

# SUPERCONVERGENCE OF DISCONTINUOUS GALERKIN METHOD FOR SCALAR NONLINEAR CONSERVATION LAWS IN ONE SPACE DIMENSION

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**Abstract.** In this paper, the analysis of the superconvergence property of the discontinuous Galerkin (DG) method applied to one-dimensional time-dependent nonlinear scalar conservation laws is carried out. We prove that the error between the DG solution and a particular projection of the exact solution achieves  $(k + \frac{3}{2})$ -th order superconvergence when upwind fluxes are used. The results hold true for arbitrary nonuniform regular meshes and for piecewise polynomials of degree  $k$  ( $k \geq 1$ ), under the condition that  $|f'(u)|$  possesses a uniform positive lower bound. Numerical experiments are provided to show that the superconvergence property actually holds true for nonlinear conservation laws with general flux functions, indicating that the restriction on  $f(u)$  is artificial.

**Key words.** discontinuous Galerkin method, superconvergence, upwind flux, error estimates

**AMS subject classifications.** 65M60, 65N12

**1. Introduction.** In this paper, we consider the semi-discrete discontinuous Galerkin (DG) method applied to one-dimensional scalar conservation laws

$$(1.1a) \quad u_t + f(u)_x = g(x, t),$$

$$(1.1b) \quad u(x, 0) = u_0(x),$$

where  $g(x, t)$  and  $u_0(x)$  are smooth functions. We assume that the nonlinear flux function  $f(u)$  is sufficiently smooth with respect to the variable  $u$ , for example,  $f \in C^3$  is enough. For the sake of simplicity and easy presentation, we only consider the periodic or compactly supported boundary conditions. We show the superconvergence property of the DG solutions towards a particular projection of the exact solution when the upwind fluxes are used.

The DG method discussed in this paper is a class of finite element methods, using discontinuous piecewise polynomials as the solution and the test functions. It was first introduced by Reed and Hill [22] for solving a steady-state linear hyperbolic equation in the framework of neutron transport, and then developed by Cockburn et al. [13, 12, 11, 14] for solving time-dependent nonlinear conservation laws, termed as Runge-Kutta discontinuous Galerkin (RKDG) method, by combining DG spatial discretization and explicit high order Runge-Kutta time discretizations. For more details of DG methods for convection-dominated problems, including algorithm design, analysis, implementation and its applications, we refer the readers to the lecture notes [8, 9, 24], the review paper [15]; see also [3] for a unified analysis for elliptic problems. Since there is no continuity requirement at cell boundaries, the DG methods have several attractive properties which are not shared by the typical finite element

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methods. For example, they can be easily constructed for any order of accuracy with the allowance of different degree of piecewise polynomials in each element; they have an extremely local data structure; they have the flexibility for arbitrarily unstructured meshes even those with hanging nodes; and they have excellent efficiency in  $h$ - $p$  adaptivity, see, e.g., [4].

We would like to mention related theoretical results including stability analysis and error estimates of the DG methods for conservation laws. For smooth solutions of linear conservation laws, optimal a priori error estimates of order  $k + 1$  for one-dimensional and some multi-dimensional cases [19, 23, 10] and a sub-optimal  $L^2$  error estimate of order  $k + \frac{1}{2}$  for arbitrary meshes [18], are obtained for steady-state solution or for the space-time DG discretization, where  $k$  is the polynomial degree of the finite element space. The results in [18] are later proven to be sharp for the most general situation [21]. For nonsmooth solutions of nonlinear conservation laws, Jiang and Shu [17] proved a cell entropy inequality for the semi-discrete DG method as well as for some fully discrete DG scheme with implicit time discretizations such as the backward Euler and Crank-Nicholson algorithms, which trivially implies the nonlinear  $L^2$  stability of the method even without limiters. More recently, the  $L^2$  stability of the fully discrete RKDG method with explicit third order total variation diminishing (TVD) Runge-Kutta time-marching method is given for the linear time-dependent conservation laws with possibly discontinuous solutions [26]. Also in [25, 26], sub-optimal a priori error estimates for general monotone numerical fluxes and optimal error estimates for upwind numerical fluxes are obtained for the RKDG schemes to sufficiently smooth solutions of nonlinear conservation laws.

Let us now mention in particular the superconvergence property of the DG methods in the literature. In [2, 1], Adjerid et al. showed that the DG solution is superconvergent at Radau points for solving ordinary differential equations and steady-state hyperbolic problems. In [5], Cheng and Shu proved superconvergence of order  $k + \frac{3}{2}$  of the DG solution towards a particular projection of the exact solution for linear conservation law when upwind numerical fluxes are used in the framework of Fourier analysis, which works for piecewise linear polynomials on uniform meshes with periodic boundary conditions. The results were later improved, using a different technique, in [6] for arbitrary nonuniform regular meshes and schemes of any order. Let us point out that the superconvergence property with linear growth in time entails that the error of the DG scheme will not grow for fine grids over a long time periods, see [5] and [20] for this excellent long time behavior of the error. The objective of this paper is to study the superconvergence property of the DG method for nonlinear scalar conservation laws, extending the results in [6] for linear problems. Superconvergence of order  $k + \frac{3}{2}$  is proved for smooth solutions of nonlinear conservation laws when upwind numerical fluxes are used, under the condition that  $|f'(u)|$  has a uniform positive lower bound, i.e., either  $f'(u(x, t)) \geq \delta > 0$  or  $f'(u(x, t)) \leq -\delta < 0, \forall (x, t) \in I \times [0, T]$ . Let us emphasize that this restriction is artificial due to the limitation of the technical proof; the superconvergence property actually holds true for nonlinear conservation laws with general flux functions, see numerical results in Section 4 below and also in [5]. As far as we know, this is the first superconvergence proof for DG methods applied to nonlinear hyperbolic equations. The generalization from linear problems in [6] to the nonlinear case in this paper involves several technical difficulties, for example, one of the most essential points is how to obtain a sharp estimate for the time derivative of the error,  $e_t$ , see appendix A.2 for a detailed proof. The main tool employed in this paper is an energy analysis. To deal with the nonlinearity of the flux, Taylor

expansion and a prior assumption about the numerical solution are used.

This paper is organized as follows. In Section 2, we review the DG scheme for conservation laws. In Section 3, we present some preliminaries about the discontinuous finite element space, state the main results and then display the main proofs. In Section 4, numerical experiments are shown to demonstrate the theoretical results. Concluding remarks and comments on future work are given in Section 5. Finally, in the appendix we provide the proofs for some of the more technical lemmas.

**2. DG scheme.** In this section, we follow [13] and review the DG scheme for the problem (1.1).

The usual notation of DG method are adopted here. Let us start by assuming the following mesh to cover the interval  $I = (0, 2\pi)$ , consisting of cells  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ , for  $1 \leq j \leq N$ , where

$$0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 2\pi.$$

The cell center and cell length, respectively, are denoted by  $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$  and  $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ . We use  $h = \max_j h_j$  and  $\rho = \min_j h_j$  to denote the length of the largest cell and the smallest cell. The mesh is assumed to be regular in the sense that the ratio of  $h/\rho$  is always bounded during mesh refinements, namely, there exists a positive constant  $\gamma$  such that  $\gamma h \leq \rho \leq h$ . We denote by  $p_{j+\frac{1}{2}}^-$  and  $p_{j+\frac{1}{2}}^+$  the values of  $p$  at the discontinuity point  $x_{j+\frac{1}{2}}$ , from the left cell,  $I_j$ , and from the right cell,  $I_{j+1}$ , respectively. Moreover, we use  $\llbracket p \rrbracket = p^+ - p^-$  and  $\{\{p\}\} = \frac{1}{2}(p^+ + p^-)$  to represent the jump and the mean of  $p$  at each element boundary point. The following piecewise polynomials space is chosen as the finite element space:

$$V_h \equiv V_h^k = \{v \in L^2(0, 2\pi) : v|_{I_j} \in P^k(I_j), \quad j = 1, \dots, N\},$$

where  $P^k(I_j)$  denotes the set of polynomials of degree up to  $k \geq 1$  defined on the cell  $I_j$ . Note that functions in  $V_h$  are allowed to have discontinuities across element interfaces. Then the approximation of the semi-discrete DG method for solving (1.1) becomes: find the unique function  $u_h = u_h(t) \in V_h$  such that

$$(2.1) \quad \int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+\frac{1}{2}}(v_h)_{j+\frac{1}{2}}^- - \hat{f}_{j-\frac{1}{2}}(v_h)_{j-\frac{1}{2}}^+ = \int_{I_j} g(x, t) v_h dx$$

holds for all  $v_h \in V_h$  and all  $j = 1, \dots, N$ . Here, in order to achieve the superconvergence property, the numerical flux  $\hat{f}_{j+\frac{1}{2}}$  is chosen to be an upwind flux defined on each boundary point, which includes the well-known Godunov flux, the Engquist-Osher flux and the Roe flux with an entropy fix. Note that we have dropped the subscript  $j + 1/2$  in the definition of upwind flux for convenience.

**3. The main results.** Before we state the main results, let us begin by introducing the following notation, definitions and some auxiliary results which will be used later in the proof of the superconvergence property.

### 3.1. Preliminaries.

**3.1.1. Notation for different constants.** We denote by  $C$  (possibly accompanied by lower or upper indices) a positive constant independent of  $h$  but may depend on the exact solution of the equation (1.1), which could have a different value in each occurrence. To emphasize the nonlinearity of the flux  $f(u)$ , we employ  $C_*$  to denote

a nonnegative constant that depending solely on the maximum of  $|f''|$  or/and  $|f'''|$ . We remark that  $C_* = 0$  for a linear flux  $f(u) = cu$ , where  $c$  is a constant.

We assume that the exact solution  $u$  is smooth enough, for example,  $\|u\|_{k+1}$ ,  $\|u_t\|_{k+1}$  and  $\|u_{tt}\|_{k+1}$  are bounded uniformly for any time  $t \in [0, T]$ . Suppose that the initial condition  $u_0(x)$  lies in  $[m_0, M_0]$ , then it follows from the maximum principle that the exact solution is also in this range. We assume that the flux function satisfies  $|f'(u)| \geq \delta$  uniformly on the interval  $[m_0, M_0]$  with  $\delta$  being a positive constant, we then follow [25] to modify the flux  $f$  such that  $|f'(u)| \geq \frac{1}{2}\delta$  on  $[m_0 - 1, M_0 + 1]$ , and for simplicity, we still denote this lower bound as  $\delta$ . We also assume the derivatives of  $f(u)$  with respect to  $u$  up to third order are globally bounded on  $[m_0 - 1, M_0 + 1]$ .

**3.1.2. Functionals related to the  $L^2$  norm.** To get the superconvergence property of the method, two functionals related to the  $L^2$  norm of a function  $\mathbb{F}(x)$  on  $I_j$  are needed as defined in [6]:

$$\begin{aligned}\mathcal{B}_j^-(\mathbb{F}) &= \int_{I_j} \mathbb{F}(x) \frac{x - x_{j-1/2}}{h_j} \frac{d}{dx} \left( \mathbb{F}(x) \frac{x - x_j}{h_j} \right) dx, \\ \mathcal{B}_j^+(\mathbb{F}) &= \int_{I_j} \mathbb{F}(x) \frac{x - x_{j+1/2}}{h_j} \frac{d}{dx} \left( \mathbb{F}(x) \frac{x - x_j}{h_j} \right) dx.\end{aligned}$$

The functionals defined above have the following properties, which are essential to the proof of the superconvergence.

LEMMA 3.1. [6] *For any function  $\mathbb{F}(x) \in C^1$  on  $I_j$ , we have*

$$(3.1) \quad \mathcal{B}_j^-(\mathbb{F}) = \frac{1}{4h_j} \int_{I_j} \mathbb{F}^2(x) dx + \frac{\mathbb{F}^2(x_{j+1/2})}{4},$$

$$(3.2) \quad \mathcal{B}_j^+(\mathbb{F}) = -\frac{1}{4h_j} \int_{I_j} \mathbb{F}^2(x) dx - \frac{\mathbb{F}^2(x_{j-1/2})}{4}.$$

The proof of this lemma is straightforward, see [6].

**3.1.3. Projections and interpolation properties.** We introduce the standard  $L^2$  projection of a function  $q \in L^2(I)$  into the finite element space  $V_h$ , denoted by  $P_h q$ , which is the unique function in  $V_h$  satisfying for each  $j$ ,

$$\int_{I_j} (P_h q(x) - q(x)) v_h dx = 0, \quad \forall v_h \in V_h.$$

In what follows, we define two kinds of Gauss-Radau projections  $P_h^\pm$  into  $V_h$  following a standard trick in the DG analysis. For any given function  $q \in L^2(I)$  and arbitrary element  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ , the special projection of  $q$ , denoted by  $P_h^\pm q$ , is the unique function in the finite element space  $V_h$  satisfying, for each  $j$ ,

$$(3.3) \quad \int_{I_j} (P_h^+ q(x) - q(x)) v_h dx = 0, \quad \forall v_h \in P^{k-1}(I_j), \quad (P_h^+ q)_{j-\frac{1}{2}}^+ = q(x_{j-\frac{1}{2}}^+);$$

$$(3.4) \quad \int_{I_j} (P_h^- q(x) - q(x)) v_h dx = 0, \quad \forall v_h \in P^{k-1}(I_j), \quad (P_h^- q)_{j+\frac{1}{2}}^- = q(x_{j+\frac{1}{2}}^-).$$

Note that these two special projections have been used to derive optimal error estimates of the DG methods in the literature, for example, in [25, 26]. We would

like to mention that the exact collocation at one of the boundary points on each cell plus the orthogonality property for polynomials of degree up to  $k - 1$  of the Gauss-Radau projections  $P_h^\pm$  play an important role and are used repeatedly in the proof of the superconvergence property. We denote by  $\eta = q(x) - \mathbb{Q}_h q(x)$  ( $\mathbb{Q}_h = P_h$ , or  $P_h^\pm$ ) the projection error, then by a standard scaling argument [7] together with trace inequality, it is easy to obtain, for smooth enough  $q(x)$ , that,

$$(3.5a) \quad \|\eta\| + h\|\eta_x\| + h^{1/2}\|\eta\|_{\Gamma_h} \leq Ch^{k+1},$$

where  $C$  is a positive constant depending solely on  $q$  and its derivatives but independent of  $h$ . Here and below, an unmarked norm  $\|\cdot\|$  is the usual  $L^2$  norm defined on the interval  $I$ , and  $\|\cdot\|_{\Gamma_h}$  denotes the  $L^2$  norm defined on the cell interfaces of the mesh. For example, for the one-dimensional case under consideration in this paper,  $\|\eta\|_{\Gamma_h}^2 = \sum_{j=1}^N \left( (\eta_{j+1/2}^+)^2 + (\eta_{j+1/2}^-)^2 \right)$ . Moreover, the Sobolev's inequality implies that

$$(3.5b) \quad \|\eta\|_\infty \leq Ch^{k+\frac{1}{2}}$$

with the positive constant  $C$  is independent of  $h$ . This inequality is important for the a priori assumption that to be used for the error estimates, see Section 3.3 below.

**3.1.4. Inverse properties.** Finally, we list some inverse properties of the finite element space  $V_h$ . For any  $p_h \in V_h$ , there exists a positive constant  $C$  independent of  $p_h$  and  $h$ , such that

$$(i) \|\partial_x p_h\| \leq Ch^{-1}\|p_h\|; \quad (ii) \|p_h\|_{\Gamma_h} \leq Ch^{-1/2}\|p_h\|; \quad (iii) \|p_h\|_\infty \leq Ch^{-1/2}\|p_h\|.$$

Here and below,  $\partial_x(\cdot)$  denotes the partial derivative with respect to the variable  $x$ , likewise for  $\partial_t(\cdot)$ . For more details of these inverse properties, we refer to [7].

**3.2. The main results.** Let us denote  $e = u - u_h$  to be the error between the exact solution and numerical solution and split it into two parts, one is the projection error denoted by  $\eta = u - \mathbb{Q}_h u$ , and the other is the part belongs to the finite element space  $V_h$ , denoted by  $\xi = \mathbb{Q}_h u - u_h$ , which is proven to be  $(k + \frac{3}{2})$ -th order superconvergence as shown in the following theorem. Here the projection  $\mathbb{Q}_h$  is defined at each time level  $t$  corresponding to the sign variation of  $f'(u)$ ; more specifically, for any  $t \in [0, T]$  and  $x \in I$ , if  $f'(u(x, t)) > 0$ , we choose  $\mathbb{Q}_h = P_h^-$ , if  $f'(u(x, t)) < 0$ , we take  $\mathbb{Q}_h = P_h^+$ .

We are now ready to state the main theorem.

**THEOREM 3.2.** *Let  $u$  be the exact solution of the problem (1.1), which is assumed to be sufficiently smooth with bounded derivatives, and assume that  $f \in C^3$  and  $|f'(u)|$  is lower bounded uniformly by any positive constant. Let  $u_h$  be the numerical solution of scheme (2.1) with initial condition  $u_h(\cdot, 0) = \mathbb{Q}_h u_0$  when the upwind flux is used. For regular triangulations of  $I = (0, 2\pi)$ , if the finite element space  $V_h^k$  of piecewise polynomials with arbitrary degree  $k \geq 1$  is used, then for small enough  $h$  there holds the following error estimate*

$$(3.6) \quad \|\xi(\cdot, t)\| \leq Ch^{k+3/2} \quad \forall t \in [0, T],$$

where the positive constant  $C$  depends on the exact solution  $u$ , the final time  $T$  and the maximum of  $|f^{(m)}|$  ( $m = 1, 2, 3$ ), but is independent of  $h$ .

**3.3. Proof of the main results.** Without loss of generality, we will only consider the case  $f'(u(x, t)) \geq \delta > 0 \forall (x, t) \in I \times [0, T]$ , the case of  $f'(u(x, t)) \leq -\delta < 0$  is similar. Therefore, we take the numerical flux as  $\hat{f} = f(u_h^-)$  on each cell interface and choose the projection as  $\mathbb{Q}_h = P_h^-$  on each cell element, the initial condition is chosen as  $u_h(\cdot, 0) = P_h^- u_0$ . Thus the DG scheme (2.1) can be written as

$$(3.7) \quad \int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h) (v_h)_x dx + f(u_h^-) v_h^-|_{j+\frac{1}{2}} - f(u_h^-) v_h^+|_{j-\frac{1}{2}} = \int_{I_j} g(x, t) v_h dx,$$

which holds for any  $v_h \in V_h$ . Since the exact solution  $u$  also satisfies the weak formulation (3.7), we thus have the error equation

$$(3.8) \quad \int_{I_j} e_t v_h dx = \int_{I_j} (f(u) - f(u_h)) (v_h)_x dx - (f(u) - f(u_h^-)) v_h^-|_{j+\frac{1}{2}} + (f(u) - f(u_h^-)) v_h^+|_{j-\frac{1}{2}}$$

for any  $v_h \in V_h$ . Summing the above error equation over  $j$  and using the periodic boundary conditions, one obtain

$$(3.9) \quad \int_I e_t v_h dx = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h)) (v_h)_x dx + \sum_{j=1}^N ((f(u) - f(u_h^-)) \llbracket v_h \rrbracket)_{j+\frac{1}{2}}$$

for all  $v_h \in V_h$ . We now take  $v_h = \xi$  to obtain the following identity

$$(3.10) \quad LHS = RHS,$$

where

$$(3.11a) \quad LHS = \int_I e_t \xi dx,$$

$$(3.11b) \quad RHS = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h)) \xi_x dx + \sum_{j=1}^N ((f(u) - f(u_h^-)) \llbracket \xi \rrbracket)_{j+\frac{1}{2}}.$$

Obviously,

$$(3.12a) \quad LHS = \frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \int_I \eta_t \xi dx.$$

To estimate  $RHS$ , let us define  $\xi = r_j + \mathbb{S}_j(x)(x - x_j)/h_j$  on each cell  $I_j$ , with  $r_j = \xi(x_j)$  being a constant and  $\mathbb{S}_j(x) \in P^{k-1}(I_j)$ . The estimate of  $RHS$  is given in the following lemma.

**LEMMA 3.3.** *Suppose that the interpolation property (3.5a) is satisfied, then we have*

$$(3.12b) \quad RHS \leq (\mathcal{C}(e) + C_* h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C_* h^{k+1} \|\mathbb{S}\| + Ch^{2k+3},$$

where  $\mathcal{C}(e) = C + C_* h^{-1} \|e\|_\infty$ , and the positive constants  $C$  and  $C_*$  are independent of  $h$  and the approximate solution  $u_h$ .

*Proof.* We start by using the second order Taylor expansion with respect to the variable  $u$  to write out the nonlinear terms, namely  $f(u) - f(u_h)$  and  $f(u) - f(u_h^-)$ , as

$$(3.13a) \quad f(u) - f(u_h) = f'(u)\xi + f'(u)\eta - \frac{1}{2} \tilde{f}_u''(\eta + \xi)^2 \triangleq \lambda_1 + \lambda_2 + \lambda_3,$$

$$(3.13b) \quad f(u) - f(u_h^-) = f'(u)\xi^- + f'(u)\eta^- - \frac{1}{2} \tilde{f}_u''(\eta^- + \xi^-)^2 \triangleq \theta_1 + \theta_2 + \theta_3,$$

where  $\tilde{f}_u''$  and  $\tilde{f}_u''$  are the mean values given by  $\tilde{f}_u'' = f''(\alpha_1 u + (1 - \alpha_1)u_h)$  and  $\tilde{f}_u'' = f''(\alpha_2 u + (1 - \alpha_2)u_h^-)$  with  $0 \leq \alpha_1, \alpha_2 \leq 1$ . Note that here we have dropped the subscript  $j + 1/2$  for notational convenience, since all quantities are evaluated at the same points (i.e., the interfaces between the cells). Thus, the identity (3.11b) can be represented by

$$RHS = \sum_{i=1}^3 (\Lambda_i + \Theta_i),$$

with  $\Lambda_i$  and  $\Theta_i$ , given by

$$\Lambda_i = \sum_{j=1}^N \int_{I_j} \lambda_i \xi_x dx \quad \text{and} \quad \Theta_i = \sum_{j=1}^N (\theta_i \llbracket \xi \rrbracket)_{j+\frac{1}{2}} \quad (i = 1, 2, 3),$$

which will be estimated separately later. A simple integration by parts gives us the estimate for  $\Lambda_1 + \Theta_1$ :

$$\begin{aligned} \Lambda_1 + \Theta_1 &= -\frac{1}{2} \sum_{j=1}^N \int_{I_j} \partial_x f'(u) \xi^2 dx - \frac{1}{2} \sum_{j=1}^N f'(u_{j+\frac{1}{2}}) \llbracket \xi \rrbracket_{j+\frac{1}{2}}^2 \\ &\leq C_\star \|\xi\|^2 - \frac{1}{2} \sum_{j=1}^N f'(u_{j+\frac{1}{2}}) \llbracket \xi \rrbracket_{j+\frac{1}{2}}^2 \\ (3.14a) \quad &\leq C_\star \|\xi\|^2, \end{aligned}$$

where we have used the positivity assumption of  $f'(u)$  to obtain the last inequality. Note that  $\eta_{j+1/2}^- = 0$ , due to the property of the projection  $P_h^-$  in (3.4), we get

$$\Theta_2 = \sum_{j=1}^N (f'(u) \eta^- \llbracket \xi \rrbracket)_{j+\frac{1}{2}} = 0.$$

To estimate  $\Lambda_2$ , we rewrite the expression for  $f'(u)$  as

$$f'(u) = f'(u_j) + (f'(u) - f'(u_j)),$$

where  $u_j$  denotes the value of the exact solution  $u$  at  $x_j$ . Note that  $\eta$  is orthogonal to any polynomials of degree up to  $k - 1$  by virtue of the property of the projection  $P_h^-$  in (3.4), therefore,  $\Lambda_2$  can be represented by

$$\begin{aligned} \Lambda_2 &= \sum_{j=1}^N \int_{I_j} (f'(u) - f'(u_j)) \eta \xi_x dx \\ &= \sum_{j=1}^N \int_{I_j} (f'(u) - f'(u_j)) \eta \left( \mathbb{S}'_j(x) \frac{x - x_j}{h_j} + \frac{\mathbb{S}_j(x)}{h_j} \right) dx. \end{aligned}$$

We now define piecewise polynomials  $\mathbb{S}(x)$  and  $\phi(x)$  such that  $\mathbb{S}(x) = \mathbb{S}_j(x)$ ,  $\phi(x) = (x - x_j)/h_j$  on each  $I_j$ . Clearly  $\|\phi\|_\infty = \frac{1}{2}$ , then Cauchy-Schwarz inequality together with the inverse property (i) yields that

$$\Lambda_2 \leq C_\star \|\eta\| \|\mathbb{S}\| \leq C_\star h^{k+1} \|\mathbb{S}\|,$$

where we have also used the interpolation property (3.5a) and the fact that  $|f'(u) - f'(u_j)| \leq C_* h$  on each element  $I_j$  due to the smoothness of the exact solution  $u$  and  $f$ . Thus, we arrive at a bound for  $\Lambda_2 + \Theta_2$ ,

$$(3.14b) \quad \Lambda_2 + \Theta_2 \leq C_* h^{k+1} \|\mathbb{S}\|.$$

It is easy to show that

$$(3.14c) \quad \begin{aligned} \Lambda_3 + \Theta_3 &\leq C_* \|e\|_\infty \|e\| \|\xi_x\| + C_* \|e\|_\infty \|e\|_{\Gamma_h} \|\xi\|_{\Gamma_h} \\ &\leq C_* h^{-1} \|e\|_\infty (\|e\| \|\xi\| + h^{\frac{1}{2}} \|\eta\|_{\Gamma_h} \|\xi\| + \|\xi\|^2) \\ &\leq C_* h^k \|e\|_\infty \|\xi\| + C_* h^{-1} \|e\|_\infty \|\xi\|^2 \\ &\leq (C_* h^{-1} \|e\|_\infty + C_* h^{-3} \|e\|_\infty^2) \|\xi\|^2 + Ch^{2k+3}, \end{aligned}$$

where for the first step we have used Cauchy-Schwarz inequality, for the second step we have used the inverse properties (i) and (ii), for the third step we have employed the interpolation properties (3.5a), and a direct application of Young's inequality is used for the last step. To complete the proof of Lemma 3.3, we need only to combine (3.14a)-(3.14c).  $\square$

We now collect (3.12a) and (3.12b) into (3.10) to get

$$(3.15) \quad \frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq \left| \int_I \eta_t \xi dx \right| + (\mathcal{C}(e) + C_* h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C_* h^{k+1} \|\mathbb{S}\| + Ch^{2k+3}.$$

It follows from the orthogonality property of the projection  $P_h^-$  in (3.4), Cauchy-Schwarz inequality and the interpolation property (3.5a) that the first term on the right-hand side of (3.15),  $|\int_I \eta_t \xi dx|$ , can be bounded by

$$(3.16) \quad \left| \int_I \eta_t \xi dx \right| \leq \|\eta_t\| \|\mathbb{S}\| \|\phi\|_\infty \leq Ch^{k+1} \|\mathbb{S}\|,$$

where  $C$  is a positive constant depending solely on the exact solution  $u$ , but is independent of  $h$ . A combination of (3.15) and (3.16) yields that

$$(3.17) \quad \frac{1}{2} \frac{d}{dt} \|\xi\|^2 \leq (\mathcal{C}(e) + C_* h^{-3} \|e\|_\infty^2) \|\xi\|^2 + C_* h^{k+1} \|\mathbb{S}\| + Ch^{2k+3}.$$

In what follows, to deal with the nonlinearity of the flux  $f(u)$  we shall make an a priori assumption that, for small enough  $h$ , there holds

$$(3.18) \quad \|\mathbb{Q}_h u - u_h\| \leq h^2.$$

Later we will justify this a priori assumption (3.18) for piecewise polynomials of degree  $k \geq 1$ .

**COROLLARY 3.4.** *Suppose that the interpolation property (3.5b) is satisfied, then the a priori assumption (3.18) implies that*

$$(3.19) \quad \|e\|_\infty \leq Ch^{\frac{3}{2}} \quad \text{and} \quad \|\xi\|_\infty \leq Ch^{\frac{3}{2}}.$$

*Proof.* This follows from the inverse property (iii), the interpolation property (3.5b) and triangle inequality.  $\square$

Under this the a priori assumption, we can first get a crude bound for  $\xi$ , which is used to derive a sharp bound for  $e_t$  in Lemma 3.7.

**COROLLARY 3.5.** *Under the same conditions as in Lemma 3.3, if the a priori assumption (3.18) holds, we have the following error estimates*

$$(3.20) \quad \|e\| \leq Ch^{k+1} \quad \text{and} \quad \|\xi\| \leq Ch^{k+1}.$$

*Proof.* The proof follows along the same lines as that in Lemma 3.3 except that we can derive a crude bound for  $\Lambda_2 + \Theta_2 \leq C_* h^{k+1} \|\xi\|$  instead of  $\Lambda_2 + \Theta_2 \leq C_* h^{k+1} \|\mathbb{S}\|$  in (3.14b) and also in (3.16). Then the results in Corollary 3.5 follow by using (3.19) implied by the a priori assumption (3.18) and a simple application of Gronwall's inequality together with the fact that  $\xi(\cdot, 0) = 0$  due to the choice of the initial condition.  $\square$

We remark that the results in Corollary 3.5 for the semi-discrete case considered in this paper can be viewed as a straightforward consequence of the fully discrete DG method for solving conservation laws when the upwind fluxes are used, see e.g., [25, 26].

The next lemma gives us a bound for  $\mathbb{S}$ , which is essential to obtain the superconvergence property.

**LEMMA 3.6.** *Under the same conditions as in Theorem 3.2, if, in addition, the a priori assumption (3.18) holds, we have*

$$(3.21) \quad \|\mathbb{S}\| \leq Ch\|e_t\| + Ch^{k+2},$$

for any  $t \in [0, T]$ , where the positive constant  $C$  is independent of  $h$  and the approximate solution  $u_h$ .

The proof of this lemma is given in the appendix, see A.1. Up to now, we see that we still need to have a bound on  $e_t$ , which is given in the following lemma.

**LEMMA 3.7.** *Under the same conditions as in Theorem 3.2, if, in addition, the a priori assumption (3.18) holds, we have*

$$(3.22) \quad \|e_t\| \leq Ch^{k+1} + C_* h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds},$$

for any  $t \in [0, T]$ , where the positive constants  $C$  and  $C_*$  are independent of  $h$  and the approximate solution  $u_h$ .

The proof of this lemma is deferred to the appendix, see A.2.

We are now ready to get the following important inequality for  $\xi$  by collecting the estimates (3.21) and (3.22) into (3.17), by employing (3.19) implied by the a priori assumption (3.18) and by virtue of Young's inequality

$$(3.23) \quad \frac{1}{2} \frac{d}{dt} \|\xi(t)\|^2 \leq C_1 \|\xi(t)\|^2 + C_2 \int_0^t \|\xi(s)\|^2 ds + C_3 h^{2k+3},$$

where  $C_1, C_2$  and  $C_3$  are positive constants independent of  $h$ . Note that there holds the following identity

$$(3.24) \quad \frac{d}{dt} \int_0^t \|\xi(s)\|^2 ds = \|\xi(t)\|^2.$$

Adding twice of (3.23) and (3.24) up, we arrive at

$$(3.25) \quad \frac{d}{dt} \left( \|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds \right) \leq C_0 \left( \|\xi(t)\|^2 + \int_0^t \|\xi(s)\|^2 ds \right) + Ch^{2k+3},$$

where  $C_0 = \max(2C_1 + 1, 2C_2)$  and  $C = 2C_3$  are positive constants independent of  $h$ .

An application of Gronwall's inequality together with the fact that  $\xi(\cdot, 0) = 0$  gives us the desired result

$$(3.26) \quad \|\xi(\cdot, t)\| \leq Ch^{k+3/2}.$$

Finally, let us complete the proof of Theorem 3.2 by verifying the a priori assumption (3.18). First of all, the a priori assumption is satisfied at  $t = 0$  since  $\xi(\cdot, 0) = 0$ . For piecewise polynomials of degree  $k$  ( $k \geq 1$ ), one can choose  $h$  small enough such that  $Ch^{k+3/2} < \frac{1}{2}h^2$ , where  $C$  is a constant in (3.6) determined by the final time  $T$ . Define  $t^* = \sup\{s \leq T : \|\mathbb{Q}_h u(t) - u_h(t)\| \leq h^2, \forall t \in [0, s]\}$ , then we have  $\|\mathbb{Q}_h u(t^*) - u_h(t^*)\| = h^2$  by continuity if  $t^* < T$ . However, our main result (3.26) implies that  $\|\mathbb{Q}_h u(t^*) - u_h(t^*)\| \leq Ch^{k+3/2} < \frac{1}{2}h^2$ , which is a contradiction. Therefore, there always holds  $t^* = T$ , and thus the a priori assumption (3.18) is justified.

**4. Numerical examples.** In this section we provide some numerical experiments to verify the superconvergence property of the DG method for hyperbolic conservation laws. To reduce the time errors, we use the five stage, fourth order strong stability preserving (SSP) Runge-Kutta discretization (see, e.g., [16]) in time and take  $\Delta t = CFL h^2$ . In the computations below, our numerical initial condition is taken by the  $L^2$  projection of the initial condition. In fact, we have observed little difference if we use the  $\mathbb{Q}_h$  projection of the initial condition instead, except that for the case with severely nonuniform mesh solving (4.1) in Example 4.1 for short time simulation (for example,  $T = 1$ ); see Table 4.2.

EXAMPLE 4.1. *First we consider the following equation*

$$(4.1) \quad \begin{cases} u_t + (u^3/3 + u)_x = g(x, t), \\ u(x, 0) = \cos(x) \\ u(0, t) = u(2\pi, t) \end{cases}$$

where  $g(x, t)$  is given by

$$g(x, t) = -(2 + \cos^2(x + t)) \sin(x + t).$$

The exact solution is

$$u(x, t) = \cos(x + t).$$

Note that  $f'(u) = u^2 + 1 \geq 1 > \delta > 0$ , we can use upwind fluxes and choose  $\mathbb{Q}_h = P_h^-$ . We test this example using  $P^k$  polynomials with  $1 \leq k \leq 3$ . Table 4.1 lists the numerical errors,  $\xi$  and  $e$ , and their orders for different final time  $T$  using  $P^1$  polynomials on a uniform mesh. We can clearly observe third order accuracy for  $\xi$  and second order for  $e$ . Also,  $\xi$  does not grow much which ensures that the error  $e$  does not grow with respect to time. Table 4.2 lists the results when using  $P^1$  polynomials on a nonuniform mesh which is a 30% random perturbation of the uniform mesh. From the table, we can still observe superconvergence (the order is around 2.5). Let us point

out that the usage of  $P_h^-$  projection of the initial condition would help to recover the third order accuracy, which are, however, not included here to save space. That is, the conclusions hold also true for this nonuniform case. The results for Example 4.1 when using  $P^2$  and  $P^3$  polynomials on a uniform mesh, respectively, are listed in Tables 4.3 and 4.4. We can clearly observe that the orders of convergence of the errors,  $\xi$  and  $e$ , are  $k + 2$  and  $k + 1$ , respectively; moreover, both errors do not grow with respect to time either.

TABLE 4.1

The errors  $\xi$  and  $e$  for Example 4.1 when using  $P^1$  polynomials on a uniform mesh of  $N$  cells. CFL = 0.5.

$P^1$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	20	2.10E-04	–	1.84E-04	–	2.45E-04	–
	40	2.65E-05	2.99	2.73E-05	2.76	3.90E-05	2.65
	80	3.31E-06	3.00	3.65E-06	2.90	5.10E-06	2.93
	160	4.14E-07	3.00	4.61E-07	2.98	6.53E-07	2.97
	320	5.17E-08	3.00	5.77E-08	3.00	8.21E-08	2.99
$e$	20	4.26E-03	–	4.26E-03	–	4.24E-03	–
	40	1.06E-03	2.00	1.06E-03	2.00	1.06E-03	2.00
	80	2.65E-04	2.00	2.66E-04	2.00	2.65E-04	2.00
	160	6.64E-05	2.00	6.64E-05	2.00	6.64E-05	2.00
	320	1.66E-05	2.00	1.66E-05	2.00	1.66E-05	2.00

TABLE 4.2

The errors  $\xi$  and  $e$  for Example 4.1 when using  $P^1$  polynomials on a random mesh of  $N$  cells. CFL = 0.5.

$P^1$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	20	5.86E-04	–	6.46E-04	–	6.21E-04	–
	40	6.19E-05	3.24	5.86E-05	3.46	5.43E-05	3.51
	80	1.18E-05	2.39	7.71E-06	2.93	8.03E-06	2.76
	160	2.30E-06	2.37	7.81E-07	3.30	9.81E-07	3.03
	320	4.65E-07	2.30	1.14E-07	2.78	1.21E-07	3.02
$e$	20	5.50E-03	–	4.98E-03	–	5.37E-03	–
	40	1.30E-03	2.08	1.23E-03	2.01	1.28E-03	2.07
	80	3.55E-04	1.88	3.52E-04	1.81	3.52E-04	1.86
	160	8.73E-05	2.02	8.32E-05	2.08	8.36E-05	2.07
	320	2.13E-05	2.03	2.10E-05	1.99	2.15E-05	1.96

EXAMPLE 4.2. In this example, we solve the following equation

$$(4.2) \quad \begin{cases} u_t + (u^3/3)_x = g(x, t), \\ u(x, 0) = \cos(x) \\ u(0, t) = u(2\pi, t) \end{cases}$$

where  $g(x, t)$  is given by

$$g(x, t) = -(1 + \cos^2(x + t)) \sin(x + t).$$

TABLE 4.3

The errors  $\xi$  and  $e$  for Example 4.1 when using  $P^2$  polynomials on a uniform mesh of  $N$  cells.  $CFL = 0.5$ .

$P^2$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	20	6.35E-06	–	6.70E-06	–	6.69E-06	–
	40	4.12E-07	3.94	4.13E-07	4.02	4.13E-07	4.02
	80	2.57E-08	4.00	2.57E-08	4.00	2.57E-08	4.00
	160	1.61E-09	4.00	1.61E-09	4.00	1.61E-09	4.00
	320	1.00E-10	4.00	1.00E-10	4.00	1.01E-10	3.99
$e$	20	1.07E-04	–	1.07E-04	–	1.07E-04	–
	40	1.34E-05	3.00	1.34E-05	3.00	1.34E-05	3.00
	80	1.67E-06	3.00	1.67E-06	3.00	1.67E-06	3.00
	160	2.09E-07	3.00	2.09E-07	3.00	2.09E-07	3.00
	320	2.61E-08	3.00	2.61E-08	3.00	2.61E-08	3.00

TABLE 4.4

The errors  $\xi$  and  $e$  for Example 4.1 when using  $P^3$  polynomials on a uniform mesh of  $N$  cells.  $CFL = 0.1$ .

$P^3$	$N$	$T = 10$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	10	2.82E-06	–	1.81E-06	–	1.98E-06	–
	20	5.47E-08	5.69	5.67E-08	5.00	5.66E-08	5.13
	40	1.74E-09	4.97	1.74E-09	5.02	1.74E-09	5.02
	80	5.42E-11	5.00	5.42E-11	5.00	5.49E-11	4.99
$e$	10	3.31E-05	–	3.30E-05	–	3.30E-05	–
	20	2.07E-06	4.00	2.07E-06	4.00	2.07E-06	4.00
	40	1.29E-07	4.00	1.29E-07	4.00	1.29E-07	4.00
	80	8.07E-09	4.00	8.07E-09	4.00	8.07E-09	4.00

The exact solution is

$$u(x, t) = \cos(x + t).$$

In this case,  $f'(u) = u^2 \geq 0$ , we can still use the upwind flux and choose  $\mathbb{Q}_h = P_h^-$ . We test this example using both  $P^1$  and  $P^2$  polynomials on a nonuniform mesh which is a 10% random perturbation of the uniform mesh. The results in Table 4.5 show that the orders of convergence of the errors,  $\xi$  and  $e$ , are  $k + \frac{3}{2}$  and  $k + 1$ , respectively.

EXAMPLE 4.3. We consider the following Burgers equation

$$(4.3) \quad \begin{cases} u_t + (u^2/2)_x = g(x, t), \\ u(x, 0) = \cos(x) \\ u(0, t) = u(2\pi, t) \end{cases}$$

where  $g(x, t)$  is given by

$$g(x, t) = -(1 + \cos(x + t)) \sin(x + t).$$

The exact solution is

$$u(x, t) = \cos(x + t).$$

TABLE 4.5

The errors  $\xi$  and  $e$  for Example 4.2 when using both  $P^1$  and  $P^2$  polynomials on a random mesh of  $N$  cells.  $CFL = 0.5$ .  $T = 1$ .

$P^k$	$k = 1$				$k = 2$			
	$\xi$		$e$		$\xi$		$e$	
	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
40	2.28E-04	–	1.08E-03	–	4.29E-06	–	1.40E-05	–
80	4.52E-05	2.33	2.75E-04	1.98	3.25E-07	3.72	1.77E-06	2.98
160	7.95E-06	2.51	6.85E-05	2.01	2.24E-08	3.86	2.18E-07	3.02
320	1.49E-06	2.42	1.72E-05	1.99	1.90E-09	3.56	2.77E-08	2.98
640	2.63E-07	2.50	4.30E-06	2.00	1.66E-10	3.52	3.48E-09	2.99

Since  $f'(u)$  changes its sign in the computational domain, we use the Godunov flux, which is an upwind flux. The projection  $\mathbb{Q}_h$  is defined element by element as follows. For  $t = T$ , if  $u(x_j, t)$  is positive, we choose  $\mathbb{Q}_h = P_h^-$  on the cell  $I_j$ ; otherwise, we use  $\mathbb{Q}_h = P_h^+$ . We test this example using  $P^k$  polynomials with  $1 \leq k \leq 3$  on a uniform mesh. The results are listed in Tables 4.6-4.8, from which we observe that  $\xi$  achieves at least  $(k + \frac{3}{2})$ -th superconvergence and it does not grow with respect to time for most meshes. Meanwhile, the error  $e$  achieves the expected  $(k + 1)$ -th order of accuracy and it does not grow with respect to time either.

TABLE 4.6

The errors  $\xi$  and  $e$  for Example 4.3 when using  $P^1$  polynomials on a uniform mesh of  $N$  cells.  $CFL = 0.5$ .

$P^1$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	20	6.31E-04	–	1.61E-03	–	1.64E-03	–
	40	9.03E-05	2.81	2.74E-04	2.56	2.65E-04	2.63
	80	1.25E-05	2.85	3.76E-05	2.86	4.24E-05	2.65
	160	1.82E-06	2.78	8.15E-06	2.21	6.67E-06	2.67
	320	2.59E-07	2.81	1.50E-06	2.44	1.04E-06	2.68
$e$	20	4.26E-03	–	4.48E-03	–	4.49E-03	–
	40	1.06E-03	2.00	1.09E-03	2.04	1.09E-03	2.04
	80	2.66E-04	2.00	2.68E-04	2.03	2.69E-04	2.02
	160	6.64E-05	2.00	6.68E-05	2.00	6.67E-05	2.01
	320	1.66E-05	2.00	1.67E-05	2.00	1.66E-05	2.00

**5. Concluding remarks.** In this paper, the superconvergence property of the DG method for nonlinear hyperbolic conservation laws is investigated. We prove that the error between the numerical solution and a particular projection of the exact solution achieves  $(k + \frac{3}{2})$ -th order superconvergence when piecewise polynomials of degree  $k$  ( $k \geq 1$ ) are used, provided that  $|f'(u)|$  is lower bounded uniformly by a positive constant. Numerical experiments are given to show that the superconvergence property holds true for nonlinear conservation laws with general flux functions, indicating that the restriction on  $f(u)$  is artificial.

Future work includes the study of superconvergence of DG method for conservation laws in multi-dimensional cases on structured and unstructured meshes. Analysis

TABLE 4.7

The errors  $\xi$  and  $e$  for Example 4.3 when using  $P^2$  polynomials on a uniform mesh of  $N$  cells.  $CFL = 0.5$ .

$P^2$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	20	7.57E-05	–	9.23E-05	–	1.05E-04	–
	40	8.19E-06	3.21	8.76E-06	3.40	9.08E-06	3.53
	80	9.76E-07	3.07	1.01E-06	3.11	9.11E-07	3.32
	160	8.72E-08	3.48	9.03E-08	3.49	8.81E-08	3.37
$e$	20	1.20E-04	–	1.31E-04	–	1.31E-04	–
	40	1.47E-05	3.03	1.49E-05	3.13	1.49E-05	3.13
	80	1.77E-06	3.05	1.78E-06	3.07	1.78E-06	3.07
	160	2.15E-07	3.04	2.15E-07	3.04	2.15E-07	3.04

TABLE 4.8

The errors  $\xi$  and  $e$  for Example 4.3 when using  $P^3$  polynomials on a uniform mesh of  $N$  cells.  $CFL = 0.2$ .

$P^3$	$N$	$T = 1$		$T = 50$		$T = 500$	
		$L^2$ error	order	$L^2$ error	order	$L^2$ error	order
$\xi$	10	1.10E-05	–	1.56E-05	–	1.50E-05	–
	20	3.94E-07	4.81	4.16E-07	5.23	4.14E-07	5.18
	40	1.49E-08	4.72	1.29E-08	5.01	1.27E-08	5.02
	80	5.39E-10	4.79	3.92E-10	5.04	3.91E-10	5.02
$e$	10	3.53E-05	–	3.63E-05	–	3.51E-05	–
	20	2.11E-06	4.06	2.11E-06	4.10	2.11E-06	4.05
	40	1.30E-07	4.02	1.30E-07	4.02	1.30E-07	4.02
	80	8.09E-09	4.01	8.08E-09	4.01	8.08E-09	4.01

of superconvergence property of the local DG (LDG) method for nonlinear diffusion equations also constitutes future work.

## Appendix A. Proof of several lemmas.

In this appendix, we give the proofs for some of the technical lemmas.

**A.1. The proof of Lemma 3.6.** A simple integration by parts of (3.8) yields that

$$(A.1) \quad \int_{I_j} e_t v_h dx + \int_{I_j} (f(u) - f(u_h))_x v_h dx + \llbracket f(u) - f(u_h) \rrbracket v_h^+ \Big|_{j-\frac{1}{2}} = 0,$$

for all  $v_h \in V_h$ . We recall that  $u - u_h = \eta + \xi$  and  $\xi = r_j + \mathbb{S}_j(x)(x - x_j)/h_j$  with  $r_j$  being a constant and  $\mathbb{S}_j(x) \in P^{k-1}(I_j)$ . If we now let  $v_h = \mathbb{S}_j(x)(x - x_{j-1/2})/h_j$ , then we have the following identity

$$(A.2) \quad \int_{I_j} e_t \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx + \int_{I_j} (f(u) - f(u_h))_x \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx = 0,$$

since  $v_h(x_{j-\frac{1}{2}}^+) = 0$ . To estimate  $\|\mathbb{S}\|$ , we would like to split the nonlinear term  $f(u) - f(u_h)$  in (A.2) into five terms by the following Taylor expansion:

$$\begin{aligned} f(u) - f(u_h) &= f'(u)\xi + f'(u)\eta - \frac{1}{2}f''_u e^2 \\ &= f'(u_j)\xi + (f'(u) - f'(u_j))\xi + f'(u_j)\eta + (f'(u) - f'(u_j))\eta - \frac{1}{2}f''_u e^2 \\ &\triangleq \pi_{1,j} + \pi_{2,j} + \pi_{3,j} + \pi_{4,j} + \pi_{5,j}, \end{aligned}$$

where  $f''_u$  is again the mean value. As we have shown before,  $|f'(u) - f'(u_j)| \leq C_\star h$  on each element  $I_j$ . Therefore, (A.2) can be written as

$$(A.3) \quad \int_{I_j} e_t \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx + \sum_{i=1}^5 \Pi_{i,j} = 0,$$

where

$$\Pi_{i,j} = \int_{I_j} (\pi_{i,j})_x \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx \quad (i = 1, \dots, 5).$$

In what follows, we will estimate each term above separately.

First, it is easy to show, by the property of  $\mathcal{B}_j^-$  in Lemma 3.1, that

$$(A.4) \quad \Pi_{1,j} = f'(u_j) \mathcal{B}_j^-(\mathbb{S}_j) = f'(u_j) \left[ \frac{1}{4h_j} \int_{I_j} \mathbb{S}_j^2(x) dx + \frac{\mathbb{S}_j^2(x_{j+1/2})}{4} \right].$$

Plugging (A.4) into (A.3) and using the assumption that  $f'(u(x, t)) \geq \delta > 0$ , we get

$$(A.5) \quad \frac{\delta}{4} \int_{I_j} \mathbb{S}_j^2(x) dx \leq h_j \left[ - \int_{I_j} e_t \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx - \sum_{i=2}^5 \Pi_{i,j} \right].$$

We now define piecewise polynomial  $\phi_1(x)$  such that  $\phi_1(x) = (x - x_{j-1/2})/h_j$  on each  $I_j$ , clearly,  $\|\phi_1\|_\infty = 1$ . A summation of (A.5) over  $j$  yields that

$$(A.6) \quad \frac{\delta}{4} \|\mathbb{S}\|^2 \leq -h \int_I e_t \mathbb{S}(x) \phi_1(x) dx - \sum_{i=2}^5 \Pi_i,$$

with  $\Pi_i = h \sum_{j=1}^N \Pi_{i,j}$  ( $i = 2, \dots, 5$ ). We shall estimate the right-hand side of (A.6) one by one below.

The first integral term can be bounded by using Cauchy-Schwarz inequality

$$(A.7a) \quad h \left| \int_I e_t \mathbb{S}(x) \phi_1(x) dx \right| \leq h \|e_t\| \|\mathbb{S}\| \|\phi_1\|_\infty \leq h \|e_t\| \|\mathbb{S}\|.$$

To estimate  $\Pi_2$ , we begin by using Cauchy-Schwarz inequality and inverse property (i) to get a bound for  $\Pi_{2,j}$ , it reads

$$\begin{aligned} \Pi_{2,j} &= \int_{I_j} [(f'(u) - f'(u_j))_x \xi + (f'(u) - f'(u_j)) \xi_x] \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx \\ &\leq C_\star \|\xi\|_{I_j} \|\mathbb{S}\|_{I_j}, \end{aligned}$$

here and below,  $\|\cdot\|_{I_j}$  denotes the usual  $L^2$  norm defined on the cell  $I_j$ . Therefore, again by Cauchy-Schwarz inequality

$$(A.7b) \quad |\Pi_2| \leq C_* h \|\xi\| \|\mathbb{S}\|.$$

A simple integration by parts together with the property of projection  $P_h^-$  in (3.4) gives us

$$\Pi_{3,j} = -f'(u_j) \int_{I_j} \eta \frac{d}{dx} \left( \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} \right) dx = 0.$$

Thus,

$$(A.7c) \quad |\Pi_3| = 0.$$

To estimate  $\Pi_4$ , we first get a bound for  $\Pi_{4,j}$ , it reads

$$\begin{aligned} \Pi_{4,j} &= \int_{I_j} [(f'(u) - f'(u_j))_x \eta + (f'(u) - f'(u_j)) \eta_x] \mathbb{S}_j(x) \frac{x - x_{j-1/2}}{h_j} dx \\ &\leq C_* (\|\eta\|_{I_j} + h \|\eta_x\|_{I_j}) \|\mathbb{S}\|_{I_j}. \end{aligned}$$

As a consequence, the Cauchy-Schwarz inequality in combination with the interpolation property (3.5a) produces a bound for  $\Pi_4$

$$(A.7d) \quad |\Pi_4| \leq C_* h (\|\eta\| + h \|\eta_x\|) \|\mathbb{S}\| \leq C_* h^{k+2} \|\mathbb{S}\|.$$

It is easy to get, for the high order term  $\Pi_5$ , that

$$(A.7e) \quad |\Pi_5| \leq C_* \|e\|_\infty (h^{k+1} + \|\xi\|) \|\mathbb{S}\|.$$

Finally, the error estimate (3.21) follows by collecting the estimates (A.7a)-(A.7e) into (A.6) and by using (3.19) implied by the a priori assumption (3.18) and a crude bound for  $\xi$  in (3.20), in Corollary 3.4 and Corollary 3.5, respectively. This completes the proof of Lemma 3.6.

**A.2. The proof of Lemma 3.7.** From the interpolation property (3.5a), we know that  $\|\eta_t(\cdot, t)\| \leq Ch^{k+1}$ , thus, to get the error estimate (3.22) we need only to prove  $\|\xi_t(\cdot, t)\| \leq Ch^{k+1} + C_* h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds}$ . To this end, we shall first get a bound for the initial error  $\|\xi_t(\cdot, 0)\|$ .

We start by noting that the error equation (3.9) still holds at  $t = 0$  for any  $v_h \in V_h$ . Using the fact that  $\xi(\cdot, 0) = 0$ , we arrive at the following representation of the nonlinear terms in (3.13a) and (3.13b) on the right-hand side of (3.9):

$$(A.8a) \quad f(u) - f(u_h) = f'(u)\eta - \frac{1}{2} \tilde{f}_u'' \eta^2$$

$$(A.8b) \quad f(u) - f(u_h^-) = f'(u)\eta^- - \frac{1}{2} \tilde{f}_u'' (\eta^-)^2.$$

By a similar analysis as that in the proof of Lemma 3.3, we can easily get a bound for the right-hand side of (3.9) at  $t = 0$ , denoted by  $\mathcal{RHS}$ , as follows

$$(A.9) \quad \mathcal{RHS} \leq C_* (h^{k+1} + h^k \|\eta(\cdot, 0)\|_\infty) \|v_h\|,$$

which holds for any  $v_h \in V_h$ . If we now let  $v_h = \xi_t(\cdot, 0)$  in (3.9) as well as in (A.9), we get, after a simple calculation, that

$$(A.10) \quad \|\xi_t(\cdot, 0)\| \leq \|\eta_t(\cdot, 0)\| + C_\star(h^{k+1} + h^k \|\eta(\cdot, 0)\|_\infty) \leq Ch^{k+1},$$

by the interpolation properties (3.5a) and (3.5b).

We then move on to estimate  $\|\xi_t(\cdot, t)\|$  for  $t > 0$ . To this end, we take the time derivative of the error equation (3.9) and let  $v_h = \xi_t$  to get

$$(A.11) \quad \int_I e_{tt} \xi_t dx = \sum_{j=1}^N \int_{I_j} (f(u) - f(u_h))_t (\xi_t)_x dx + \sum_{j=1}^N ((f(u) - f(u_h^-))_t \llbracket \xi_t \rrbracket)_{j+\frac{1}{2}}.$$

To estimate the right-hand side of (A.11), we would like to use the following Taylor expansion for the nonlinear terms:

$$(A.12a) \quad \begin{aligned} (f(u) - f(u_h))_t &= (f'(u)\xi)_t + (f'(u)\eta)_t - (R_1 e^2)_t \\ &= \partial_t f'(u)\xi + f'(u)\xi_t + \partial_t f'(u)\eta + f'(u)\eta_t - \partial_t R_1 e^2 - 2R_1 e e_t \\ &\triangleq \varphi_1 + \cdots + \varphi_6, \end{aligned}$$

$$(A.12b) \quad \begin{aligned} (f(u) - f(u_h^-))_t &= (f'(u)\xi^-)_t + (f'(u)\eta^-)_t - (R_2 (e^-)^2)_t \\ &= \partial_t f'(u)\xi^- + f'(u)\xi_t^- + \partial_t f'(u)\eta^- + f'(u)\eta_t^- - \partial_t R_2 (e^-)^2 - 2R_2 e^- e_t^- \\ &\triangleq \psi_1 + \cdots + \psi_6, \end{aligned}$$

where  $R_1 = \int_0^1 (1 - \mu) f''(u + \mu(u_h - u)) d\mu$  and  $R_2 = \int_0^1 (1 - \nu) f''(u + \nu(u_h^- - u)) d\nu$ . Therefore, the right-hand side of (A.11), denoted by  $\Upsilon$ , can be formulated as

$$(A.13) \quad \Upsilon = \mathcal{K}_1 + \cdots + \mathcal{K}_6,$$

where

$$\mathcal{K}_i = \sum_{j=1}^N \int_{I_j} \varphi_i(\xi_t)_x dx + \sum_{j=1}^N (\psi_i \llbracket \xi_t \rrbracket)_{j+\frac{1}{2}} \quad (i = 1, \dots, 6)$$

which will be estimated one by one below. Accordingly, (A.11) can be represented by

$$(A.14) \quad \frac{1}{2} \frac{d}{dt} \|\xi_t\|^2 \leq \Upsilon + \|\eta_{tt}\| \|\xi_t\| \leq \Upsilon + Ch^{k+1} \|\xi_t\|,$$

by the interpolation error estimates (3.5a). We estimate the term  $\mathcal{K}_1$  firstly, it follows from Young's inequality and the inverse property (ii) that

$$(A.15a) \quad \mathcal{K}_1 = \sum_{j=1}^N \int_{I_j} \partial_t f'(u) \xi (\xi_t)_x dx + \varepsilon \sum_{j=1}^N \llbracket \xi_t \rrbracket_{j+\frac{1}{2}}^2 + C_\star h^{-1} \|\xi\|^2,$$

where  $\varepsilon$  is a small positive constant. We would like to point out that the first integral term on the right-hand side of (A.15a) is intractable to get a sharp bound due to the hybrid of  $\xi$  and  $(\xi_t)_x$  and is left to be estimated later, together with other terms. A simple integration by parts gives us a bound for  $\mathcal{K}_2$

$$\begin{aligned} \mathcal{K}_2 &= -\frac{1}{2} \sum_{j=1}^N \int_{I_j} \partial_x f'(u) (\xi_t)^2 dx - \frac{1}{2} \sum_{j=1}^N f'(u_{j+\frac{1}{2}}) \llbracket \xi_t \rrbracket_{j+\frac{1}{2}}^2 \\ &\leq C_\star \|\xi_t\|^2 - \frac{\delta}{2} \sum_{j=1}^N \llbracket \xi_t \rrbracket_{j+\frac{1}{2}}^2, \end{aligned}$$

where we have used the assumption that  $f'(u(x, t)) \geq \delta > 0$ . Combining the above estimates for  $\mathcal{K}_1$  and  $\mathcal{K}_2$ , we arrive at

$$\begin{aligned} \mathcal{K}_1 + \mathcal{K}_2 &\leq C_\star \|\xi_t\|^2 + \sum_{j=1}^N \int_{I_j} \partial_t f'(u) \xi(\xi_t)_x dx + C_\star h^{-1} \|\xi\|^2 - \left(\frac{\delta}{2} - \varepsilon\right) \sum_{j=1}^N [\xi_t]_{j+\frac{1}{2}}^2 \\ \text{(A.15b)} \quad &\leq C_\star \|\xi_t\|^2 + \sum_{j=1}^N \int_{I_j} \partial_t f'(u) \xi(\xi_t)_x dx + C_\star h^{-1} \|\xi\|^2, \end{aligned}$$

where we have chosen  $\varepsilon$  to be small enough, for example,  $\varepsilon = \delta/4$ , to obtain the last inequality. Noting that  $\partial_t f'(u) = \partial_t f'(u_j) + (\partial_t f'(u) - \partial_t f'(u_j))$  and  $|\partial_t f'(u) - \partial_t f'(u_j)| \leq C_\star h$  on each element  $I_j$ , we thus have, by the property of the projection  $P_h^-$ , that

$$\text{(A.15c)} \quad \mathcal{K}_3 = \sum_{j=1}^N \int_{I_j} (\partial_t f'(u) - \partial_t f'(u_j)) \eta(\xi_t)_x dx \leq C_\star \|\eta\| \|\xi_t\| \leq C_\star h^{k+1} \|\xi_t\|,$$

where we have used Cauchy-Schwarz inequality, the inverse inequality (i) and the interpolation property (3.5a). Similarly, it follows from the property of the projection  $P_h^-$  and the inverse property (i), that

$$\text{(A.15d)} \quad \mathcal{K}_4 \leq C_\star \|\eta_t\| \|\xi_t\| \leq C_\star h^{k+1} \|\xi_t\|.$$

It is easy to show, for the high order term  $\mathcal{K}_5$ , that

$$\text{(A.15e)} \quad \mathcal{K}_5 \leq C_\star h^{-1} \|e\|_\infty (\|\xi\| + h^{k+1}) \|\xi_t\| \leq C_\star h^k \|e\|_\infty \|\xi_t\|,$$

where we have employed (3.20) in Corollary 3.5 to obtain the last inequality. For the last term, namely  $\mathcal{K}_6$ , we have

$$\text{(A.15f)} \quad \mathcal{K}_6 \leq C_\star h^{-1} \|e\|_\infty \|\xi_t\|^2 + C_\star h^k \|e\|_\infty \|\xi_t\|.$$

Therefore, by collecting the estimates (A.15b)-(A.15f) and (A.13) into (A.14), we get, after a straightforward application of Young's inequality, that

$$\text{(A.16)} \quad \frac{1}{2} \frac{d}{dt} \|\xi_t\|^2 \leq \mathcal{C}^2(e) \|\xi_t\|^2 + \sum_{j=1}^N \int_{I_j} \partial_t f'(u) \xi(\xi_t)_x dx + C_\star h^{-1} \|\xi\|^2 + Ch^{2k+2},$$

where  $\mathcal{C}(e) = C + C_\star h^{-1} \|e\|_\infty$  has been defined in (3.12b). Now we integrate the above inequality (A.16) with respect to time between 0 and  $t$  and take into account the initial error estimate (A.10) to obtain

$$\text{(A.17)} \quad \frac{1}{2} \|\xi_t\|^2 \leq \mathcal{C}^2(e) \int_0^t \|\xi_t\|^2 dt + \mathcal{Q} + C_\star h^{-1} \int_0^t \|\xi\|^2 dt + Ch^{2k+2},$$

where

$$\mathcal{Q} = \int_0^t \sum_{j=1}^N \int_{I_j} \partial_t f'(u) \xi(\xi_t)_x dx dt.$$

Let us work on the term  $\mathcal{Q}$ . To do that, we begin by using integration by parts with respect to time to get

$$\begin{aligned} \mathcal{Q} &= \sum_{j=1}^N \int_{I_j} \int_0^t \partial_t f'(u) \xi(\xi_x)_t dt dx \\ &= \sum_{j=1}^N \int_{I_j} \left\{ - \int_0^t \xi_x \partial_t (\partial_t f'(u) \xi) dt + \partial_t f'(u) \xi \xi_x \Big|_0^t \right\} dx \\ &= \sum_{j=1}^N \int_{I_j} \left\{ - \int_0^t \xi_x \partial_t (\partial_t f'(u) \xi) dt + (\partial_t f'(u) \xi \xi_x)(t) \right\} dx, \end{aligned}$$

since  $\xi(\cdot, 0) = 0$ . Next, by a similar analysis as that in the proof of (3.14b), we have that

$$\begin{aligned} \mathcal{Q} &\leq C_\star h^{-1} \int_0^t \|\mathbb{S}\| (\|\xi\| + \|\xi_t\|) dt + C_\star h^{-1} \|\mathbb{S}\| \|\xi\| \\ &\leq C_\star \int_0^t \|\xi_t\|^2 dt + C_\star h^{k+1} \|\xi_t\| + Ch^{2k+2} \\ \text{(A.18)} \quad &\leq C_\star \int_0^t \|\xi_t\|^2 dt + \frac{1}{4} \|\xi_t\|^2 + Ch^{2k+2}, \end{aligned}$$

where for the second inequality we have used a crude bound for  $\xi$  in (3.20) and a compact bound for  $\mathbb{S}$ , namely,  $\|\mathbb{S}\| \leq Ch\|\xi_t\| + Ch^{k+2}$  by taking into account the interpolation properties (3.5a); the last inequality is a direct application of Young's inequality.

Plugging the estimate (A.18) into (A.17) and taking into account (3.19) implied by the a priori assumption (3.18), we get that, for small enough  $h$

$$\text{(A.19)} \quad \frac{1}{4} \|\xi_t\|^2 \leq \tilde{C} \int_0^t \|\xi_t\|^2 dt + C_\star h^{-1} \int_0^t \|\xi\|^2 dt + Ch^{2k+2},$$

where  $C, C_\star$  and  $\tilde{C}$  are positive constants independent of  $h$ . Finally, a direct application of Gronwall's inequality yields that

$$\|\xi_t\| \leq Ch^{k+1} + C_\star h^{-\frac{1}{2}} \sqrt{\int_0^t \|\xi(s)\|^2 ds}.$$

This completes the proof of Lemma 3.7.

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