\mathcal{F}_n(l)\}$ is generated by its sections for all $l \geq l_1$: i.e., there is a set of surjections:

$$\mathcal{O}^N_{\mathbb{P}^m_R}/I^{n+1} \cdot \mathcal{O}^N_{\mathbb{P}^m_R} \longrightarrow \mathcal{F}_n(l) \longrightarrow 0 \tag{2.22}$$

commuting with restriction from $n+1$ to n. To see this, take $l_1 \geq l_0$ so that $\mathcal{F}_0(l)$ is generated by its sections for $l \geq l_1$. This means there is a surjection:

$$\mathcal{O}^N_{\mathbb{P}^m_k} \longrightarrow \mathcal{F}_0(l) \longrightarrow 0.$$

By (2.21), this lifts successively to compatible surjections as in the third step of the GAGA Theorem 2.8. In other words, we have a surjection of formal coherent sheaves:

$$\mathcal{O}^N_{\mathbb{P}^m_R}(-l)_{\text{for}} \longrightarrow \{\mathcal{F}_n\}. \tag{2.23}$$

Next, as in the fourth step of the GAGA Theorem 2.8, we prove

$$\begin{aligned}\varprojlim_n H^0(\mathcal{O}_{\mathbb{P}^m_R}(l)/I^{n+1} \cdot \mathcal{O}_{\mathbb{P}^m_R}(l)) &\cong (R\text{-module of homogeneous forms of degree } l) \\ &\cong H^0(\mathcal{O}_{\mathbb{P}^m_R}(l)).\end{aligned}$$

This is obvious since $\mathcal{O}_{\mathbb{P}^m_R}(l)/I^{n+1} \cdot \mathcal{O}_{\mathbb{P}^m_R}(l)$ is just the structure sheaf of $\mathbb{P}^m_{R_n}$, where $R_n = R/I^{n+1} \cdot R$. Then the fifth step follows GAGA in Theorem 2.8 precisely: given $\{\mathcal{F}_n\}$, we take the kernel of Corollary 2.20 and repeat the construction, obtaining a presentation:

$$\mathcal{O}^{N_1}_{\mathbb{P}^m_R}(-l_1)_{\text{for}} \xrightarrow{\phi} \mathcal{O}^{N_0}_{\mathbb{P}^m_R}(-l_0)_{\text{for}} \longrightarrow \{\mathcal{F}_n\} \longrightarrow 0.$$

By the fourth step, ϕ is given by a matrix of homogeneous forms, hence we can form the *algebraic* coherent sheaf:

$$\mathcal{F} = \text{Coker}\left[\phi\colon \mathcal{O}^{N_1}_{\mathbb{P}^m_R}(-l_1) \longrightarrow \mathcal{O}^{N_0}_{\mathbb{P}^m_R}(-l_0)\right]$$

and it follows immediately that $\mathcal{F}_n \cong \mathcal{F}/I^{n+1} \cdot \mathcal{F}$, i.e., $\{\mathcal{F}_n\} \cong \mathcal{F}_{\text{for}}$.

The rest of the proof follows that of GAGA in Theorem 2.8 precisely with $H^q(\mathcal{F}_{\text{an}})$ replaced by $\varprojlim_n H^q(\mathcal{F}/I^{n+1} \cdot \mathcal{F})$, once one checks that

$$\mathcal{F} \longmapsto \varprojlim_n H^q(\mathcal{F}/I^n \cdot \mathcal{F})$$

is a "cohomological δ-functor" of coherent algebraic sheaves $\mathcal{F}$, i.e., if $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is exact, then one has a long exact sequence

$$\begin{aligned}0 \longrightarrow \varprojlim_n H^0(\mathcal{F}/I^n \cdot \mathcal{F}) &\longrightarrow \varprojlim_n H^0(\mathcal{G}/I^n \cdot \mathcal{G}) \longrightarrow \varprojlim_n H^0(\mathcal{H}/I^n \cdot \mathcal{H}) \\ &\xrightarrow{\delta} \varprojlim_n H^1(\mathcal{F}/I^n \cdot \mathcal{F}) \longrightarrow \cdots\cdots .\end{aligned}$$

But this follows by looking at the exact sequences:

$$0 \longrightarrow \mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G}) \longrightarrow \mathcal{G}/I^n \cdot \mathcal{G} \longrightarrow \mathcal{H}/I^n \cdot \mathcal{H} \longrightarrow 0.$$

By (2.21), the cohomology groups

$$\begin{aligned}&\varprojlim_n H^q(\mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G})) \\ &\varprojlim_n H^q(\mathcal{G}/I^n \cdot \mathcal{G}) \\ &\varprojlim_n H^q(\mathcal{H}/I^n \cdot \mathcal{H})\end{aligned}$$

fit into a long exact sequence (since for each n, the n-th terms of these limits are finitely generated $(R/I^n \cdot R)$-modules, hence are of finite length). But by the Artin-Rees lemma (Zariski-Samuel

[**119**, vol. II, Chapter VIII, §2, Theorem 4′, p. 255]), the sequence of subsheaves $\mathcal{F} \cap I^n \cdot \mathcal{G}$ of $\mathcal{F}$ is cofinal with the sequence of subsheaves $I^n \cdot \mathcal{F}$: in fact $\exists l$ such that for all $n \geq l$:

$$I^n \cdot \mathcal{F} \subset \mathcal{F} \cap I^n \cdot \mathcal{G} = I^{n-l} \cdot (\mathcal{F} \cap I^l \cdot \mathcal{G}) \subset I^{n-l} \cdot \mathcal{F}.$$

Therefore

$$\varprojlim_n H^q(\mathcal{F}/(\mathcal{F} \cap I^n \cdot \mathcal{G})) = \varprojlim_n H^q(\mathcal{F}/I^n \cdot \mathcal{F}).$$

□

Corollary 2.24. *Every formal closed subscheme Y_{for} of X (i.e., the set of closed subschemes $Y_n \subset X_n$ such that $Y_{n-1} = Y_n \times_{X_n} X_{n-1}$) is induced by a unique closed subscheme Y of X (i.e., $Y_n = Y \times_X X_n$).*

Corollary 2.25. *Every formal étale covering $\pi\colon Y_{\text{for}} \to X$ (i.e., a set of coverings $\pi_n\colon Y_n \to X_n$ plus isomorphisms $Y_{n-1} \cong Y_n \times_{X_n} X_{n-1}$) is induced by a unique étale covering $\pi\colon Y \to X$ (i.e., $Y_n \cong Y \times_X X_n$).*

In fact, it turns out that an étale covering $\pi_0\colon Y_0 \to X_0$ already defines uniquely the whole formal covering, so that it follows that $\pi_1^{\text{alg}}(X_0) \cong \pi_1^{\text{alg}}(X)$: See Corollary 5.9 below.[11] Another remarkable fact is that the GAGA and GFGA comparison theorems are closer than it would seem at first. In fact, if R is a complete discrete valuation ring with absolute value $|\ \ |$, note that for $\mathbb{A}^m_R$:

$$\begin{aligned} \varprojlim_n H^0(\mathbb{A}^m_{(R/I^{n+1})}, \mathcal{O}_{\mathbb{A}^m}) &= \varprojlim_n (R/I^{n+1})[X_1, \ldots, X_m] \\ &\cong \text{ring of "convergent power series" } \sum c_\alpha X^\alpha \\ &\quad \text{where } c_\alpha \in R \text{ and } |c_\alpha| \to 0 \text{ as } |\alpha| \to \infty. \end{aligned}$$

This is the basis of a connection between the above formal geometry and a so-called "rigid" or "global" analytic geometry over the quotient field K of R. For an introduction to this, see Tate [**112**].

Remark. (Added in publication) (Chai) Grothendieck's GFGA theorem (Theorem 2.17) is proved here when X is projective over a complete local ring R and I is the maximal ideal of R. The proof for the case X proper over any complete noetherian ring (R, I) follows from the case X projective over (R, I) again by an argument using Chow's lemma and noetherian induction similar to that in the proof of Grothendieck's coherency theorem (Theorem VII.6.5).

The assumption here that I is the maximal ideal makes the Mittag-Leffler condition automatically satisfied and simplifies the proof.

Grothendieck's original proof of GFGA (Theorem 2.17) does not seem to have been published. The folklore is that the original proof uses downward induction on the degree of cohomology as in Serre's proof of GAGA theorem (Theorem 2.8). In the published proof in EGA [**1**, Chap. III, §4.1], the degree is fixed, and the Artin-Rees/Mittag-Leffler type conditions are deduced from the finiteness theorem for proper morphism applied to the base change of $f\colon X \to \operatorname{Spec} R$ to the spectrum of the Artin-Rees algebra $\bigoplus_\nu^\infty I^\nu$.

Mumford's proof of the GAGA theorem (Theorem 2.8) does not use downward induction on the degree of cohomology, but uses downward induction on the integers d and m for $\mathbb{P}^d$ and $\mathcal{F}(m)$. In Mumford's proof here of the GFGA theorem (Theorem 2.17), the required uniform

[11] (Added in publication) See §5 for other applications of GFGA in connection with deformations (e.g., Theorem 5.5 on algebraization). See also Illusie's account in FAG [**3**, Chapter 8].

vanishing is proved by the usual vanishing theorem applied to the associated graded ring $\mathrm{gr}(R) = \bigoplus_{\nu=0}^{\infty} I^{\nu}/I^{\nu+1}$.

It might be of interest to note that this trick of base change to the associated graded ring also gives a proof that the projective system

$$\big(H^i(\mathcal{F}_n)\big)_{n\in\mathbb{N}}$$

attached to any compatible family $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of coherent $\mathcal{O}_{X_n}$-modules satisfies the Mittag-Leffler condition and that the topology induced on the projective limit

$$\varprojlim_n H^i(\mathcal{F}_n)$$

is equal to the I-adic topology on $\varprojlim_n H^i(\mathcal{F}_n)$ *without* the assumption that I is the maximal ideal. With this additional ingredient, Mumford's proof of the GFGA theorem (Theorem 2.17) becomes valid for a general open (not necessarily maximal) ideal I in a complete noetherian local ring R. The detailed proof for

> the finiteness of
> $$\bigoplus_{n=0}^{\infty} H^i(I^n\mathcal{F}_n) \quad \text{as an } \Big(\bigoplus_{n=0}^{\infty} I^n/I^{n+1}\Big)\text{-module for } i=i_0 \text{ and } i_0+1$$
> implies the uniform Mittag-Leffler/Artin-Rees for
> $$\big(H^i(\mathcal{F}_n)\big)_{n\in\mathbb{N}} \quad \text{for } i=i_0 \text{ and } i_0+1$$

can be found in EGA [**1**, 0_{III}, 13.7.7] (with correction). Here is a sketch of the spectral sequence argument for the proof:

For any triple of natural numbers p, i and n, define

$$\begin{aligned} Z_r^{p,i-p}(H^\bullet(\mathcal{F}_n)) &:= \mathrm{Image}\left(H^i(I^p\mathcal{F}_n/I^{p+r}\mathcal{F}_n) \to H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n)\right) \\ B_r^{p,i-p}(H^\bullet)(\mathcal{F}_n) &:= \mathrm{Image}\left(H^{i-1}(I^{p-r+1}\mathcal{F}_n/I^p\mathcal{F}_n) \to H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n)\right). \end{aligned}$$

Note that $H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n) = H^i(I^p\mathcal{F}_p)$ for $\forall n \geq p$. For each fixed n we have natural isomorphisms

$$(*) \qquad d_r^{p,i-p}\colon \frac{Z_r^{p,i-p}(H^\bullet(\mathcal{F}_n))}{Z_{r+1}^{p,i-p}(H^\bullet(\mathcal{F}_n))} \xrightarrow{\sim} \frac{B_{r+1}^{p+r,i-p-r+1}(H^\bullet(\mathcal{F}_n))}{B_r^{p+r,i-p-r+1}(H^\bullet(\mathcal{F}_n))},$$

inclusion relations

$$\begin{aligned} (0) = B_1^{p,i-p}(H^\bullet(\mathcal{F}_n)) &\subset B_2^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subset \cdots \subset B_{p+1}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \\ &= B_{p+2}^{p,i-p}(H^\bullet(\mathcal{F}_n)) = \cdots = B_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n)) \\ &\subset Z_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n)) = \cdots = Z_{n-p+2}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \\ &= Z_{n-p+1}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subset Z_{n-p}^{p,i-p}(H^\bullet(\mathcal{F}_n)) \subset \cdots \subset Z_1^{p,i-p}(H^\bullet(\mathcal{F}_n)) = H^i(I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n), \end{aligned}$$

and isomorphisms

$$\frac{\mathrm{Image}\left(H^i(I^p\mathcal{F}_n) \to H^i(\mathcal{F}_n)\right)}{\mathrm{Image}\left(H^i(I^{p+1}\mathcal{F}_n) \to H^i(\mathcal{F}_n)\right)} \xrightarrow{\sim} \frac{Z_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n))}{B_\infty^{p,i-p}(H^\bullet(\mathcal{F}_n))},$$

which comes from the map between cohomology long exact sequences associated to the map

$$\begin{CD} 0 @>>> I^p\mathcal{F}_n @>>> \mathcal{F}_n @>>> \mathcal{F}_n/I^p\mathcal{F}_n @>>> 0 \\ @. @VVV @VVV @VVV @. \\ 0 @>>> I^p\mathcal{F}_n/I^{p+1}\mathcal{F}_n @>>> \mathcal{F}_n/I^{p+1}\mathcal{F}_n @>>> \mathcal{F}_n/I^p\mathcal{F}_n @>>> 0 \end{CD}$$

between short exact sequences. If we fix p and r, then we have

$$\cdots \xrightarrow{\sim} B_r^{p,i-p}(H^\cdot(\mathcal{F}_{p+2})) \xrightarrow{\sim} B_r^{p,i-p}(H^\cdot(\mathcal{F}_{p+1})) \xrightarrow{\sim} B_r^{p,i-p}(H^\cdot(\mathcal{F}_p))$$

and

$$\cdots \xrightarrow{\sim} Z_r^{p,i-p}(H^\cdot(\mathcal{F}_{p+r+1})) \xrightarrow{\sim} Z_r^{p,i-p}(H^\cdot(\mathcal{F}_{p+r})) \xrightarrow{\sim} Z_r^{p,i-p}(H^\cdot(\mathcal{F}_{p+r-1})).$$

Denote by $B_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))$ and $Z_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))$ the respective projective limits, and let

$$E_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty)) := \frac{Z_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))}{B_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))}.$$

We record here that

$$Z_r^{p,q}(H^\cdot(\mathcal{F}_n)) \quad \text{stabilizes} \quad \begin{cases} \text{in the } r\text{-direction for} & r \geq n-p+1 \\ \text{in the } n\text{-direction for} & n \geq p+r-1 \end{cases}$$

and

$$B_r^{p,q}(H^\cdot(\mathcal{F}_n)) \quad \text{stabilizes} \quad \begin{cases} \text{in the } r\text{-direction for} & r \geq p+1 \\ \text{in the } n\text{-direction for} & n \geq p. \end{cases}$$

For each i and r, the direct sum

$$\bigoplus_{p\geq 0} B_r^{p,i-p}(H^i(\mathcal{F}_\infty))$$

has a natural structure as a graded $(\bigoplus_{p\geq 0} I^p/I^{p+1})$-submodule of $\bigoplus_{p\geq 0} H^i(I^p\mathcal{F}_p)$ and increases with r. Since $\bigoplus_{p\geq 0} H^i(I^p\mathcal{F}_p)$ is finitely generated over $\bigoplus_{p\geq 0} I^p/I^{p+1}$, the increasing chain of submodules $\bigoplus_{p\geq 0} B_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))$ stabilizes for $r \geq r(i)$, where $r(i)$ is a positive integer depending on i. (This is where the properness assumption is used.) So the differentials $d_r^{p,q} = 0$ for all p, q with $p+q = i_0 - 1$ and all $r \geq r(i-1)$. Hence the decreasing chain of submodules $\bigoplus_{p\geq 0} Z_r^{p,i-p}(H^\cdot(\mathcal{F}_\infty))$ stabilizes for $r \geq r(i-1)$. Let r_0 be the maximum of $r(0), r(1), \ldots, r(d)$, where $d = \dim(X_0/\operatorname{Spec}(R/I))$.

Since the graded $(\bigoplus_{p\geq 0} I^p/I^{p+1})$-modules

$$\bigoplus_{p\geq 0} B_{r_0}^{p,i-p}(H^\cdot(\mathcal{F}_\infty)), \quad \bigoplus_{p\geq 0} Z_{r_0}^{p,i-p}(H^\cdot(\mathcal{F}_\infty)) \quad \text{and} \quad \bigoplus_{p\geq 0} E_{r_0}^{p,i-p}(H^\cdot(\mathcal{F}_\infty))$$

are finitely generated, we have:

$$(1) \qquad B_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) \xrightarrow{\sim} B_{r_0+1}^{p,q}(H^\cdot(\mathcal{F}_\infty)) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \varinjlim_r B_r^{p,q}(H^\cdot(\mathcal{F}_\infty)) \qquad \forall p,q,$$

$$(2) \qquad Z_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) \xleftarrow{\sim} Z_{r_0+1}^{p,q}(H^\cdot(\mathcal{F}_\infty)) \xleftarrow{\sim} \cdots \xleftarrow{\sim} \varprojlim_r Z_r^{p,q}(H^\cdot(\mathcal{F}_\infty)) \qquad \forall p,q,$$

(3) there exists a positive integer p_0 such that

$$\begin{aligned} I \cdot Z_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) &= Z_{r_0}^{p+1,q}(H^\cdot(\mathcal{F}_\infty)), \quad \forall p \geq p_0 \\ I \cdot B_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) &= B_{r_0}^{p+1,q}(H^\cdot(\mathcal{F}_\infty)), \quad \forall p \geq p_0 \\ I \cdot E_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) &= E_{r_0}^{p+1,q}(H^\cdot(\mathcal{F}_\infty)), \quad \forall p \geq p_0. \end{aligned}$$

From (1) and the stabilization range for $Z_r^{p,q}(H^\cdot(\mathcal{F}_n))$, we see that

$$Z_r^{p,q}(H^\cdot(\mathcal{F}_n)) = Z_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty))$$

if $p \leq p_0$, $n \geq p + r_0 - 1$ and $r \geq r_0$. Note that the family $(Z_r^{p,q}(H^\cdot(\mathcal{F}_n)))_{r,n\in\mathbb{N}}$ is projective in both directions r and n, and the transition maps are injective for $n \geq p$. Hence we see from (3), (2) and $(*)$ that

$$(4) \qquad Z_r^{p,q}(H^\cdot(\mathcal{F}_n)) \xrightarrow{\sim} Z_{r_0}^{p,q}(H^\cdot(\mathcal{F}_\infty)) \quad \text{if } n \geq p + r_0 - 1 \text{ and } r \geq r_0,$$

and

$$(5)\qquad B_r^{p,q}(H^\bullet(\mathcal{F}_n)) \cong B_{r_0}^{p,q}(H^\bullet(\mathcal{F}_\infty)) \quad \text{if } n \geq p \text{ and } r \geq r_0.$$

It is not difficult to deduce from these the following:
(6) (uniform Mittag-Leffler)

$$\mathrm{Image}\left(H^i(\mathcal{F}_{n+r_0-1}) \to H^i(\mathcal{F}_n)\right) = \mathrm{Image}\left(H^i(\mathcal{F}_m) \to H^i(\mathcal{F}_n)\right) \quad \text{for all } m \geq n + r_0 - 1.$$

(7) For each i, the topology on $H^i(\mathcal{F}_\infty) := \varprojlim_n H^i(\mathcal{F}_n)$ induced by the projective limit coincides with the I-adic topology on $H^i(\mathcal{F}_\infty)$.

3. De Rham cohomology

As in §2 we wish to work in this section only with varieties X over $\mathbb{C}$. For any such X, we have the topological space (X in the classical topology) and for any group G, we can consider the "constant sheaf G_X" on this:

$$G_X(U) = \left\{ \begin{array}{l} \text{functions } f\colon U \to G, \text{ constant on each} \\ \text{connected component of } U. \end{array} \right\}$$

It is a standard fact from algebraic topology (cf. for instance, Spanier [**109**, Chapter 6, §9]; or Warner [**114**]) that if a topological space Y is nice enough — e.g., if it is a finite simplicial complex — then the sheaf cohomology $H^i(Y, G_Y)$ and the singular cohomology computed by G-valued cochains on all singular simplices of Y as in Part I [**87**, §5C] are canonically isomorphic. One may call these the *classical cohomology groups of* Y. I would like in this part to indicate the basic connection between these groups for $G = \mathbb{C}$, and the coherent sheaf cohomology studied above. This connection is given by the ideas of De Rham already mentioned in Part I [**87**, §5C].

We begin with a completely general definition: let $f\colon X \to Y$ be a morphism of schemes. We have defined the Kähler differentials $\Omega_{X/Y}$ in Chapter V. We now go further and set:

$$\Omega^k_{X/Y} \underset{\text{def}}{=} \bigwedge^k(\Omega_{X/Y}), \text{ i.e., the sheafification of the pre-sheaf}$$

$$U \mapsto \bigwedge^k \text{ of the } \mathcal{O}_X(U)\text{-module } \Omega_{X/Y}(U).$$

One checks by the methods used above that this is quasi-coherent and that

$$\Omega^k_{X/Y}(U) = \bigwedge^k \text{ over } \mathcal{O}_X(U) \text{ of } \Omega_{X/Y}(U) \text{ for } U \text{ affine.}$$

In effect, this means that for U affine in X lying over V affine in Y:

$$\begin{aligned}
\Omega^k_{X/Y}(U) = &\text{free } \mathcal{O}_X(U)\text{-module on generators } dg_1 \wedge \cdots \wedge dg_k, \\
&(g_i \in \mathcal{O}_X(U)), \text{ modulo} \\
&\text{a) } d(g_1 + g_1') \wedge \cdots \wedge dg_k = dg_1 \wedge \cdots \wedge dg_k + dg_1' \wedge \cdots \wedge dg_k \\
&\text{b) } d(g_1 g_1') \wedge \cdots \wedge dg_k = g_1 dg_1' \wedge \cdots \wedge dg_k + g_1' dg_1 \wedge \cdots \wedge dg_k \\
&\text{c) } dg_{\epsilon 1} \wedge \cdots \wedge dg_{\epsilon k} = \mathrm{sgn}(\epsilon) \cdot dg_1 \wedge \cdots \wedge dg_k \ (\epsilon = \text{permutation}) \\
&\text{d) } dg_1 \wedge dg_2 \wedge \cdots \wedge dg_k = 0 \text{ if } g_1 = g_2 \\
&\text{d) } dg_1 \wedge \cdots \wedge dg_k = 0 \text{ if } g_1 \in \mathcal{O}_Y(V).
\end{aligned}$$

The derivation $d\colon \mathcal{O}_X \to \Omega_{X/Y}$ extends to maps:

$$d\colon \Omega^k_{X/Y} \longrightarrow \Omega^{k+1}_{X/Y} \qquad (\textit{not } \mathcal{O}_X\text{-linear})$$

given on affine U by:

$$d(f dg_1 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge \cdots \wedge dg_k, \quad f, g_i \in \mathcal{O}_X(U).$$

(Check that this is compatible with relations (a)–(e) on Ω^k and Ω^{k+1}, hence d is well-defined.) It follows immediately from the definition that $d^2 = 0$, i.e.,

$$\Omega^{\boldsymbol{\cdot}}_{X/Y} : 0 \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega^1_{X/Y} \xrightarrow{d} \Omega^2_{X/Y} \xrightarrow{d} \cdots$$

is a *complex*. Therefore as in §VII.3 we may define the hypercohomology $\mathbb{H}^i(X, \Omega^{\boldsymbol{\cdot}}_{X/Y})$ of this complex, which is known as the De Rham cohomology $H^i_{\mathrm{DR}}(X/Y)$ of X over Y. Grothendieck [**45**], putting together more subtly earlier ideas of Serre, Atiyah and Hodge with Hironaka's resolution theorems [**61**] has proven the very beautiful:

THEOREM 3.1 (De Rham comparison theorem). *If X is a variety smooth (but not necessarily proper) over $\mathbb{C}$, then there is a canonical isomorphism:*

$$H^i_{\mathrm{DR}}(X/\mathbb{C}) \cong H^i((X \textit{ in the classical topology}), \mathbb{C}_X).$$

We will only prove this for projective X referring the reader to Grothendieck's elegant paper [**45**] for the general case. Combining Theorem 3.1 with the spectral sequence of hypercohomology gives:

COROLLARY 3.2. *There is a spectral sequence with*

$$E_1^{pq} = H^q(X, \Omega^p_{X/\mathbb{C}})$$

and d_1^{pq} being induced by $d \colon \Omega^p \to \Omega^{p+1}$ abutting to $H^\nu((X$ in the classical topology$), \mathbb{C})$. In particular, if X is affine, then

$$\frac{\{\textit{closed } \nu\textit{-forms}\}}{\{\textit{exact } \nu\textit{-forms}\}} \cong H^\nu((X \textit{ in the classical topology}), \mathbb{C}).$$

To prove the theorem in the projective case, we must simply combine the GAGA comparison theorem (Theorem 2.8) with the so-called Poincaré lemma on analytic differentials. First we recall the basic facts about analytic differentials. If X is an n-dimensional complex manifold, then the tangent bundle T_X has a structure of a rank n complex analytic vector bundle over X, i.e.,

$$T_X \cong \{(P, D) \mid P \in X,\ D \colon (\mathcal{O}_{X,\mathrm{an}})_P \to \mathbb{C} \text{ a derivation over } \mathbb{C} \text{ centered at } P\}$$

(cf. Part I [**87**, SS1A, 5C, 6B]). Thus if $U \subset X$ is an open set with analytic coordinates $z_1, \ldots, z_n$, then the inverse image of U in T_X has coordinates

$$(P, D) \longmapsto (z_1(P), \ldots, z_n(P), D(z_1), \ldots, D(z_n))$$

under which it is analytically isomorphic to $U \times \mathbb{C}^n$. We then define the sheaves $\Omega^p_{X,\mathrm{an}}$ of holomorphic p-forms by:

$$\Omega^p_{X,\mathrm{an}}(U) = \{\text{holomorphic sections over } U \text{ of the complex vector bundle } \bigwedge^p(T_X^*)\}.$$

(Here $E^* = \mathrm{Hom}(E, \mathbb{C})$ is the dual bundle.) Locally such a section ω is written as usual by an expression

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} c_{i_1,\ldots,i_p} dz_{i_1} \wedge \cdots \wedge dz_{i_p}, \quad c_{i_1,\ldots,i_p} \in \mathcal{O}_{X,\mathrm{an}}(U),$$

and we get the first order differential operators:

$$d \colon \Omega^p_{X,\mathrm{an}} \longrightarrow \Omega^{p+1}_{X,\mathrm{an}}$$

given by

$$d\omega = \sum_{1\le i_1<\cdots<i_{p+1}\le n} \sum_{k=1}^{p+1} (-1)^{k+1} \frac{\partial c_{i_1,\dots,\widehat{i_k},\dots,i_{p+1}}}{\partial z_{i_k}} dz_{i_1}\wedge\cdots\wedge dz_{i_{p+1}}.$$

The map $(\omega,\eta)\mapsto\omega\wedge\eta$ makes $\bigoplus_p \Omega^p_{X,\mathrm{an}}$ into a skew-commutative algebra in which d is a derivation.

LEMMA 3.3 (Poincaré's lemma). *The sequence of sheaves:*

$$0\longrightarrow \mathcal{O}_{X,\mathrm{an}} \xrightarrow{d} \Omega^1_{X,\mathrm{an}} \xrightarrow{d} \Omega^2_{X,\mathrm{an}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{X,\mathrm{an}} \longrightarrow 0$$

is exact, except at the 0*-th place where* $\mathrm{Ker}(d\colon \mathcal{O}_{X,\mathrm{an}}\to\Omega^1_{X,\mathrm{an}})$ *is the sheaf of constant functions* $\mathbb{C}_X$*.*

For an elementary proof of this see Hartshorne [**57**, Remark after Proposition (7.1), p. 54]. (See also Wells [**115**, Chapter II, §2, Example 2.13, p. 49] as well as the proof of the Dolbeault Lemma in Gunning-Rossi [**54**, Chapter I, §D, 3.Theorem, p. 27].)

Now if X is a variety smooth over $\mathbb{C}$, an essential point to check is that the general functor $\mathcal{F}\mapsto\mathcal{F}_{\mathrm{an}}$ of §2 takes the Kähler p-forms $\Omega^p_{X/\mathbb{C}}$ to the above-defined sheaf of holomorphic p-forms $\Omega^p_{X,\mathrm{an}}$. This is virtually a tautology but to tie things together properly, we can proceed like this. For the sake of this argument, we write $\Omega^p_{X,\mathrm{alg}}$ for Kähler differentials on the scheme X, parallel to $\Omega^p_{X,\mathrm{an}}$ defined above:

a) For all $U\subset X$ affine, $\bigoplus_p \Omega^p_{X,\mathrm{alg}}(U)$ is the universal skew-commutative $\mathcal{O}_X(U)$-algebra with derivation (i.e., the free algebra on elements df, $f\in\mathcal{O}_X(U)$, modulo the standard identities); since $\bigoplus_p \Omega^p_{X,\mathrm{an}}(U)$ is another skew-commutative algebra with derivation over $\mathcal{O}_X(U)$ (via the inclusion $\mathcal{O}_X(U)\subset\mathcal{O}_{X,\mathrm{an}}(U)$), there is a unique collection of maps:
$$\Omega^p_{X,\mathrm{alg}}(U)\longrightarrow\Omega^p_{X,\mathrm{an}}(U)$$
commuting with $\wedge$ and d.

b) From a general sheaf theory argument, such a collection of maps factors through a map of sheaves of $\mathcal{O}_{X,\mathrm{an}}$-modules (on X in the classical topology):
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_{\mathrm{an}}\longrightarrow\Omega^p_{X,\mathrm{an}}.$$

c) If $z_1,\dots,z_n\in\mathfrak{m}_{X,x}$ induce a basis of $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$, then we have:
$$\left(\Omega_{X,\mathrm{alg}}\right)_x\cong\bigoplus_{i=1}^n \mathcal{O}_{X,x}\cdot dz_i$$
hence
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_x\cong\bigoplus_{1\le i_i<\cdots<i_p\le n}\mathcal{O}_{X,x}\cdot dz_{i_1}\wedge\cdots\wedge dz_{i_p},$$
hence
$$\left(\Omega^p_{X,\mathrm{alg}}\right)_{\mathrm{an},x}\cong\bigoplus_{1\le i_1<\cdots<i_p\le n}\left(\mathcal{O}_{X,\mathrm{an}}\right)_x dz_{i_1}\wedge\cdots\wedge dz_{i_p}.$$

While $z_1,\dots,z_n$ are local analytic coordinates near x, so

$$\left(\Omega^p_{X,\mathrm{an}}\right)_x\cong\bigoplus_{1\le i_1<\cdots<i_p\le n}\left(\mathcal{O}_{X,\mathrm{an}}\right)_x dz_{i_1}\wedge\cdots\wedge dz_{i_p}$$

too! So we have the following situation: with respect to the identity map

$$\epsilon\colon (X\text{ in the classical topology})\longrightarrow(X\text{ in the Zariski topology})$$

we have a map backwards from the De Rham complex $(\Omega^{\cdot}_X, d)$ of the scheme X to the analytic De Rham complex $(\Omega^{\cdot}_{X,\mathrm{an}}, d)$ of the analytic manifold X. This induces:

a) a map of hypercohomology

$$\mathbb{H}^i(X \text{ in the Zariski topology}, \Omega^{\cdot}_{X/\mathbb{C}}) \longrightarrow \mathbb{H}^i(X \text{ in the classical topology}, \Omega^{\cdot}_{X,\mathrm{an}})$$

and

b) a map of the spectral sequences abutting to these too:

$$\begin{array}{ccc} \text{algebraic } E_1^{pq} & \xrightarrow{\hspace{5em}} & \text{analytic } E_1^{pq} \\ \| & & \| \\ H^q(X \text{ in the Zariski topology}, \Omega^p_{X/\mathbb{C}}) & & H^q(X \text{ in the classical topology}, \Omega^p_{X,\mathrm{an}}). \end{array}$$

But by the GAGA comparison theorem (Theorem 2.8), the map on E_1^{pq}'s is an isomorphism. Now quite generally, if

$$\begin{aligned} E_2^{pq} &\Longrightarrow E^\nu \\ \widetilde{E}_2^{pq} &\Longrightarrow \widetilde{E}^\nu \end{aligned}$$

are two spectral sequences, and

$$\begin{aligned} \phi^{p,q}\colon E_2^{pq} &\longrightarrow \widetilde{E}_2^{pq} \\ \phi^\nu\colon E^\nu &\longrightarrow \widetilde{E}^\nu \end{aligned}$$

are homomorphisms "compatible with the spectral sequences", i.e., commuting with the d's, taking $F^l(E^\nu)$ into $F^l(\widetilde{E}^\nu)$ and commuting with the isomorphisms of E_∞^{pq} with $F^p(E^{p+q})/F^{p+1}(E^{p+q})$, then it follows immediately that

$$\phi^{p,q} \text{ isomorphisms, all } p,q \Longrightarrow \phi^\nu \text{ isomorphisms, all } \nu.$$

In our case, this means that the map in (a) is an isomorphism.

Now compute $\mathbb{H}^\nu(X \text{ in the classical topology}, \Omega^{\cdot}_{X,\mathrm{an}})$ by the second spectral sequence of hypercohomology (cf. (VII.3.11)). Since X in its classical topology is paracompact Hausdorff, we get (cf. §VII.1)

$$H^p\left(X \text{ in the classical topology, sheaf } \frac{\mathrm{Ker}\left(d\colon \Omega^q_{\mathrm{an}} \to \Omega^{q+1}_{\mathrm{an}}\right)}{\mathrm{Image}\left(d\colon \Omega^{q-1}_{\mathrm{an}} \to \Omega^q_{\mathrm{an}}\right)}\right) \Longrightarrow \mathbb{H}^\nu(X, \Omega^{\cdot}_{X,\mathrm{an}}).$$

By Poincaré's lemma (Lemma 3.3), all but one of these sheaves are (0) and the spectral sequence degenerates to an isomorphism:

$$H^\nu(X \text{ in the classical topology}, \mathbb{C}_X) \cong \mathbb{H}^\nu(X \text{ in the classical topology}, \Omega^{\cdot}_{X,\mathrm{an}}).$$

This proves Theorem 3.1 in the projective case.

In the projective case and more generally for any complete variety X, the spectral sequence of Corollary 3.2:

$$E_1^{pq} = H^q(X, \Omega^p_{X/\mathbb{C}}) \Longrightarrow H^\nu(X \text{ in the classical topology}, \mathbb{C})$$

simplifies quite remarkably. In fact the Theory of Hodge implies:

FACT. I: *All* d_r^{pq}*'s are* 0.

This implies that

$$H^q(X, \Omega^p_{X/\mathbb{C}}) \cong E^{pq}_\infty \cong p\text{-th graded piece: } F^p(H^{p+q})/F^{p+1}(H^{p+q}) \text{ of } H^{p+q}(X, \mathbb{C}).$$

Note that $H^{p+q}(X, \mathbb{C}) \cong H^{p+q}(X, \mathbb{Z}) \otimes \mathbb{C}$, hence there is a natural complex conjugation $x \mapsto \overline{x}$ on $H^{p+q}(X, \mathbb{C})$.

FACT. II: *In $H^{p+q}(X, \mathbb{C})$, $\overline{F^{q+1}(H^{p+q})}$ is a complement to the subspace $F^p(H^{p+q})$.*

This implies that H^{p+q} splits canonically into a direct sum:

$$H^\nu(X, \mathbb{C}) = \bigoplus_{\substack{p+q=\nu \\ p,q\geq 0}} H^{p,q}$$

such that

a) $H^{q,p} = \overline{H^{p,q}}$.
b) $F^p(H^{p+q}) = \bigoplus_{p'\geq p} H^{p',q'}$.

Combining both facts,

$$H^{p,q} \cong H^q(X, \Omega^p_{X/\mathbb{C}})$$

hence

$$H^\nu(X, \mathbb{C}) \cong \bigoplus_{p+q=\nu} H^q(X, \Omega^p_{X/\mathbb{C}}). \tag{3.4}$$

FACT. III: *If we calculate $H^\nu(X, \mathbb{C})$ by C^∞ differential forms, then*

$$H^{p,q} \cong \left\{ \begin{array}{l} \textit{set of cohomology classes representable by forms } \omega \\ \textit{of type } (p,q)\textit{, i.e., in local coordinates } z_1, \dots, z_n, \\ \omega = \displaystyle\sum_{\substack{1\leq i_1<\cdots<i_p\leq n \\ 1\leq j_1<\ldots<j_q\leq n}} c_{i_1,\dots,i_p,j_1,\dots,j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q} \end{array} \right\}.$$

(See Kodaira-Morrow [**72**]).

4. Characteristic p phenomena

The theory of De Rham cohomology in characteristic p is still in its infancy[12] and rather than trying to discuss the situation at all generality, I would like instead to fix on one of the really new features of characteristic p and discuss this: namely the Hasse-Witt matrix. To set the stage, if X is a complete non-singular variety over a field k of characteristic $p > 0$, then the De Rham groups

$$\mathbb{H}^\nu(X, \Omega^\cdot_{X/k})$$

are finite-dimensional k-vector spaces which usually behave quite like their counterparts in characteristic 0 and are "reasonable" candidates for the cohomology of X with coefficients in k[13].

For instance, if X is a complete non-singular curve of genus g, then

$$\dim H^1(\mathcal{O}_X) = \dim H^0(\Omega^1) = g, \quad \dim \mathbb{H}^1(\Omega^\cdot_X) = 2g$$

[12](Added in publication) There have been considerable advances, since the original manuscript was written. See the footnote at the end of this section.

[13]There are some cases where their dimension is larger than the expected n-th Betti number B_n and there are also cases where the spectral sequence

$$H^q(\Omega^p) \Longrightarrow \mathbb{H}^n(\Omega^\cdot)$$

does not degenerate. This is apparently connected with the presence of p-torsion on X. And if X is affine instead of complete, these groups are not even finite-dimensional.

in all characteristics. However the De Rham groups have a much richer structure in characteristic p even in the case of curves. The simplest examples of this are the Frobenius cohomology operations:

$$F\colon H^n(X,\mathcal{O}_X) \longrightarrow H^n(X,\mathcal{O}_X),$$
$$X = \text{any scheme in which } p\cdot\mathcal{O}_X \equiv 0$$

given by

$$F(\{a_{i_0,\dots,i_n}\}) = \{a^p_{i_0,\dots,i_n}\}$$

on the cocycle level. Note that if X is a scheme over k, $\operatorname{char} k = p$, so that $H^\nu(X,\mathcal{O}_X)$ is a k-vector space, then F is not k-linear; in fact $F(\alpha\cdot x) = \alpha^p\cdot F(x)$, $\forall\alpha\in k,\ x\in H^\nu(\mathcal{O}_X)$. Such a map we call p-linear. Expanded in terms of a basis of $H^\nu(\mathcal{O}_X)$, F is given by a matrix which is called the ν-th *Hasse-Witt matirix* of X. p-linear maps do *not* have eigenvalues; instead they have the following canonical form:

Lemma 4.1. *Let k be an algebraically closed field of characteristic p, let V be a finite-dimensional vector space over k and let $T\colon V\to V$ be a p-linear transformation. Then V has a unique decomposition:*

$$V = V_s \oplus V_n$$

where

a) *$T(V_n)\subset V_n$ and T is nilpotent on V_n.*
b) *$T(V_s)\subset V_s$ and V_s has a basis $e_1,\dots,e_n$ such that $T(e_i)=e_i$. Furthermore,*

$$\{e\in V_s \mid Te = e\} = \{\sum m_ie_i \mid m_i\in\mathbb{Z}/p\mathbb{Z}\}.$$

Proof. Let $V_s = \bigcap_{\nu=1}^\infty \operatorname{Image} T^\nu$ and $V_n = \bigcup_{\nu=1}^\infty \operatorname{Ker} T^\nu$. Since $\dim V < +\infty$, $V_s = \operatorname{Image} T^\nu$, $V_n = \operatorname{Ker} T^\nu$ for $\nu \gg 0$. Now if $\nu\gg 0$:

$$\begin{aligned} x\in V_s\cap V_n &\Longrightarrow T^\nu x = 0 \text{ and } x = T^\nu y\\ &\Longrightarrow T^{2\nu}y = 0\\ &\Longrightarrow T^\nu y = 0\\ &\Longrightarrow x = 0\end{aligned}$$

and since $\dim V = \dim\operatorname{Ker} T^\nu + \dim\operatorname{Image} T^\nu$, it follows that $V\cong V_s\oplus V_n$. Then $T|_{V_n}$ is nilpotent and $T|_{V_s}$ is bijective. Now choose $x\in V_s$ and take ν minimal such that there is a relation

$$T^\nu x = a_0x + a_1T(x) + \cdots + a_{\nu-1}T^{\nu-1}(x).$$

If $a_0 = 0$, then

$$T^{\nu-1}x = a_1'x + \cdots + a_{\nu-1}'T^{\nu-2}(x)$$

and ν would not be minimal. Now try to solve the equation:

$$T(\lambda_0x + \cdots + \lambda_{\nu-1}T^{\nu-1}(x)) = \lambda_0x + \cdots + \lambda_{\nu-1}T^{\nu-1}(x).$$

This leads to

$$\begin{aligned} \lambda^p_{\nu-1}\cdot a_0 &= \lambda_0\\ \lambda_0^p + \lambda^p_{\nu-1}\cdot a_1 &= \lambda_1\\ &\cdots\cdots\\ \lambda^p_{\nu-2} + \lambda^p_{\nu-1}\cdot a_{\nu-1} &= \lambda_{\nu-1}.\end{aligned}$$

By substitution, we get:

$$\lambda^{p^\nu}_{\nu-1}\cdot a_0^{p^{\nu-1}} + \lambda^{p^{\nu-1}}_{\nu-1}\cdot a_1^{p^{\nu-2}} + \cdots + \lambda^p_{\nu-1}\cdot a_{\nu-1} - \lambda_{\nu-1} = 0$$

which has a non-zero solution. Solving backwards, we find $\lambda_{\nu-2}, \dots, \lambda_0$ as required, hence an $x \in V_s$ with $Tx = x$. Now take a maximal independent set of solutions $e_1, \dots, e_l$ to the equation $Tx = x$. If $W = \sum k \cdot e_i$, then $T\colon W \to W$ is bijective, hence $T\colon V_s/W \to V_s/W$ is also bijective. If $W \subsetneqq V_s$, the argument above then shows $\exists\, \overline{x} \in V_s/W$ such that $T\overline{x} = \overline{x}$. Lifting $\overline{x}$ to $x \in V_s$, we find

$$Tx = x + \sum \lambda_i e_i.$$

Let $\mu_i \in k$ satisfy $\mu_i^p - \mu_i = \lambda_i$. Then $e_{l+1} = x - \sum \mu_i e_i$ also lifts $\overline{x}$ but it satisfies $Te_{l+1} = e_{l+1}$. This proves that e_i span V_s. $\square$

We can apply this decomposition in particular to $H^1(X, \mathcal{O}_X)$ and we find the following interpretations of the eigenvectors:

THEOREM 4.2. *Let X be a complete variety over an algebraically closed field k of characteristic p. Consider F acting on $H^1(X, \mathcal{O}_X)$. Then:*

a) *There is a one-to-one correspondence between $\{\alpha \in H^1(\mathcal{O}_X) \mid F\alpha = \alpha\}$ and pairs (π, ϕ):*

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[loop right, "\phi"] \\ X \end{tikzcd}$$

*π étale, proper, $\pi \circ \phi = \pi$, $\phi^p = \mathrm{id}_Y$, such that $\forall x \in X$ closed, $\#\pi^{-1}(x) = p$ and ϕ permutes these points cyclically: we call this, for short, a p-*cyclic étale convering[14].

b) *If X is non-singular, there is an isomorphism:*

$$\{a \in H^1(\mathcal{O}_X) \mid F\alpha = 0\} \cong \{\omega \in H^0(\Omega^1_X) \mid \omega = df, \text{ some } f \in \mathbf{R}(X)\}.$$

PROOF. (a) Given α with $F\alpha = \alpha$, represent α by a cocycle $\{f_{ij}\}$. Then $F\alpha$ is represented by $\{f_{ij}^p\}$ and since this is cohomologous to α:

$$\begin{aligned} f_{ij} &= f_{ij}^p + g_i - g_j \\ g_i &\in \mathcal{O}_X(U_i). \end{aligned}$$

But then define a sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras by:

$$\begin{aligned} \mathcal{A}|_{U_i} &= \mathcal{O}_X[t_i]/(f_i) \\ f_i(t_i) &= t_i^p - t_i + g_i \end{aligned}$$

and by the glueing:

$$t_i = t_j + f_{ij}$$

over $U_i \cap U_j$. Let $Y_\alpha = \mathrm{Spec}_X(\mathcal{A})$. Since $(df_i/dt_i)(t_i) = -1$, Y_α is étale over X. Since $\mathcal{A}$ is integral and finitely generated over $\mathcal{O}_X$, Y_α is proper over X (cf. Corollary II.6.7). Define $\phi_\alpha\colon Y_\alpha \to Y_\alpha$ by

$$\phi_\alpha^*(t_i) = t_i + 1.$$

For all closed points $x \in U_i$, let a be one solution of $t_i^p - t_i + g_i(x) = 0$. Then $\pi^{-1}(x)$ consists of the p points $t_i = a, a+1, \dots, a+p-1$ which are permuted cyclically by ϕ_α. Finally, and this is where we use the completeness of X, note that Y_α depends only on α:

$$\begin{aligned} \text{if} \quad f'_{ij} &= f_{ij} + h_i - h_j \\ \text{and} \quad f'_{ij} &= (f'_{ij})^p + g'_i - g'_j \end{aligned}$$

[14]Compare the statement of (a) and the proof with Exercise (3) in Chapter V, which treats the Kummer theory of n-cyclic étale coverings for n not divisible by char(k).

is another solution to the above requirements, then

$$\begin{aligned} g_i' - g_j' &= f_{ij}' - (f_{ij}')^p \\ &= f_{ij} - f_{ij}^p + h_i - h_j - (h_i - h_j)^p \\ &= (g_i + h_i - h_i^p) - (g_j + h_j - h_j^p), \end{aligned}$$

hence

$$g_i' = g_i + h_i - h_i^p + \xi, \qquad \xi \in \Gamma(\mathcal{O}_X).$$

Thus $\xi \in k$, hence $\xi = \eta - \eta^p$ for some $\eta \in k$ and we get an isomorphism

$$\begin{array}{ccc} \mathcal{O}_X[t_i]/(f_i^p - t_i + g_i) & \xrightarrow{\approx} & \mathcal{O}_X[t_i']/((t_i')^p - t_i' + g_i') \\ t_i & \longmapsto & t_i' - (h_i + \eta). \end{array}$$

We leave it to the reader to check that $(Y_\alpha, \phi_\alpha) \cong (Y_\beta, \phi_\beta)$ only if $\alpha = \beta$.

Conversely, suppose $\pi\colon Y \to X$ and $\phi\colon Y \to Y$ are a p-cyclic étale covering. By Proposition II.6.5, $Y = \operatorname{Spec}_X \mathcal{A}$, $\mathcal{A}$ a coherent sheaf of $\mathcal{O}_X$-algebras.

Now π étale $\Longrightarrow$ π flat $\Longrightarrow$ $\mathcal{A}_x$ is a flat $\mathcal{O}_{x,X}$-module. A finitely presented flat module over a local ring is free (cf. Bourbaki [**27**, Chapter II, §3.2]), hence $\mathcal{A}_x$ is a free $\mathcal{O}_{x,X}$-module. In fact

$$\mathcal{A}_x/\mathfrak{m}_{x,X} \cdot \mathcal{A}_x \cong \Gamma(\mathcal{O}_{\pi^{-1}(x)}) \cong \bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y)$$

so $\mathcal{A}_x$ is free of rank p, and the function $1 \in \mathcal{A}_x$, since it is not in $\mathfrak{m}_{x,X} \cdot \mathcal{A}_x$, may be taken as a part of a basis. Moreover, ϕ induces an automorphism $\phi^*\colon \mathcal{A} \to \mathcal{A}$ in terms of which we can characterize the subsheaf $\mathcal{O}_X \subset \mathcal{A}$:

$$\mathcal{O}_X(U) = \{f \in \mathcal{A}(U) \mid \phi^* f = f\}.$$

In fact for all closed points $x \in X$, we get an inclusion:

$$\begin{array}{ccc} \mathcal{O}_{x,X}/\mathfrak{m}_{x,X} & \hookrightarrow & \mathcal{A}_x/\mathfrak{m}_{x,X} \cdot \mathcal{A}_x \\ \| & & \| \\ \Bbbk(x) & & \Gamma(\mathcal{O}_{\pi^{-1}(x)}) \\ & & \| \\ & & \bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y) \end{array}$$

and clearly $\Bbbk(x)$ is characterized as the set of ϕ^*-invariant functions in $\bigoplus_{y \in \pi^{-1}(x)} \Bbbk(y)$. So if U is affine and $f \in \mathcal{A}(U)$ is ϕ^*-invariant, then

$$(*) \qquad f \in \bigcap_{\substack{x \in U \\ \text{closed}}} [\mathcal{O}_X(U) + \mathfrak{m}_{x,X} \cdot \mathcal{A}(U)].$$

But if U is small enough, $\mathcal{A}|_U$ has a free basis:

$$\mathcal{A}|_U = \mathcal{O}_X|_U \oplus \sum_{i=2}^{p} \mathcal{O}_X|_U \cdot e_i$$

and if we expand $f = f_1 + \sum_{i=2}^p f_i \cdot e_i$, then $(*)$ means that $f_i(x) = 0$, $2 \le i \le p$, $\forall x \in U$ closed. By the Nullstellensatz, $f_i = 0$, hence $f \in \mathcal{O}_X(U)$. Let $x \in X$ be a closed point and let $\pi^{-1}(x) = \{y, \phi y, \dots, \phi^{p-1} y\}$. We can find a function $e_x \in \mathcal{A}_x$ such that $e_x(y) = 1$, $e_x(\phi^i y) = 0$, $1 \le i \le p-1$. Then

- $\operatorname{Tr} e_x = \sum_{i=0}^{p-1} (\phi^i)^* e_x$ satisfies $\phi^*(\operatorname{Tr} e_x) = \operatorname{Tr} e_x$ and has value 1 at all points of $\pi^{-1}(x)$, hence is invertible in $\mathcal{A}_x$.

- Set

$$f_x = \frac{-1}{\operatorname{Tr} e_x} \cdot \sum_{i=0}^{p-1} i \cdot (\phi^i)^* e_x.$$

A small calculation shows that $\phi^*(f_x) = f_x + 1$ and $f_x(\phi^i y) = i$. Let $g_x = f_x - f_x^p$. Then $\phi^* g_x = g_x$, hence $g_x \in \mathcal{O}_{x,X}$. Define a homomorphism

$$\begin{aligned} \lambda_x \colon \mathcal{O}_{x,X}[t_x]/(t_x^p - t_x + g_x) &\longrightarrow \mathcal{A}_x \\ t_x &\longmapsto f_x. \end{aligned}$$

Note that since f_x has *distinct* values at all points $\phi^i y$, f_x generates

$$\mathcal{A}_x / \mathfrak{m}_{x,X} \cdot \mathcal{A}_x \cong \bigoplus_{i=0}^{p-1} \Bbbk(\phi^i y),$$

hence by Nakayama's lemma, λ_x is surjective. But as λ_x is a homomorphism of free $\mathcal{O}_{x,X}$-modules of rank p, it must be injective too. Now λ_x extends to an isomorphism in some neighborhood of x and covering X by such neighborhoods, we conclude that X has a covering $\{U_i\}$ such that

$$\mathcal{A}|_{U_i} \cong \mathcal{O}_X|_{U_i}[t_i]/(t_i^p - t_i + g_i), \quad g_i \in \mathcal{O}_X(U_i).$$

Over $U_i \cap U_j$, $\phi^*(t_i - t_j) = t_i - t_j$, hence $t_i = t_j + f_{ij}$, $f_{ij} \in \mathcal{O}_X(U_i \cap U_j)$. Then

$$\begin{aligned} f_{ij} - f_{ij}^p &= (t_i - t_j) - (t_i - t_j)^p \\ &= g_i - g_j, \end{aligned}$$

so $\alpha = \{f_{ij}\}$ is a cohomology class in $\mathcal{O}_X$ such that $F\alpha = \alpha$. This completes the proof of (a).

(b) Given α with $F\alpha = 0$, represent α by a cocycle $\{f_{ij}\}$. Then

$$\begin{aligned} f_{ij}^p &= g_i - g_j \\ g_i &\in \mathcal{O}_X(U_i) \end{aligned}$$

hence $dg_i = dg_j$ on $U_i \cap U_j$. Therefore the dg_i's define a global section ω_α of Ω_X^1 of the form df, $f \in \mathbf{R}(X)$. If

$$\begin{aligned} f'_{ij} &= f_{ij} + h_i - h_j \\ (f'_{ij})^p &= g'_i - g'_j \end{aligned}$$

is another solution to the above requirements, then

$$\begin{aligned} g'_i - g'_j &= (f'_{ij})^p \\ &= f_{ij}^p + h_i^p - h_j^p \\ &= (g_i + h_i^p) - (g_j + h_j^p) \end{aligned}$$

hence

$$g'_i = g_i + h_i^p + \xi, \qquad \xi \in \Gamma(\mathcal{O}_X) = k.$$

Thus $dg'_i = dg_i$ and ω_α depends only on α. Conversely, if we are given $\omega \in \Gamma(\Omega_X^1)$, $\omega = df$, $f \in \mathbf{R}(X)$, the first step is to show that for all $x \in X$, $\omega = df_x$ for some $f_x \in \mathcal{O}_{x,X}$ too. We use the following important lemma:

Lemma 4.3. *Let X be a smooth n-dimensional variety over an algebraically closed field k, and assume $\exists z_1, \ldots, z_n \in \Gamma(\mathcal{O}_X)$ such that*

$$\Omega_{X/k}^1 \cong \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot dz_i.$$

Consider $\mathcal{O}_X$ as a sheaf of $\mathcal{O}_X^p$-modules: $\mathcal{O}_X$ is a free $\mathcal{O}_X^p$-module with basis consisting of monomials $\prod_{i=1}^n z_i^{\alpha_i}$, $0 \le \alpha \le p-1$.

Another way to view this is to consider the pair $Y = (X, \mathcal{O}_X^p)$ consisting of the topological space X and the sheaf of rings $\mathcal{O}_X^p$: this is a scheme too, in fact it is isomorphic to X as scheme — but not as scheme over k — via:

$$\text{identity}\colon X \xrightarrow{\quad\approx\quad} Y$$

$$p\text{-th power}\colon \mathcal{O}_X \xrightarrow{\approx} \mathcal{O}_X^p = \mathcal{O}_Y.$$

Thus Y is in fact an irreducible regular scheme, ant it is of finite type over k, i.e., a smooth k-variety. Now

$$\text{identity}\colon X \xrightarrow{\quad\approx\quad} X$$

$$\text{inclusion}\colon \mathcal{O}_Y = \mathcal{O}_X^p \longrightarrow \mathcal{O}_X$$

induces a k-morphism

$$\pi\colon X \longrightarrow Y$$

which is easily seen to be bijective and proper. Thus $\pi_*\mathcal{O}_X$ is a coherent $\mathcal{O}_Y$-module, and we are asserting that it is free with basis $\prod_{i=1}^n z_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$.

Proof of Lemma 4.3. To check that $\prod z_i^{\alpha_i}$ generate $\pi_*\mathcal{O}_X$, it suffices to prove that for all closed points $x \in Y$, $\prod z_i^{\alpha_i}$ generate $(\pi_*\mathcal{O}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\mathcal{O}_X)_x$ over k. But identifying $\mathcal{O}_{x,Y}$ with $\mathcal{O}_{x,X}^p$, $\mathfrak{m}_{x,Y} = \{f^p \mid f \in \mathfrak{m}_{x,X}\}$: write this $\mathfrak{m}_{x,X}^{[p]}$. Then

$$(\pi_*\mathcal{O}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\mathcal{O}_X)_x \cong \mathcal{O}_{x,X}/\mathfrak{m}_{x,X}^{[p]}\cdot\mathcal{O}_{x,X}.$$

Let $a_i = z_i(x)$ and $y_i = z_i - a_i$. Then $y_1, \dots, y_n$ generate $\mathfrak{m}_{x,X}$ and $\widehat{\mathcal{O}}_{x,X} \cong k[[y_1, \dots, y_n]]$ by Proposition V.3.8. Thus

$$(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x \cong k[[y_1, \dots, y_n]]/(y_1^p, \dots, y_n^p)$$

and the latter has a basis given by the monomials $\prod y_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$, hence by $\prod z_i^{\alpha_i}$, $0 \le \alpha_i \le p-1$.

But now suppose there was a relation over $U \subset X$:

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha^p \cdot z^\alpha = 0, \qquad c_\alpha \in \mathcal{O}_X(T) \text{ not all zero.}$$

Then for some closed point $x \in U$, $c_\alpha(x) \ne 0$ for some α, hence there would be relation over k:

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha(x)^p \cdot z^\alpha = 0$$

in $(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x$. But the above proof showed that the z^α were k-independent in $(\pi_*\widehat{\mathcal{O}}_X)_x/\mathfrak{m}_{x,Y}\cdot(\pi_*\widehat{\mathcal{O}}_X)_x$. □

To return to the proof of Theorem 4.2, let $x \in X$, $f \in \mathbf{R}(X)$ and suppose $df \in (\Omega^1_{X/k})_x$. Write $f = g/h^p$, $g, h \in \mathcal{O}_{x,X}$, and by Lemma 4.3 expand:

$$g = \sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ 0\le\alpha_i\le p-1}} c_\alpha^p z^\alpha, \qquad \{z_1, \dots, z_n\} \text{ a generator of } \mathfrak{m}_{x,X}.$$

Then

$$df = \sum_{l=1}^{n} \left(\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n) \\ 0\le\alpha_i\le p-1}} \left(\frac{c_\alpha}{h}\right)^p \alpha_l z_1^{\alpha_1} \cdots z_l^{\alpha_l-1} \cdots z_n^{\alpha_n} \right) dz_l$$

hence

$$\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n) \\ 0\le\alpha_i\le p-1 \\ 0<\alpha_l}} c_\alpha^p \alpha_l z_1^{\alpha_1} \cdots z_l^{\alpha_l-1} \cdots z_n^{\alpha_n} = h^p \cdot b_l, \quad b_l \in \mathcal{O}_{x,X}.$$

Expanding b_l by Lemma 4.3, and equating coefficients of z^α, it follows that $c_\alpha^p \in h^p \cdot \mathcal{O}_{x,X}^p$ if $\alpha_l > 0$. Since this is true for all $l = 1, \dots, n$, it follows:

$$g = c_{(0,\dots,0)}^p + h^p \cdot f_x, \qquad f_x \in \mathcal{O}_{x,X}.$$

Therefore

$$df = d(g/h^p) = df_x.$$

Now we can find a covering $\{U_i\}$ of X and $f_i \in \mathcal{O}_X(U_i)$ such that $\omega = df_i$. Then in $U_i \cap U_j$, $d(f_i - f_j) = 0$, hence $f_i - f_j = g_{ij}^p$, $g_{ij} \in \mathcal{O}(U_i \cap U_j)$ (prove this either by Lemma 4.3 again, or by field theory since $d\colon \mathbf{R}(X)/\mathbf{R}(X)^p \to \Omega^1_{X/k}$ is injective and $\mathcal{O}_x \cap \mathbf{R}(X)^p = \mathcal{O}_x^p$ by the normality of X). Then $\{g_{ij}\}$ defines $\alpha \in H^1(\mathcal{O}_X)$ such that $F\alpha = 0$. This completes the proof of Theorem 4.2. □

The astonishing thing about (b) is that any $f \in [\mathbf{R}(X) \setminus k]$ must have poles and *in characteristic* 0,

$$f \notin \mathcal{O}_{x,X} \Longrightarrow df \notin (\Omega^1_{X/k})_x.$$

In fact, if f has an l-fold pole along an irreducible divisor D, then df has an $(l+1)$-fold pole along D. But *in characteristic* p, if $p \mid l$ then the expected pole of df may sometimes disappear! Nonetheless, this is relatively rare phenomenon even in characteristic p.

For instance, in char $\neq 2$, consider a hyperelliptic curve C. This is defined to be the normalization of $\mathbb{P}^1$ in a quadratic field extension $k(X, \sqrt{f(X)})$. Explicitly, if we take $f(X)$ to be a polynomial with no multiple roots and assume its degree is odd: say $2n+1$, then C is covered by two affine pieces:

$$\begin{aligned} C_1 &= \operatorname{Spec} k[X,Y]/(Y^2 - f(X)) \\ C_2 &= \operatorname{Spec} k[\widetilde{X},\widetilde{Y}]/(\widetilde{Y}^2 - g(\widetilde{X})) \end{aligned}$$

where

$$\begin{aligned} \widetilde{X} &= 1/X \\ \widetilde{Y} &= Y/X^{n+1} \\ g(\widetilde{X}) &= (\widetilde{X})^{2n+2} \cdot f(1/\widetilde{X}). \end{aligned}$$

Then consider $\omega = dX/Y$:

$$\begin{aligned}
\text{On } C_1\colon 2Y\,dY = f'(X)\cdot dX, &\text{ so}\\
&\omega = dX/Y = 2dY/f'(X) \text{ and since } Y, f'(X)\\
&\text{have no common zeroes, } \omega \text{ has no poles.}\\
\text{On } C_2\colon 2\widetilde{Y}\,d\widetilde{Y} = g'(\widetilde{X})\cdot d\widetilde{X}, &\text{ and one checks}\\
&\omega = -(\widetilde{X})^{n-1}d\widetilde{X}/\widetilde{Y} = -2(\widetilde{X})^{n-1}d\widetilde{Y}/g'(\widetilde{X}) \text{ and since}\\
&\widetilde{Y}, g'(\widetilde{X}) \text{ have no common zeroes, } \omega \text{ has no poles.}
\end{aligned}$$

But now say

$$f(X) = h(X)^p + X.$$

Then $f'(X) = 1$, so $\omega = d(2Y)$ is exact!

The area of characteristic p De Rham theory is far from being completely understood. For further developments, see Serre [**102**, p.24] (from which our theorem has been taken), Grothendieck [**46**] and Monsky [**80**, p.451][15].

5. Deformation theory

We want to study here some questions of a completely new type: given an artin local ring R, with maximal ideal M, residue field $k = R/M$ and some other ideal I such that $I\cdot M = (0)$, we get

$$\operatorname{Spec} R \supset \operatorname{Spec} R/I \supset \operatorname{Spec} k.$$

Then

a) Suppose X_1 is a scheme smooth and of finite type over R/I. How many schemes X_2 are there, smooth and of finite type over R, such that $X_1 \cong X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$?

$$\begin{array}{ccc}
X_2 & \supset & X_1\\
\downarrow & & \downarrow\\
\operatorname{Spec} R & \supset & \operatorname{Spec} R/I
\end{array}$$

Such an X_2 we call a *deformation of X_1 over R*.

b) Suppose X_2, Y_2 are two schemes smooth and of finite type over R, and let $X_1 = X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$, $Y_1 = Y_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$. Suppose $f_1\colon X_1 \to Y_1$ is an R/I-morphism. How many R-morphisms $f_2\colon X_2 \to Y_2$ are there lifting f_1?

In fact the methods that we use to study these questions can be extended to the case where the X's and Y's are merely *flat* over R or R/I (this is another reason why flat is such an important concept). We can state the results in the smooth case as follows:

In case (a), let $X_0 = X_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$. As in §V.3, let

$$\begin{aligned}
\Theta_{X_0} &= \mathcal{H}om(\Omega^1_{X_0/k}, \mathcal{O}_{X_0})\\
&\cong \text{sheaf of } k\text{-derivations from } \mathcal{O}_{X_0} \text{ to } \mathcal{O}_{X_0}
\end{aligned}$$

be the tangent sheaf to X_0. Then

a$_{\text{i}}$) In order that at lease one X_2 exist, it is necessary and sufficient that a canonically defined obstruction $\alpha \in H^2(X_0, \Theta_{X_0}) \otimes_k I$ vanishes. (α will be denoted by $\operatorname{obstr}(X_1)$ below.)

[15] (Add in publication) There have been considerable developments since the manuscript was written. See, for instance, Chambert-Lior [**53**], Astérisque volumes [**51**], [**52**] on "p-adic cohomology" related to "crystalline cohomology" initiated by Grothendieck [**46**].

a$_{\text{ii}}$) If one X_2 exists, consider the set of pairs (X_2, ϕ), with X_2 as above and $\phi\colon X_1 \xrightarrow{\approx} X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I$ an isomorphism, modulo the equivalence relation

$$(X_2, \phi) \sim (X_2', \phi') \quad \text{if } \exists \text{an } R\text{-isomorphism } X_2 \underset{\psi}{\xrightarrow{\approx}} X_2'$$

such that

$$\begin{tikzcd} X_2 \times_{\operatorname{Spec} R} \operatorname{Spec} R/I \arrow[rr, "\psi \times 1_{R/I}"] \arrow[dr] & & X_2' \times_{\operatorname{Spec} R} \operatorname{Spec} R/I \arrow[dl] \\ & X_1 & \end{tikzcd}$$

commutes.

Denote this set $\operatorname{Def}(X_1/R)$: then $\operatorname{Def}(X_1/R)$ is a principal homogeneous space over the group $H^1(X_0, \Theta_{X_0}) \otimes_k I$: i.e., the group acts freely and transitively on the set.

a$_{\text{iii}}$) Given two smooth schemes X_1 and Y_1 over R/I and a morphism over R/I:

$$f_1\colon X_1 \longrightarrow Y_1$$

the obstructions to deforming X_1 and Y_1 are connected by having the same image in $H^2(X_0, f^*\Theta_{Y_0}) \otimes_k I$:

$$\begin{tikzcd} \operatorname{obstr}(X_1) \in H^2(X_0, \Theta_{X_0}) \otimes_k I \arrow[dr, "df_0"] & \\ & H^2(X_0, f_0^*\Theta_{Y_0}) \otimes_k I \\ \operatorname{obstr}(Y_1) \in H^2(Y_0, \Theta_{Y_0}) \otimes_k I \arrow[ur, "f_0^*"'] & \end{tikzcd}$$

where $f_0 = f_1 \otimes_{R/I} k\colon X_0 \to Y_0$ and $df_0\colon \Theta_{X_0} \to f_0^*\Theta_{Y_0}$ is the differential of f_0.

In case (b), let $X_0 = X_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$, $Y_0 = Y_1 \times_{\operatorname{Spec} R/I} \operatorname{Spec} k$ and let f_1 induce $f_0\colon X_0 \to Y_0$. We have:

b$_{\text{i}}$) In order that at least one lifting f_2 exist, it is necessary and sufficient that a canonically defined obstruction $\alpha \in H^1(X_0, f_0^*\Theta_{Y_0}) \otimes_k I$ vanishes.

b$_{\text{ii}}$) If one lifting f_2 exists, denote the set of all lifts by $\operatorname{Lift}(f_1/R)$. Then $\operatorname{Lift}(f_1/R)$ is a principal homogeneous space over the group $H^0(X_0, f_0^*\Theta_{Y_0}) \otimes_k I$.

b$_{\text{iii}}$) The action of $H^1(X_0, \Theta_{X_0}) \otimes_k I$ on $\operatorname{Def}(X_1/R)$ is a special case of the obstructions in (i): namely, if X_2, X_2' are two deformations of X_1 over R, then the element of $H^1(X_0, \Theta_{X_0}) \otimes_k I$ by which they differ is the obstruction to lifting $1_{X_1}\colon X_1 \to X_1$ to a morphism from X_2 to X_2'.

b$_{\text{iv}}$) Given three schemes and two morphisms:

$$X_1 \xrightarrow{f_1} Y_1 \xrightarrow{g_1} Z_1,$$

the obstructions to lifting compose as follows: if

$$\begin{aligned} \alpha &= (\text{obstruction for } f_1) \in H^1(X_0, f_0^*\Theta_{Y_0}) \\ \beta &= (\text{obstruction for } g_1) \in H^1(Y_0, g_0^*\Theta_{Z_0}) \\ \gamma &= (\text{obstruction of } g_1 \circ f_1) \in H^1(X_0, (g_0 \circ f_0)^*\Theta_{Z_0}) \end{aligned}$$

then

$$\gamma = dg_0(\alpha) + f_0^*(\beta)$$

where $dg_0\colon \Theta_{Y_0} \to g_0^*\Theta_{Z_0}$ is the differential of g_0.

Note, in particular, what these say in the affine case[16]

Affine a) If X_0 is affine, $\exists!$ deformation X_2 of X_1 smooth over R.

Affine b) If X_0 and Y_0 are affine, then every f_1 lifts to some $f_2\colon X_2 \to Y_2$ and if $X_0 = \operatorname{Spec} A_0$, $Y_0 = \operatorname{Spec} B_0$ then these liftings are a principal homogeneous space under:

$$\Gamma(X_0, f_0^*\Theta_{Y_0}) \otimes_k I \cong \mathrm{Der}_k(B_0, A_0) \otimes_k I.$$

If one is interested only in the existence of a lifting in (b), then the smoothness of X_2 is irrelevant and one can prove:

Lifting Property for smooth morphisms: : If X_2, Y_2 are of finite type over R, Y_2 smooth and X_2 affine, then any $f_1\colon X_1 \to Y_1$ lifts to an $f_2\colon X_2 \to Y_2$.

Variants of this lifing property have been used by Grothendiek to *characterize* smooth morphisms (cf. “formal smoothness” in Criterion V.4.10, EGA [**1**, Chapter IV, §17] and SGA1 [**4**, Exposé III]). Our method of proof will be to analyze the deformation problem in an even more local case and then to analyze the patching problem via Čech cocycles. In fact if Z is smooth over $\operatorname{Spec} R'$, then we know that locally Z is isomorphic to U where

$$\begin{array}{c} U = (\operatorname{Spec} R'[X_1, \ldots, X_{n+l}]/(f_1, \ldots, f_l))_g \\ \downarrow \\ \operatorname{Spec} R' \end{array}$$

where in $R'[X_1, \ldots, X_{n+l}]$

$$\det_{1\le i,j\le l}\left(\frac{\partial f_i}{\partial X_{n+j}}\right) \cdot h = g, \qquad \text{some } h \in R'[X].$$

Let’s call such U *special smooth affine* schemes over R'.

Step. I: If X_1 is a special smooth affine over R/I, then $\exists$ a deformation X_2 of X_1 over R which is again a special smooth affine.

Proof. Write $X_1 = (\operatorname{Spec}(R/I)[X]/(f))_g$ as above, with $\det \cdot h = g$. Simply choose any polynomials f_i', h' with coefficients in R which reduce mod I to f_i, h. Let $X_2 = (\operatorname{Spec} R[X]/(f'))_{g'}$, where $g' = \det' \cdot h'$. $\square$

Step. II: If X_2 is any affine over R (not even necessarily smooth) and Y_2 is a special smooth affine over R, then any $f_1\colon X_1 \to Y_1$ lifts to an $f_2\colon X_2 \to Y_2$.

Proof. If $X_2 = \operatorname{Spec} A_2$ and $Y_2 = (\operatorname{Spec} R[X]/(f))_g$ as above, then the problem is to define a homomorphism ϕ_2 indicated by the dotted arrow.

$$\begin{tikzcd} & R[X]_g \arrow[dl] \\ A_2 \arrow[d] & R[X]_g/(f) \arrow[l, dashed, "\phi_2"'] \arrow[d] \\ A_2/I\cdot A_2 & (R/I)[X]_g/(f). \arrow[l, "\phi_1"'] \end{tikzcd}$$

If we choose any element $a_j \in A_2$ which reduce mod I to $\phi_1(X_j)$, then we get a homomorphism

$$\phi_2'\colon R[X]_g \longrightarrow A_2$$

[16] It is a theorem that for any noetherian scheme X, X affine $\Longleftrightarrow X_{\mathrm{red}}$ affine (EGA [**1**, Chapter I, (5.1.10)]). Hence in our case, X_2 affine $\Longleftrightarrow X_1$ affine $\Longleftrightarrow X_0$ affine.

by setting $\phi_2'(X_j) = a_j$ (since $\phi_2'(g)$ mod $I \cdot A_2$ equals $\phi_1(g)$ which is a unit; hence $\phi_2'(g)$ is a unit in A_2). However $\phi_2'(f_i) = f_i(a)$ may not be zero. But we may alter a_j to $a_j + \delta a_j$ provided $\delta a_j \in I \cdot A_2$. Then since $I^2 = (0)$, $\phi_2'(f_i)$ changes to

$$f_i(a + \delta a) = f_i(a) + \sum_{j=1}^{n+l} \frac{\partial f_i}{\partial X_j}(a) \cdot \delta a_j.$$

Note that since $\delta a_j \in I \cdot A_2$ and $I \cdot M = (0)$, $\dfrac{\partial f_i}{\partial X_j}(a) \cdot \delta a_j$ depends only on the image of $\dfrac{\partial f_i}{\partial X_j}(a)$ in $k[X]$. Multiplying the adjoint matrix to $\left(\dfrac{\partial f_i}{\partial X_{n+j}}\right)_{1 \le i,j \le l}$ by h, we obtain an $(l \times l)$-matrix $(h_{ij}) \in k[X]$ such that

$$\sum_{j=1}^{l} \frac{\partial f_i}{\partial X_{n+j}} \cdot h_{jq} = g \cdot \delta_{iq}.$$

Now set

$$\begin{aligned} \delta a_j &= 0, \quad 1 \le j \le n \\ \delta a_{n+j} &= -g(a)^{-1} \sum_{q=1}^{l} h_{jq} f_q(a), \quad 1 \le j \le l. \end{aligned}$$

Then:

$$\begin{aligned} f_i(a + \delta a) &= f_i(a) - \sum_{n+j}^{l} \frac{\partial f_i}{\partial X_{n+j}}(a) \cdot g(a)^{-1} \cdot \sum_{q=1}^{l} h_{jq} f_q(a) \\ &= f_i(a) - g(a)^{-1} \sum_{q=1}^{l} f_q(a) \cdot g(a) \delta_{iq} \\ &= 0. \end{aligned}$$

Therefore if we define ϕ_2 by $\phi_2(X_j) = a_j + \delta a_j$, we are through. $\square$

STEP. III: Suppose X_2 and Y_2 are affines over R, $X_2 = \operatorname{Spec} A_2$, $Y_2 = \operatorname{Spec} B_2$. $A_0 = A_2/M \cdot A_2$, $B_0 = B_2/M \cdot B_2$. Let $f_2 \colon X_2 \to Y_2$ be a morphism and let $f_1 = \operatorname{res}_{X_1} f_2$. Then $\operatorname{Lift}(f_1/R)$ is a principal homogeneous space over

$$\operatorname{Der}_k(B_0, I \cdot A_2).$$

PROOF. We are given a homomorphism $\phi_1 \colon B_1 \to A_1$ and we wish to study

$$L = \{\phi_2 \colon B_2 \longrightarrow A_2 \mid \phi_2 \text{ mod } I = \phi_1\},$$

which we assume is non-empty. If $\phi_2, \phi_2' \in L$, then $\phi_2' - \phi_2$ factors via D:

$$\begin{tikzcd} B_2 \arrow[rr, "\phi_2' - \phi_2"] \arrow[dr] & & A_2 \\ & B_2/M \cdot B_2 \arrow[r, dashed, "D"] \arrow[d, equal] & I \cdot A_2 \arrow[u, phantom, "\cup"] \\ & B_0 & \end{tikzcd}$$

One checks immediately that D is a derivation. And conversely for any such derivation D, $\phi_2 \in L \Longrightarrow \phi_2 + D \in L$. $\square$

STEP. IV: Globalize Step III: Let X_2, Y_2 be two schemes of finite type over R. Let $f_2\colon X_2 \to Y_2$ be a morphism, and let $f_1 = \mathrm{res}_{X_1} f_2$. Then $\mathrm{Lift}(f_1/R)$ is a principal homogeneous space over

$$\Gamma(X_2, \mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I \cdot \mathcal{O}_X)).$$

(Note that $I \cdot \mathcal{O}_X$ is really an $\mathcal{O}_{X_0}$-module).

PROOF. Take affine coverings $\{U_\alpha\}$, $\{V_\alpha\}$ of X_2 and Y_2 such that $f_2(U_\alpha) \subset V_\alpha$. If $U_\alpha = \operatorname{Spec} A_2^{(\alpha)}$, $V_\alpha = \operatorname{Spec} B_2^{(\alpha)}$, $f_1^{(\alpha)} = \mathrm{res}_{U_\alpha} f_1$, then as in Step III,

$$\begin{aligned}\mathrm{Lift}(f_1^{(\alpha)}/R) = \text{principal homogeneous space under}\quad & \mathrm{Der}_k(B_0^{(\alpha)}, I \cdot A_2^{(\alpha)})\\ & \qquad\| \\ & \mathrm{Hom}_{B_0^{(\alpha)}}(\Omega^1_{B_0^{(\alpha)}/k}, I \cdot A_2^{(\alpha)})\\ & \qquad\| \\ & \mathrm{Hom}_{A_0^{(\alpha)}}(\Omega^1_{B_0^{(\alpha)}/k} \otimes_{B_0^{(\alpha)}} A_0^{(\alpha)}, I \cdot A_2^{(\alpha)})\\ & \qquad\| \\ & \Gamma(U_\alpha, \mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I \cdot \mathcal{O}_{X_2})).\end{aligned}$$

Therefore on the one hand, one can "add" a morphism $f_2\colon X_2 \to Y_2$ and a global section D of $\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I \cdot \mathcal{O}_{X_2})$ by adding them locally on the U_α's and noting that the "sums" agree on overlaps $U_\alpha \cap U_\beta$. Again given two lifts f_2, f_2', their "difference" $f_2 - f_2'$ defines locally on the U_α's a section D_α of $\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, I \cdot \mathcal{O}_{X_2})$, hence a global section D.

Note that if Y_0 is smooth over k, $\Omega^1_{Y_0/k}$ is locally free with dual Θ_{Y_0}, hence

$$\mathcal{H}om(f_0^*\Omega^1_{Y_0/k}, \mathcal{F}) \cong f_0^*\Theta_{Y_0} \otimes_{\mathcal{O}_{X_0}} \mathcal{F} \quad \text{for any sheaf } \mathcal{F};$$

and if X_2 is flat over R, then $I \cdot \mathcal{O}_{X_2} \cong I \otimes_k \mathcal{O}_{X_0}$. Thus case ($\mathrm{b_{ii}}$) of our main result is proven! □

STEP. V: Proof of case ($\mathrm{b_i}$): viz. construction of the obstruction to lifting $f_1\colon X_1 \to Y_1$.[17]

PROOF. Choose affine open coverings $\{U_\alpha\}$, $\{V_\alpha\}$ of X_2, Y_2 such that

- $f_1(U_\alpha) \subset V_\alpha$
- V_α is a special smooth affine.

Then by Step II, there exists a lift $f_2^{(\alpha)}\colon U_\alpha \to V_\alpha$ of $\mathrm{res}_{U_\alpha} f_1$. By Step III, $\mathrm{res}\, f_2^{(\alpha)}\colon U_\alpha \cap U_\beta \to V_\alpha \cap V_\beta$ and $\mathrm{res}\, f_2^{(\beta)}\colon U_\alpha \cap U_\beta \to V_\alpha \cap V_\beta$ differ by an element

$$D_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, f_0^*\Theta_{Y_0} \otimes_k I).$$

But on $U_\alpha \cap U_\beta \cap U_\gamma$ we may write somewhat loosely:

$$\begin{aligned} D_{\alpha\beta} + D_{\beta\gamma} &= [\mathrm{res}\, f_2^{(\alpha)} - \mathrm{res}\, f_2^{(\beta)}] + [\mathrm{res}\, f_2^{(\beta)} - \mathrm{res}\, f_2^{(\gamma)}]\\ &= \mathrm{res}\, f_2^{(\alpha)} - \mathrm{res}\, f_2^{(\gamma)}\\ &= D_{\alpha\gamma}.\end{aligned}$$

(Check the proof in Step III to see that this does make sense.) Thus

$$\{D_{\alpha\beta}\} \in Z^1(\{U_\alpha\}, f_0^*\Theta_{Y_0} \otimes_k I).$$

Now if the lifts $f_2^{(\alpha)}$ are changed, this can only be done by adding to them elements $E_\alpha \in \Gamma(U_\alpha, f_0^*\Theta_{Y_0} \otimes_k I)$ and then $D_{\alpha\beta}$ is changed to $D_{\alpha\beta} + E_\alpha - E_\beta$. Moreover, if the covering $\{U_\alpha\}$

[17]Note that we use, in fact, only that Y_2 is smooth over R and that the same proof gives the Lifting Property for smooth morphisms.

is refined and one restricts the lifts $f_2^{(\alpha)}$, then the cocycle we get is just the refinement of $D_{\alpha\beta}$. Thus we have a well-defined element of $H^1(X_0, f_0^*\Theta_{Y_0} \otimes_k I)$. Moreover it is zero if and only if for some coverings $\{U_\alpha\}$, $\{V_\alpha\}$, $D_{\alpha\beta}$ is homologous to zero, i.e.,

$$D_{\alpha\beta} = E_\alpha - E_\beta, \qquad E_\alpha \in \Gamma(U_\alpha, f_0^*\Theta_{Y_0} \otimes_k I).$$

Then changing $f_2^{(\alpha)}$ by E_α as in Step III, we get $\widetilde{f}_2^{(\alpha)}$'s, lifting f_1 such that on $U_\alpha \cap U_\beta$, $\widetilde{f}_2^{(\alpha)} - \widetilde{f}_2^{(\beta)}$ is represented by the zero derivation, i.e., the $\widetilde{f}_2^{(\alpha)}$'s agree on overlaps and give an f_2 lifting f_1. □

The assertion ($\mathrm{b_{iv}}$) is a simple calculation that we leave to the reader.

Step. VI: Proof of ($\mathrm{a_{ii}}$) and ($\mathrm{b_{ii}}$) simultaneously.

Proof. Suppose we are given X_1 smooth over $\operatorname{Spec} R/I$ and at least one deformation X_2 of X_1 over R exists. If X_2, X_2' are any two deformations, we can apply the construction of Step V to the lifting of $1_{X_1}\colon X_1 \to X_1$ to an R-morphism $X_2 \to X_2'$, getting an obstruction in $H^1(X_0, \Theta_{X_0}) \otimes_k I$. This gives us a map:

$$\operatorname{Def}(X_1/R) \times \operatorname{Def}(X_1/R) \longrightarrow H^1(X_0, \Theta_{X_0}) \otimes_k I$$

which we write:

$$(X, X') \longmapsto X - X'.$$

The functorial property ($\mathrm{b_{iv}}$) proves that:

$$(*) \qquad (X - X') + (X' - X'') = (X - X'').$$

Moreover, $X - X' = 0 \Longrightarrow X = X'$: because if $1_{X_1}\colon X_1 \to X_1$ lifts to an R-morphism $f\colon X_2 \to X_2'$, f is automatically an isomorphism in view of the easy:

Lemma 5.1. *Let A and B be R-algebras, B flat over R. If $\phi\colon A \to B$ is an R-homomorphism such that*

$$\overline{\phi}\colon A/I \cdot A \xrightarrow{\approx} B/I \cdot B$$

is an isomorphism, then ϕ is an isomorphism.

(Proof left to the reader.)

If we now show that $\forall$ deformation X_2 and $\forall \alpha \in H^1(X_0, \Theta_{X_0}) \otimes_k I$, $\exists$ a deformation X_2' with $X_2' - X_2 = \alpha$, we will have proven that $\operatorname{Def}(X_1/R)$ is a principal homogeneous space over $H^1(X_0, \Theta_{X_0}) \otimes_k I$ as required. To construct X_2', represent α by a Čech cocycle $\{D_{ij}\}$, for any open covering $\{U_i\}$ of X_2, where

$$D_{ij} \in \Gamma(U_i \cap U_j, \Theta_{X_0} \otimes_k I).$$

As in Step IV, we then have an automorphism of $U_i \cap U_j$ (as a *subscheme* of X_2):

$$1_{U_i \cap U_j} + D_{ij}\colon U_i \cap U_j \longrightarrow U_i \cap U_j.$$

X_2' is obtained by glueing together the subschemes U_i of X_2 by these new automorphisms between $U_i \cap U_j$ regarded as part of U_i and $U_i \cap U_j$ regarded as part of U_j. The cocycle condition $D_{ij} + D_{ji} = D_{ik}$ guarantees that these glueings are consistent and one checks easily that for this X_2', $X_2' - X_2$ is indeed α. □

Step. VII: Proof of ($\mathrm{a_i}$): viz. construction of the obstruction to deforming X_1 over R.

Proof. Starting with X_1, take a special affine covering $\{U_{i,1}\}$ of X_1. By Step I, $U_{i,1}$ deforms to a special affine $U_{i,2}$ over R. This gives us two deformations of the affine scheme $U_{i,1} \cap U_{j,1}$ over R, viz. the open subschemes

$$\begin{aligned} {}_jU_{i,2} &\subset U_{i,2} \\ {}_iU_{j,2} &\subset U_{j,2}. \end{aligned}$$

By Step VI, these must be isomorphic so choose

$$\phi_{ij} \colon {}_jU_{i,2} \xrightarrow{\approx} {}_iU_{j,2}.$$

If we try to glue the schemes $U_{i,2}$ together by these isomorphisms, consistency requires that the following commutes:

$$\begin{tikzcd} & {}_iU_{j,2} \cap {}_kU_{j,2} \arrow[dr, "\operatorname{res}\phi_{jk}"] & \\ {}_jU_{i,2} \cap {}_kU_{i,2} \arrow[ur, "\operatorname{res}\phi_{ij}"] \arrow[rr, "\operatorname{res}\phi_{ik}"'] & & {}_jU_{k,2} \cap {}_iU_{k,2}. \end{tikzcd}$$

But, in general, $(\operatorname{res}\phi_{ij}) \circ (\operatorname{res}\phi_{ik})^{-1} \circ (\operatorname{res}\phi_{jk})$ will be an automorphism of ${}_iU_{j,2} \cap {}_kU_{j,2}$ given by a derivation $D_{ijk} \in \Gamma(U_i \cap U_j \cap U_k, \Theta_{X_0} \otimes_k I)$. One checks easily (1) that D_{ijk} is a 2-cocycle, (2) that altering the ϕ_{ij}'s adds to the D_{ijk} a 2-coboundary, and conversely that any D'_{ijk} cohomologous to D_{ijk} in $H^2(\{U_i\}, \Theta_{X_0} \otimes_k I)$ is obtained by altering the ϕ_{ij}'s, and (3) that refining the covering $\{U_i\}$ replaces D_{ijk} by the refined 2-cocycle. Thus $\{D_{ijk}\}$ defines an element $\alpha \in H^2(X_0, \Theta_{X_0} \otimes_k I)$ depending only on X_1, and $\alpha = 0$ if and only if X_2 exists. $\square$

Step. VIII Proof of ($\mathrm{a_{iii}}$).

Proof. Given X_1, Y_1 and f, take special affine coverings $\{U_{i,1}\}$, $\{V_{i,1}\}$ of X_1 and Y_1 such that $f(U_{i,1}) \subset V_{i,1}$. Deform $U_{i,1}$ (resp. $V_{i,1}$) to $U_{i,2}$ (resp. $V_{i,2}$) over R. By Step II, lift f to $f_i \colon U_{i,2} \to V_{i,2}$. Consider the diagram:

$$\begin{tikzcd} {}_jU_{i,2} \arrow[r, "\operatorname{res} f_i"] \arrow[d, "\phi_{ij}"'] & {}_jV_{i,2} \arrow[d, "\psi_{ij}"'] \\ {}_iU_{j,2} \arrow[r, "\operatorname{res} f_j"] & {}_iV_{j,2}. \end{tikzcd}$$

It need not commute, so let

$$(\operatorname{res} f_j) \circ \phi_{ij} = \psi_{ij} \circ (\operatorname{res} f_i) + F_{ij}$$

where $F_{ij} \in \Gamma(U_i \cap U_j, f_0^*\Theta_{Y_0} \otimes_k I)$. It is a simple calculation to check now that if the ϕ_{ij}'s define a 2-cocycle D_{ijk} representing $\operatorname{obstr}(X_1)$ and the ψ_{ij}'s similarly define E_{ijk}, then

$$df_0(D_{ijk}) - f_0^* E_{ijk} = F_{ij} - F_{ik} + F_{jk}.$$

$\square$

This completes the proof of the main results of infinitesimal deformation theory. We get some important corollaries:

Corollary 5.2. *Let R be an artin local ring with maximal ideal M and residue field k and let $I \subset R$ be* any *ideal contained in M. If X_1 is a scheme smooth of finite type over* $\operatorname{Spec} R/I$ *such that $H^2(X_0, \Theta_{X_0}) = (0)$ — e.g., if* $\dim X_0 = 1$ *— then a deformation X_2 of X_1 over R exists.*

PROOF. Filter I as follows: $I \supset MI \supset M^2 I \supset \cdots \supset M^\nu I = (0)$. Then deform X_1 successively as follows:

$$\begin{array}{ccccccccc} X_1 & \subset & X_1^{(1)} & \subset & \cdots & \subset & X_1^{(l)} & & \\ \downarrow & & \downarrow & & & & \downarrow & & \\ \operatorname{Spec} R/I & \subset & \operatorname{Spec} R/MI & \subset & \cdots & \subset & \operatorname{Spec} R/M^l I & \subset \quad \cdots & \subset \operatorname{Spec} R \end{array}$$

using case ($\mathrm{a_i}$) of each stage to show that $X_1^{(l)}$ can be deformed to a $X_1^{(l+1)}$. Set $X_2 = X_1^{(\nu)}$. $\square$

COROLLARY 5.3. *Let R be an artin local ring with residue field k and let X be a scheme smooth*[18] *of finite type over R. Let $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and let*

$$f_0 \colon Y_0 \longrightarrow X_0$$

be an étale morphism. Then there exists a unique deformation Y of Y_0 over R such that f_0 lifts to $f \colon Y \to X$.

PROOF. Let M be the maximal ideal of R. Deform Y_0 successively as follows:

$$\begin{array}{ccccccccccc} Y_0 & \subset & Y_1 & \subset & \cdots & \subset & Y_{l-1} & & & & \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & & & {\scriptstyle f_{l-1}}\downarrow & & & & \\ X_0 & \subset & X_1 & \subset & \cdots & \subset & X_{l-1} & \subset & \cdots & \subset & X \\ \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\ \operatorname{Spec} k & \subset & \operatorname{Spec}(R/M^2) & \subset & \cdots & \subset & \operatorname{Spec}(R/M^l) & \subset & \cdots & \subset & \operatorname{Spec} R \end{array}$$

where $X_{l-1} = X \times_{\operatorname{Spec} R} \operatorname{Spec}(R/M^l)$. Because f_0 is étale, $df_0 \colon \Theta_{Y_0} \to f_0^* \Theta_{X_0}$ is an isomorphism. Therefore at each stage, the existence of the deformation X_l of X_{l-1} gurantees by ($\mathrm{a_{iii}}$) the existence of a deformation Y_l' of Y_{l-1}. Then choose any Y_l' and ask whether f_{l-1} lifts. We get an obstruction α:

$$H^1(Y_0, \Theta_{Y_0}) \otimes_k (M^l/M^{l+1}) \xrightarrow[df_0]{\approx} H^1(Y_0, f_0^* \Theta_{X_0}) \otimes_k (M^l/M^{l+1}).$$

Then alter the deformation Y_l' by $df_0^{-1}(\alpha)$, giving a new deformation Y_l. By functoriality ($\mathrm{b_{iv}}$), f_{l-1} lifts to $f_l \colon Y_l \to X_l$ and by injectivity of df_0 this is the only deformation for which this is so. $\square$

The most exciting applications of deformation theory, however, are those cases when one can construct deformations not only over artin local rings, but over complete local rings. If the ring R is actually an integral domain, then one has constructed, by taking fibre product, a scheme over the quotient field K of R as well. A powerful tool for extending constructions to this case is Grothendieck's GFGA Theorem (Theorem 2.17). This is applied as follows:

DEFINITION 5.4. Let R be a complete local noetherian ring with maximal ideal M. Then a *formal scheme* $\mathcal{X}$ over R is a system of schemes and morphisms:

$$\begin{array}{ccccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow \cdots \longrightarrow & X_n & \longrightarrow & \cdots\cdots & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ \operatorname{Spec}(R/M) & \to & \operatorname{Spec}(R/M^2) & \to \cdots \to & \operatorname{Spec}(R/M^{n+1}) & \to & \cdots\cdots & & \operatorname{Spec} R \end{array}$$

[18] A more careful proof shows that the smoothness of X is not really needed here and that Corollary 5.3 is true for any X of finite type over R. It is even true for comparing étale coverings of X and X_{red}, any noetherian scheme X (cf. SGA1 [**4**, Exposé I, Théorème 8.3, p. 14]).

where $X_{n-1} \cong X_n \times_{\mathrm{Spec}(R/M^{n+1})} \mathrm{Spec}(R/M^n)$. $\mathcal{X}$ is *flat* over R if each X_n is flat over $\mathrm{Spec}(R/M^{n+1})$. If X is a scheme over $\mathrm{Spec}\, R$, the associated formal scheme $\widehat{X}$ is the system of schemes $X_n = X \times_{\mathrm{Spec}\, R} \mathrm{Spec}(R/M^{n+1})$ together with the obvious morphisms $X_n \to X_{n+1}$.

THEOREM 5.5. *Let R be a complete local noetherian ring and let $\mathcal{X} = \{X_n\}$ be a formal scheme flat over R. If X_0 is smooth and projective over $k = R/M$, and if $H^2(X_0, \mathcal{O}_{X_0}) = (0)$, then there exists a scheme X smooth and proper over R such that:*

$$\mathcal{X} = \widehat{X}.$$

PROOF. Since X_0 is projective over k, there exists a very ample invertible sheaf $\mathcal{L}_0$ on X_0. By Exercise (3) in Chapter III, $\mathcal{L}_0^n$ is very ample for all $n \geq 1$; by Theorem VII.8.1, $H^1(\mathcal{L}_0^n) = (0)$ if $n \gg 0$. So we may replace $\mathcal{L}_0$ by $\mathcal{L}_0^n$ and assume that $H^1(\mathcal{L}_0) = (0)$ too. The first step is to "lift" $\mathcal{L}_0$ to a sequence of invertible sheaves $\mathcal{L}_n$ on X_n such that

$$\mathcal{L}_n \cong \mathcal{L}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}, \qquad \text{all } n \geq 0.$$

To do this, recall that the isomorphism classes of invertible sheaves on any scheme X are classified by $H^1(X, \mathcal{O}_X^*)$. Therefore to construct the $\mathcal{L}_n$'s inductively, it will suffice to show that the natural map:

$$H^1(X_n, \mathcal{O}_{X_n}^*) \longrightarrow H^1(X_{n-1}, \mathcal{O}_{X_{n-1}}^*)$$

(given by restriction of functions from X_n to X_{n-1}) is surjective. But consider the map of sheaves:

$$\begin{aligned} \exp\colon M^n \cdot \mathcal{O}_{X_n} &\longrightarrow \mathcal{O}_{X_n}^* \\ a &\longmapsto 1 + a. \end{aligned}$$

Since $M^n \cdot M^n \equiv 0$ in $\mathcal{O}_{X_n}$, this map is a homomorphism from $M^n \cdot \mathcal{O}_{X_n}$ in its additive structure to $\mathcal{O}_{X_n}^*$ in its multiplicative structure, and the image is obviously $\mathrm{Ker}\left(\mathcal{O}_{X_n}^* \to \mathcal{O}_{X_{n-1}}^*\right)$, i.e., we get an exact sequence:

$$0 \longrightarrow M^n \cdot \mathcal{O}_{X_n} \xrightarrow{\exp} \mathcal{O}_{X_n}^* \longrightarrow \mathcal{O}_{X_{n-1}}^* \longrightarrow 1.$$

But now the flatness of X_n over R/M^{n+1} implies that for any ideal $\mathfrak{a} \subset R/M^{n+1}$,

$$\mathfrak{a} \otimes_{R/M^{n+1}} \mathcal{O}_{X_n} \longrightarrow \mathfrak{a} \cdot \mathcal{O}_{X_n}$$

is an isomorphism. Apply this with $\mathfrak{a} = M^n/M^{n+1}$:

$$\begin{aligned} M^n \cdot \mathcal{O}_{X_n} &\cong (M^n/M^{n+1}) \otimes_{R/M^{n+1}} \mathcal{O}_{X_n} \\ &\cong (M^n/M^{n+1}) \otimes_{R/M} \left((R/M) \otimes_{R/M^{n+1}} \mathcal{O}_{X_n}\right) \\ &\cong (M^n/M^{n+1}) \otimes_k \mathcal{O}_{X_0} \\ &\cong \mathcal{O}_{X_0}^{\oplus \nu_n}, \text{ if } \nu_n = \dim_k M^n/M^{n+1}. \end{aligned}$$

Therefore we get an exact sequence[19]:

$$0 \longrightarrow \mathcal{O}_{X_0}^{\oplus \nu_n} \longrightarrow \mathcal{O}_{X_n}^* \longrightarrow \mathcal{O}_{X_{n-1}}^* \longrightarrow 1$$

hence an exact cohomology sequence:

$$\begin{tikzcd} H^1(\mathcal{O}_{X_n}^*) \arrow[r] & H^1(\mathcal{O}_{X_{n-1}}^*) \arrow[r, "\delta"] & H^2(\mathcal{O}_{X_0})^{\oplus \nu_n}. \arrow[d, equal] \\ & & (0) \end{tikzcd}$$

[19]Note that all X_n are topologically the same space, hence this exact sequence makes sense as a sequence of sheaves of abelian groups on X_0.

This proves that the sheaves $\mathcal{L}_n$ exist.

The second step is to lift the projective embedding of X_0. Let $s_0, \ldots, s_N$ be a basis of $\Gamma(X_0, \mathcal{L}_0)$, so that $(\mathcal{L}_0, s_0, \ldots, s_N)$ defines a projective embedding of X_0 in $\mathbb{P}^N_k$. I claim that for each n there are sections $s_0^{(n)}, \ldots, s_N^{(n)}$ of $\mathcal{L}_n$ such that via the isomorphism:

$$\mathcal{L}_{n-1} \cong \mathcal{L}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{X_{n-1}} \cong \mathcal{L}_n / M^n \cdot \mathcal{L}_n,$$

$s_i^{(n-1)} = \text{image of } s_i^{(n)}$. To see this, use the exact sequence:

$$0 \longrightarrow M^n \cdot \mathcal{L}_n \longrightarrow \mathcal{L}_n \longrightarrow \mathcal{L}_{n-1} \longrightarrow 0$$

and note that because $\mathcal{L}_n$ is flat over R/M^{n+1} too,

$$\begin{aligned} M^n \cdot \mathcal{L}_n &\cong (M^n/M^{n+1}) \otimes_{R/M^{n+1}} \mathcal{L}_n \\ &\cong (M^n/M^{n+1}) \otimes_{R/M} \big((R/M) \otimes_{R/M^{n+1}} \mathcal{L}_n\big) \\ &\cong (M^n/M^{n+1}) \otimes_k \mathcal{L}_0, \end{aligned}$$

hence we get an exact cohomology sequence:

$$\begin{array}{ccccc} H^0(\mathcal{L}_n) & \longrightarrow & H^0(\mathcal{L}_{n-1}) & \xrightarrow{\delta} & H^1(\mathcal{L}_0)^{\oplus \nu_n}. \\ & & & & \| \\ & & & & (0) \end{array}$$

This allows us to define $s_i^{(n)}$ inductively on n. Now for each n, $(\mathcal{L}_n, s_0^{(n)}, \ldots, s_N^{(n)})$ defines a morphism

$$\phi_n \colon X_n \longrightarrow \mathbb{P}^N_{(R/M^{n+1})}$$

such that the diagram:

$$\begin{tikzcd} X_{n-1} \arrow[r] \arrow[d] & \mathbb{P}^N_{(R/M^n)} \arrow[d] \\ X_n \arrow[r] & \mathbb{P}^N_{(R/M^{n+1})} \end{tikzcd}$$

commutes. I claim that ϕ_n is a closed immersion for each n. Topologically it is closed and injective because topologically $\phi_n = \phi_0$ and ϕ_0 is by assumption a closed immersion. As for structure sheaves, ϕ_n^* lies in a diagram:

$$\begin{tikzcd} 0 \arrow[r] & (M^n/M^{n+1}) \otimes_k \mathcal{O}_{\mathbb{P}^N_k} \arrow[r] \arrow[d, "1_{M^n} \otimes \phi_0^*"] & \mathcal{O}_{\mathbb{P}^N_{(R/M^{n+1})}} \arrow[r] \arrow[d, "\phi_n^*"] & \mathcal{O}_{\mathbb{P}^N_{(R/M^n)}} \arrow[r] \arrow[d, "\phi_{n-1}^*"] & 0 \\ 0 \arrow[r] & (M^n/M^{n+1}) \otimes_k \mathcal{O}_{X_0} \arrow[r] & \mathcal{O}_{X_n} \arrow[r] & \mathcal{O}_{X_{n-1}} \arrow[r] & 0. \end{tikzcd}$$

Since ϕ_0^* is surjective, this shows ϕ_{n-1}^* surjective $\implies$ ϕ_n^* surjective. So by induction, all the ϕ_n are closed immersions.

Finally, let ϕ_n induce an isomorphism of X_n with the closed subscheme $Y_n \subset \mathbb{P}^N_{(R/M^{n+1})}$. Then the sequence of coherent sheaves $\{\mathcal{O}_{Y_n}\}$ is a formal coherent sheaf on $\mathbb{P}^N_R$ in the sense of §2 above. By the GFGA theorem (Theorem 2.17), there is a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N_R$ such that

$$\mathcal{O}_{Y_n} \cong \mathcal{F} \otimes_R (R/M^{n+1})$$

for every n. Moreover since $\{\mathcal{O}_{Y_n}\}$ is a quotient of the formal sheaf $\left\{\mathcal{O}_{\mathbb{P}^N_{(R/M^{n+1})}}\right\}$, $\mathcal{F}$ is quotient of $\mathcal{O}_{\mathbb{P}^N}$, i.e., $\mathcal{F} = \mathcal{O}_Y$ for some closed subscheme $Y \subset \mathbb{P}^N$. Therefore since:

$$X_n \cong Y_n \cong Y \times_{\operatorname{Spec} R} \operatorname{Spec}(R/M^{n+1}),$$

it follows that $\widehat{Y} \cong \mathcal{X}$. $\square$

COROLLARY 5.6. *Let R be a complete local noetherian ring and let $\mathcal{X} = \{X_n\}$ be a formal scheme flat over R. If X_0 is a smooth complete curve, then $\mathcal{X} = \widehat{X}$ for some X smooth and proper over R.*

PROOF. By Proposition V.5.11, X_0 is projective over k and because $\dim X_0 = 1$, $H^2(\mathcal{O}_{X_0}) = (0)$. $\square$

COROLLARY 5.7. (Severi[20] -Grothendieck) *Let R be a complete local noetherian ring and let X_0 be a smooth complete curve over $k = R/M$. Then X_0 has a deformation over R, i.e., there exists a scheme X, smooth and proper over R such that $X_0 \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} k$.*

PROOF. Corollaries 5.2 and 5.6. $\square$

An important supplementary remark here is that if for simplicity X_0 is geometrically irreducible (also said to be absolutely irreducible), i.e., $X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is irreducible ($\overline{k} =$ algebraic closure of k), then $H^1(\mathcal{O}_X)$ is a free R-module such that

$$\begin{aligned} H^1(\mathcal{O}_X) \otimes_R k &\cong H^1(\mathcal{O}_{X_0}) \\ H^1(\mathcal{O}_X) \otimes_R K &\cong H^1(\mathcal{O}_{X_\eta}) \end{aligned}$$
$$(K = \text{ quotient field of } R,\ X_\eta = X \times_{\operatorname{Spec} R} \operatorname{Spec} K).$$

Since the *genus* of a curve Y over k is nothing but $\dim_k H^1(\mathcal{O}_Y)$, this shows that $\operatorname{genus}(X_\eta) = \operatorname{genus}(X_0)$. The proof in outline is this:

a) $X_0 \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ irreducible and X_0 smooth over k implies k algebraically closed in $\mathbf{R}(X)$, hence k algebraically closed in $H^0(\mathcal{O}_{X_0})$. Thus
$$k \xrightarrow{\approx} H^0(\mathcal{O}_{X_0}).$$
b) Show that there are exact sequences
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^n \cdot \mathcal{O}_{X_n} & \longrightarrow & \mathcal{O}_{X_n} & \longrightarrow & \mathcal{O}_{X_{n-1}} & \longrightarrow & 0. \\ & & \| & & & & & & \\ & & \mathcal{O}_{X_0}^{\oplus \nu_n} & & & & & & \end{array}$$
c) Show by induction on n that if $g = \dim_k H^1(\mathcal{O}_{X_0})$, then $R/M^{n+1} \xrightarrow{\approx} H^0(\mathcal{O}_{X_n})$ and $H^1(\mathcal{O}_{X_n})$ is a free (R/M^{n+1})-module of rank g such that
$$H^1(\mathcal{O}_{X_{n-1}}) \cong H^1(\mathcal{O}_{X_n})/M^n \cdot H^1(\mathcal{O}_{X_n}).$$
d) Apply GFGA (Theorem 2.17) to prove that $H^1(\mathcal{O}_X)$ is a free R-module of rank g such that $H^1(\mathcal{O}_{X_n}) \cong H^1(\mathcal{O}_X)/M^{n+1} \cdot H^1(\mathcal{O}_X)$ for all n.
e) Use the flatness of K over R to prove that
$$H^1(\mathcal{O}_{X_\eta}) \cong H^1(\mathcal{O}_X) \otimes_R K.$$

Corollary 5.7 is especially interesting when k is a perfect field of characteristic p and R is the Witt vectors over k (see, for instance, Mumford [**84**, Lecture 26 by G. Bergman]), in which case one summarizes Corollary 5.7 by saying: "non-singular curves can be lifted from characteristic p to characteristic 0". On the other hand, Serre [**104**][21] has found non-singular projective varieties X_0 over algebraically closed fields k of characteristic p such that for every

[20] Modulo translating Italian style geometry into the theory of schemes, a rigorous proof of this is contained in Severi [**107**, Anhang]. This approach was worked out by Popp [**91**].

[21] (Added in Publication) See also Illusie's account in FAG [**3**, Chapter 8].

complete local characteristic 0 domain R with $R/M = k$, no such X exists: such an X_0 is called a non-liftable variety!

One can strengthen the application of deformation theory to coverings in the same way:

THEOREM 5.8. *Let R be a complete local noetherian ring with residue field k and let X be a scheme smooth*[22] *and proper over R. Let $X_0 = X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ and let*

$$f_0 \colon Y_0 \longrightarrow X_0$$

be a finite étale morphism. Then there exists a unique finite étale morphism

$$f \colon Y \longrightarrow X$$

such that f_0 is obtained from f by fibre product $\times_{\operatorname{Spec} R} \operatorname{Spec} k$.

PROOF. By Corollary 5.3 we can lift $f_0 \colon Y_0 \to X_0$ to a unique formal finite étale scheme $F \colon \mathcal{Y} \to \widehat{X}$, i.e., $\mathcal{Y} = \{Y_n\}$, $F = \{f_n\}$ where $f_n \colon Y_n \to X_n$ is finite and étale, where $X_n = X \times_{\operatorname{Spec} R} \operatorname{Spec} R/M^{n+1}$ and the diagram:

$$\begin{tikzcd} Y_n \arrow[r] \arrow[d, "f_n"'] & Y_{n+1} \arrow[d, "f_{n+1}"] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd}$$

commutes (the inclusion $Y_n \to Y_{n+1}$ being part of the definition of a formal scheme $\mathcal{Y}$). If $\mathcal{A}_n = f_{n,*}(\mathcal{O}_{Y_n})$, then

$$\begin{aligned} \operatorname{Spec}_{X_n}(\mathcal{A}_n) &\cong Y_n \\ &\cong Y_{n+1} \times_{X_{n+1}} X_n \\ &\cong \operatorname{Spec}_{X_n}\left(\mathcal{A}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}\right) \end{aligned}$$

hence $\mathcal{A}_n \cong \mathcal{A}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n}$. Therefore $\{\mathcal{A}_n\}$ is a coherent formal sheaf on X, hence by GFGA (Theorem 2.17) there is a unique coherent sheaf $\mathcal{A}$ on X such that $\mathcal{A}_n \cong \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$ for all n. Using the fact that

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}, \mathcal{A}) \cong \text{Formal} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{A}_{\text{for}} \otimes_{\mathcal{O}_X} \mathcal{A}_{\text{for}}, \mathcal{A}_{\text{for}})$$

and similar facts with $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, we see immediately that $\mathcal{A}$ is a sheaf of commutative algebras. Let $Y = \operatorname{Spec}_X(\mathcal{A})$, and let $f \colon Y \to X$ be the canonical morphism. f is obviously proper and finite to one. Moreover since for all $x \in X_0$, $\mathcal{A}_{n,x}$ is a free $\mathcal{O}_{x,X_n}$-module, it follows immediately that $\mathcal{A}_x$ is a free $\mathcal{O}_{x,X}$-module, i.e., f is flat at x. And f_0 étale implies $\Omega_{Y/X} \otimes_{\mathcal{O}_X} \Bbbk(x) = (0)$, hence $(\Omega_{Y/X})_x = (0)$ by Nakayama's lemma. Therefore by Criterion V.4.1, f is étale at x. Since this holds for all x, f is étale in an open set $U \subset X$, with $U \supset X_0$. But X proper over $\operatorname{Spec} R$ implies that every such open set U equals X. Thus f is étale. Finally if $f' \colon \operatorname{Spec}_X \mathcal{A}' \to X$ is another such lifting with $\mathcal{A}'_0 = \mathcal{A}_0$, then by GFGA (Theorem 2.17) there is a unique isomorphism

$$\phi_n \colon \mathcal{A}_n \xrightarrow{\approx} \mathcal{A}'_n$$

of $\mathcal{O}_{X_n}$-algebras inducing the identity $\mathcal{A}_0 \xrightarrow{\approx} \mathcal{A}'_0$. Then these ϕ_n patch together into a *formal* isomorphism $\mathcal{A}_{\text{for}} \xrightarrow{\approx} \mathcal{A}'_{\text{for}}$, which comes by GFGA (Theorem 2.17) from a unique algebraic isomorphism $\phi \colon \mathcal{A} \xrightarrow{\approx} \mathcal{A}'$. □

[22]As mentioned above, Corollary 5.3 is actually true without assuming smoothness and hence since smoothness is not used in proving this result from Corollary 5.3, it is unnecessary here too.

Corollary 5.9. *In the situation of Theorem* 5.8*, there is an isomorphism of pro-finite groups:*

$$\pi_1^{\mathrm{alg}}(X_0) \xrightarrow{\approx} \pi_1^{\mathrm{alg}}(X)$$

canonical up to inner automorphism.

Proof. If $f\colon Y \to X$ is any connected covering, i.e., f finite and étale, then by fibre product $\times_X X_0$, we get a covering $f_0\colon Y_0 \to X_0$. Note that Y_0 is connected (if not, we could lift its connected components separately by Theorem 5.8, hence find a disconnected covering $f'\colon Y' \to X$ lifting f_0, thus contradicting the uniqueness in the theorem). By Theorem 5.8 every connected covering $f_0\colon Y_0 \to X_0$, up to isomorphism, arises in this way. Moreover, if $\mathbf{R}(Y)$ is Galois over $\mathbf{R}(X)$, then we get a homomorphism:

$$\mathrm{Gal}(\mathbf{R}(Y)/\mathbf{R}(X)) \xrightarrow{\approx} \mathrm{Aut}(Y/X) \longrightarrow \mathrm{Aut}(Y_0/X_0) \xrightarrow{\approx} \mathrm{Gal}(\mathbf{R}(Y_0)/\mathbf{R}(X_0))$$

which is easily seen to be an isomorphism. Now fix separable algebraic closures $\mathbf{R}(X)^*$ of $\mathbf{R}(X)$ and $\mathbf{R}(X_0)^*$ of $\mathbf{R}(X_0)$, and let $\widetilde{\mathbf{R}(X)} \subset \mathbf{R}(X)^*$, $\widetilde{\mathbf{R}(X_0)} \subset \mathbf{R}(X_0)^*$ be the maximal subfields such that normalization in finitely generated subfields of these is étale over X or over X_0. Now write $\widetilde{\mathbf{R}(X)}$ as an increasing union of finite Galois extensions L_n of $\mathbf{R}(X)$; we get a tower of coverings $Y_{L_n} =$ normalization of X in L_n; let $Z_n = Y_{L_n} \times_X X_0$ (this is a tower of connected coverings of X_0); choose inductively in n isomorphisms:

$$\mathbf{R}(Z_n) \xrightarrow{\approx} K_n \subset \widetilde{\mathbf{R}(X_0)}.$$

It follows readily that $\bigcup K_n = \widetilde{\mathbf{R}(X_0)}$, and that

$$\begin{aligned}\pi_1^{\mathrm{alg}}(X) &\cong \varprojlim \mathrm{Gal}(L_n/\mathbf{R}(X))\\ &\cong \varprojlim \mathrm{Gal}(K_n/\mathbf{R}(X_0))\\ &\cong \pi_1^{\mathrm{alg}}(X_0).\end{aligned}$$

□

This result can be used to partially compute π_1 of liftable characteristic p varieties in terms of π_1 of varieties over $\mathbb{C}$, hence in terms of classical topology. This method is due to Grothendieck and illustrates very beautifully the Kroneckerian idea of §IV.1: Let

$$\begin{aligned}k &= \text{algebraically closed field of characteristic } p,\\ R &= \text{complete local domain of characteristic 0 with } R/M = k,\\ K &= \text{quotient field of } R,\\ \overline{K} &= \text{algebraic closure of } K.\end{aligned}$$

Choose an isomorphism (embedding?):

$$\overline{K} \cong \mathbb{C}.$$

Let

$$\begin{aligned}X &= \text{scheme proper and smooth over } R,\\ X_0 &= X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k: \ \text{ we assume this is irreducible},\\ X_\eta &= X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, K,\\ \overline{X}_\eta &= X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \overline{K}: \ \text{ we assume this is irreducible},\\ \widetilde{X}_\eta &= X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, \mathbb{C}.\end{aligned}$$

THEOREM 5.10. *There is a surjective homomorphism*

$$\pi_1^{\mathrm{alg}}(\overline{X}_\eta) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X_0)$$

canonical up to inner automorphism, and hence, fixing an isomorphism $\overline{K} \approx \mathbb{C}$, a surjective homomorphism:

$$\widehat{\pi}_1^{\mathrm{top}}(\widetilde{X}_\eta) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X_0).$$

PROOF. By Theorem 2.16 and Corollary 5.9, it suffices to compare $\pi_1^{\mathrm{alg}}(\overline{X}_\eta)$ and $\pi_1^{\mathrm{alg}}(X)$. Let $\Omega \supset \mathbf{R}(\overline{X}_\eta)$ be an algebraic closure. Note that

i) $\mathbf{R}(X) = \mathbf{R}(X_\eta)$
ii) $\mathbf{R}(\overline{X}_\eta) = \mathbf{R}(X_\eta) \otimes_K \overline{K}$ is algebraic over $\mathbf{R}(X_\eta)$, hence Ω is an algebraic closure of $\mathbf{R}(X_\eta)$ too.

Thus we may consider the maximal subfields of Ω such that the normalization of any of the schemes X, X_η and $\overline{X}_\eta$ is étale. Note that:

iii) $L \subset \Omega$ finite over $\mathbf{R}(X)$, normalization of X in L étale over $X \implies$ normalization of X_η in L étale over X_η,
iv) $K_0 \subset \overline{K}$ finite over K, normalization of X_η in $\mathbf{R}(X_\eta) \otimes_K K_0$ is $X_\eta \times_{\operatorname{Spec} K} \operatorname{Spec} K_0$ which is étale over X_η.

Thus we get a diagram

$$\begin{tikzcd}
& & \Omega \arrow[d, dash] & \\
& & \Omega_1 \arrow[dl, dash] \arrow[dr, dash] & \\
& \overline{K}\cdot\mathbf{R}(X_\eta) = \mathbf{R}(\overline{X}_\eta) \arrow[dl, dash] \arrow[dr, dash] & & \Omega_2 \arrow[dl, dash] \\
\overline{K} \arrow[dr, dash] & & \mathbf{R}(X_\eta) = \mathbf{R}(X) \arrow[dl, dash] & \\
& K & &
\end{tikzcd}$$

where

$$\Omega_1/\mathbf{R}(\overline{X}_\eta) = \text{maximal extension étale over } \overline{X}_\eta$$
$$\Omega_1/\mathbf{R}(X_\eta) = \text{maximal extension étale over } X_\eta$$
$$\Omega_2/\mathbf{R}(X_\eta) = \text{maximal extension étale over } X$$

i.e.,

$$\pi_1^{\mathrm{alg}}(\overline{X}_\eta) \cong \mathrm{Gal}(\Omega_1/\mathbf{R}(\overline{X}_\eta))$$
$$\pi_1^{\mathrm{alg}}(X_\eta) \cong \mathrm{Gal}(\Omega_1/\mathbf{R}(X_\eta))$$
$$\pi_1^{\mathrm{alg}}(X) \cong \mathrm{Gal}(\Omega_2/\mathbf{R}(X_\eta)).$$

Since

v) $\mathrm{Gal}(\mathbf{R}(\overline{X}_\eta)/\mathbf{R}(X_\eta)) \cong \mathrm{Gal}(\overline{K}/K)$, we get homomorphisms:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1^{\mathrm{alg}}(\overline{X}_\eta) \arrow[r] \arrow[dr, "\phi"'] & \pi_1^{\mathrm{alg}}(X_\eta) \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{K}/K) \arrow[r] & 1 \\ & & \pi_1^{\mathrm{alg}}(X) & & \end{tikzcd}$$

To finish the proof of Theorem 5.10, we must show that ϕ is surjective. A small consideration of this diagram of fields shows that this amounts to saying:

vi) $\Omega_2 \otimes_K \overline{K} \to \Omega$ is injective; or equivalently (cf. §IV.2) K is algebraically closed in Ω_2.

If this is not true, then suppose $L \subset \overline{K}$ is finite over K and $L \subset \Omega_2$. Let S be the integral closure of R in L. Then $X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S$ is smooth over $\mathrm{Spec}\, S$, hence is normal; since $\mathbf{R}(X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S) = \mathbf{R}(X) \otimes_K L$, $X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S$ is the normalization of X in $\mathbf{R}(X) \otimes_K L$. Now $f\colon \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is certainly *not* étale unless $R = S$: because if $[M] \in \mathrm{Spec}\, R$ is the closed point, then (a) by Hensel's lemma (Lemma IV.6.1), $f^{-1}([M]) =$ one point, so (b) if f is étale, the closed subscheme $f^{-1}([M])$ is isomorphic to $\mathrm{Spec}\, k$, hence (c) f is a closed immersion, i.e., $R \to S$ is surjective. But then neither can $g\colon X \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, S \to X$ be étale because I claim there is a section s:

$$\begin{tikzcd} X \arrow[d, shift right] \\ \mathrm{Spec}\, R \arrow[u, shift right, "s"'] \end{tikzcd}$$

hence g étale implies by base change via s that f is étale. To construct s, just take a closed point $x \in X_0$, let $\overline{a}_1, \ldots, \overline{a}_n$ be generators of $\mathfrak{m}_{x,X_0}$ in the regular local ring $\mathcal{O}_{x,X_0}$, lift these to $a_1, \ldots, a_n \in \mathfrak{m}_{x,X}$ and set $Z = \mathrm{Spec}\,(\mathcal{O}_{x,X}/(a_1, \ldots, a_n))$. By Hensel's lemma (Lemma IV.6.1), Z is finite over R, hence is isomorphic to a closed subscheme of X. Since the projection $Z \to \mathrm{Spec}\, R$ is immediately seen to be étale, $Z \to \mathrm{Spec}\, R$ is an isomorphism, hence there is a unique s with $Z = \mathrm{Image}(s)$. □

At this point we can put together Parts I and II to deduce the following famous result of Grothendieck.

Corollary 5.11. *Let k be an algebraically closed field of characteristic p and let X be a non-singular complete curve over k. Let $g = \dim_k H^1(\mathcal{O}_X)$, the genus of X. Then*

$$\exists a_1, \ldots, a_g, b_1, \ldots, b_g \in \pi_1^{\mathrm{alg}}(X)$$

satisfying:

$$(*) \qquad a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = e$$

and generating a dense subgroup: equivalently $\pi_1^{\mathrm{alg}}(X)$ is a quotient of the pro-finite completion of the free group on the a_i's and b_i's modulo a normal subgroup containing at least $(*)$.

Proof. Lift X to a scheme Y over the ring of Witt vectors $W(k)$, and via an isomorphism of $\mathbb{C}$ with (embedding into $\mathbb{C}$ of?) the algebraic closure of the quotient field of $W(k)$ (see, for instance, Mumford [**84**, Lecture 26 by G. Bergman]), let Y induce a curve Z over $\mathbb{C}$. Note that $g = \dim_{\mathbb{C}} H^1(\mathcal{O}_Z)$ by the remark following Corollary 5.7. By Part I [**87**, §7B], we know that topologically Z is a compact orientable surface with g handles. It is a standard result in elementary topology that π_1^{top} of such a surface is free on a_i's and b_i's modulo the one relation $(*)$. Thus everything follows from Theorem 5.10. □

What is the kernel of $\pi_1^{\text{alg}}(\overline{X}_\eta) \to \pi_1^{\text{alg}}(X_0)$? A complete structure theorem is not known, even for curves, but the following two things have been discovered:

a) Grothendieck has shown that the kernel is contained in the closed normal subgroup generated by the p-Sylow subgroups: i.e., if H is finite such that $p \nmid \#H$ and $\pi_1^{\text{alg}}(\overline{X}_\eta) \to H$ is a continuous map, then this map factors through $\pi_1^{\text{alg}}(X_0)$.
b) If you abelianize the situation, and look at the p-part of these groups, the kernel tends to be quite large. In fact
$$\pi_1^{\text{alg}}(\overline{X}_\eta)/\left[\pi_1^{\text{alg}}, \pi_1^{\text{alg}}\right] \cong \prod_{\text{primes } l} \mathbb{Z}_l^{2g} \times T_0$$
while
$$\pi_1^{\text{alg}}(X_0)/\left[\pi_1^{\text{alg}}, \pi_1^{\text{alg}}\right] \cong \prod_{l \neq p} \mathbb{Z}_l^{2g} \times \mathbb{Z}_p^r \times T_p$$
where $0 \leq r \leq g$ and T_0, T_p are finite groups, (0) in the case of curves. In fact, Shafarevitch has shown for curves that the maximal pro-p-nilpotent quotient of $\pi_1^{\text{alg}}(X_0)$ is a free pro-p-group on r generators.

Going back now to general deformation theory, it is clear that the really powerful applications are in situations where one can apply the basic set-up: $R \to R/I$ $(I \cdot M = (0))$ inductively and get statements over general artin rings and via GFGA (Theorem 2.17) to complete local ring. In the two cases examined above, we could do this by proving that there were no obstructions. However even if obstructions may be present, one can seek to build up inductively a *maximal* deformation of the original variety X_0/k. This is the point of view of moduli, which we want to sketch briefly.

Start with an arbitrary scheme X_0 over k. Then for all artin local rings R with residue field k, define

$$\begin{array}{l} \mathrm{Def}(X_0/R) \\ \text{or} \\ \text{the deformations} \\ \text{of } X_0 \text{ over } R \end{array} = \left\{ \begin{array}{l} \text{the set of triples } (X, \phi, \pi), \text{ where} \\ \pi\colon X \to \operatorname{Spec} R \text{ is a flat morphism} \\ \text{and } \phi\colon X \times_{\operatorname{Spec} R} \operatorname{Spec} k \xrightarrow{\approx} X_0 \text{ is a} \\ k\text{-isomorphism, modulo } (X,\phi,\pi) \sim (X',\phi',\pi') \\ \text{if } \exists \text{ an } R\text{-isomorphism } \psi\colon X \xrightarrow{\approx} X' \text{ such that} \\ \begin{tikzcd} X \times_{\operatorname{Spec} R} \operatorname{Spec} k \arrow[rr, "\psi \times 1_k"] \arrow[dr, "\phi"'] & & X' \times_{\operatorname{Spec} R} \operatorname{Spec} k \arrow[dl, "\phi'"] \\ & X_0 & \end{tikzcd} \\ \text{commutes.} \end{array} \right\}$$

Note that
$$R \longmapsto \mathrm{Def}(X_0/R)$$
is a covariant functor for all homomorphisms $f\colon R \to R'$ inducing the identities on the residue fields. In fact, if $(X, \phi, \pi) \in \mathrm{Def}(X_0/R)$, let

$X' = X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$

$\pi' =$ projection of X' onto $\operatorname{Spec} R'$

$\phi' =$ the composition:
$$X' \times_{\operatorname{Spec} R'} \operatorname{Spec} k = (X \times_{\operatorname{Spec} R} \operatorname{Spec} R') \times_{\operatorname{Spec} R'} \operatorname{Spec} k \cong X \times_{\operatorname{Spec} R} \operatorname{Spec} k \xrightarrow{\phi} X_0.$$

Then $(X', \phi', \pi') \in \mathrm{Def}(X_0/R')$ depends only on the equivalence class of (X, ϕ, π) and on f. One says that X'/R' is the deformation obtained from X/R by the base change f. What one wants to do next is to *build up inductively the biggest possible deformation of* X_0 so that any other is obtained from it by base change! More precisely, suppose that $\mathfrak{R}$ is a complete noetherian local ring with maximal ideal $\mathfrak{M}$ and residue field $\mathfrak{R}/\mathfrak{M} = k$. Then we get a sequence of artin local rings $R_n = \mathfrak{R}/\mathfrak{M}^{n+1}$. Then, by definition, a *formal deformation* $\mathcal{X}$ *of* X_0 *over* $\mathfrak{R}$ is a sequence of deformations X_n and closed immersions ϕ_n:

$$\begin{tikzcd} & \cdots & X_n \arrow[l] \arrow[d] & X_{n-1} \arrow[l, "\phi_n"'] \arrow[d] & \cdots \arrow[l] & X_0 \arrow[l, "\phi_1"'] \arrow[d] \\ \operatorname{Spec}\mathfrak{R} & \cdots \arrow[l] & \operatorname{Spec}R_n \arrow[l] & \operatorname{Spec}R_{n-1} \arrow[l] & \cdots \arrow[l] & \operatorname{Spec}k \arrow[l] \end{tikzcd}$$

where ϕ_n induces an isomorphism:

$$X_{n-1} \xrightarrow{\approx} X_n \times_{\operatorname{Spec}R_n} \operatorname{Spec}R_{n-1}.$$

Note that if S is artin local with residue field k, $\mathfrak{R} \to S$ is a homomorphism inducing identity on residue fields and $\mathcal{X}/\mathfrak{R}$ is a formal deformation, we can define a real deformation $\mathcal{X} \times_{\operatorname{Spec}\mathfrak{R}} \operatorname{Spec}S$ by base change, since $\mathfrak{R} \to S$ factors through R_n if n is large enough. Then a formal deformation $\mathcal{X}/\mathfrak{R}$ is said to be *versal* or *semi-universal* if:

(1) every deformation Y of X_0 over S is isomorphic to the one obtained by base change $\mathcal{X} \times_{\operatorname{Spec}\mathfrak{R}} \operatorname{Spec}S$ for a suitable $\alpha\colon \mathfrak{R} \to S$, and
(2) if the maximal ideal $N \subset S$ satisfies $N^2 = (0)$, then one asks that there be only one α for which (1) holds.
(2′) $\mathcal{X}/\mathfrak{R}$ is *universal* if α is always unique. It is clear that a universal deformation is unique if it exists, and it is not hard to prove that a versal one is also unique, but only up to a non-canonical isomorphism.

A theorem of Grothendieck and Schlessinger [**97**] asserts the following:

a) If X_0 is smooth and proper over k, then a versal deformation $\mathcal{X}/\mathfrak{R}$ exists and there is a canonical isomorphism:

$$\left.\begin{array}{ll} \operatorname{char}k = 0: & \operatorname{Hom}_k(\mathfrak{M}/\mathfrak{M}^2, k) \\ \operatorname{char}k = p: & \operatorname{Hom}_k(\mathfrak{M}/(\mathfrak{M}^2 + (p)), k) \end{array}\right\} \cong H^1(X_0, \Theta_{X_0}).$$

b) If $H^0(X_0, \Theta_{X_0}) = (0)$, then $\mathcal{X}/\mathfrak{R}$ is universal.
c) If $H^2(X_0, \Theta_{X_0}) = (0)$, then

$$\left\{\begin{array}{lll} \operatorname{char}k = 0: & \mathfrak{R} \cong k[[t_1, \dots, t_n]], & n = \dim H^1(X_0, \Theta_{X_0}) \\ \operatorname{char}k = p,\ k \text{ perfect}: & \mathfrak{R} \cong W(k)[[t_1, \dots, t_n]], & n = \dim H^1(X_0, \Theta_{X_0}). \end{array}\right.$$

A further development of these ideas leads us to the global problem of moduli. Starting with any X_0 smooth and proper over k, suppose you drop the restriction that R be an artin local ring and for any pair $(R, \mathfrak{m})$, R a ring, $\mathfrak{m} \subset R$ a maximal ideal such that $R/\mathfrak{m} = k$ you define $\operatorname{Def}(X_0/R)$ to be the pairs (X, ϕ) as before, but now X is assumed smooth and proper over $\operatorname{Spec}R$. If moreover you isolate the main qualitative properties that X_0 and its deformations have, it is natural to cut loose from the base point $[\mathfrak{m}] \in \operatorname{Spec}R$ and consider instead functors like:

$$\mathfrak{M}_{\mathbf{P}}(S) = \left\{\begin{array}{l} \text{set of smooth proper morphisms } f\colon X \to S \text{ such that} \\ \text{all the fibres } f^{-1}(s) \text{ of } f \text{ have property } \mathbf{P}, \\ \text{modulo } f \sim f' \text{ if } \exists \text{ an } S\text{-isomorphism } g\colon X \xrightarrow{\approx} X' \end{array}\right\},$$

where S is any scheme and $\mathbf{P}$ is some property of schemes X over fields k. Provided that $\mathbf{P}$ satisfies: [if X/k has $\mathbf{P}$ and $k' \supset k$, then $X \times_{\operatorname{Spec}k} \operatorname{Spec}k'$ has $\mathbf{P}$], then $\mathfrak{M}_{\mathbf{P}}$ is a functor in S, i.e., given $g\colon S' \to S$ and $X/S \in \mathfrak{M}_{\mathbf{P}}(s)$, then $X \times_S S' \in \mathfrak{M}_{\mathbf{P}}(S')$.

For instance, take $\mathbf{P}(X/k)$ to mean

$$\begin{aligned} &\dim X = 1 \\ &H^0(X, \mathcal{O}_X) \xrightarrow{\approx} k \\ &\dim_k H^1(X, \mathcal{O}_X) = g; \end{aligned}$$

then $\mathfrak{M}_{\mathbf{P}}$ is the usual moduli functor for curves of genus g. The "problem of moduli" is just the question of describing $\mathfrak{M}_{\mathbf{P}}$ as explicitly as possible and in particular asking how far it deviates from a representable functor. The best case, in other words, would be that as functors in S, $\mathfrak{M}_{\mathbf{P}}(S) \cong \mathrm{Hom}(S, M_{\mathbf{P}})$ for some scheme $M_{\mathbf{P}}$ which would then be called the moduli space. For an introduction to these questions, see Mumford et al. [**83**].

Exercise

(1) Let X be a normal irreducible noetherian scheme and let $L \supset \mathbf{R}(X)$ be a *Galois* extension such that the normalization Y_L of X in L is étale over X. Let $\pi\colon Y_L \to X$ be the canonical morphism. Let $G = \mathrm{Gal}(L/\mathbf{R}(X))$. Then G acts on Y_L over X: show that for all $y \in Y_L$, if $x = \pi(y)$, then:
   a) G acts transitively on $\pi^{-1}(x)$.
   b) If $G_y \subset G$ is the subgroup leaving y fixed, then G_y acts naturally on $\Bbbk(y)$ leaving $\Bbbk(x)$ fixed.
   c) $\Bbbk(y)$ is Galois over $\Bbbk(x)$ and, via the action in (b),
   $$G_y \xrightarrow{\approx} \mathrm{Gal}(\Bbbk(y)/\Bbbk(x)).$$

   [*Hint*: Let $n = [L\colon \mathbf{R}(X)]$. Using the fact that $L \otimes_{\mathbf{R}(X)} L \cong \overbrace{L \times \cdots \times L}^{n \text{ times}}$ and that $Y_L \times_X Y_L$ is normal, prove that $Y_L \times_X Y_L =$ disjoint union of n copies of Y_L. Prove that if G acts on $Y_L \times_X Y_L$ non-trivially on the first factor but trivially on the second, then it permutes these components simply transitively.]

(2) Note that the first part of the GFGA theorem (Theorem 2.17) would be trivial *if* the following were true:

$$\begin{aligned} &X \text{ a scheme over } \operatorname{Spec} A \\ &\mathcal{F} \text{ a quasi-coherent sheaf of } \mathcal{O}_X\text{-modules} \\ &B \text{ an } A\text{-algebra.} \end{aligned}$$

   Then for all i, the canonical map
   $$H^i(X, \mathcal{F}) \otimes_A B \longrightarrow H^i(X \times_{\operatorname{Spec} A} \operatorname{Spec} B, \mathcal{F} \otimes_A B)$$
   is an isomorphism. Show that if B is flat over A, this is correct.

(3) Using (2), deduce the more elementary form of GFGA:

$$\begin{gathered} f\colon Z \longrightarrow X \text{ proper, } X \text{ noetherian} \\ \mathcal{F} \text{ a coherent sheaf of } \mathcal{O}_X\text{-modules.} \end{gathered}$$

   Then for all i, and for all $x \in X$,
   $$\varprojlim_n R^i f_*(\mathcal{F})_x / \left(\mathfrak{m}_x^n \cdot R^i f_*(\mathcal{F})_x\right) \cong \varprojlim_n H^i(f^{-1}(x), \mathcal{F}/\mathfrak{m}_x^n \cdot \mathcal{F}).$$

CHAPTER IX

Applications

1. Mori's existence theorem of rational curves

(Added in publication)

In this section X is a smooth projective variety over an algebraically closed field k of characteristic $p \geq 0$. For simplicity, we omit k and $\operatorname{Spec}(k)$ from tensor products over k, X/k, fibre products over $\operatorname{Spec}(k)$, etc.

The highest nonzero exterior power

$$\mathcal{K}_X := \det \Omega_X^{\dim X} = \bigwedge^{\dim X} \Omega_X^1$$

is an invertible $\mathcal{O}_X$-module called the *canonical sheaf*. The canonical divisor K_X is the divisor on X defined up to linear equivalence by

$$\mathcal{K}_X = \mathcal{O}_X(K_X).$$

The tangent sheaf $\Theta_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ thus gives rise to

$$\det \Theta_X = \mathcal{O}_X(-K_X) = \mathcal{K}_X^{-1}.$$

We already encountered canonical divisors in the Riemann-Roch theorem for curves in §VIII.1.

The canonical sheaf and divisor play pivotal roles especially in birational geometry. As a nontrivial application of what we have seen so far, we prove the following theorem, which provided a breakthrough in higher-dimensional birational geometry since the 1980's:

THEOREM 1.1 (Mori). (Mori [**81**, Theorem 5], Mori [**82**, Theorem 1.4], Kollár-Mori [**74**, Theorem 1.13]) *Let X be a smooth projective variety over k with a closed irreducible curve $C \subset X$ such that*

$$(\mathcal{K}_X^{-1}.C) > 0.$$

Then for any fixed ample divisor $\mathcal{H}$ on X, there exists a rational curve l on X such that

$$\dim X + 1 \geq (\mathcal{K}_X^{-1}.l) > 0 \qquad \textit{and} \qquad \frac{(\mathcal{K}_X^{-1}.l)}{(\mathcal{H}.l)} \geq \frac{(\mathcal{K}_X^{-1}.C)}{(\mathcal{H}.C)}.$$

Here a *rational curve* is an irreducible and reduced curve proper over k with the normalization $\mathbb{P}^1$, while $(\mathcal{L}.C)$ denotes the intersection number of an inverstible sheaf $\mathcal{L}$ and a curve C defined in §VII.11.

Why is a rational curve so important? As in §III.3, consider the blow up of a smooth variety Z over k along a closed smooth subvariety $Y \subset Z$ of codimension $r \geq 2$ defined by an ideal sheaf $\mathcal{I}$:

$$\pi \colon Z' := \operatorname{Bl}_Y(X) \longrightarrow Z.$$

Since Y is supposed to be smooth, hence is a local complete intersection, $\mathcal{I}/\mathcal{I}^2$ is $\mathcal{O}_Y$-locally free of rank r. Thus by Theorem III.3.5, the *exceptional divisor*

$$\pi^{-1}(Y) = Z' \times_Z Y = \mathbb{P}_Y(\mathcal{I}/\mathcal{I}^2)$$

is a $\mathbb{P}^{r-1}$-bundle over Y, hence contains lots of rational curves in the fibres that are contracted to points by π.

In particular when Z is a smooth surface and $Y = \{z\}$ is a closed point, the exceptional curve $\pi^{-1}(z) = \mathbb{P}^1$ is contracted to the point z by π.

Mori's result was the starting point of looking for rational curves that can be contracted.

Mori produces a rational curve out of a given curve C in the theorem via the "bend and break" technique. To do so, he needs the following nontrivial result on deformation of morphisms, which combine what we had in §I.8 and §VIII.5.

We work over $S = \operatorname{Spec}(k)$ with an algebraically closed field k. For simplicity, we omit subscripts S and k for $/k$, $\times_S$, $\otimes_k$, etc.

Let V be a projective variety over k, while W is a smooth quasi-projective variety over k.

By Grothendieck's theorem (Theorem I.8.14), there exists a locally noetherian scheme $\mathbf{Hom}(V, W)$ over k parametrizing morphisms from V to W, that is, for any locally noetherian k-scheme T, the set $\mathbf{Hom}(V, W)(T)$ of its T-valued points is canonically isomorphic to the set of T-morphisms $V \times T \to W \times T$.

Fixing a morphism $f\colon V \to W$, let us consider its infinitesimal deformations. For an artin local k-algebra R, denote by $\mathfrak{m}$ its maximal ideal (hence $R/\mathfrak{m} = k$) and let I be an ideal of R such that $\mathfrak{m}I = 0$.

Given an infinitesimal deformation f_1 over $\operatorname{Spec}(R/I)$ of f, we would like to see if it lifts to an infinitesimal deformation f_2 over $\operatorname{Spec}(R)$, that is,

$$\begin{array}{ccccc}
\mathbf{Hom}(V,W)(\operatorname{Spec}(R)) & \longrightarrow & \mathbf{Hom}(V,W)(\operatorname{Spec}(R/I)) & \longrightarrow & \mathbf{Hom}(V,W)(\operatorname{Spec}(k)) \\
f_2 & \longmapsto & f_1 & \longmapsto & f
\end{array}$$

In the description of §VIII.5, we are in the situation:

$$\begin{array}{ccc}
V_2 = V \times \operatorname{Spec}(R) & \overset{f_2}{\dashrightarrow} & W \times \operatorname{Spec}(R) = W_2 \\
\cup & & \cup \\
V_1 = V \times \operatorname{Spec}(R/I) & \xrightarrow{f_1} & W \times \operatorname{Spec}(R/I) = W_1 \\
\cup & & \cup \\
V_0 = V & \xrightarrow{\qquad f \qquad} & W = W_0.
\end{array}$$

As we saw in $\mathrm{b_i}$) and $\mathrm{b_{ii}}$), the obstruction for lifting f_1 to f_2 lies in

$$H^1(V, f^*\Theta_W) \otimes I.$$

If the lifting exists, then the set $\operatorname{Lift}(f_1/R)$ of all liftings is a principal homogeneous space over

$$H^0(V, f^*\Theta_W) \otimes I.$$

In Mori's applications, we have an additional information: Fix a closed immersion $j\colon Z \hookrightarrow V$ and a morphism $\zeta\colon Z \to W$, and consider a morphism $f\colon V \to W$ whose restriction to Z is ζ, that is, $f \circ j = \zeta$. The subfunctor of $\mathbf{Hom}(V, W)$, defined by

$$\mathbf{Hom}(V, W; \zeta)(T) := \{g\colon V \times T \to W \times T \mid T\text{-morphism with } g \circ (j \times \mathrm{id}) = \zeta \times \mathrm{id}\}$$

for k-schemes T, is represented by a closed subscheme of $\mathbf{Hom}(V, W)$ obtained as the fibre product of the natural restriction morphism $\mathbf{Hom}(V, W) \to \mathbf{Hom}(Z, W)$ and the morphism $\operatorname{Spec}(k) \to \mathbf{Hom}(Z, W)$ corresponding to $\zeta \in \mathbf{Hom}(Z, W)(\operatorname{Spec}(k))$.

In terms of the ideal sheaf $\mathcal{I}_Z$ on V defining the subscheme Z, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Tensoring the locally free $\mathcal{O}_V$-module $f^*\Theta_W$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W \longrightarrow f^*\Theta_W \longrightarrow \zeta^*\Theta_W \longrightarrow 0,$$

hence the associated cohomology long exact sequence

$$\begin{aligned} 0 \longrightarrow & H^0(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) \longrightarrow H^0(V, f^*\Theta_W) \longrightarrow H^0(Z, \zeta^*\Theta_W) \\ & \longrightarrow H^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) \longrightarrow H^1(V, f^*\Theta_W) \longrightarrow H^1(Z, \zeta^*\Theta_W) \longrightarrow \cdots . \end{aligned}$$

Given $f_1 \in \mathbf{Hom}(V, W; \zeta)(\mathrm{Spec}(R/I))$, the obstruction for lifting it to $f_2 \in \mathbf{Hom}(V, W; \zeta)(\mathrm{Spec}(R))$ lies in $H^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W)$, since the image in $\mathbf{Hom}(Z, W)(\mathrm{Spec}(R/I))$ of f_1, regarded as an element of $\mathbf{Hom}(V, W)(\mathrm{Spec}(R/I))$, is $\zeta \times \mathrm{id}$, which is lifted to $\zeta \times \mathrm{id} \in \mathbf{Hom}(Z, W)(\mathrm{Spec}(R))$. When f_1 is liftable to $\mathrm{Spec}(R)$, the set of all liftings is a principal homogeneous space over $H^0(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W)$, since the liftings as elements of $\mathbf{Hom}(V, W)(\mathrm{Spec}(R))$ have to be mapped to $\zeta \times \mathrm{id} \in \mathbf{Hom}(Z, W)(\mathrm{Spec}(R))$.

Proposition 1.2 (Mori). (cf. Mori [**81**, Proposition 2]) *Let k be an algebraically closed field. For a projective variety V over k and a smooth quasi-projective variety W over k, consider a k-morphism $f\colon V \to W$ whose restriction to a closed subscheme $j\colon Z \hookrightarrow V$ is $\zeta = f \circ j\colon Z \to W$. Then the tangent space of $\mathbf{H} := \mathbf{Hom}(V, W; \zeta)$ at its point $[f]$ corresponding to $f \in \mathbf{Hom}(V, W; \zeta)(\mathrm{Spec}(k))$ is given by*

$$\mathbf{T}_{\mathbf{H},[f]} = H^0(V, \mathcal{I}_Z \otimes_V f^*\Theta_W),$$

while the dimension of $\mathbf{H}$ at $[f]$ satisfies

$$\dim_{[f]} \mathbf{H} \geq h^0(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) - h^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W).$$

Here, $h^i(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W)$ is the customary notation for the dimension of $H^i(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W)$ as a vector space over k.

Proof. For the first assertion, apply what we have seen to the situation $R = k[\varepsilon]$ with $\varepsilon^2 = 0$ and $I = k\varepsilon$ (cf. Definition V.1.3).

To prove the second assertion, let us simplify the notation as

$$\mathcal{O} := \mathcal{O}_{\mathbf{H},[f]}, \quad \mathfrak{m} := \mathfrak{m}_{\mathbf{H},[f]}.$$

There certainly exists a formal power series ring A over k with maximal ideal M such that the $\mathfrak{m}$-adic completion $\widehat{\mathcal{O}}$ is of the form

$$\widehat{\mathcal{O}} = A/\mathfrak{a}, \quad \text{with} \quad M^2 \supset \mathfrak{a}.$$

For any positive integer $\nu \geq 2$, consider the canonical surjective homomorphism

$$A/(M\mathfrak{a} + M^\nu) \longrightarrow \mathcal{O}/\mathfrak{m}^\nu = A/(\mathfrak{a} + M^\nu)$$

whose kernel $(\mathfrak{a} + M^\nu)/(M\mathfrak{a} + M^\nu)$ is killed by M. The canonical surjection

$$\varphi_1\colon \mathcal{O} = A/\mathfrak{a} \longrightarrow \mathcal{O}/\mathfrak{m}^\nu = A/(\mathfrak{a} + M^\nu)$$

corresponds to $f_1 \in \mathbf{H}(\mathrm{Spec}(A/(\mathfrak{a} + M^\nu)))$. Thus by what we have seen above, the obstruction for lifting f_1 to $f_2 \in \mathbf{H}(\mathrm{Spec}(A/(M\mathfrak{a} + M^\nu)))$, hence the obstruction ψ for lifting φ_1 to

$$\varphi_2\colon \mathcal{O} = A/\mathfrak{a} \longrightarrow A/(M\mathfrak{a} + M^\nu),$$

lies in

$$H^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) \otimes \left(\frac{\mathfrak{a} + M^\nu}{M\mathfrak{a} + M^\nu} \right).$$

In terms of a basis $\{\psi_1, \ldots, \psi_a\}$ of $H^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W)$, the obstruction is of the form

$$\psi = \psi_1 \otimes \overline{r}_1 + \psi_2 \otimes \overline{r}_2 + \cdots + \psi_a \otimes \overline{r}_a$$

for the residue classes $\overline{r}_1, \ldots, \overline{r}_a$ modulo $M\mathfrak{a} + M^\nu$ of $r_1, \ldots, r_a \in \mathfrak{a} \subset \mathfrak{a} + M^\nu = \mathfrak{a} + (M\mathfrak{a} + M^\nu)$. This obstruction thus lies in

$$H^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) \otimes \left(\frac{(r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu}{M\mathfrak{a} + M^\nu} \right).$$

Hence there exists a lifting

$$\mathbf{H}(\mathrm{Spec}(A/((r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu))) \ni f_2' \longmapsto f_1 \in \mathbf{H}(\mathrm{Spec}(A/(\mathfrak{a} + M^\nu))),$$

or equivalently, there exists a homomorphism $\alpha \colon \mathcal{O} \to A/((r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu)$ such that the diagram

$$\begin{tikzcd} A/(M\mathfrak{a} + M^\nu) \arrow[r, two heads] & A/((r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu) \arrow[r, two heads] & \mathcal{O}/\mathfrak{m}^\nu = A/(\mathfrak{a} + M^\nu) \\ & \mathcal{O} = A/\mathfrak{a} \arrow[u, "\alpha"] \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

is commutative. Obviously α is surjective, since $\mathfrak{a} \subset M^2$. Hence there exists a k-algebra automorphism σ of A such that the diagram

$$\begin{tikzcd} A \arrow[r, two heads] & A/((r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu) \arrow[r, two heads] & A/(\mathfrak{a} + M^\nu) = \mathcal{O}/\mathfrak{m}^\nu \\ A \arrow[u, "\sigma"] \arrow[r, two heads] & A/\mathfrak{a} = \mathcal{O} \arrow[u, "\alpha"] \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

is commutative. We automatically have $\sigma(M) = M$. By the commutativity of the diagram, we have $r - \sigma(r) \in \mathfrak{a} + M^\nu$ for any $r \in A$. In particular, $r - \sigma(r) \in \mathfrak{a} + M^\nu$ for all $r \in \sigma^{-1}(\mathfrak{a})$. Thus $\sigma^{-1}(\mathfrak{a}) \subset \mathfrak{a} + M^\nu$, hence $\mathfrak{a} \subset \sigma(\mathfrak{a}) + M^\nu$. Again by the commutativity of the diagram, we thus have

$$\sigma(\mathfrak{a}) \subset (r_1, \ldots, r_a) + M\mathfrak{a} + M^\nu \subset (r_1, \ldots, r_a) + M\sigma(\mathfrak{a}) + M^\nu.$$

On the other hand, as an easy consequence of the Artin-Rees lemma (cf., e.g., Zariski-Samuel [**119**, Chap. VIII, §2, Theorem 4′], or Matsumura [**78**, Theorem 8.5]) we have

$$\mathfrak{a} \cap M^\nu = M(\mathfrak{a} \cap M^{\nu-1}) \subset M\mathfrak{a}, \qquad \text{for } \nu \gg 0.$$

Consequently, we have $\sigma(\mathfrak{a}) \cap M^\nu \subset M\sigma(\mathfrak{a})$. Thus the images of $r_1, \ldots, r_a \in \mathfrak{a} \subset \sigma(\mathfrak{a}) + M^\nu$ modulo $M\sigma(\mathfrak{a}) + M^\nu$ generate

$$\frac{\sigma(\mathfrak{a}) + M^\nu}{M\sigma(\mathfrak{a}) + M^\nu} \cong \frac{\sigma(\mathfrak{a})}{M\sigma(\mathfrak{a})}.$$

Thus $r_1, \ldots, r_a$ generate $\sigma(\mathfrak{a})$ by Nakayama's lemma, hence $\sigma^{-1}(r_1), \ldots, \sigma^{-1}(r_a)$ generate $\mathfrak{a}$. Consequently, we get

$$\begin{aligned} \text{Krull dim}\, \mathcal{O} &= \text{Krull dim}\, A/\mathfrak{a} \\ &\geq \text{Krull dim}\, A - a \\ &= h^0(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W) - h^1(V, \mathcal{I}_Z \otimes_{\mathcal{O}_V} f^*\Theta_W). \end{aligned}$$

□

Recall that for a locally free $\mathcal{O}_Y$-module $\mathcal{E}$ of rank $r = \mathrm{rk}\, \mathcal{E}$ on a scheme Y, we denote

$$\det \mathcal{E} := \bigwedge^r \mathcal{E},$$

which is an invertible sheaf on Y.

Let C be a smooth projective curve over k of genus g. As in the Remark immediately after the proof of Proposition VIII.1.7, the Riemann-Roth theorem says

$$\chi(C, \mathcal{E}) = \deg(\det \mathcal{E}) + (1-g) \operatorname{rk} \mathcal{E}$$

for any locally free $\mathcal{O}_C$-module $\mathcal{E}$. By the filtration for $\mathcal{E}$ mentioned there, it is easy to see that

$$\det(\mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{E}) = \mathcal{L}^{\otimes \operatorname{rk} \mathcal{E}} \otimes_{\mathcal{O}_C} \det \mathcal{E}.$$

COROLLARY 1.3. (Mori [**81**, Proof of Theorem 4]) *Suppose W is a smooth quasi-projective variety over an algebraically closed field k, and let $f\colon \mathbb{P}^1 \to W$ be a morphism such that $f(0) \neq f(\infty)$. Denote by $j\colon Z := \{0, \infty\} \hookrightarrow \mathbb{P}^1$ the closed immersion of the reduced subscheme consisting of two closed points. Then*

$$\dim_{[f]} \mathbf{Hom}(\mathbb{P}^1, W; f \circ j) \geq \deg(f^* \det \Theta_W) - \dim W.$$

PROOF. In the situation of Proposition 1.2, we have

$$\mathcal{I}_Z = \mathcal{O}_{\mathbb{P}^1}(-2)$$

and

$$\det\left(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^*\Theta_W\right) = \mathcal{O}_{\mathbb{P}^1}(-2\dim W) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^* \det \Theta_W.$$

Thus by Proposition 1.2, we have

$$\begin{aligned}
\dim_{[f]} \mathbf{Hom}(\mathbb{P}^1, W; f \circ j) &\geq h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^*\Theta_W) - h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^*\Theta_W) \\
&= \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^*\Theta_W) \\
&= \deg\Big(\det\left(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^*\Theta_W\right)\Big) + (1-0)\dim W \\
&= \deg(\mathcal{O}_{\mathbb{P}^1}(-2\dim W) \otimes_{\mathcal{O}_{\mathbb{P}^1}} f^* \det \Theta_W) + \dim W \\
&= -2\dim W + \deg(f^* \det \Theta_W) + \dim W \\
&= \deg(f^*\Theta_W) - \dim W.
\end{aligned}$$

□

COROLLARY 1.4. (Mori [**81**, Proof of Theorem 5]) *Let W be a smooth quasi-projective variety over k, and C a smooth projective curve over k of genus g. Fix a closed point $P_0 \in C$ and denote by $j\colon Z = \{P_0\} \hookrightarrow C$ the closed immersion of the reduced subvariety consisting of one point. For a nonconstant morphism $f\colon C \to W$, we have*

$$\dim_{[f]} \mathbf{Hom}(C, W; f \circ j) \geq \deg(f^*\Theta_W) - g \dim W.$$

PROOF. In the situation of Proposition 1.2, we have $\mathcal{I}_Z = \mathcal{O}_C(P_0)$. Hence

$$\begin{aligned}
\dim_{[f]} \mathbf{Hom}(C, W; f \circ j) &\geq \chi(C, \mathcal{O}_C(-P_0) \otimes_{\mathcal{O}_C} f^*\Theta_W) \\
&= \deg\Big(\det\left(\mathcal{O}_C(-P_0) \otimes_{\mathcal{O}_C} f^*\Theta_W\right)\Big) + (1-g)\dim W \\
&= \deg\left(\mathcal{O}_C((-\dim W)P_0) \otimes_{\mathcal{O}_C} f^* \det \Theta_W\right) + (1-g)\dim W \\
&= -\dim W + \deg(f^* \det \Theta_W) + (1-g)\dim W \\
&= \deg(f^*\Theta_W) - g\dim W.
\end{aligned}$$

□

For simplicity, let us mean by a *curve on X* a closed irreducible reduced subscheme of X of dimension one, unless otherwise specified.

Definition 1.5. A 1-*cycle* on X is a finite linear combination

$$a_1Y_1 + a_2Y_2 + \cdots + a_mY_m$$

of curves $Y_1, \ldots, Y_m$ on X with coefficients $a_1, \ldots, a_m \in \mathbb{Z}$. It is said to be *effective* if $a_j \geq 0$ for all j. We denote by $Z_1(X)$ the free abelian group of 1-cycles on X.

Recall that for an invertible sheaf $\mathcal{L}$ on X and a closed 1-dimensional subscheme $Y \subset X$, we introduced in §VII.11 the intersection symbol

$$(\mathcal{L}.\mathcal{O}_Y) := \begin{pmatrix} \text{the coefficient of } n \text{ in the polynomial} \\ \chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_Y) \text{ in } n \text{ of degree} \leq 1, \end{pmatrix}$$

which was also denoted $(\mathcal{L}.Y)$ in §VII.12. We denote the intersection symbol for 1-cycles as above by

$$(\mathcal{L}.a_1Y_1 + \cdots + a_mY_m) := a_1(\mathcal{L}.\mathcal{O}_{Y_1}) + \cdots + a_m(\mathcal{L}.\mathcal{O}_{Y_m}).$$

More generally for a coherent $\mathcal{O}_X$-module $\mathcal{F}$ with $\dim \operatorname{Supp}(\mathcal{F}) = 1$, the intersection symbol

$$(\mathcal{L}.\mathcal{F}) := \begin{pmatrix} \text{the coefficient of } n \text{ in the polynomial} \\ \chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{F}) \text{ in } n \text{ of degree} \leq 1 \end{pmatrix}$$

was defined in §VII.11.

Definition 1.6. Invertible sheaves $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic}(X)$ are said to be *numerically equivalent* and denoted $\mathcal{L} \equiv \mathcal{L}'$ if $(\mathcal{L}.C) = (\mathcal{L}'.C)$ for any curve C on X. On the other hand, 1-cycles $Z, Z' \in Z_1(X)$ are said to be *numerically equivalent* and denoted $Z \equiv Z'$ if $(\mathcal{L}.Z) = (\mathcal{L}.Z')$ for all $\mathcal{L} \in \operatorname{Pic}(X)$.

Proposition 1.7. *The intersection number* $(\mathcal{L}.Z)$ *defines a perfect pairing*

$$(\operatorname{Pic}(X)/\equiv) \times (Z_1(X)/\equiv) \longrightarrow \mathbb{Z}$$

between the group $\operatorname{Pic}(X)/\equiv$ *of invertible sheaves modulo numerical equivalence and the group* $Z_1(X)/\equiv$ *of* 1*-cycles modulo numerical equivalence. These groups are free* $\mathbb{Z}$*-modules of finite rank* $\rho = \rho(X)$ *called the* Picard number*.*

The proof can be found in Kleiman [**70**, Chapter IV, §1, Propositions 1 and 4].

Proposition 1.8. *To every morphism* $\varphi\colon Y \to X$ *from a purely* 1*-dimensional proper scheme* Y *over* k *is associated a unique effective* 1*-cycle* $(\varphi_*Y)_{\mathrm{cycle}}$ *on* X *such that*

$$(\mathcal{L}.(\varphi_*Y)_{\mathrm{cycle}}) = (\mathcal{L}.\varphi_*\mathcal{O}_Y), \quad \textit{for any invertible sheaf } \mathcal{L} \textit{ on } X,$$

which is the coefficient of n *in the polynomial* $\chi(Y, \varphi^*\mathcal{L}^n)$ *in* n*. If* $\operatorname{Ass}(\mathcal{O}_Y) = \{y_1, \ldots, y_l\}$ *and* Y_j *is the irreducible subscheme of* Y *with the generic point* y_j *and* $\operatorname{Ass}(\mathcal{O}_{Y_j}) = \{y_j\}$ *as in the global primary decomposition (cf. Theorem* II.3.12*), then*

$$(\varphi_*Y)_{\mathrm{cycle}} = \sum_{\substack{1 \leq j \leq l \\ \dim \varphi(Y_j) = 1}} \operatorname{length}(\mathcal{O}_{Y_j,y_j})[\Bbbk(y_j) \colon \Bbbk(\varphi(y_j))]\varphi(Y_j)_{\mathrm{red}}.$$

Proof. We proceed in several steps.

(1) Suppose Y is irreducible and smooth, $Y' := \varphi(Y) \subset X$ with the reduced structure is a curve, and $\varphi\colon Y \to Y'$ is the resolution of singularities (i.e., the normalization in the function field). Then $(\varphi_*X)_{\mathrm{cycle}} := Y'$. It suffices to show

$$(\mathcal{L}.\mathcal{O}_{Y'}) = (\mathcal{L}.\varphi_*\mathcal{O}_Y) = \deg(\varphi^*\mathcal{L}).$$

Indeed, $(\mathcal{L}.\varphi_*\mathcal{O}_Y)$ is the coefficient of n in

$$\chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \varphi_*\mathcal{O}_Y) = \chi(Y, \varphi^*\mathcal{L}^n) = n\deg(\varphi^*\mathcal{L}) + \chi(\mathcal{O}_Y),$$

by the projection formula (cf. the footnote after Proposition VII.9.4) and the Riemann-Roch theorem (Theorem VIII.1.1). Since the support of the cokernel of the canonical injection $\mathcal{O}_{Y'} \to \varphi_*\mathcal{O}_Y$ is 0-dimensional, the additivity of χ (cf. Proposition VII.10.1) implies that the coefficient in question coincides with the coefficient $(\mathcal{L}.\mathcal{O}_{Y'})$ of $\chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \mathcal{O}_{Y'})$.

(2) Suppose Y is a reduced and irreducible curve over k and $Y' = \varphi(Y) \subset X$ with the reduced structure. Then

$$(\varphi_*Y)_{\text{cycle}} := \begin{cases} 0 & \text{if } \dim Y' = 0 \\ [\mathbf{R}(Y)\colon \mathbf{R}(Y')]Y' & \text{if } \dim Y' = 1. \end{cases}$$

It suffices to show that

$$(\mathcal{L}.\varphi_*\mathcal{O}_Y) = \begin{cases} 0 & \text{if } \dim Y' = 0 \\ [\mathbf{R}(Y)\colon \mathbf{R}(Y')](\mathcal{L}.\mathcal{O}_{Y'}) & \text{if } \dim Y' = 1. \end{cases}$$

Since the first case is obvious, we assume $\dim Y' = 1$. The left hand side is the coefficient of n in

$$\chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \varphi_*\mathcal{O}_Y) = \chi(Y, \varphi^*\mathcal{L}^n),$$

which is the same as the coefficient of n in $\chi(\widetilde{Y}, \sigma^*\varphi^*\mathcal{L}^n)$, where $\sigma\colon \widetilde{Y} \to Y$ is the resolution of singularities (cf. the proof of (1) above). Thus we may replace Y by $\widetilde{Y}$ and assume Y to be smooth. Then φ factors through the resolution of singularities $\pi\colon \widetilde{Y'} \to Y'$ so that $\varphi = \pi \circ \psi$ with $\psi\colon Y \to \widetilde{Y'}$. By the projection formula and the Riemann-Roch theorem for locally free sheaves (cf. Remark after Proposition VIII.1.7), we have

$$\begin{aligned}
\chi(Y, \varphi^*\mathcal{L}^n) &= \chi(\widetilde{Y'}, \psi_*\varphi^*\mathcal{L}^n) \\
&= \chi(\widetilde{Y'}, \pi^*\mathcal{L}^n \otimes_{\mathcal{O}_{\widetilde{Y'}}} \psi_*\mathcal{O}_Y) \\
&= \deg(\det(\pi^*\mathcal{L}^n \otimes_{\mathcal{O}_{Y'}} \psi_*\mathcal{O}_Y)) + [\mathbf{R}(Y)\colon \mathbf{R}(Y')]\chi(\mathcal{O}_{\widetilde{Y'}}) \\
&= n[\mathbf{R}(Y)\colon \mathbf{R}(Y')]\deg(\pi^*\mathcal{L}) + \deg(\det\psi_*\mathcal{O}_Y) + [\mathbf{R}(Y)\colon \mathbf{R}(Y')]\chi(\mathcal{O}_{\widetilde{Y'}}).
\end{aligned}$$

On the other hand, $(\mathcal{L}.\mathcal{O}_{Y'}) = (\mathcal{L}.\pi_*\mathcal{O}_{\widetilde{Y'}})$ (by (1)) is the coefficient of n in

$$\chi(\widetilde{Y'}, \pi^*\mathcal{L}^n) = n\deg(\pi^*\mathcal{L}) + \chi(\mathcal{O}_{\widetilde{Y'}}).$$

(3) Suppose Y is irreducible but not necessarily reduced with the generic point y, and $\varphi\colon Y \to Y' = \varphi(Y) \subset X$ with the reduced structure on Y'. Then

$$(\varphi_*Y)_{\text{cycle}} = \begin{cases} 0 & \text{if } \dim Y' = 0 \\ \text{length}(\mathcal{O}_{Y,y})[\mathbf{R}(Y_{\text{red}})\colon \mathbf{R}(Y')]Y' & \text{if } \dim Y' = 1. \end{cases}$$

Since the first case is obvious, we assume $\dim Y' = 1$, and show

$$(\mathcal{L}.\varphi_*\mathcal{O}_Y) = \text{length}(\mathcal{O}_{Y,y})[\mathbf{R}(Y_{\text{red}})\colon \mathbf{R}(Y')](\mathcal{L}.\mathcal{O}_{Y'}).$$

The left hand side is the coefficient of n in $\chi(Y, \varphi^*\mathcal{L}^n)$. Denote by $\mathfrak{n}$ the nilradical sheaf of $\mathcal{O}_Y$ so that $\mathcal{O}_{Y_{\text{red}}} = \mathcal{O}_Y/\mathfrak{n}$. If we denote $\nu := \text{length}(\mathcal{O}_{Y,y})$, then we have a filtration

$$\mathcal{O}_Y \supset \mathfrak{n} \supset \mathfrak{n}^2 \supset \cdots \supset \mathfrak{n}^\nu = (0).$$

By the additivity of χ, we obviously have

$$(\mathcal{L}.\varphi_*\mathcal{O}_Y) = \text{length}(\mathcal{O}_{Y,y})(\mathcal{L}.\varphi_*\mathcal{O}_{Y_{\text{red}}}),$$

hence we are done by (2).

(4) In the general case, if $\mathrm{Ass}(\mathcal{O}_Y) = \{y_1, \ldots, y_l\}$ and Y_j is the irreducible subscheme of Y with the generic point y_j and $\mathrm{Ass}(\mathcal{O}_{Y_j}) = \{y_j\}$ as in the global primary decomposition (cf. Theorem II.3.12), then we have a surjective homomorphism

$$\mathcal{O}_Y \longrightarrow \bigoplus_{j=1}^{l} \mathcal{O}_{Y_j}$$

with the support of the kernel having dimension less than 1. Thus by the additivity of χ, the coefficient $(\mathcal{L}.\varphi_*\mathcal{O}_Y)$ of n in $\chi(X, \mathcal{L}^n \otimes_{\mathcal{O}_X} \varphi_*\mathcal{O}_Y) = \chi(Y, \varphi^*\mathcal{L}^n)$ coincides with the coefficient of n in

$$\sum_{j=1}^{l} \chi(Y, (\varphi^*\mathcal{L})^n \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_j}) = \sum_{j=1}^{l} \chi(Y_j, \varphi_j^*\mathcal{L}^n),$$

where $\varphi_j \colon Y_j \to X$ is the composite of $\varphi \colon Y \to X$ with the canonical closed immersion $Y_j \to Y$. Thus

$$(\mathcal{L}.\varphi_*\mathcal{O}_Y) = \sum_{j=1}^{l} (\mathcal{L}.(\varphi_j)_*\mathcal{O}_{Y_j}),$$

and we are done by (3). $\square$

PROPOSITION 1.9 (Bend and break with a fixed point). (Kollar-Mori [**74**, Corollary 1.7], Mori [**81**, proof of Theorem 5]) *Let P be a closed point in an irreducible smooth proper curve C, and $f \colon C \to X$ a non-constant morphism. Suppose there exists a smooth connected curve T, a closed point $t_0 \in T$ and a morphism $\varphi \colon C \times T \to X$ such that*

$$\begin{aligned} \varphi|_{C \times \{t_0\}} &= f \\ \varphi(\{P\} \times T) &= f(P) \\ \varphi|_{C \times \{t\}} &\neq f \quad \textit{for general } t \in T. \end{aligned}$$

Then there exists a closed point t_1 in the smooth compactification $\overline{T}$ of T with $t_1 \notin T$, and morphisms $\pi \colon Y \to \overline{T}$ and $\psi \colon Y \to X$ such that

$$(\psi_* Y_t)_{\mathrm{cycle}} = \begin{cases} (f_*C)_{\mathrm{cycle}} & \textit{for } t = t_0 \\ (f'_*C)_{\mathrm{cycle}} + Z & \textit{for } t = t_1, \end{cases}$$

for a (possibly constant) morphism $f' \colon C \to X$, and a nonzero effective 1-cycle of rational curves with $f(P)$ contained in the support of Z, where

$$Y_t = \pi^{-1}(t) = Y \times_{\overline{T}} \mathrm{Spec}(\Bbbk(t)).$$

In particular,

$$(f_*C)_{\mathrm{cycle}} \equiv (f'_*C)_{\mathrm{cycle}} + Z.$$

PROOF. $\varphi \colon C \times T \to X$ gives rise to a rational map $\overline{\varphi} \colon C \times \overline{T} \cdots \to X$. We first claim that $\overline{\varphi}$ is not defined at (P, t_1) for some $t_1 \in \overline{T} \setminus T$. Otherwise, we would have a morphism $\overline{\varphi} \colon U \to X$ from a neighborhood U of $\{P\} \times \overline{T}$ such that $\overline{\varphi}(\{P\} \times \overline{T}) = f(P)$, hence by the rigidity lemma (cf. Remark below), we have $\overline{\varphi} = f \circ p_1$ on U with the projection $p_1 \colon C \times \overline{T} \to C$, a contradiction to the assumption.

Let $r \colon Y \to C \times \overline{T}$ be a succession of point blow ups eliminating the indeterminacy of $\overline{\varphi}$ (cf. Remark below) and giving a morphism $\psi \colon Y \to X$. Denote $\pi = p_2 \circ r \colon Y \to C \times \overline{T} \to \overline{T}$. We have $\pi^{-1}(t_0) = C \times \{t_0\}$, while

$$\pi^{-1}(t_1) = (\text{strict transform in } Y \text{ of } C \times \{t_1\}) + (\text{exceptional divisor for } r).$$

Since $\{P\} \times \overline{T}$ and $C \times \{t_1\}$ intersect transversally, the strict transform in Y of $\{P\} \times \overline{T}$ (which is mapped by ψ to the point $f(P)$) does not intersect the strict transform in Y of $C \times \{t_1\}$. Hence $f(P)$ is contained in the exceptional divisor.

Denote

$$\Psi := (\psi, \pi) \colon Y \to X \times \overline{T},$$

which is a proper $\overline{T}$-morphism. For each closed point $t \in \overline{T}$, denote by the subscript t the base change with respect to $\mathrm{Spec}(\Bbbk(t)) \to \overline{T}$. Then

$$(\Psi_* \mathcal{O}_Y)_t = \psi_* \mathcal{O}_{Y_t},$$

since for affine open sets $U \subset X$ and $t \in V \subset \overline{T}$ we have

$$\mathcal{O}_Y\left(\Psi^{-1}(U \times V)\right) \otimes_{\mathcal{O}_{V,t}} \Bbbk(t) = \mathcal{O}_{Y_t}(\psi^{-1}(U))$$

by the flatness of Y over $\overline{T}$. Hence by Proposition 1.8

$$(\mathcal{L}.(\psi_* Y_t)_{\mathrm{cycle}}) = (\mathcal{L}.\psi_* \mathcal{O}_{Y_t}) = (\mathcal{L}.(\Psi_* \mathcal{O}_Y)_t)$$

for any invertible sheaf $\mathcal{L}$ on X and for any closed point $t \in \overline{T}$. We are done by what we remarked in §VII.11, since $\Psi_* \mathcal{O}_Y$ is a coherent $\mathcal{O}_{X \times \overline{T}}$-module flat over $\overline{T}$. $\square$

Proposition 1.10 (Bend and break with two fixed points). (Kollár-Mori [**74**, Lemma 1.9], Mori [**81**, Proof of Theorem 4]) *Let $f \colon \mathbb{P}^1 \to X$ be a morphism such that $f(0) \neq f(\infty)$. Suppose there exists a smooth connected curve T, a closed point $t_0 \in T$ and a morphism $\varphi \colon \mathbb{P}^1 \times T \to X$ such that*

$$\begin{aligned} \varphi|_{\mathbb{P}^1 \times \{t_0\}} &= f \\ \varphi(\{0\} \times T) &= f(0) \\ \varphi(\{\infty\} \times T) &= f(\infty) \\ \dim \varphi(\mathbb{P}^1 \times T) &= 2. \end{aligned}$$

Then there exists a closed point t_1 in the smooth compactification $\overline{T}$ of T with $t_1 \notin T$ and morphisms $\pi \colon Y \to \overline{T}$ and $\psi \colon Y \to X$ such that

$$(\psi_* Y_t)_{\mathrm{cycle}} = \begin{cases} (f_* \mathbb{P}^1)_{\mathrm{cycle}} & \text{for } t = t_0 \\ \left(\begin{array}{l} \text{cycle with at least two rational curves} \\ \text{or with multiple rational curves} \end{array} \right) & \text{for } t = t_1. \end{cases}$$

In particular, $(f_* \mathbb{P}^1)_{\mathrm{cycle}}$ is numerically equivalent to a cycle with at least two rational curves or multiple rational curves as components.

Proof. For the proof by induction, we use a ruled surface $q \colon S \to \overline{T}$ (cf. Remark below) to compactify $p_2 \colon \mathbb{P}^1 \times T \to T$. We denote by C_0 (resp. C_∞) the section of the ruled surface S extending $\{0\} \times T$ (resp. $\{\infty\} \times T$).

$\varphi \colon \mathbb{P}^1 \times T \to X$ gives rise to a rational map $\overline{\varphi} \colon S \cdots \to X$. We first claim that $\overline{\varphi}$ is not a morphism. Otherwise, $\overline{\varphi}$ maps the sections C_0 and C_∞ to distinct points $f(0)$ and $f(\infty)$. Let $\mathcal{H}$ be an ample invertible sheaf on X. Then, since $\dim \overline{\varphi}(S) = \dim \varphi(\mathbb{P}^1 \times T) = 2$ by assumption, while $\overline{\varphi}(C_0)$ and $\overline{\varphi}(C_\infty)$ are points, we have

$$(\overline{\varphi}^* \mathcal{H})^2 > 0, \quad (\overline{\varphi}^* \mathcal{H}.C_0) = 0, \quad (\overline{\varphi}^* \mathcal{H}.C_\infty) = 0.$$

By the Hodge index theorem (cf. Remark below), $(C_0^2) < 0$ and $(C_\infty^2) < 0$ but $(C_0.C_\infty) = 0$, hence $\overline{\varphi}^* \mathcal{H}$, C_0 and C_∞ are linearly independent modulo numerical equivalence. However, the Picard number of the ruled surface S is two (cf. Remark below), a contradiction.

Let $r\colon Y \to S$ be a succession of point blow ups eliminating the indeterminacy of $\overline{\varphi}$ and giving a morphism $\psi\colon Y \to X$. Denote $\pi = q \circ r\colon Y \to S \to \overline{T}$. Our proof is by induction on the number of point blow ups appearing in r.

Let $\sigma\colon \widetilde{S} \to S$ be the blow up of $P \in S$ appearing as the first blow up in the succession r. Denote $t_1 = q(P) \in \overline{T} \setminus T$. The irreducible components of $\pi^{-1}(t_1)$ are rational curves, since they arise either as exceptional divisors or as the strict transform of $q^{-1}(t_1) = \mathbb{P}^1$. Then $\psi(\pi^{-1}(t_1))$ is a union of rational curves by Lüroth's theorem (cf. Remark below).

We are done if $\psi(\pi^{-1}(t))$ is reducible or non-reduced. Thus we assume

$$(*) \qquad \psi(\pi^{-1}(t_1)) \text{ is reduced and irreducible.}$$

We claim that $\overline{\varphi}$ is defined at the points in $q^{-1}(t_1) \setminus \{P\}$. For, if $\overline{\varphi}$ were not defined at $P' \in q^{-1}(t_1) \setminus \{P\}$, then

$$\psi(\pi^{-1}(t_1)) \supset \psi(r^{-1}(P)) \cup \psi(r^{-1}(P')),$$

a contradiction to the assumption $(*)$.

Let E (resp. F) be the exceptional curve for $\sigma\colon \widetilde{S} \to S$ (resp. the strict transform by σ of $q^{-1}(t_1)$). Thus

$$(q \circ \sigma)^{-1}(t_1) = F \cup E \qquad \text{with} \quad Q := E \cap F.$$

$\varphi\colon \mathbb{P}^1 \times T \to X$ gives rise to a rational map $\widetilde{\varphi}\colon \widetilde{S} \cdots \to X$. Denote $r = r' \circ \sigma$ with $r'\colon Y \to \widetilde{S}$ being the composite of the point blow ups other than σ.

We claim that $\widetilde{\varphi}$ is defined at $Q = E \cap F$. For otherwise, the blow up of Q would appear in the succession $r'\colon Y \to \widetilde{S}$ so that every irreducible component of $(r')^{-1}(Q)$ has multiplicity ≥ 2 in $\pi^{-1}(t_1)$, a contradiction to the assumption $(*)$.

We have $F = \mathbb{P}^1$ with $(F^2) = -1$. Hence by Castelnuovo's criterion (cf. Remark below) F can be contracted to a point by $\sigma'\colon \widetilde{S} \to S'$ giving rise to another ruled surface $q'\colon S' \to \overline{T}$ (S' is said to be obtained from S by an elementary transformation.) The resulting rational map $\varphi'\colon S' \cdots \to X$ needs one less point blow ups for the elimination of indeterminacy, since $\widetilde{\varphi}$ is defined along the exceptional divisor F of $\sigma'\colon \widetilde{S} \to S$. Thus we are done by induction.

The proof of the final assertion is exactly the same as that for Proposition 1.9. $\square$

Remark. Here are the results used in the proofs of Propositions 1.9 and 1.10 and their references:

Elimination of indeterminacy of a rational map: Although there are many variations, here is the one we need (whose proof over $\mathbb{C}$ in Part I [**87**, Chapter 8, §8B, Corollary (8.8)] works over any k as well): Let $\varphi\colon S \cdots \to \mathbb{P}^m$ be a rational map from a smooth surface over k. Then there exists a sequence

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0 = S$$

of point blow ups such that the induced rational map $S_n \cdots \to \mathbb{P}^m$ is a morphism.

Rigidity lemma: (cf. Mumford [**85**, Chapter II, §4]) Let U, V, W be varieties over k with V proper, and $\varphi\colon U \times V \to W$ a morphism such that $\varphi(u_0 \times V) =$ point for a closed point $u_0 \in U$. Then there exists a morphism $\psi\colon U \to W$ such that $\varphi(u, v) = \psi(u)$ for all closed points $u \in U$ and $v \in V$.

Lüroth's theorem: (cf., e.g., Hartshorne [**58**, Chapter IV, Example 2.5.5]) A curve C (i.e., an irreducible reduced proper scheme over k of dimension one) is a rational curve (i.e., its normalization in its function field $\mathbf{R}(C)$ is $\mathbb{P}^1$), if there exists a surjective morphism $\psi\colon \mathbb{P}^1 \to C$.

Here is a sketch of the proof. Without loss of generality, we may assume C to be smooth and show $C \cong \mathbb{P}^1$. If the finite extension $\mathbf{R}(\mathbb{P}^1) \supset \mathbf{R}(C)$ is purely inseparable, we show

ψ to be an iteration of the Frobenius morphism $F\colon C \to C^{(p)}$ as in Definition VI.1.15 (see also the proof of Theorem 1.13 below), hence $C \cong \mathbb{P}^1$. Thus we may assume the $\mathbf{R}(\mathbb{P}^1) \supset \mathbf{R}(C)$ to be a finite separable extension. Then what was discussed in §V.2–4 gives rise to an exact sequence

$$0 \longrightarrow \psi^*\Omega^1_{C/k} \longrightarrow \Omega^1_{\mathbb{P}^1/k} \longrightarrow \Omega^1_{\mathbb{P}^1/C} \longrightarrow 0$$

with $\dim \operatorname{Supp}(\Omega^1_{\mathbb{P}^1/C}) = 0$. If g is the genus of C, then by Corollary VIII.1.5

$$\begin{aligned} -2 = \deg \Omega^1_{\mathbb{P}^1/k} \geq \deg \psi^*\Omega^1_{C/k} &= [\mathbf{R}(\mathbb{P}^1)\colon \mathbf{R}(C)] \deg \Omega^1_{C/k} \\ &= [\mathbf{R}(\mathbb{P}^1)\colon \mathbf{R}(C)](2g-2), \end{aligned}$$

hence $g = 0$, and we are done by Corollary VIII.1.8.

Hodge index theorem: (cf., e.g., Mumford [**84**, Lecture 18], Hartshorne [**58**, Chapter V, Theorem 1.9]) For a smooth projective surface S, divisors and 1-cycles coincide. Hence the free $\mathbb{Z}$-module

$$\Lambda := (\operatorname{Pic}(S)/\equiv) = (Z_1(S)/\equiv) \cong \mathbb{Z}^\rho$$

is endowed with a non-degenerate symmetric bilinear pairing

$$(\quad.\quad)\colon \Lambda \times \Lambda \longrightarrow \mathbb{Z}.$$

The Hodge index theorem says that its scalar extension to $\Lambda_\mathbb{R} \cong \mathbb{R}^\rho$ has one positive eigenvalue and $\rho-1$ negative eigenvalues. More specifically, let $h = [\mathcal{H}]$ be the numerical equivalence class of an ample invertible sheaf on S, hence $(h.h) > 0$. Then

$$\Lambda_\mathbb{R} = \mathbb{R}h \oplus (\mathbb{R}h)^\perp$$

with the restriction of $(\quad.\quad)$ to $(\mathbb{R}h)^\perp$ being negative definite.

Castelnuovo's theorem: (cf., e.g., Hartshorne [**58**, Chapter V, Theorem 5.7]) Let C be a smooth rational curve on a smooth proper surface S. Then $C = \sigma^{-1}(P')$ for the blow up $\sigma\colon S \to S'$ of a smooth proper surface S' at a closed point $P' \in S'$ if and only if $(C^2) = -1$.

Ruled surfaces: (cf., e.g., Hartshorne [**58**, Chapter V, §2]) Let $\mathcal{E}$ be a locally free sheaf of rank 2 on a smooth proper curve C. Then the $\mathbb{P}^1$-bundle $S = \mathbb{P}(\mathcal{E})$ over C is called a ruled surface. Let $\pi\colon S \to C$ be the projection. By (V.2.16), we have an exact sequence

$$0 \longrightarrow \Omega^1_{S/C} \longrightarrow \mathcal{O}_S(-1) \otimes_{\mathcal{O}_S} \pi^*\mathcal{E} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

where $\mathcal{O}_S(1)$ is the tautological invertible sheaf, while by what was discussed in §V.2–4, we have an exact sequence

$$0 \longrightarrow \pi^*\Omega^1_C \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/C} \longrightarrow 0.$$

Hence we have the canonical sheaf formula

$$\mathcal{K}_S = \det(\Omega^1_S) = \mathcal{O}_S(-2) \otimes_{\mathcal{O}_S} \pi^*(\mathcal{K}_C \otimes_{\mathcal{O}_C} \det \mathcal{E}).$$

$\Lambda := (\operatorname{Pic}(S)/\equiv) = (Z_1(S)/\equiv)$ can be shown to be a free $\mathbb{Z}$-module generated by the numerical equivalence classes $[\mathcal{O}_S(1)]$ and $[f]$, where f is a fibre of π. Clearly, we have

$$(\mathcal{O}_S(1).f) = 1 \quad \text{and} \quad (f.f) = 0.$$

Let $\sigma\colon \widetilde{S} \to S$ be the blow up of a point $P \in S$. Let $f := \pi^{-1}(\pi(P))$ be the fibre of π passing through P. The total transform of f is

$$\sigma^{-1}(f) = E + F,$$

where $E = \sigma^{-1}(P)$ is the exceptional divisor and F is the strict transform of f hence $F \cong \mathbb{P}^1$. Since $(E + F.E + F) = 0$, $(E.E) = -1$ and $(E.F) = 1$, we have $(F.F) = -1$. Hence by Castelnuovo's theorem, F is obtained as the exceptional divisor of a blow up $\sigma' : \widetilde{S} \to S'$. It turns out that S' is another ruled surface over C with $\sigma'(E)$ as the fibre passing through the center of the blow up σ'. The process of obtaining S' from S is called an elementary transformation.

THEOREM 1.11. (Mori [**82**, Theorem (1.6)]) *For a smooth projective curve C of genus g and a morphism $f : C \to X$, there exists a morphism $h : C \to X$ and an effective 1-cycle Z with the properties*

(a) $(\mathcal{K}_X^{-1}.(h_*C)_{\mathrm{cycle}}) \le g \dim X$,
(b) *every irreducible component E of Z is a rational curve with* $(\mathcal{K}_X^{-1}.E) \le \dim X + 1$,
(c) $(f_*C)_{\mathrm{cycle}} \equiv (h_*C)_{\mathrm{cycle}} + Z$.

PROOF. Let $\mathcal{H}$ be a fixed ample invertible sheaf on X.
(Case $g = 0$, hence $C = \mathbb{P}^1$) We proceed by induction on $(\mathcal{H}.(f_*\mathbb{P}^1)_{\mathrm{cycle}})$.
If $(\mathcal{K}_X^{-1}.(f_*\mathbb{P}^1)_{\mathrm{cycle}}) \le \dim X + 1$, then take h to be a constant map and $Z = (f_*C)_{\mathrm{cycle}}$.
If $(\mathcal{K}_X^{-1}.(f_*\mathbb{P}^1)_{\mathrm{cycle}}) \ge \dim X + 2$, then by Corollary 1.3, $\mathbf{H} := \mathbf{Hom}(\mathbb{P}^1, X; f \circ j)$ satisfies

$$\dim_{[f]} \mathbf{H} \ge \deg(f^*\mathcal{K}_X^{-1}) - \dim X \ge 2,$$

where $j : \{0, \infty\} \hookrightarrow \mathbb{P}^1$ for any pair of distinct points $0, \infty \in \mathbb{P}^1$ with $f(0) \ne f(\infty)$. The group of automorphisms of $\mathbb{P}^1$ fixing 0 and ∞, which is the multiplicative group $\mathbb{G}_m$, has a natural action on $\mathbf{H}$ with 1-dimensional orbit $\mathbb{G}_m[f]$ through $[f]$. Since $\dim \mathbf{H} \ge 2$, there exists a curve (possibly not proper over k) $T \subset \mathbf{H}$ passing through $[f]$ and $T \not\subset \mathbb{G}_m[f]$. The embedding $T \hookrightarrow \mathbf{H}$ induces a T-morphism $\mathbb{P}^1 \times T \to X \times T$, hence its composite $\varphi : \mathbb{P}^1 \times T \to X$ with the projection $p_1 : X \times T \to X$ gives rise to the situation of the "bend and break with two fixed points" (Proposition 1.10). Hence

$$(f_*\mathbb{P}^1)_{\mathrm{cycle}} \equiv Z_1 + Z_2$$

for nonzero effective 1-cycles Z_1 and Z_2 with rational curves as components. We are done by induction, since $(\mathcal{H}.(f_*\mathbb{P}^1)_{\mathrm{cycle}}) = (\mathcal{H}.Z_1) + (\mathcal{H}.Z_2)$.

(Case $g > 0$) We proceed by induction on $(\mathcal{H}.(f_*C)_{\mathrm{cycle}})$.
If $(\mathcal{K}_X^{-1}.(f_*C)_{\mathrm{cycle}}) \le g \dim X$, then we just take $h = f$ and $Z = 0$.
If $(\mathcal{K}_X^{-1}.(f_*C)_{\mathrm{cycle}}) \ge g \dim X + 1$, then by Corollary 1.4, $\mathbf{H} := \mathbf{Hom}(C, X; f \circ j)$ satisfies

$$\dim_{[f]} \mathbf{H} \ge \deg(f^*\mathcal{K}_X^{-1}) - g \dim X \ge 1,$$

where $j : \{P_0\} \hookrightarrow C$ for a closed point $P_0 \in C$. Hence there exists a curve (possibly not proper over k) $T \subset \mathbf{H}$ passing through $[f]$. The embedding $T \hookrightarrow \mathbf{H}$ induces a T-morphism $C \times T \to X \times T$, whose composite $\varphi : C \times T \to X$ with the projection $p_1 : X \times T \to X$ gives rise the situation of the "bend and break with a fixed point" (Proposition 1.9). Hence

$$(f_*C)_{\mathrm{cycle}} \equiv (f'_*C)_{\mathrm{cycle}} + Z',$$

where $f' : C \to X$ is a morphism and Z' is a nonzero effective 1-cycle with rational curves as components. We are done by induction because of $(\mathcal{H}.(f'_*C)_{\mathrm{cycle}}) < (\mathcal{H}.(f_*C)_{\mathrm{cycle}})$ as well as by our earlier result in the case $g = 0$ applied to each component of Z'. $\square$

DEFINITION 1.12. We denote

$$N_1(X) := (Z_1(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R},$$

which is an $\mathbb{R}$-vector space of dimension ρ, the Picard number of X (cf. Proposition 1.7). For a curve $C \subset X$, we denote its image in $N_1(X)$ by $[C]$.

THEOREM 1.13. (Mori [**82**, Theorem (1.4), (1.4.1)]) *Suppose k is of characteristic $p > 0$. For a fixed ample invertible sheaf $\mathcal{L}$ on X and $\varepsilon > 0$, there exist rational curves $l_1, \dots, l_r$ (possibly $r \geq 0$) on X satisfying*

$$(\mathcal{K}_X^{-1}.l_i) \leq \dim X + 1 \quad \textit{for} \quad i = 1, \dots, r$$

such that for any effective 1-cycle Z, its numerical equivalence class satisfies

$$[Z] \in \sum_{i=1}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \textit{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}, y) \right\},$$

where $\mathbb{Q}_{\geq 0}$ denotes the set of nonnegative rational numbers.

PROOF. Let

$$\Phi := \left\{ l \;\middle|\; \begin{array}{l} \text{rational curves on } X \\ \varepsilon(\mathcal{L}.l) < (\mathcal{K}_X^{-1}.l) \leq \dim X + 1 \end{array} \right\}.$$

Since $(\mathcal{L}.l) < (\dim X + 1)/\varepsilon$ for $l \in \Phi$, the Hilbert polynomials of $l \in \Phi$ with respect to $\mathcal{L}$ have only a finite number of possibilities. (Φ is a so-called bounded family.) Hence by Grothendieck's decomposition (cf. FGA [**2**, Exposé 221]) of the Hilbert scheme Hilb_X into the components Hilb_X^P corresponding to Hilbert polynomials P, we see that the points of Hilb_X corresponding to $l \in \Phi$ belong to the union of a finite number of those components Hilb_X^P, which are projective. Thus by the invariance of intersection numbers under connected flat family of curves (possibly reducible or non-reduced) on X as in the proofs of Propositions 1.9 and 1.10, we see that Φ modulo numerical equivalence is a finite set. We claim that a complete set $\{l_1, \dots, l_r\}$ of representatives of Φ up to numerical equivalence satisfies the requirements. It certainly suffices to show that for any smooth projective curve C of genus g and a morphism $f: C \to X$,

$$[(f_*C)_{\mathrm{cycle}}] \in \sum_{i=1}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}.$$

(Case $g = 0$) By Theorem 1.11, C is numerically equivalent to an effective 1-cycle whose components E are rational curves such that $(\mathcal{K}_X^{-1}.E) \leq \dim X + 1$. We are done, since either E is in Φ or satisfies $(\mathcal{K}_X^{-1}.E) \leq \varepsilon(\mathcal{L}.E)$.

(Case $g > 0$) Denote by $\sigma: k \to k$ the p-th power automorphism. As in §IV.3 denote by $\phi: C \to C$ the p-th power morphism defined by

a) set-theoretically, $\phi =$ identity,
b) $\forall U$ and $\forall a \in \mathcal{O}_C(U)$, define $\phi^* a = a^p$.

ϕ is not a k-morphism, but the *Frobenius morphism*

$$C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\sigma^{-1}, k) \longrightarrow C$$

induced by ϕ is a k-morphism, where $\mathrm{Spec}(\sigma^{-1}, k)$ is the k-scheme for the k-algebra (σ^{-1}, k), which is k regarded as a k-algebra through σ^{-1}. Likewise, we define

$$C_j := C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\sigma^{-j}, k).$$

The p-th power morphism ϕ induces a k-morphism $\pi_j: C_j \to C_{j-1}$ for each $j > 0$, with $C_0 = C$. Starting from $f_0 = f$, we inductively find a k-morphism

$$f_j: C_j \to X, \qquad \text{with} \quad D_j := (f_{j*}C_j)_{\mathrm{cycle}}$$

such that

$$(\mathcal{K}_X^{-1}.D_{j+1}) \leq g \dim X$$

$$p[D_j] - [D_{j+1}] \in \sum_{i=0}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}$$

for all $j \geq 0$. Indeed, applying Theorem 1.11 to $f_j \circ \pi_{j+1} \colon C_{j+1} \to C_j \to X$, we get a morphism $f_{j+1} \colon C_{j+1} \to X$ such that

$$(\mathcal{K}_X^{-1}.D_{j+1}) \leq g \dim X \qquad \text{where } D_{j+1} := (f_{j+1*}C_{j+1})_{\text{cycle}}$$

$$((f_j \circ \pi_{j+1})_* C_{j+1})_{\text{cycle}} = D_{j+1} + Z,$$

where Z is an effective 1-cycle whose irreducible components E are rational curves satisfying $(\mathcal{K}^{-1}.E) \leq \dim X + 1$. However,

$$((f_j \circ \pi_{j+1})_* C_{j+1})_{\text{cycle}} = pD_j,$$

since $(\pi_{j+1*}C_{j+1})_{\text{cycle}} = pC_j$. Hence

$$p[D_j] - [D_{j+1}] = [Z] \in \sum_{i=1}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}$$

by what we have seen in Case $g = 0$ above. Thus for any positive integer a, we obviously have

$$\begin{aligned}[D_0] - p^{-a}[D_a] &= \sum_{j=0}^{a-1} p^{-j-1}(p[D_j] - [D_{j+1}]) \\ &\in \sum_{i=1}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}.\end{aligned}$$

Thus it suffice to show that $[D_a]$ belongs to the right hand side for some $a > 0$, since $[D_0] = [C]$.

We are done if

$$[D_a] \in \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}$$

for some a. Otherwise, we have

$$(\mathcal{L}.D_j) < \frac{(\mathcal{K}_X^{-1}.D_j)}{\varepsilon} \leq \frac{g \dim X}{\varepsilon}, \qquad \text{for any } j.$$

Thus $(\mathcal{L}.D_j)$'s are uniformly bounded above. Hence by the same argument as before on the finiteness of such bounded family of 1-cycles modulo numerical equivalence, there exist $b < c$ such that $[D_b] \equiv [D_c]$. Then

$$\begin{aligned}(p^{c-b} - 1)[D_b] &= p^{c-b}[D_b] - [D_c] \\ &= \sum_{j=b}^{c-1} p^{c-1-j}(p[D_j] - [D_{j+1}]) \\ &\in \sum_{i=1}^{r} \mathbb{Q}_{\geq 0}[l_i] + \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\},\end{aligned}$$

and we are done. □

Finally we are ready to prove Mori's theorem on the existence of rational curves. *It is noteworthy that the proof of the result in characteristic zero requires those in positive characteristics.*

PROOF OF THEOREM 1.1.
(Positive characteristic cases $\Longrightarrow$ characteristic zero case) First of all, suppose we have proven the assertion when the characteristic of k is positive. Then the assertion holds in characteristic zero as well. Here is why:

As in Proposition IV.1.4, we can find a subring $R \subset k$ finitely generated over $\mathbb{Z}$ and a smooth projective morphism $\pi\colon \mathfrak{X} \to S = \operatorname{Spec}(k)$, a π-ample invertible sheaf $\mathfrak{H}$ on $\mathfrak{X}$ and a closed subscheme $\mathfrak{C} \subset \mathfrak{X}$ flat over S such that their base extensions by $\operatorname{Spec}(k) \to S$ are X, $\mathcal{H}$ and C, respectively. Indeed, we could express X as a closed subscheme of a projective space over k defined by a finite set of homogeneous equations. We obtain a subring $R \subset k$ generated over $\mathbb{Z}$ by these coefficients as well as the coefficients of those homogeneous polynomials needed to define $\mathcal{H}$ and C. Thus we obtain a projective scheme $\mathfrak{X}$ over R, an invertible sheaf $\mathfrak{H}$ on $\mathfrak{X}$ and a closed subscheme $\mathfrak{C}$ whose base extensions by $\operatorname{Spec}(k) \to \operatorname{Spec}(R)$ are X, $\mathcal{H}$ and C, respectively. We then replace R by an appropriate localization to guarantee that $\mathfrak{X}$ is smooth over R and $\mathfrak{C}$ flat over R.

By Grothendieck's theorem (Theorem I.8.14), there exists a locally noetherian S-scheme

$$\mathbf{Hom}_S(\mathbb{P}^1_S, \mathfrak{X})$$

representing the functor

$$\text{locally noetherian } S\text{-schemes } T \longmapsto \operatorname{Hom}_T(\mathbb{P}^1_T, \mathfrak{X} \times_S T).$$

In fact, it is obtained as an open subscheme of the Hilbert scheme

$$\operatorname{Hilb}_{(\mathbb{P}^1_S \times_S \mathfrak{X})/S}$$

parametrizing flat families of subschemes of $\mathbb{P}^1_S \times_S \mathfrak{X}$: a T-morphism $\mathbb{P}^1_T \to \mathfrak{X} \times_S T$ is dealt with as its graph in $\mathbb{P}^1_T \times_T (\mathfrak{X} \times_S T)$.

Denote by $\Xi \subset \mathbf{Hom}_S(\mathbb{P}^1_S, \mathfrak{X})$ the subscheme such that for any morphism $\operatorname{Spec}(K) \to S$ with an algebraically closed field K, the base extention Ξ_K parametrizes K-morphisms $f\colon \mathbb{P}^1_K \to \mathfrak{X}_K$ such that

$$\dim X + 1 \geq \deg(f^*\mathcal{K}_{\mathfrak{X}_K}^{-1}), \qquad \text{and} \qquad \frac{\deg(f^*\mathcal{K}_{\mathfrak{X}_K}^{-1})}{\deg(f^*\mathfrak{H}_K)} \geq \frac{(\mathcal{K}_{\mathfrak{X}_K}^{-1}.\mathfrak{C}_K)}{(\mathfrak{H}_K.\mathfrak{C}_K)} = \frac{(\mathcal{K}_X^{-1}.C)}{(\mathcal{H}.C)}.$$

In particular, we have

$$(\dim X + 1)\frac{(\mathcal{H}.C)}{(\mathcal{K}_X^{-1}.C)} \geq \deg(f^*\mathfrak{H}_K) > 0.$$

We claim that Ξ is of finite type, hence quasi-projective, over S. Indeed, since the left hand side of the above inequality is a constant, $\deg(f^*\mathfrak{H}_K)$ can take only a finite number of positive integral values. Consequently, the Hilbert polynomial of the graph Γ_f of f with respect to the ample invertible sheaf $\mathcal{O}_{\mathbb{P}^1_K}(1) \otimes_K \mathfrak{H}_K$ on $\mathbb{P}^1_K \times_K \mathfrak{X}_K$ can have only a finite number of possibilities. (Γ_f's form a so-called bounded family.) Hence by Grothendieck's decomposition of Hilb into components corresponding to Hilbert polynomials (cf. FGA [**2**, Exposé 221]), we see that Ξ is contained in the union of a finite number of these components, which are projective over S.

Denote by $\xi\colon \Xi \to S = \operatorname{Spec}(R)$ the structure morphism. Then the fibre of ξ over $\operatorname{Spec}(K) \to S$ for an algebraically closed field K consists of K-valued points $f\colon \mathbb{P}^1_K \to \mathfrak{X}_K$ of Ξ_K, which gives rise to a possibly multiple rational curve $(f_*\mathbb{P}^1_K)_{\text{cycle}} = [\mathbf{R}(\mathbb{P}^1_K)\colon \mathbf{R}(f(\mathbb{P}^1_K))]f(\mathbb{P}^1_K) \subset \mathfrak{X}_K$

satisfying

$$\dim X + 1 \geq (\mathcal{K}_{\mathfrak{X}_K}^{-1}.(f_*\mathbb{P}^1_K)_{\text{cycle}}) \qquad \text{and} \qquad \frac{(\mathcal{K}_{\mathfrak{X}_K}^{-1}.(f_*\mathbb{P}^1_K)_{\text{cycle}})}{(\mathfrak{H}_K.(f_*\mathbb{P}^1_K)_{\text{cycle}})} \geq \frac{(\mathcal{K}_{\mathfrak{X}_K}^{-1}.\mathfrak{C}_K)}{(\mathfrak{H}_K.\mathfrak{C}_K)}.$$

We now claim $\xi\colon \Xi \to S = \operatorname{Spec}(R)$ to be surjective. Indeed, if s is a closed point of S, then the residue field $\Bbbk(s)$ is of positive characteristic, since R is finitely generated over $\mathbb{Z}$. Hence the geometric fibre of ξ over s is non-empty, since we can take $f\colon \mathbb{P}^1_K \to E \subset \mathfrak{X}_K$ to be the resolution of singularities of a rational curve E satisfying the required inequalities for the algebraic closure K of $\Bbbk(s)$. Thus the image of ξ contains all the closed points of S. By Chevalley's Nullstellensatz (Theorem II.2.9), the image thus contains the generic point of S, hence ξ is surjective.

Consequently, the fibre of ξ over the generic geometric point $\operatorname{Spec}(k) \to S$ is non-empty. Thus there exists a possibly multiple rational curve $(f_*\mathbb{P}^1_k)_{\text{cycle}}$ on $\mathfrak{X}_k$ for some $f\colon \mathbb{P}^1_k \to \mathfrak{X}_k = X$ satisfying the required inequalities. Obviously, we are done.

(Characteristic $p > 0$) Let $\mathcal{L}$ be another ample invertible sheaf on X and let $\varepsilon = 1/N$ for a large enough positive integer N such that

$$\mathcal{H}^{\otimes N} \otimes_{\mathcal{O}_X} \mathcal{L}^{-2(\mathcal{H}.C)} \quad \text{is ample.}$$

Then by Theorem 1.13, there exist rational curves $l_1, \ldots, l_r$ with

$$(\mathcal{K}_X^{-1}.l_i) \leq \dim X + 1, \quad \text{for } i = 1, \ldots, r$$

such that

$$[C] = \sum_{i=1}^{r} a_i [l_i] + z$$

for $a_1, \ldots, a_r \in \mathbb{Q}_{\geq 0}$ and

$$z \in \left\{ y \in \sum_{\substack{\Gamma \subset X \\ \text{curves}}} \mathbb{Q}_{\geq 0}[\Gamma] \;\middle|\; (\mathcal{K}_X^{-1}.y) \leq \varepsilon(\mathcal{L}.y) \right\}.$$

By the choice of $\varepsilon = 1/N$, we have

$$0 \leq \frac{(\mathcal{H}^{\otimes N} \otimes_{\mathcal{O}_X} \mathcal{L}^{-2(\mathcal{H}.C)}.z)}{N} = (\mathcal{H}.z) - 2\varepsilon(\mathcal{H}.c)(\mathcal{L}.z).$$

Hence

$$(\mathcal{K}_X^{-1}.z) \leq \varepsilon(\mathcal{L}.z) \leq \frac{(\mathcal{H}.z)}{2(\mathcal{H}.C)}.$$

Consequently,

$$\begin{aligned}
\frac{(\mathcal{K}_X^{-1}.C)}{(\mathcal{H}.C)} &= \frac{\sum_i a_i(\mathcal{K}_X^{-1}.l_i) + (\mathcal{K}_X^{-1}.z)}{\sum_i a_i(\mathcal{H}.l_i) + (\mathcal{H}.z)} \\
&\leq \frac{\sum_i a_i(\mathcal{K}_X^{-1}.l_i) + \frac{(\mathcal{H}.z)}{2(\mathcal{H}.C)}}{\sum_i a_i(\mathcal{H}.l_i) + (\mathcal{H}.z)} \\
&\leq \max\left\{ \max_i \frac{(\mathcal{K}_X^{-1}.l_i)}{(\mathcal{H}.l_1)}, \frac{1}{2(\mathcal{H}.C)} \right\}.
\end{aligned}$$

Since $(\mathcal{K}_X^{-1}.C) \geq 1$, by assumption, we thus have

$$\frac{(\mathcal{K}_X^{-1}.C)}{(\mathcal{H}.C)} \leq \max_i \frac{(\mathcal{K}_X^{-1}.l_i)}{(\mathcal{H}.l_i)}.$$

Hence there exists i such that

$$\frac{(\mathcal{K}_X^{-1}.C)}{(\mathcal{H}.C)} \leq \frac{(\mathcal{K}_X^{-1}.l_i)}{(\mathcal{H}.l_i)}.$$

□

Definition 1.14. Denote by $\mathbb{R}_{\geq 0}$ the semigroup of nonnegative real numbers, and let

$$\begin{aligned} NE(X) &:= \sum_{\substack{C \subset X \\ \text{curves}}} \mathbb{R}_{\geq 0}[C] \\ &= \{\text{finite nonnegative linear combinations of } [C]\text{'s for curves } C \subset X\}. \end{aligned}$$

The *Kleiman-Mori cone* $\overline{NE}(X)$ is defined to be the closure of $NE(X)$ in the usual topology of $N_1(X) \cong \mathbb{R}^\rho$.

$\overline{NE}(X)$ plays pivotal roles in birational geometry. The following theorem implies that $\overline{NE}(X)$ is a "strictly convex cone", that is, it does not contain a straight line.

Theorem 1.15 (Kleiman's criterion for ampleness). (Kleiman [**70**, Chapter IV, Theorem 2 and Proposition 2]) *An invertible sheaf $\mathcal{L}$ on X is ample if and only if*

$$(\mathcal{L}.z) > 0, \qquad \forall z \in \overline{NE}(X) \setminus \{0\}.$$

This theorem holds even if X is a singular projective variety. The Nakai-Moishezon criterion (Theorem VII.12.4) plays a crucial role in the proof. (See also Kollár-Mori [**74**, Theorem 1.18].)

Mori further formulated the existence of rational curves in the following Cone Theorem, which gives an entirely new perspective even to classical results on nonsingular surfaces including Castelnuovo's theorem (cf. Remark immediately before Theorem 1.11) as Mori explains in [**82**, Chapter 2].

Theorem 1.16 (The cone theorem). (Mori [**82**, Theorem (1.4)]. See also Kollár-Mori [**74**, Theorem 1.24]) *For any ample invertible sheaf $\mathcal{H}$ and any $0 < \varepsilon \in \mathbb{R}$, there exist rational curves $l_1, \ldots, l_r$ (possibly $r = 0$) on X satisfying*

$$0 < (\mathcal{K}_X^{-1}.l_i) \leq \dim X + 1, \qquad i = 1, \ldots, r$$

such that

$$\overline{NE}(X) = \mathbb{R}_{\geq 0}[l_1] + \cdots + \mathbb{R}_{\geq 0}[l_r] + \{z \in \overline{NE}(X) \mid (\mathcal{K}_X^{-1}.z) \leq \varepsilon(\mathcal{H}.z)\}.$$

This means that the part

$$\{z \in \overline{NE}(X) \mid (\mathcal{K}_X^{-1}.z) \geq \varepsilon(\mathcal{H}.z)\}$$

of $\overline{NE}(X)$, if non-empty, is a polyhedral cone spanned by a finite number of *extremal rays* $\mathbb{R}_{\geq 0}[l]$ for *extremal rational curves* l, i.e., rational curves l satisfying

$$0 < (\mathcal{K}_X^{-1}.l) \leq \dim X + 1.$$

For the proof, the reader is referred to Mori [**82**, pp.139–140].

For the Minimal Model Program (also called the Mori Program) in higher dimension, however, we need to prove an analog for X with "terminal singularities", since the contraction of an extremal ray may give rise to varieties with such singularities. In fact, it is essential to prove an analog even for "projective pairs" (X, Δ) with "Kawamata log terminal" singularities. "Relativization" is crucial as well. Entirely different methods are needed in these general cases. See Kawamata [**68**] and Kollár [**73**]. See also Kawamata-Matsuda-Matsuki [**67**] and Kollár-Mori [**74**, Chapter 3, especially Theorem 3.7], for instance.

2. Belyi's three point theorem

(Added in publication)

The following result is due to Belyi [**22**], [**23**]:

THEOREM 2.1 (Belyi's three point theorem). *Let C be an irreducible proper smooth curve over $\mathbb{C}$. Then C is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers (that is, $C = C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$ for some C_0 over $\overline{\mathbb{Q}}$) if and only if it can be represented as a covering of the projective line $\mathbb{P}^1_{\mathbb{C}}$ branched only at 0, 1, ∞.*

Let C and C' be irreducible curves proper and smooth over an algebraically closed field k, and $f: C \to C'$ a finite surjective separable morphism. The *ramification locus* of f is the finite set of closed points of C at which f is not étale, and coincides with $\mathrm{Supp}(\Omega_{C/C'})$ by Definition V.3.1 and Criterion V.4.1.

$$\Delta(f) = f(\mathrm{Supp}(\Omega_{C/C'}))$$

is called the *branch locus* of f.

REMARK. (Added in Publication) This result is closely related to "dessins d'enfants" introduced by Grothendieck [**47**]. See, for instance, Luminy Proceedings [**50**].

PROOF OF THE "ONLY IF" PART OF THEOREM 2.1. We show that if C is an irreducible curve proper and smooth over $\overline{\mathbb{Q}}$, then there exists a finite surjective morphism $f: C \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ such that $\Delta(f)(\overline{\mathbb{Q}}) \subset \{0, 1, \infty\}$.

Since the function field $\mathbf{R}(C)$ is an extension of $\overline{\mathbb{Q}}$ of transcendence degree 1, choose $f_0 \in \mathbf{R}(C) \setminus \overline{\mathbb{Q}}$, which gives a finite surjective morphism

$$f_0: C \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}.$$

Without loss of generality, we may assume $\Delta(f_0)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\overline{\mathbb{Q}})$ contains ∞.

We now show the existence of a non-constant polynomial $g(t) \in \mathbb{Q}[t]$ such that the composite morphism

$$g \circ f_0: C \xrightarrow{f_0} \mathbb{P}^1_{\overline{\mathbb{Q}}} \xrightarrow{g} \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

satisfies $\Delta(g \circ f_0)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\mathbb{Q})$ by induction on

$$\delta(f_0) = \sum_{y \in \Delta(f_0)(\overline{\mathbb{Q}})} ([\Bbbk(y) : \mathbb{Q}] - 1).$$

There is nothing to prove if $\delta(f_0) = 0$. If $\delta(f_0) > 0$, choose $y_1 \in \Delta(f_0)(\overline{\mathbb{Q}})$ with $n = [\Bbbk(y_1) : \mathbb{Q}] > 1$. Let $g_1(t)$ be the minimal polynomial over $\mathbb{Q}$ of y_1. We then replace f_0 with $g_1 \circ f_0$. Since $(g_1 \circ f_0)' = ((g_1)' \circ f_0) \cdot f_0'$, the new morphism is ramified where g_1 or f_0 is. But since $g_1(y_1) = 0$, y_1 no longer adds to $\delta(g_1 \circ f_0)$, decreasing it by $n-1$. But the sum of the degrees of the zeros of g_1' is $n-1$, so $\delta(g_1) \le n-2$ and we have $\delta(g_1 \circ f_0) < \delta(f_0)$.

Thus it suffices to show the following:

LEMMA 2.2 (Belyi). *If $f_1: \mathbb{P}^1_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is a finite surjective morphism with $\Delta(f_1)(\overline{\mathbb{Q}}) \subset \mathbb{P}^1(\mathbb{Q})$, then there exists a finite surjective morphism*

$$h: \mathbb{P}^1_{\overline{\mathbb{Q}}} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

such that $\Delta(h \circ f_1)(\overline{\mathbb{Q}}) \subset \{0, 1, \infty\}$.

The first proof of Lemma 2.2. We prove the existence of $h(t) \in \mathbb{Q}[t]$ by induction on the cardinality $\#\Delta(f_1)(\overline{\mathbb{Q}})$.

If $\#\Delta(f_1)(\overline{\mathbb{Q}}) \leq 3$, we choose h to be a linear fractional transformation with coefficients in $\mathbb{Q}$ that sends $\Delta(f_1)(\overline{\mathbb{Q}})$ to $\{0, 1, \infty\}$.

If $\#\Delta(f_1)(\overline{\mathbb{Q}}) > 3$, we may choose a suitable linear fractional transformation with coefficients in $\mathbb{Q}$ and assume that

$$\Delta(f_1)(\overline{\mathbb{Q}}) \supset \{0, 1, \frac{n}{m+n}, \infty\}$$

for positive integers m, n. Let

$$h(t) = \frac{(m+n)^{m+n}}{m^m n^n} t^m (1-t)^n \in \mathbb{Q}[t],$$

which gives a morphism $h \colon \mathbb{P}^1_{\mathbb{Q}} \longrightarrow \mathbb{P}^1_{\mathbb{Q}}$ with

$$\begin{aligned} h(0) &= 0 \\ h(1) &= 0 \\ h(\frac{n}{m+n}) &= 1. \end{aligned}$$

Thus we have $\#\Delta(h \circ f_1)(\overline{\mathbb{Q}}) < \#\Delta(f_1)(\overline{\mathbb{Q}})$. □

The second proof of Lemma 2.2. By linear fractional transformation with coefficients in $\mathbb{Q}$ we may assume

$$\Delta(f_1)(\overline{\mathbb{Q}}) = \{\lambda_1, \dots, \lambda_n, \infty\} \subset \mathbb{P}^1(\mathbb{Q})$$

with $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ such that

$$0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n, \qquad \gcd(\lambda_2, \dots, \lambda_n) = 1.$$

Denote the Vandermonde determinant by

$$w = W(\lambda_1, \dots, \lambda_n) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j>l} (\lambda_j - \lambda_l).$$

Similarly, denote the Vandermonde determinant for each $i = 1, \dots, n$ by

$$w_i = (-1)^{n-i} W(\lambda_1, \dots, \widehat{\lambda_i}, \dots, \lambda_n) = (-1)^{n-i} \prod_{\substack{j>l \\ j,l \neq i}} (\lambda_j - \lambda_l),$$

where $\widehat{\lambda_i}$ means λ_i deleted. It is easy to check that

$$\begin{aligned} \sum_{i=1}^{n} \frac{w_i}{t - \lambda_i} &= \frac{w}{\prod_{i=1}^{n}(t - \lambda_i)} \\ \sum_{i=1}^{n} w_i &= 0 \\ \sum_{i=1}^{n} \lambda_i^{n-1} w_i &= w. \end{aligned}$$

Let $r_i = w_i / \gcd(w_1, \dots, w_n) \in \mathbb{Z}$ and

$$h(t) = \prod_{i=1}^{n} (t - \lambda_i)^{r_i} \in \mathbb{Q}(t).$$

Note that $\sum_{i=1}^n r_i = 0$. Since

$$\frac{h'(t)}{h(t)} = \sum_{i=1}^n \frac{r_i}{t-\lambda_i} = \frac{w/\gcd(w_1,\ldots,w_n)}{\prod_{i=1}^n (t-\lambda_i)},$$

the ramification locus of $h\colon \mathbb{P}^1_{\overline{\mathbb{Q}}} \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is contained in $\{\lambda_1,\ldots,\lambda_n,\infty\}$, while $\Delta(h)(\overline{\mathbb{Q}}) \subset \{0,1,\infty\}$. We see that

$$\begin{aligned} h(\lambda_i) &= 0, \qquad n-i \text{ even} \\ h(\lambda_i) &= \infty, \qquad n-i \text{ odd} \\ h(\infty) &= 1 \\ \Delta(h)(\overline{\mathbb{Q}}) &= \{0,1,\infty\}. \end{aligned}$$

Then the composite

$$\mathbb{P}^1_{\overline{\mathbb{Q}}} \xrightarrow{f_1} \mathbb{P}^1_{\overline{\mathbb{Q}}} \xrightarrow{h} \mathbb{P}^1_{\overline{\mathbb{Q}}}$$

has the property $\Delta(h\circ f_1)(\overline{\mathbb{Q}}) \subset \{0,1,\infty\}$. □

□

Proof of the "if" part of Theorem 2.1. We show that if $g'\colon C \to \mathbb{P}^1_{\mathbb{C}}$ is a finite covering with $\Delta(g')(\mathbb{C}) \subset \{0,1,\infty\}$, then there exists a curve C_0 over $\overline{\mathbb{Q}}$ such that $C = C_0 \times_{\mathrm{Spec}(\overline{\mathcal{Q}})} \mathrm{Spec}(\mathbb{C}) = (C_0)_{\mathbb{C}}$.

Here is what we are going to do: We construct a "deformation" $f\colon X \to S$ of C parametrized by an irreducible affine smooth variety S over $\overline{\mathbb{Q}}$. Then the fibre over a $\overline{\mathbb{Q}}$-rational point $s_0 \in S$ turns out to be C_0 we are looking for.

Since C is projective (cf. Proposition V.5.11), we have a closed immersion $C \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$. In view of the covering $g'\colon C \to \mathbb{P}^1_{\mathbb{C}}$ and the Segre embedding (cf. Example I.8.11 and Proposition II.1.2), we have closed immersions

$$C \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^{2N+1}_{\mathbb{C}}.$$

Using an idea similar to that in the proof of Proposition IV.1.4, we have a subring $R \subset \mathbb{C}$ generated over $\overline{\mathbb{Q}}$ by the coefficients of the finite number of homogeneous equations defining C as well as $\mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^{2N+1}_{\mathbb{C}}$ and a scheme X of finite type over R with closed immersions

$$X \hookrightarrow \mathbb{P}^N_R \times_{\mathrm{Spec}(R)} \mathbb{P}^1_R \hookrightarrow \mathbb{P}^{2N+1}_R$$

such that the base extension by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(R)$ gives rise to

$$C \hookrightarrow \mathbb{P}^N_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{P}^1_{\mathbb{C}} \hookrightarrow \mathbb{P}^{2N+1}_{\mathbb{C}}.$$

$S = \mathrm{Spec}(R)$ is an integral scheme of finite type over $\overline{\mathbb{Q}}$. Replacing S by a suitable non-empty affine open subset, we may assume S to be smooth over $\overline{\mathbb{Q}}$. Moreover, S is endowed with a fixed $\mathbb{C}$-valued point $\mathrm{Spec}(\mathbb{C}) \to S$. Denote the structure morphism of X by $f\colon X \to S$. By construction, we have a factorization

$$f\colon X \xrightarrow{g} \mathbb{P}^1_S \longrightarrow S$$

with f and g projective. Moreover, the base extension by $\mathrm{Spec}(\mathbb{C}) \to S$ gives rise to

$$C = X_{\mathbb{C}} \xrightarrow{g'} \mathbb{P}^1_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C}).$$

We now show the following:

LEMMA 2.3. *There exists a non-empty affine open subset $S_0 \subset S$ such that the restriction*

$$f\colon X_0 = f^{-1}(S_0) \xrightarrow{g} \mathbb{P}^1_{S_0} \longrightarrow S_0$$

to S_0 satisfies the following conditions:

i) *X_0 is integral.*
ii) *$f\colon X_0 \to S_0$ is surjective and smooth of relative dimension* 1.
iii) *$g\colon X_0 \to \mathbb{P}^1_{S_0}$ is étale outside $\{0,1,\infty\} \times S_0$. We may further assume*

$$f(X_0 \setminus (\{0,1,\infty\} \times S_0)) = S_0.$$

PROOF OF LEMMA 2.3.

Proof of (i): Let K be the function field of S so that K is the field of fractions of R and is a subfield of $\mathbb{C}$. The fibre of f over the generic point η_S of S is $f^{-1}(\eta_S) = X_K$, whose base extension by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(K)$ gives rise to the original curve $C = X_{\mathbb{C}}$. Hence X_K is integral.

Let $X = \bigcup_i X_i$ be the irreducible decomposition with the generic point η_i of X_i for each i. Let U_i's be mutually disjoint neighborhoods of η_i. Since $X_K = f^{-1}(\eta_S)$ is irreducible, at most one U_i intersects $f^{-1}(\eta_S)$. If none of the η_i's were in $f^{-1}(\eta_S)$, then for each i we would have $f(\eta_i) \neq \eta_S$ so that $\eta_S \notin \overline{f(\eta_i)}$ and $f^{-1}(\eta_S) \cap f^{-1}(\overline{f(\eta_i)}) = \emptyset$. Since closed $f^{-1}(\overline{f(\eta_i)})$ contains X_i, we would have $f^{-1}(\eta_S) \cap X_i = \emptyset$ for all i, a contradiction. Thus there exists exactly one i such that $\eta_i \in f^{-1}(\eta_S)$. Hence $f^{-1}(\eta_S) \subset X_i$ and $\eta_S \notin f(X \setminus X_i)$. By Chevalley's Nullstellensatz (cf. Theorem II.2.9) $f(X \setminus X_i)$ is constructible. Thus there exists an open neighborhood S_0 of η_S with $S_0 \cap f(X \setminus X_i) = \emptyset$. Hence $f^{-1}(S_0) \cap (X \setminus X_i) = \emptyset$ so that $f^{-1}(S_0) \subset X_i$ is irreducible. Obviously, we may replace S_0 by a non-empty affine open subset.

Let us replace S and X by this S_0 and $f^{-1}(S_0)$, respectively so that we may now assume X to be irreducible.

We next show that there exists a non-empty affine open subset $\mathrm{Spec}(R_t) \subset S = \mathrm{Spec}(R)$ for some $t \in R$ such that $f^{-1}(\mathrm{Spec}(R_t))$ is reduced. Indeed, let $X = \bigcup_i \mathrm{Spec}(A_i)$ be a finite affine open covering. Since $X_K = f^{-1}(\eta_S)$ is reduced, $A_i \otimes_R K$ is reduced for all i. Obviously, there exists a non-zero divisor $t_i \in R$ such that $A_i \otimes_R R_{t_i}$ is reduced. Letting $t = \prod_i t_i$, we see that $X \times_S \mathrm{Spec}(R_t) = f^{-1}(\mathrm{Spec}(R_t))$ is reduced.

Proof of (ii): Let us replace S and X by $\mathrm{Spec}(R_t)$ and $f^{-1}(\mathrm{Spec}(R_t))$ in (i), respectively so that we may assume X to be integral with the generic point η_X of X mapped by f to η_S.

Since $X_{\mathbb{C}} = C$ is smooth of relative dimension 1 over $\mathbb{C}$, so is X_K smooth of relative dimension 1 over K. By what we saw in §V.3, the stalks of $\Omega_{X/S}$ at points $x \in f^{-1}(\eta_S)$ are locally free of rank 1. Thus we find an open neighborhood U of $f^{-1}(\eta_S)$ such that $f\colon U \to S$ is smooth of relative dimension 1. Since f is projective, $f(X \setminus U)$ is closed and does not contain η_S. Hence $S_0 = S \setminus f(X \setminus U)$ is an open neighborhood of η_S such that $f^{-1}(S_0) \to S_0$ is smooth of relative dimension 1.

Replacing S and X by this S_0 and $f^{-1}(S_0)$, respectively, we may thus assume $f\colon X \to S$ to be smooth of relative dimension 1.

Proof of (iii): The base extension of $g\colon X \to \mathbb{P}^1_S$ by $\mathrm{Spec}(\mathbb{C}) \to S$ is $g'\colon C = X_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$, which is étale outside $\{0,1,\infty\}$. Hence the base extension

$$g_K\colon X_K = f^{-1}(\eta_S) \to \mathbb{P}^1_K$$

by $\mathrm{Spec}(K) \to S$ is étale outside $\{0,1,\infty\}$. By what we saw in §V.3, we have

$$\mathrm{Supp}(\Omega_{X/\mathbb{P}^1_S}) \cap f^{-1}(\eta_S) \subset \{0,1,\infty\} \times \{\eta_S\}.$$

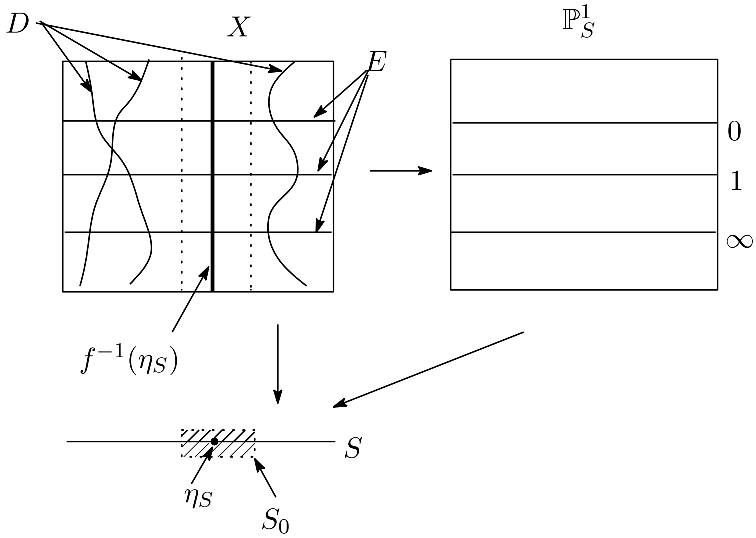

FIGURE IX.1

Denote

$$E = g^{-1}(\{0,1,\infty\} \times S)$$
$$D = \text{the closure of } \Big(\mathrm{Supp}(\Omega_{X/\mathbb{P}^1_S}) \setminus E\Big).$$

Thus $\eta_S \notin f(D \setminus E)$, which is constructible again by Chevalley's Nullstellensatz. Hence $\eta_S \notin \overline{f(D \setminus E)} = f(D)$, and there exists an open affine neighborhood of η_S such that $S_0 \cap f(D) = \emptyset$. Consequently, $f^{-1}(S_0) \cap D = \emptyset$ so that $g\colon f^{-1}(S_0) \to \mathbb{P}^1_{S_0}$ is étale outside $\{0,1,\infty\} \times S_0$, and

$$g\colon f^{-1}(S_0) \setminus E \longrightarrow \mathbb{P}^1_{S_0}$$

is étale.

We may thus replace S and X by this S_0 and $f^{-1}(S_0)$, respectively. If $f(X \setminus E) \neq S$, then since $f(X \setminus E)$ contains η_S and is constructible again by Chevalley's Nullstellensatz, there exists an affine open neighborhood S_0 of η_S such that for $X_0 = f^{-1}(S_0)$, we have $f(X_0 \setminus E) = S_0$. Thus we are in the situation as in Figure IX.1. $\square$

To continue the proof of the "if" part of Theorem 2.1, we denote X_0 and S_0 obtained in Lemma 2.3 by X and S, respectively.

Choose a closed point $s_0 \in S$. Obviously, we have $\Bbbk(s_0) = \overline{\mathbb{Q}}$. Thus $C_0 = f^{-1}(s_0)$ is an irreducible projective smooth curve over $\overline{\mathbb{Q}}$. We now show

$$C \cong C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C}) \quad \text{as algebraic curves,}$$

which would finish the proof of the "if" part of Theorem 2.1.

The base change by $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\overline{\mathbb{Q}})$ of what we obtained in Lemma 2.3 gives rise to

$$f_{\mathbb{C}}\colon X_{\mathbb{C}} \xrightarrow{g_{\mathbb{C}}} \mathbb{P}^1_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C})} S_{\mathbb{C}} \longrightarrow S_{\mathbb{C}}.$$

We also have two $\mathbb{C}$-valued points of $S_{\mathbb{C}}$:

$$t_0\colon \mathrm{Spec}(\mathbb{C}) \longrightarrow S_{\mathbb{C}} \quad \text{induced by } \Bbbk(s_0) = \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$
$$t_1\colon \mathrm{Spec}(\mathbb{C}) \longrightarrow S_{\mathbb{C}} \quad \text{induced by } \Bbbk(\eta_S) = K \hookrightarrow \mathbb{C}$$

so that

$$(f_{\mathbb{C}})^{-1}(t_0) = C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$$
$$(f_{\mathbb{C}})^{-1}(t_1) = C.$$

As we have explained in §VIII.2, let us consider the associated complex analytic spaces and holomorphic maps. For simplicity, we denote

$$M = X_{\mathbb{C}}^{\mathrm{an}}, \quad T = S_{\mathbb{C}}^{\mathrm{an}}, \quad \mathbb{P}^1(\mathbb{C}) = (\mathbb{P}^1_{\mathbb{C}})^{\mathrm{an}}, \quad \varphi = f_{\mathbb{C}}^{\mathrm{an}}, \quad \psi = g_{\mathbb{C}}^{\mathrm{an}}.$$

Thus we have

$$\varphi\colon M \xrightarrow{\psi} \mathbb{P}^1(\mathbb{C}) \times T \longrightarrow T,$$

where M and T are connected complex manifolds, $\varphi\colon M \to T$ is a proper smooth holomorphic map of relative dimension 1, $\psi\colon M \to \mathbb{P}^1(\mathbb{C}) \times T$ is a finite covering unramified outside $\{0, 1, \infty\} \times T$. We can regard t_0 and t_1 as points of T so that

$$\varphi^{-1}(t_0) = \Big(C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})\Big)^{\mathrm{an}}$$
$$\varphi^{-1}(t_1) = C^{\mathrm{an}}.$$

Lemma 2.4. *For any pair of points $t, t' \in T$, one has*

$$\varphi^{-1}(t) \cong \varphi^{-1}(t') \qquad \textit{as complex manifolds.}$$

As a consequence of this lemma, one has

$$(C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C}))^{\mathrm{an}} = \varphi^{-1}(t_0) \cong \varphi^{-1}(t_1) = C^{\mathrm{an}}.$$

In view of a GAGA result given as Corollary VIII.2.11, we have

$$C_0 \times_{\mathrm{Spec}(\overline{\mathbb{Q}})\,\mathrm{Spec}(\mathbb{C})} \cong C \qquad \text{as algebraic curves.}$$

□

Proof of Lemma 2.4. For simplicity, denote

$$P^\circ = \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, \qquad M^\circ = \psi^{-1}(P^\circ \times T)$$

so that the restriction of ψ to M° induces a finite surjective unramifield covering

$$\psi^\circ\colon M^\circ \longrightarrow P^\circ \times T.$$

For each $t \in T$, let

$$\psi_t^\circ\colon \varphi^{-1}(t) \cap M^\circ \longrightarrow P^\circ \times \{t\} \xrightarrow{\sim} P^\circ$$

be the restriction of ψ° to the fibre over t.

We claim that for any pair of points $t, t' \in T$, there exists a homeomorphism

$$h\colon \varphi^{-1}(t) \cap M^\circ \longrightarrow \varphi^{-1}(t') \cap M^\circ$$

such that the diagram

$$\begin{tikzcd} \varphi^{-1}(t) \cap M^\circ \arrow[rr, "h"] \arrow[dr, "\psi_t^\circ"'] & & \varphi^{-1}(t') \cap M^\circ \arrow[dl, "\psi_{t'}^\circ"] \\ & P^\circ & \end{tikzcd}$$

is commutative.

Before proving this claim, let us continue the proof of Lemma 2.4. Since ψ_t° and $\psi_{t'}^\circ$ are finite unramifield coverings of P°, they are local analytic isomorphisms. Hence

$$h\colon \varphi^{-1}(t) \cap M^\circ \xrightarrow{\sim} \varphi^{-1}(t') \cap M^\circ$$

is necessarily an analytic isomorphism. Examining h on a disjoint open disc at each point of the finite ramification loci $\varphi^{-1}(t) \setminus (\varphi^{-1}(t) \cap M^\circ)$ and $\varphi^{-1}(t') \setminus (\varphi^{-1}(t') \cap M^\circ)$, we see by the Riemann Extension Theorem that h extends to a unique analytic isomorphism $h\colon \varphi^{-1}(t) \xrightarrow{\sim} \varphi^{-1}(t)$.

It remains to prove the above claim. Since T is path-connected, it suffices to show the claim for t' in a contractible open neighborhood (e.g., open ball) U of t. Denote by $\psi_U^\circ\colon \varphi^{-1}(U)\cap M^\circ \to P^\circ \times U$ the restriction of ψ°. Thus we have a commutative diagram

$$\begin{tikzcd}
\varphi^{-1}(t)\cap M^\circ \arrow[rr, hook] \arrow[d, "\psi_t^\circ"'] & & \varphi^{-1}(U)\cap M^\circ \arrow[d, "\psi_U^\circ"] \\
P^\circ \arrow[r, "\sim"] & P^\circ \times \{t\} \arrow[r, hook] & P^\circ \times U.
\end{tikzcd}$$

The finite surjective unramified covering ψ_U° corresponds, in terms of the fundamental groups, to a subgroup

$$\pi_1(\varphi^{-1}(U)\cap M^\circ) \subset \pi_1(P^\circ \times U)$$

of finite index. The restriction of this covering to the covering ψ_t° along the fibre corresponds to a subgroup

$$\pi_1(\varphi^{-1}(t)\cap M^\circ) \subset \pi_1(P^\circ).$$

Since U is assumed to be contractible, the restriction to the fibre induces isomorphisms

$$\begin{tikzcd}
\pi_1(\varphi(U)\cap M^\circ) \arrow[r, phantom, "\subset"] \arrow[d, "\sim"'] & \pi_1(P^\circ \times U) \arrow[d, "\sim"] \\
\pi_1(\varphi^{-1}(t)\cap M^\circ) \arrow[r, phantom, "\subset"] & \pi_1(P^\circ),
\end{tikzcd}$$

hence a commutative diagram

$$\begin{tikzcd}
(\varphi^{-1}(t)\cap M^\circ)\times U \arrow[rr, "\text{homeo}"] \arrow[dr, "p_1"'] & & \varphi^{-1}(U)\cap M^\circ \arrow[dl, "\psi_U^\circ"] \\
& U. &
\end{tikzcd}$$

$\square$

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