

Fourier analysis for discontinuous Galerkin and related methods

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Abstract

In this paper we review a series of recent work on using a Fourier analysis technique to study the stability and error estimates for the discontinuous Galerkin method and other related schemes. The advantage of this approach is that it can reveal instability of certain “bad” schemes; it can verify stability for certain good schemes which are not easily amendable to standard finite element stability analysis techniques; it can provide quantitative error comparisons among different schemes; and it can be used to study superconvergence and time evolution of errors for the discontinuous Galerkin method. We will briefly describe this Fourier analysis technique, summarize its usage in stability and error estimates for various schemes, and indicate the advantages and disadvantages of this technique in comparison with other finite element techniques.

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1 Introduction

In this review paper we are concerned with the stability analysis and error estimates for the discontinuous Galerkin (DG) method [10] and related methods, such as the spectral finite volume (SV) method [24, 25].

The DG method is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions to solve various partial differential equations (PDEs). The original design of the DG method is for solving the hyperbolic conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

where we have written the equation in one space dimension for simplicity. The DG method uses the finite dimensional space

$$\mathcal{V}_{\Delta x} = \{v : v \text{ is a polynomial of degree at most } p \text{ for } x \in I_j, j = 1, \dots, N\} \tag{1.2}$$

for both the numerical solution and the test function, where $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \dots, N$, is a mesh for the computational interval and we denote by $x_j = \frac{1}{2} (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ the center of the cell and by $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ the cell sizes, with the maximum size cell denoted by $\Delta x = \max_j \Delta x_j$. To be more specific, a semi-discrete DG method for solving (1.1) is defined by: find $u \in \mathcal{V}_{\Delta x}$ such that, for $j = 1, \dots, N$,

$$\int_{I_j} u_t v dx - \int_{I_j} f(u) v_x dx + \hat{f}_{j+\frac{1}{2}} v_{j+\frac{1}{2}}^- - \hat{f}_{j-\frac{1}{2}} v_{j-\frac{1}{2}}^+ = 0 \tag{1.3}$$

for all test functions $v \in \mathcal{V}_{\Delta x}$. Here v^\pm denotes the left and right limits of the discontinuous function v at the interface, and $\hat{f} = \hat{f}(u^-, u^+)$ is a monotone numerical flux. For the simple case $f'(u) \geq 0$, the numerical flux can be taken as the upwind flux $\hat{f} = f(u^-)$. The time discretization in (1.3) can be implemented by the nonlinearly stable Runge-Kutta methods [22, 13].

We can write the DG scheme in an equivalent form by performing an integration by parts

to the second term in (1.3), resulting in finding $u \in \mathcal{V}_{\Delta x}$ such that, for $j = 1, \dots, N$,

$$\int_{I_j} (u_t + f(u)_x) v dx + \left(\hat{f}_{j+\frac{1}{2}} - f(u_{j+\frac{1}{2}}^-) \right) v_{j+\frac{1}{2}}^- - \left(\hat{f}_{j-\frac{1}{2}} - f(u_{j-\frac{1}{2}}^+) \right) v_{j-\frac{1}{2}}^+ = 0 \quad (1.4)$$

for all test functions $v \in \mathcal{V}_{\Delta x}$. This equivalent formulation is more convenient for some situations, e.g. in the definition of the SV scheme in Section 4.

The first DG method was introduced in 1973 by Reed and Hill [19], in the framework of neutron transport equations. Cockburn et al. in a series of papers [7, 6, 5, 8] have established a framework to easily solve *nonlinear* time dependent hyperbolic conservation laws (1.1) using explicit, nonlinearly stable high order Runge-Kutta time discretizations [22] and DG discretization in space with exact or approximate Riemann solvers as interface numerical fluxes and TVB (total variation bounded) nonlinear limiters to achieve non-oscillatory properties for strong shocks. Later, the DG method is extended to solve convection-diffusion equations [1, 2, 9] and PDEs with higher derivatives, such as the KdV equations and other nonlinear dispersive type equations [27, 26]. For more information, we refer to the review paper [10] and the two special journal issues devoted exclusively to the recent development of DG methods [11, 12].

Being a finite element method, the DG method usually allows for stability analysis and error estimates using standard finite element techniques, which are easier and more general than that for finite difference schemes. For example, it is usually quite difficult to prove a cell entropy inequality (which implies and is stronger than L^2 stability of the numerical solution) for a finite difference scheme. One would need to assume one space dimension, convex conservation laws, and special limiters for at most second order accurate TVD (total variation diminishing) schemes, in order to prove such an inequality [15, 18]. However, it is quite easy to prove such a cell entropy inequality for the DG method, which applies to arbitrary triangulations in multi-dimensions, general nonlinear conservation laws, and arbitrary order of accuracy, without the need to use any limiters [14]. The technique in [14] is later generalized to prove similar cell entropy inequality and the resulting L^2 stability for DG methods applied to more general PDEs, such as the convection-diffusion equations [9]

and nonlinear dispersive wave PDEs [27, 26]. For more details, see for example the lecture notes [21]. Likewise, standard finite element techniques usually can be used to obtain error estimates for DG methods, e.g. [9, 30, 31].

On the other hand, Fourier analysis is a tool commonly used for finite difference methods. This analysis has more restrictions than the finite element techniques mentioned above: the Fourier analysis can only be used on linear PDEs with constant coefficients, uniform meshes, and periodic boundary conditions. It is also technically difficult to carry out except for low order methods (small p in the space $\mathcal{V}_{\Delta x}$ in (1.2)), because of the difficulty in manipulating large matrices. However, as we will review in this paper, the Fourier analysis does have several advantages over the standard finite element techniques. It can be used to analyze some of the “bad” schemes; it can be used for stability analysis for some of the non-standard methods such as the SV method, which belongs to the class of Petrov-Galerkin methods and cannot be easily amended to the standard finite element analysis framework; it can provide quantitative error comparisons among different schemes; and it can be used to prove superconvergence and time evolution of errors for the DG method.

The organization of this paper is as follows. In Section 2, we will briefly describe the Fourier analysis procedure. The application of the Fourier analysis to study “bad” DG and SV schemes for the heat equation is reviewed in Section 3. In Section 4, we review the application of the Fourier analysis in providing stability analysis for the SV method. Section 5 is devoted to a review on quantitative error estimates and comparison between DG, SV and central DG methods. In Section 6, we review the application of the Fourier analysis in the study of superconvergence of DG methods for hyperbolic and convection-diffusion equations. Finally, we give concluding remarks in Section 7.

2 A description of the Fourier analysis

We demonstrate the Fourier analysis using the linear version of the PDE (1.1), namely $f(u) = u$ (it would of course be the same if $f(u) = cu$ for a constant c):

$$u_t + u_x = 0. \quad (2.1)$$

We also assume periodic boundary conditions and uniform mesh. We first notice that, after picking a local basis for the space $\mathcal{V}_{\Delta x}$ in (1.2) and inverting a local $(p+1) \times (p+1)$ mass matrix (which could be done by hand), the DG scheme (1.3) can be written as

$$\frac{d}{dt}u_j = \frac{1}{\Delta x} (Au_{j-1} + Bu_j) \quad (2.2)$$

where u_j is a small vector of length $p+1$ containing the coefficients of the solution u in the local basis inside cell I_j , and A and B are $(p+1) \times (p+1)$ constant matrices. If we choose the degrees of freedom for the p -th degree polynomial inside the cell I_j as the point values of the solution, denoted by

$$u_{j+\frac{2i-p}{2(p+1)}}, \quad i = 0, \dots, p,$$

at the $p+1$ equally spaced points

$$\left(j + \frac{2i-p}{2(p+1)} \right) \Delta x, \quad i = 0, \dots, p,$$

then the DG scheme written in terms of these degrees of freedom becomes a finite difference scheme on a globally uniform mesh (with a mesh size $\Delta x/(p+1)$), however they are not standard finite difference schemes because each point in the group of $p+1$ points belonging to the cell I_j obeys a different form of the finite difference scheme. Let us now demonstrate the procedure using the piecewise linear case $p = 1$. The degrees of freedom are now the point values at the $2N$ uniformly spaced points

$$u_{j-\frac{1}{4}}, \quad u_{j+\frac{1}{4}}, \quad j = 1, \dots, N.$$

The solution inside the cell I_j is then represented by

$$u(x) = u_{j-\frac{1}{4}}\phi_{j-\frac{1}{4}}(x) + u_{j+\frac{1}{4}}\phi_{j+\frac{1}{4}}(x)$$

where $\phi_{j-\frac{1}{4}}(x)$ is the linear polynomial which equals 1 at the point $(j - \frac{1}{4})\Delta x$ and equals 0 at the point $(j + \frac{1}{4})\Delta x$, and similarly $\phi_{j+\frac{1}{4}}(x)$ is the linear polynomial which equals 0 at the point $(j - \frac{1}{4})\Delta x$ and equals 1 at the point $(j + \frac{1}{4})\Delta x$. With this representation, taking the test functions v also as $\phi_{j-\frac{1}{4}}(x)$ and $\phi_{j+\frac{1}{4}}(x)$, respectively, and inverting the small 2×2 mass matrix by hand, we obtain the finite difference representation of the DG method as (2.2) with

$$u_j = \begin{pmatrix} u_{j-\frac{1}{4}} \\ u_{j+\frac{1}{4}} \end{pmatrix}, \quad A = \frac{1}{4} \begin{pmatrix} -5 & 15 \\ 1 & -3 \end{pmatrix}, \quad B = \frac{1}{4} \begin{pmatrix} -7 & -3 \\ 11 & -9 \end{pmatrix}. \quad (2.3)$$

For a Fourier analysis, we make an ansatz of the form

$$\begin{pmatrix} u_{j-\frac{1}{4}}(t) \\ u_{j+\frac{1}{4}}(t) \end{pmatrix} = \begin{pmatrix} \hat{u}_{k,-\frac{1}{4}}(t) \\ \hat{u}_{k,\frac{1}{4}}(t) \end{pmatrix} e^{ikx_j} \quad (2.4)$$

and substitute the ansatz (2.4) into the DG scheme (2.2)-(2.3) to find the evolution equation for the coefficient vector as

$$\begin{pmatrix} \hat{u}'_{k,-\frac{1}{4}}(t) \\ \hat{u}'_{k,\frac{1}{4}}(t) \end{pmatrix} = G(k, \Delta x) \begin{pmatrix} \hat{u}_{k,-\frac{1}{4}}(t) \\ \hat{u}_{k,\frac{1}{4}}(t) \end{pmatrix} \quad (2.5)$$

where $'$ denotes the time derivative, and the amplification matrix $G(k, \Delta x)$ is given by

$$G(k, \Delta x) = \frac{1}{\Delta x} (A e^{-ik\Delta x} + B). \quad (2.6)$$

with the matrices A and B defined by (2.3). Now, both the stability analysis and error estimates can be performed based on the matrix $G(k, \Delta x)$. For stability, we would need $\|e^{G(k, \Delta x)t}\|$ to be uniformly bounded with respect to the two parameters k and Δx . Usually, instability shows up for the higher frequency modes with larger k . For accuracy, we would only need to look low modes with smaller k , e.g. $k = 1$.

3 Analysis for “bad” schemes

For quite a long time, the DG method was used only for hyperbolic conservation laws (1.1). However, in applications, one often encounters convection dominated PDEs which contain

higher order derivatives, such as the convection diffusion equation

$$u_t + f(u)_x = (a(u)u_x)_x \tag{3.1}$$

containing second order derivatives, where $a(u) \geq 0$, and convection dispersion equations containing third or higher order derivatives. The recent development of stable DG methods for such PDEs is one of the main reasons that the DG method has been becoming more popular [11, 12].

For the simple heat equation

$$u_t = u_{xx} \tag{3.2}$$

which is a special case of (3.1) with $f(u) = 0$ and $a(u) = 1$, a direct and seemingly logical adaptation of the DG method for the conservation law (1.1) can be formulated as follows. If we formally identify $f(u)$ in (1.1) to be $-u_x$, then the heat equation (3.2) “becomes” a conservation law (1.1). Thus the DG scheme given by (1.3) can be adapted to a DG scheme for the heat equation (3.2) if we change $f(u)$ to $-u_x$ and replace the numerical flux $\hat{f}_{j+\frac{1}{2}}$ by $-\hat{u}_{x_{j+\frac{1}{2}}}$. Of course, the upwinding principle, which guided the design of the numerical flux $\hat{f}_{j+\frac{1}{2}}$ for the hyperbolic conservation law (1.1), is no longer valid for the heat equation (3.2). Considering that information propagates in both directions equally for the heat equation (3.2), it would be reasonable to take the numerical flux $\hat{u}_{x_{j+\frac{1}{2}}}$ as a central flux

$$\hat{u}_{x_{j+\frac{1}{2}}} = \frac{1}{2} \left((u_x)_{j+\frac{1}{2}}^- + (u_x)_{j+\frac{1}{2}}^+ \right).$$

The resulting scheme can be implemented similar to (2.2) and takes the form

$$\frac{d}{dt}u_j + \frac{1}{\Delta x^2} (Au_{j-1} + Bu_j + Cu_{j+1}) = 0 \tag{3.3}$$

where u_j is again a small vector of length $p + 1$ containing the coefficients of the solution u in the local basis inside cell I_j , and A, B, C are $(p + 1) \times (p + 1)$ constant matrices.

This scheme was first described in [20]. It is verified numerically in [20], see also [10, 28], that this scheme leads to numerically stable but inconsistent solutions. In Figure 3.1 we

plot the numerical solution with 40 and 320 cells versus the exact solution, for the two cases $p = 1$ and 2 (piecewise linear and piecewise quadratic cases) at $t = 0.7$. We can see that the numerical solutions seem to converge with mesh refinements but have $O(1)$ errors when comparing with the exact solution.

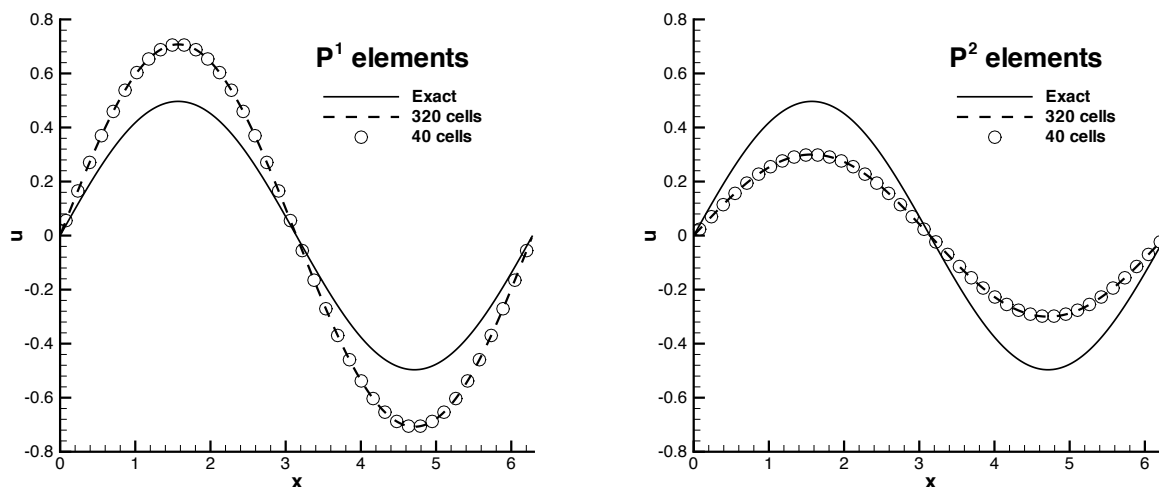


Figure 3.1: The numerically inconsistent DG method applied to the heat equation (3.2) with an initial condition $u(x, 0) = \sin(x)$. $t = 0.7$. Numerical solutions with 40 cells (circles) and 320 cells (dashed lines), versus the exact solution (solid line). Left: $p = 1$; Right: $p = 2$.

It is very dangerous that the scheme produces numerically stable but completely incorrect solution. If one does not know the exact solution, even if one does a mesh refinement study, one could still conclude incorrectly that the method is convergent.

It is not easy to use standard finite element techniques to analyze this or similar “bad” schemes. On the other hand, the Fourier analysis technique outlined in Section 2 can be readily applied to analyze this scheme [28], with a somewhat surprising result. One would expect, based on the numerical evidence in Figure 3.1, that the scheme should be stable but inconsistent. However, the analysis shows that it is completely the other way round: the scheme is consistent, but it is unstable. In fact, for the $p = 1$ case, we can easily obtain

$$A = \begin{pmatrix} 5 & -5 \\ -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \\ -5 & 5 \end{pmatrix}. \quad (3.4)$$

for the matrices in the scheme (3.3). It can be shown [28] that this scheme is consistent by the usual truncation error analysis for finite difference schemes based on Taylor expansions. We can also easily verify that the two eigenvalues of the amplification matrix $G(k, \Delta x)$ (see (2.6)) are

$$\lambda_1 = -\frac{6}{\Delta x^2} (1 - \cos(k\Delta x)), \quad \lambda_2 = 0. \quad (3.5)$$

which are both non-positive. However, the non-positivity of (the real part of) the eigenvalues of $G(k, \Delta x)$ is only a necessary but not a sufficient condition for stability. With a tedious computation in [28], it is possible to explicitly write out the matrix $e^{G(k, \Delta x)t}$, and compute its L^2 norm

$$\|e^{G(k, \Delta x)t}\| = \sqrt{\frac{1}{18} \left(5 + 8\alpha + 5\alpha^2 + \frac{1-\alpha}{\beta} \left[8(1-\alpha) + \sqrt{(8+5\beta) [8(1-\alpha)^2 + \beta(5+26\alpha+5\alpha^2)]} \right] \right)},$$

where

$$\alpha = e^{\lambda_1 t}, \quad \beta = 1 - \cos(\xi)$$

with $\xi = k\Delta x$. If we take $\beta = \frac{\Delta x^2}{t}$, then it is easy to see that

$$\|e^{G(k, \Delta x)t}\| = O\left(\frac{1}{\Delta x}\right)$$

which is unbounded when $\Delta x \rightarrow 0$. Hence the semi-discrete scheme is not stable.

We remark that this instability is very mild, and it grows at most linearly with a mesh refinement. Also, further analysis, by looking at the eigenvectors, shows that this instability only occurs when the initial condition is chosen so that the slope of the linear function in each cell is of order $O\left(\frac{1}{\Delta x}\right)$. Since such initial conditions are not physical, they can only occur at the round-off level and they grow slower than linearly with the number of time steps. This explains why we have never seen such instability in the numerical experiments: our meshes are simply not refined enough. However, this instability accounts for the apparent contradiction of a consistent local truncation error and a global $O(1)$ error of the numerical solution.

In [23], a similar analysis is performed to a “bad” SV scheme applied to the heat equation (3.2), with similar conclusions as those for the “bad” DG scheme mentioned above.

4 Stability analysis for good schemes

While the Fourier analysis played a crucial role in the analysis of instability of the “bad” scheme in the previous section, in this section we review its application in the stability analysis of good schemes. For most DG schemes, it will be more convenient to use finite element techniques, such as the one in [14], to obtain stability results, which would apply to more general setting (nonlinear PDEs, multi-dimensions, arbitrary order of accuracy). However, for some schemes, especially those not based on a Galerkin framework, such as the SV schemes in [24, 25], the standard finite element technique is difficult to apply. The Fourier analysis technique, on the other hand, can be handily applied.

The SV method [24, 25] is a Petrov-Galerkin method, namely, the solution space and the test function space are different. The solution space is still $\mathcal{V}_{\Delta x}$ given by (1.2). However the test function space is the collection of piecewise constant functions on a sub-partition of the cell I_j :

$$\mathcal{W}_{\Delta x} = \{v : v \text{ is a constant for } x \in I_{i,j}, \quad i = 0, 1, \dots, p, \quad j = 1, \dots, N\} \quad (4.1)$$

where $I_{0,j}, \dots, I_{p,j}$ is a partition of cell I_j into $p+1$ subcells. The semidiscrete SV scheme is thus: find $u \in \mathcal{V}_{\Delta x}$ such that, for $j = 1, \dots, N$, (1.4) holds for all test functions $v \in \mathcal{W}_{\Delta x}$. Clearly, the method strongly depends on the specific choice of the partition of cell I_j into $p+1$ subcells.

It is not easy to use standard finite element technique to prove the stability of the SV method. In fact, to our knowledge, the only stability result about the SV method is obtained using the Fourier analysis in [29]. After choosing a basis for the finite element space and with some algebraic manipulation, the SV method for the linear conservation law (2.1) can also be written in the form (2.2). For the piecewise linear $p = 1$ case, the SV scheme is given

by (2.2) with

$$u_j = \begin{pmatrix} u_{j-\frac{1}{4}} \\ u_{j+\frac{1}{4}} \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}. \quad (4.2)$$

It can be verified [29] that the amplification matrix $G(k, \Delta x)$ (see (2.6)) has two eigenvalues with non-positive real parts. Also, the matrix $G(k, \Delta x)$ is diagonalizable, with the eigenvector matrix R and its inverse R^{-1} satisfying

$$\|R\| = \sqrt{2(-\cos \xi + 9 + d + \sqrt{(\cos \xi - 9 - d)^2 - 32d})}$$

and

$$\|R^{-1}\| = \frac{1}{8} \sqrt{\frac{1}{d}(-\cos \xi + 9 + d + \sqrt{(\cos \xi - 9 - d)^2 - 32d})},$$

with

$$d = \sqrt{137 - 66 \cos \xi - 7 \cos^2 \xi}.$$

It is easy to see that both $\|R\|$ and $\|R^{-1}\|$ are uniformly bounded with respect to the parameter ξ . Thus the stability of the semi-discrete scheme is established.

In [23], a similar analysis is performed to two good SV schemes, proving their stability when applied to the heat equation (3.2).

5 Quantitative error comparisons for different schemes

Another advantage of the Fourier analysis technique is that it can provide quantitative comparison of errors for different schemes.

For example, in [29], such a comparison is performed between the DG and SV methods for solving the linear conservation law (2.1) with an initial condition

$$u(x, 0) = \sin(x). \quad (5.1)$$

Both schemes are given by (2.2), with the matrices A and B given by (2.3) and (4.2), respectively. For a low mode initial condition

$$u_{j\pm\frac{1}{4}}(0) = e^{ix_{j\pm\frac{1}{4}}}, \quad (5.2)$$

whose imaginary part is our initial condition (5.1), algebraic manipulations in [29] indicate that the DG solution satisfies

$$\text{Im}\{u_{j-\frac{1}{4}}(t)\} = \left(1 - \frac{\Delta x^2}{24}\right) \sin(x_{j-\frac{1}{4}} - t) + O(\Delta x^3),$$

indicating a second order convergence to the exact solution $u(x, t) = \sin(x - t)$ of (2.1) under the initial condition (5.1), with a leading error at the size of

$$\frac{\Delta x^2}{24} \sin(x_{j-\frac{1}{4}} - t),$$

while the SV solution satisfies

$$\text{Im}\{u_{j-\frac{1}{4}}(t)\} = \left(1 - \frac{\Delta x^2}{16}\right) \sin(x_{j-\frac{1}{4}} - t) + O(\Delta x^3),$$

indicating a second order convergence to the exact solution with a leading error at the size of

$$\frac{\Delta x^2}{16} \sin(x_{j-\frac{1}{4}} - t).$$

This leads to the conclusion that both methods are second order accurate, at least for the linear problem (2.1) with a uniform mesh. Their leading errors for the first mode (i.e. for the $\sin(x)$ initial condition) have a ratio 2:3, that is the SV method has a 50% larger error than the DG method on the same mesh. Numerical results in [29] verify very accurately this quantitative analysis. Similar analysis and comparison have also been performed in [29] for the third order schemes ($p = 2$).

Such analysis can reveal the relative efficiency of different methods. It has also been used in [28] to compare the DG method of Baumann and Oden [2] and the local DG (LDG) method of Cockburn and Shu [9] for solving the heat equation (3.2). Another application is in [17], in which this technique is used to compare the regular DG method and the central DG method [16] for their errors. In [23], this analysis is used to compare the accuracy of two different formulations of the SV method for solving the heat equation (3.2).

6 Analysis for superconvergence and error evolution

A very interesting application of the Fourier analysis technique is the recent work of Cheng and Shu [3, 4] on the analysis of superconvergence and error evolution for the DG method used on the hyperbolic conservation law (1.1) and the convection diffusion equation (3.1). The analysis itself is performed only for the linear PDEs (2.1) and (3.2), however numerical experiments performed in [3, 4] indicate that the conclusions hold true for more general cases including variable coefficient and nonlinear PDEs, systems and two dimensions.

We define a special projection $P_h^- u$ to be a projection of the exact solution u into the finite element space $\mathcal{V}_{\Delta x}$, such that

$$\int_{I_j} P_h^- u v dx = \int_{I_j} u v dx \quad (6.1)$$

for any $v \in P^{p-1}$ on I_j , where p is the polynomial degree of the DG solution and P^m denotes the set of polynomials of degree at most m , and

$$(P_h^- u)^- = u^- \quad \text{at } x_{j+1/2}. \quad (6.2)$$

Similarly, we can define the projection $P_h^+ u$. These special projections are used in the error estimates of the DG methods to derive optimal L^2 error bounds in the literature, e.g. in [30]. Cheng and Shu have shown in [3, 4] that indeed the numerical solution is closer to one of these special projections of the exact solution than to the exact solution itself. If we denote e to be the error between the exact solution and numerical solution, and \bar{e} to be the error between the numerical solution and the projection of the exact solution, then, using the Fourier analysis technique, Cheng and Shu have shown in [3] that, for the hyperbolic PDE (2.1) and the piecewise linear case $p = 1$,

$$\|\bar{e}(\cdot, t)\|_{L^2} \leq C_1 (t + 1) \Delta x^{5/2}, \quad (6.3)$$

and

$$\|e(\cdot, t)\|_{L^2} \leq C_1 t \Delta x^{5/2} + C_2 \Delta x^2, \quad (6.4)$$

where C_1 and C_2 are constants which do not depend on t or Δx . This leads to the following conclusions:

1. The error \bar{e} , which is the difference between the numerical solution and the projection $P_h^- u$ of the exact solution, is superconvergent by half an order (it converges at 2.5 order rather than second order), and it grows with time at most linearly.
2. The error e between the numerical solution and the exact solution consists of two parts, a second order part which does not grow with time, and a superconvergent 2.5 order part which grows with time at most linearly.
3. Consequently, the error e does not grow with time during a long period $0 < t < O\left(\frac{1}{\sqrt{h}}\right)$.

Numerical results in [3] have verified these conclusions, and have shown that they hold true for more general cases: non-uniform meshes, higher order schemes, variable coefficient and nonlinear problems, systems, and two dimensions.

In [4], similar analysis and numerical experiments are performed for the convection diffusion equation (3.1).

7 Concluding remarks

We have reviewed the application of a Fourier analysis technique for analyzing certain unstable “bad” discontinuous Galerkin (DG) schemes, for giving stability analysis to certain Petrov-Galerkin schemes such as the spectral finite volume (SV) schemes which are not easily amendable to standard finite element stability analysis approaches, for quantitative comparison of errors among different schemes, and for analysis of superconvergence and time evolution of errors for DG schemes. Even though the Fourier analysis technique suffers from restrictions such as the requirement of uniform meshes, periodic boundary conditions, linear PDEs with constant coefficients, and applicability only to lower order schemes (because of the complexity of analyzing large matrices), this review indicates that it can still play an

important role in many situations where standard finite element techniques do not easily apply.

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