

## Discontinuous Galerkin method for a class of elliptic multi-scale problems

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### SUMMARY

In this paper we develop a discontinuous Galerkin (DG) method for solving a class of second order elliptic multi-scale problems. The main ingredient of this method is to use a non-polynomial multi-scale approximation space in the DG method to capture the multi-scale solutions using coarse meshes without resolving the fine-scale structure of the solution. We perform analysis on the approximation, stability and error estimates, and provide numerical results to demonstrate the proposed method.

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KEY WORDS: Discontinuous Galerkin method; multi-scale approximation

### 1. Introduction

In this paper, we design a discontinuous Galerkin (DG) method based on non-polynomial basis functions [12] for solving a class of multi-scale second order elliptic partial differential equations

$$-\nabla \cdot (a^\varepsilon(x)\nabla u) = f(x) \quad \text{in } \Omega \quad (1)$$

with the boundary condition

$$u = 0 \quad \text{on } \partial\Omega,$$

where  $\Omega$  is a rectangular domain and the coefficient  $a^\varepsilon(x)$  is an oscillatory function involving a small scale  $\varepsilon$ , for example it could be  $a^\varepsilon(x) = a(x, \frac{x}{\varepsilon})$ , however the force function  $f(x)$  does not involve this small scale. A standard discontinuous Galerkin method [1, 10] using piecewise polynomial approximation spaces can be used to solve (1), however it would require a very fine mesh to resolve the small scale  $\varepsilon$  in the solution, which might be too costly or impossible on today's computers.

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<sup>\dagger</sup>Research of this author is partially supported by ARO grant W911NF-04-1-0291 and NSF grant DMS-0510345.

We will explore the usage of non-polynomial multi-scale approximation spaces in the DG method, consisting of special basis functions constructed from the PDEs to capture the micro-scale structure information of the solutions so that the solutions of multi-scale PDEs can be well approximated even under coarse meshes. Similar approaches have been used by Babuška *et al.* in [3, 4, 5] and by Hou and Wu in [11] for continuous finite element methods, in which both theoretical proofs and numerical experiments were provided to show that more accurate results can be obtained by using the multi-scale approximation spaces instead of piecewise polynomial spaces. We extend the methodology to the DG methods, providing analysis on the approximation, stability and error estimates, and numerical results to demonstrate the proposed method. The DG method, comparing with the continuous finite element methods, has the advantage of more flexibility on non-conforming meshes with hanging nodes, easier  $h$ - $p$  adaptivity, and more importantly easier patching of the local multi-scale approximation spaces in different cells, especially for multi-dimensional problems, since no continuity across cell interfaces is required.

This paper is organized as follows. In Section 2, we give a brief review of the DG method for elliptic PDEs. In Section 3, the new multi-scale approximation spaces are introduced and some approximation results are given. In Section 4, stability and approximation properties are proven for the DG method based on the multi-scale approximation spaces. Then, error estimates are obtained in Section 5. In Section 6, a numerical example in one space dimension is provided to demonstrate the performance of the DG method. Concluding remarks and plans for future work are given in Section 7.

## 2. The discontinuous Galerkin method

In this section we give a brief review of some discontinuous Galerkin finite element methods for elliptic problems [1, 2, 7, 8]. For the sake of simplicity, we only consider the second order elliptic problem in 1-D

$$-(a(x)u_x)_x = f(x), \quad 0 \leq x \leq 1 \quad (2)$$

where  $a(x) \geq 0$ . We first rewrite the problem as a first order system

$$-(aw)_x = f, \quad w - u_x = 0. \quad (3)$$

Assuming  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ ,  $j = 1, \dots, N$ , is a partition of  $[0, 1]$ , we multiply both equations by test functions  $v$  and  $\tau$ , respectively, and integrate over the cell  $I_j$ , followed by an integration by parts to obtain the weak formulation

$$\int_{I_j} awv_x dx - a_{j+\frac{1}{2}}w_{j+\frac{1}{2}}v_{j+\frac{1}{2}} + a_{j-\frac{1}{2}}w_{j-\frac{1}{2}}v_{j-\frac{1}{2}} = \int_{I_j} fvdx, \quad (4)$$

$$\int_{I_j} w\tau dx + \int_{I_j} u\tau_x dx - u_{j+\frac{1}{2}}\tau_{j+\frac{1}{2}} + u_{j-\frac{1}{2}}\tau_{j-\frac{1}{2}} = 0. \quad (5)$$

The general formulation of the DG method for the elliptic problem (2) is: Find  $U, W \in V_h$  such that

$$\int_{I_j} aWv_x dx - a_{j+\frac{1}{2}}\hat{W}_{j+\frac{1}{2}}v_{j+\frac{1}{2}}^- + a_{j-\frac{1}{2}}\hat{W}_{j-\frac{1}{2}}v_{j-\frac{1}{2}}^+ = \int_{I_j} fvdx, \quad (6)$$

$$\int_{I_j} W \tau dx + \int_{I_j} U \tau_x dx - \hat{U}_{j+\frac{1}{2}} \tau_{j+\frac{1}{2}}^- + \hat{U}_{j-\frac{1}{2}} \tau_{j-\frac{1}{2}}^+ = 0 \quad (7)$$

for all test functions  $v, \tau \in V_h$ . Here  $V_h$  is a finite dimensional finite element space containing functions which are discontinuous across cell interfaces. For traditional DG methods, these functions are piecewise polynomials. In this paper, we consider multi-scale approximation spaces. The equations (6)-(7) are called the *flux formulation*. We can also eliminate the auxiliary variable  $W$  from the *flux formulation* (6)-(7) to obtain a typical finite element formulation, which is called the *primal formulation*.

There are many choices for the fluxes  $\hat{U}$  and  $\hat{W}$ . Different choices will give us different DG methods. We will just give two examples below.

First, the *flux formulation* of the LDG method [9, 7, 8] is: (6)-(7) with the following choice of fluxes

$$\hat{U}_{j+\frac{1}{2}} = U_{j+\frac{1}{2}}^-, \quad \hat{W}_{j+\frac{1}{2}} = W_{j+\frac{1}{2}}^+ + [U]_{j+\frac{1}{2}}; \quad \text{or} \quad \hat{U}_{j+\frac{1}{2}} = U_{j+\frac{1}{2}}^+, \quad \hat{W}_{j+\frac{1}{2}} = W_{j+\frac{1}{2}}^- + [U]_{j+\frac{1}{2}}$$

where  $[U] = U^+ - U^-$ .

Secondly, the *primal formulation* of the Babuška-Zlámal method [6] is: Find  $U \in V_h$  such that

$$\sum_{j=1}^N \int_{I_j} U_x v_x dx + \sum_{j=0}^N \left( a_{j+\frac{1}{2}} \frac{\eta_{j+\frac{1}{2}}}{\Delta x} [U]_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \right) = \sum_{j=1}^N \int_{I_j} f v dx,$$

for all test functions  $v \in V_h$ , where  $\eta_{j+\frac{1}{2}}$  is a positive constant. There is a theoretical requirement for the lower bound  $\eta_0 = \inf_j \eta_{j+\frac{1}{2}}$  for maintaining stability and rates of convergence of this method:  $\eta_0 \approx \Delta x^{-2k}$  if  $\dim V_h|_{I_j} = k + 1$ .

For the multi-scale problem (1), a direct numerical solution of this problem by the DG method based on standard piecewise polynomial spaces is difficult when  $\varepsilon$  is very small, even with modern supercomputers. The reason is that piecewise polynomial spaces do not approximate well the solution of (1) when the mesh is not refined enough. We need to use a very fine mesh in the computation to get good numerical results, hence the computational cost will be huge.

The core idea of the DG method in this paper is to construct the finite element basis functions which can capture the small scale information. In this DG method, the basis functions are constructed from the elliptic differential operator, hence they are adapted to the local properties of the differential operator in each cell. This idea of the construction of basis functions was first used by Babuška *et al.* to 1-D problems [4, 5] and to a special class of 2-D problems [3], and also by Hou and Wu to some 2-D elliptic problems with rapidly oscillating coefficients [11], in the context of continuous finite elements. For continuous finite element methods, there exists difficulty in enforcing continuity of basis functions at the 2-D cell interfaces. However, for the DG methods, continuity at the element interfaces is not needed, thus eliminating this difficulty.

### 3. Multi-scale approximation spaces

The multi-scale approximation spaces are constructed as below:

$$S^1 = \{\phi : \nabla \cdot (a^\varepsilon(x) \nabla \phi)|_K = 0\} \quad (8)$$

and

$$S^k = \{\phi : \nabla \cdot (a^\varepsilon(x)\nabla\phi)|_K \in P^{k-2}(K)\} \quad \text{for } k \geq 2 \quad (9)$$

where  $K$  denotes the cell in the space discretization. If we define  $P^{-1}(K) = \{0\}$ , (8) can also be denoted by (9) for  $k = 1$ .

First consider the 1-D elliptic problem

$$-(a^\varepsilon(x)u_x)_x = f(x), \quad 0 \leq x \leq 1 \quad (10)$$

with the boundary condition

$$u(0) = u(1) = 0, \quad (11)$$

where

$$0 < \alpha \leq a^\varepsilon(x) \leq \beta < +\infty. \quad (12)$$

The multi-scale approximation space (9) for this problem is explicitly given

$$S^k = \left\{ v : v|_{I_j} \in \text{span} \left\{ 1, \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi, \int_{x_j}^x \frac{\xi - x_j}{a^\varepsilon(\xi)} d\xi, \dots, \int_{x_j}^x \frac{(\xi - x_j)^{k-1}}{a^\varepsilon(\xi)} d\xi \right\} \right\} \quad (13)$$

where  $x_j = \frac{1}{2} (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ . This multi-scale approximation space (13) has a very good approximation property for the solution of the problem (10). This can be shown in the following lemmas.

**Lemma 3.1:** Let  $u(x)$  be the exact solution of (10). There exists some  $v(x) \in S^k (k \geq 1)$  such that for all  $j$ :

$$|u(x) - v(x)| \leq C(\alpha, \beta) \|f\|_{H^{k-1}(I_j)} (\Delta x_j)^{k+1/2}, \quad \forall x \in I_j \quad (14)$$

where  $C(\alpha, \beta)$  is a constant depending on  $\alpha, \beta$  and independent of  $\Delta x_j$  and  $\varepsilon$ .

**Proof:** The proof is inspired by the fundamental theorem of calculus

$$\begin{aligned} u(x) &= u(x_j) + \int_{x_j}^x u'(y) dy = u(x_j) + \int_{x_j}^x a^\varepsilon(y) u'(y) \frac{1}{a^\varepsilon(y)} dy \\ &= u(x_j) + \int_{x_j}^x a^\varepsilon(y) u'(y) d \left( \int_x^y \frac{1}{a^\varepsilon(\xi)} d\xi \right) \\ &= u(x_j) + a^\varepsilon(x_j) u'(x_j) \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi - \int_{x_j}^x \left( \int_x^y \frac{1}{a^\varepsilon(\xi)} d\xi \right) d(a^\varepsilon(y) u'(y)) \\ &= u(x_j) + a^\varepsilon(x_j) u'(x_j) \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi - \int_{x_j}^x \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) f(y) dy \end{aligned}$$

where in the last equality we have used the differential equation (10). When  $k = 1$ , let  $v = u(x_j) + a^\varepsilon(x_j) u'(x_j) \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi$ . We can see  $v \in S^1$ . And we have

$$\begin{aligned} |u(x) - v(x)| &= \left| - \int_{x_j}^x \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) f(y) dy \right| \leq \left| \int_{x_j}^x \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right)^2 dy \right|^{1/2} \left| \int_{x_j}^x f^2(y) dy \right|^{1/2} \\ &\leq \frac{1}{\alpha} (\Delta x_j)^{3/2} \|f\|_{L^2(I_j)} \end{aligned} \quad (15)$$

for all  $x \in I_j$ .

When  $k \geq 2$ , we can find some  $p \in P^{k-2}(I_j)$  such that

$$\|f - p\|_{L^2(I_j)} \leq C|f|_{H^{k-1}(I_j)}(\Delta x_j)^{k-1}.$$

Here and below  $C$  is a generic constant independent of the functions and meshes sizes. Let  $v = u(x_j) + a^\varepsilon(x_j)u'(x_j) \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi - \int_{x_j}^x \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) p(y) dy$ . We can easily verify that  $v \in S^k$  since

$$\int_{x_j}^x \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) p(y) dy = \int_{x_j}^x \left( \int_{x_j}^\xi p(y) dy \right) \frac{1}{a^\varepsilon(\xi)} d\xi$$

and  $\int_{x_j}^\xi p(y) dy \in P^{k-1}(I_j)$ . Then we can obtain

$$\begin{aligned} |u(x) - v(x)| &= \left| - \int_{x_j}^x \left( \int_x^y \frac{1}{a^\varepsilon(\xi)} d\xi \right) (f(y) - p(y)) dy \right| \\ &\leq \left| \int_{x_j}^x \left( \int_x^y \frac{1}{a^\varepsilon(\xi)} d\xi \right)^2 dy \right|^{1/2} \left| \int_{x_j}^x (f(y) - p(y))^2 dy \right|^{1/2} \\ &\leq \frac{1}{\alpha} (\Delta x_j)^{3/2} \|f - p\|_{L^2(I_j)} \leq C(\alpha, \beta) (\Delta x_j)^{k+1/2} |f|_{H^{k-1}(I_j)} \end{aligned} \quad (16)$$

for all  $x \in I_j$ .

Combining (15) and (16) will finish the proof.  $\square$

Next, we estimate the approximation rate in the  $L^2$  norm.

**Lemma 3.2:** Let  $u(x)$  be the exact solution of (10) and  $P_h$  be the  $L^2$  projection operator into the space  $S^k$ . There exists a constant  $C(\alpha, \beta)$  such that:

$$\|u - P_h u\|_{L^2(0,1)} \leq C(\alpha, \beta) |f|_{H^{k-1}(0,1)} (\Delta x)^{k+1}. \quad (17)$$

**Proof:** We choose the same  $v$  as that in Lemma 3.1. Squaring both sides of (14) and then integrating in the cell  $I_j$ , we obtain, for any  $j$ ,

$$\|u - v\|_{L^2(I_j)}^2 \leq C(\alpha, \beta) |f|_{H^{k-1}(I_j)}^2 (\Delta x_j)^{2k+2}.$$

Therefore,

$$\begin{aligned} \|u - P_h u\|_{L^2(0,1)}^2 &\leq \|u - v\|_{L^2(0,1)}^2 = \sum_j \|u - v\|_{L^2(I_j)}^2 \\ &\leq C(\alpha, \beta) \sum_j |f|_{H^{k-1}(I_j)}^2 (\Delta x_j)^{2k+2} \leq C(\alpha, \beta) |f|_{H^{k-1}(0,1)}^2 (\Delta x)^{2k+2}. \end{aligned}$$

Taking square roots on both sides finishes the proof.  $\square$

We now show a numerical example in Table I for the approximation to the solution of the elliptic problem (10). We choose

$$f = x, \quad a = \frac{1}{2 + x + \sin\left(\frac{2\pi x}{\varepsilon}\right)}. \quad (18)$$

Table I.  $L^2$  and  $L^\infty$ -errors of the approximation to the solution of problem (10) with the choice of (18). Uniform mesh with  $N$  cells.

$N$	$S^1$ space				$S^2$ space			
	$\varepsilon = 0.1$				$\varepsilon = 0.1$			
	$L^2$ -error	order	$L^\infty$ -error	order	$L^2$ -error	order	$L^\infty$ -error	order
10	5.58E-04		2.42E-03		7.82E-06		2.59E-05	
20	1.49E-04	1.91	6.38E-04	1.92	1.00E-06	2.97	3.43E-06	2.92
40	3.71E-05	2.01	1.75E-04	1.87	1.26E-07	2.99	4.65E-07	2.88
80	9.36E-06	1.99	4.65E-05	1.91	1.60E-08	2.98	6.24E-08	2.90
160	2.35E-06	1.99	1.18E-05	1.98	2.00E-09	3.00	7.94E-09	2.97
320	5.87E-07	2.00	2.96E-06	2.00	2.51E-10	2.99	9.97E-10	2.99
	$\varepsilon = 0.01$				$\varepsilon = 0.01$			
	$L^2$ -error	order	$L^\infty$ -error	order	$L^2$ -error	order	$L^\infty$ -error	order
10	5.92E-04		2.38E-03		7.94E-06		2.60E-06	
20	1.48E-04	2.00	6.37E-04	1.90	9.97E-07	2.99	3.38E-06	2.94
40	3.73E-05	1.99	1.68E-04	1.92	1.27E-07	2.97	4.47E-07	2.92
80	9.18E-06	2.02	4.86E-05	1.79	1.51E-08	3.07	5.27E-08	3.08
160	2.26E-06	2.02	9.86E-06	2.30	1.87E-09	3.01	6.27E-09	3.07
320	5.85E-07	1.95	3.02E-06	1.71	2.44E-10	2.97	9.48E-10	2.73

with different choices of  $\varepsilon$ . We can see that we obtain the optimal order of the approximation rate (equal to the dimension of the local approximation space) when using the approximation spaces  $S^k$  ( $k = 1, 2$ ).

We also show that the usual inverse inequality holds for the multi-scale approximation spaces.

**Lemma 3.3:** (Inverse Inequality) For any  $\phi \in S^k$ , we have the following inverse inequality

$$\|\phi\|_{L^\infty(I_j)} \leq C(\Delta x_j)^{-1/2} \|\phi\|_{L^2(I_j)} \quad (19)$$

for some constant  $C$ .

**Proof:** Only the proof for  $k = 1$  will be given here.  $\left\{1, \int_{x_j}^x \frac{1}{a^\varepsilon(\xi)} d\xi\right\}$  is a set of basis functions of the local space  $S^1|_{I_j}$ . We can transfer it to an orthogonal basis  $\{b_0, b_1\}$  with

$$b_0 = 1, \quad b_1 = \frac{1}{\Delta x_j} \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy.$$

First we prove that for each  $i$ ,  $b_i$  satisfy the inverse inequality (19). The proof is trivial for  $i = 0$ . For  $i = 1$ , we have

$$|b_1| = \frac{1}{\Delta x_j} \left| \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right| \leq \frac{1}{\Delta x_j} \int_{I_j} \left| \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right| dy \leq \frac{1}{\alpha \Delta x_j} \int_{I_j} |x - y| dy \leq \frac{\Delta x_j}{\alpha},$$

that is

$$\|b_1\|_{L^\infty(I_j)} \leq \frac{\Delta x_j}{\alpha}. \quad (20)$$

From symmetry, we have

$$\int_{I_j} \left( \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right) dx = 0.$$

Hence there exists some  $x_0 \in I_j$  such that

$$\int_{I_j} \left( \int_y^{x_0} \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy = 0.$$

We claim that for  $x > x_0$ ,  $\int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy > 0$ , since

$$\int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy = \int_{I_j} \left( \int_{x_0}^x \frac{1}{a^\varepsilon(\xi)} d\xi + \int_y^{x_0} \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy = \int_{I_j} \left( \int_{x_0}^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy > 0.$$

Similarly, for  $x < x_0$ , we have  $\int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy < 0$ .

Without loss of generality, we may assume  $x_0 \leq x_j$ . We then have

$$\begin{aligned} (\Delta x_j)^2 \|b_1\|_{L^2(I_j)}^2 &= \int_{I_j} \left( \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right)^2 dx \\ &\geq \int_{x_0}^{x_j + \Delta x_j/2} \left( \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right)^2 dx \\ &\geq \frac{1}{x_j + \frac{\Delta x_j}{2} - x_0} \left( \int_{x_0}^{x_j + \Delta x_j/2} \left( \int_{I_j} \left( \int_y^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right) dx \right)^2 \\ &\geq \frac{1}{\Delta x_j} \left( \int_{x_0}^{x_j + \Delta x_j/2} \left( \int_{I_j} \left( \int_{x_0}^x \frac{1}{a^\varepsilon(\xi)} d\xi \right) dy \right) dx \right)^2 \\ &\geq \frac{1}{\Delta x_j} \left( \int_{x_0}^{x_j + \Delta x_j/2} \frac{\Delta x_j}{\beta} (x - x_0) dx \right)^2 \geq \frac{(\Delta x_j)^5}{64\beta^2}. \end{aligned}$$

That is, we have

$$\|b_1\|_{L^2(I_j)} \geq \frac{(\Delta x_j)^{3/2}}{8\beta}. \quad (21)$$

Combining the two inequalities (20) and (21), we obtain

$$\|b_1\|_{L^\infty(I_j)} \leq C(\Delta x_j)^{-1/2} \|b_1\|_{L^2(I_j)}.$$

For any  $\phi \in S^1$ , we can write  $\phi = c_0 b_0 + c_1 b_1$ , and we have

$$\begin{aligned} \|\phi\|_{L^\infty(I_j)}^2 &\leq 2c_0^2 \|b_0\|_{L^\infty(I_j)}^2 + 2c_1^2 \|b_1\|_{L^\infty(I_j)}^2 \leq C(\Delta x_j)^{-1/2} \left( c_0^2 \|b_0\|_{L^2(I_j)}^2 + c_1^2 \|b_1\|_{L^2(I_j)}^2 \right) \\ &= C(\Delta x_j)^{-1/2} \|\phi\|_{L^2(I_j)}^2. \end{aligned}$$

Now the proof is complete for  $k = 1$ .  $\square$

We now consider a special 2-D elliptic multi-scale problem

$$-(a^\varepsilon(x)u_x)_x - (b^\varepsilon(y)u_y)_y = f(x, y), \quad (x, y) \in \Omega \quad (22)$$

with the boundary condition

$$u(x, y) = 0, \quad (x, y) \in \partial\Omega$$

where

$$0 < \alpha \leq a^\varepsilon(x), \quad b^\varepsilon(y) \leq \beta < +\infty. \quad (23)$$

We propose a 2-D multi-scale approximation space for this 2-D elliptic problem to be

$$\begin{aligned} S_2^k = & \left\{ v : v|_K \in \text{span} \left\{ 1, \int_{x_K}^x \frac{1}{a^\varepsilon(\xi)} d\xi, \int_{y_K}^y \frac{1}{b^\varepsilon(\eta)} d\eta, \int_{x_K}^x \frac{\xi - x_K}{a^\varepsilon(\xi)} d\xi, \right. \right. \\ & \int_{x_K}^x \frac{1}{a^\varepsilon(\xi)} d\xi \int_{y_K}^y \frac{1}{b^\varepsilon(\eta)} d\eta, \int_{y_K}^y \frac{\eta - y_K}{b^\varepsilon(\eta)} d\eta, \dots, \\ & \left. \int_{x_K}^x \frac{(\xi - x_K)^{k-1}}{a^\varepsilon(\xi)} d\xi, \int_{x_K}^x \frac{(\xi - x_K)^{k-2}}{a^\varepsilon(\xi)} d\xi \int_{y_K}^y \frac{1}{b^\varepsilon(\eta)} d\eta, \dots, \right. \\ & \left. \left. \int_{x_K}^x \frac{1}{a^\varepsilon(\xi)} d\xi \int_{y_K}^y \frac{(\eta - y_K)^{k-2}}{b^\varepsilon(\eta)} d\eta, \int_{y_K}^y \frac{(\eta - y_K)^{k-1}}{b^\varepsilon(\eta)} d\eta \right\} \right\}. \end{aligned}$$

Here  $K$  is a two dimensional cell with the barycenter at the point  $(x_K, y_K)$ .

In the following lemma we will prove the approximation property for  $S_2^1$ .

**Lemma 3.4:** Let  $u(x, y)$  be the exact solution of (22). There exists some  $v(x, y) \in S_2^1$  such that for all cell  $K$ :

$$\|u - v\|_{L^2(K)} \leq C(\alpha, \beta) \|f\|_{L^2(K)} (\Delta_K)^2, \quad \forall (x, y) \in K \quad (24)$$

where  $\Delta_K = \text{diam}(K)$  and  $C(\alpha, \beta)$  is a constant depending on  $\alpha, \beta$  and independent of  $\Delta_K$  and  $\varepsilon$ .

**Proof:** This lemma is a generalization of the theorems in [3] and so is the procedure of its proof. We define  $\tilde{x} = \int_{x_K}^x \frac{1}{a^\varepsilon(\xi)} d\xi$ ,  $\tilde{y} = \int_{y_K}^y \frac{1}{b^\varepsilon(\eta)} d\eta$ ,  $\tilde{u}(\tilde{x}, \tilde{y}) = u(x, y)$ ,  $\tilde{a}^\varepsilon(\tilde{x}) = a^\varepsilon(x)$ ,  $\tilde{b}^\varepsilon(\tilde{y}) = b^\varepsilon(y)$ , and  $\tilde{f}(\tilde{x}, \tilde{y}) = f(x, y)$ . Change the variable  $x$  and  $y$  in (22) to  $\tilde{x}$  and  $\tilde{y}$ , we obtain a new elliptic PDE

$$-\tilde{b}^\varepsilon \tilde{u}_{\tilde{x}\tilde{x}} - \tilde{a}^\varepsilon \tilde{u}_{\tilde{y}\tilde{y}} = \tilde{a}^\varepsilon \tilde{b}^\varepsilon \tilde{f}, \quad (\tilde{x}, \tilde{y}) \in \tilde{K} \quad (25)$$

From the Bernstein Theorem in [3], we have

$$\|\tilde{u}\|_{H^2(\tilde{K})} \leq C(\alpha, \beta) \|\tilde{a}^\varepsilon \tilde{b}^\varepsilon \tilde{f}\|_{L^2(\tilde{K})} \leq C(\alpha, \beta) \|f\|_{L^2(K)}$$

From the approximation theory of polynomial spaces, we can find a linear polynomial  $\tilde{v} = c_0 + c_1 \tilde{x} + c_2 \tilde{y}$  such that

$$\|\tilde{u} - \tilde{v}\|_{L^2(\tilde{K})} \leq C \|\tilde{u}\|_{H^2(\tilde{K})} (\Delta_{\tilde{K}})^2.$$

Now we denote  $v(x, y) = \tilde{v}(\tilde{x}, \tilde{y}) = c_0 + c_1 \int_{x_K}^x \frac{1}{a^\varepsilon(\xi)} d\xi + c_2 \int_{y_K}^y \frac{1}{b^\varepsilon(\eta)} d\eta$ . We also have  $\Delta_{\tilde{K}} \leq C \Delta_K$ . The approximation property can then be obtained

$$\|u - v\|_{L^2(K)} = \|\tilde{u} - \tilde{v}\|_{L^2(\tilde{K})} \leq C \|\tilde{u}\|_{H^2(\tilde{K})} (\Delta_{\tilde{K}})^2 \leq C(\alpha, \beta) \|f\|_{L^2(K)} (\Delta_K)^2.$$

This complete the proof.  $\square$

Next, we will prove the  $L^2$  approximation property for  $S_2^1$ .

**Lemma 3.5:** Let  $u(x, y)$  be the exact solution of (22) and  $P_h$  be the  $L^2$  projection operator into the space  $S_2^1$ . There exists a constant  $C(\alpha, \beta)$  such that:

$$\|u - P_h u\|_{L^2(\Omega)} \leq C(\alpha, \beta) \|f\|_{L^2(\Omega)} \Delta^2 \quad (26)$$

where  $\Delta = \max_K \Delta_K$ .

**Proof:** We choose the same  $v$  as that in Lemma 3.4. Squaring both sides of (24) and then integrating over the cell  $K$ , we obtain

$$\|u - v\|_{L^2(K)}^2 \leq C(\alpha, \beta) \|f\|_{L^2(K)}^2 (\Delta_K)^4, \quad \forall K.$$

Therefore,

$$\|u - P_h u\|_{L^2(\Omega)}^2 \leq \|u - v\|_{L^2(\Omega)}^2 \leq C(\alpha, \beta) \sum_K \|f\|_{L^2(K)}^2 (\Delta_K)^4 \leq C(\alpha, \beta) \|f\|_{L^2(\Omega)}^2 \Delta^4.$$

Taking square roots on both sides finishes the proof.  $\square$

#### 4. Boundedness and stability of the multi-scale DG operator, and approximation properties

Here we denote the *primal form*  $B_h(\cdot, \cdot)$  as the left-hand-side of the *primal formulation* of the DG method. In this section, we discuss the boundedness and stability of  $B_h$  and the approximation properties of the space  $S^k$  with respect to some appropriate norm. For simplicity, we will only provide the proof for the Babuška-Zlámal DG method in 1-D following the proof in [2]. The proof can also be easily generalized to other DG methods and to 2-D along the lines of [2].

##### 4.1. Boundedness

**Lemma 4.1:** (Boundedness) There exists some constant  $C_b$  such that

$$B_h(w, v) \leq C_b \|w\| \|v\|, \quad \forall w, v \in S^k \quad (27)$$

where the norm  $\|\cdot\|$  is defined as

$$\|v\|^2 = |v|_{H^1(0,1)}^2 + |v|_*^2$$

with the norm  $|\cdot|_*$  defined as

$$|v|_*^2 = \sum_{j=0}^N (\Delta x)^{-2k-1} [v]_{j+\frac{1}{2}}^2.$$

**Proof:** For the Babuška-Zlámal method, we have

$$B_h(w, v) = \sum_{j=1}^N \int_{I_j} a^\varepsilon w_x v_x dx + \sum_{j=0}^N \left( \frac{a^\varepsilon(x_{j+\frac{1}{2}}) \eta_{j+\frac{1}{2}}}{\Delta x} [w]_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \right).$$

From (12), we can easily obtain

$$\sum_{j=1}^N \int_{I_j} a^\varepsilon w_x v_x dx \leq C |w|_{H^1(0,1)} |v|_{H^1(0,1)} \quad (28)$$

for some constant  $C$ . We also have

$$\sum_{j=0}^N \left( \frac{a^\varepsilon(x_{j+\frac{1}{2}}) \eta_{j+\frac{1}{2}}}{\Delta x} [w]_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \right) \leq C_1 (\Delta x)^{2k} (\sup_j \eta_{j+\frac{1}{2}}) |w|_* |v|_* \quad (29)$$

for some constant  $C_1$ . Combining (28) and (29) completes the proof.  $\square$

#### 4.2. Stability

**Lemma 4.2:** (Stability) There exists some constant  $C_s$  such that

$$B_h(v, v) \leq C_s \|v\|^2, \quad \forall v \in S^k. \quad (30)$$

**Proof:** From (12), we can easily obtain

$$\sum_{j=1}^N \int_{I_j} a^\varepsilon v_x v_x dx \geq C |v|_{H^1(0,1)}^2 \quad (31)$$

for some constant  $C$ . We also have

$$\sum_{j=0}^N \left( \frac{a^\varepsilon(x_{j+\frac{1}{2}}) \eta_{j+\frac{1}{2}}}{\Delta x} [v]_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} \right) \geq C_1 (\Delta x)^{2k} (\inf_j \eta_{j+\frac{1}{2}}) |v|_*^2 \quad (32)$$

for some constant  $C_1$ . The combination of (31) and (32) gives us (30). To make  $C_s$  independent of  $\Delta x$ , the lower bound of  $\eta_{j+\frac{1}{2}}$  must be sufficiently large, i.e.  $\inf_j \eta_{j+\frac{1}{2}} = \mathcal{O}((\Delta x)^{-2k})$ .  $\square$

#### 4.3. Approximation

**Lemma 4.3:** (Approximation) Let  $u(x)$  be the exact solution of (10). There exists some interpolant  $u_I(x) \in S^k (k \geq 1)$  such that

$$\|u - u_I\| \leq C_a (\Delta x)^k |f|_{H^{k-1}(0,1)} \quad (33)$$

**Proof:** We choose  $u_I$  as the usual continuous interpolant, then the jumps of  $u - u_I$  will vanish on the boundaries. We also have  $|u - u_I|_{H^1(0,1)} \leq C (\Delta x)^k |f|_{H^{k-1}(0,1)}$ , proven in [4]. Then

$$\|u - u_I\| = |u - u_I|_{H^1(0,1)} \leq C (\Delta x)^k |f|_{H^{k-1}(0,1)}.$$

The proof is completed.  $\square$

## 5. Error estimates

We now prove an error estimate for the DG method based on the multi-scale approximation spaces by using the properties of consistency, boundedness, stability and approximation discussed previously. Again, only the results for the Babuška-Zlámal DG method is provided.

**Lemma 5.1:** (Error estimates) Let  $u(x)$  be the exact solution of (10) and  $u_h$  be the numerical solution computed by the multi-scale Babuška-Zlámal DG method. There exists some constant  $C$  independent of  $\varepsilon$  such that

$$\|u - u_h\|_{L^2(0,1)} \leq C(\Delta x)^{k+1} \|f\|_{H^{k-1}(0,1)}. \quad (34)$$

**Proof:** The proof here mainly follows the lines in [2]. The Babuška-Zlámal DG method is not consistent. Instead of satisfying the consistency condition, it satisfies

$$B_h(u, v) = \int_0^1 f v dx + \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) \{u_x\}_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}}, \quad \forall v|_{I_j} \in H^2(I_j), \quad \forall j. \quad (35)$$

where  $u$  is the exact solution of (10) and  $\{w\} = \frac{1}{2}(w^- + w^+)$ .

This method is neither adjoint consistent. We also have

$$B_h(v, \psi) = \int_0^1 v g dx + \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) [v]_{j+\frac{1}{2}} \{\psi_x\}_{j+\frac{1}{2}}, \quad \forall v|_{I_j} \in H^2(I_j), \quad \forall j,$$

where  $\psi$  is the exact solution of the adjoint problem of (10), which is

$$-(a^\varepsilon(x)\psi_x)_x = g(x), \quad 0 \leq x \leq 1 \quad (36)$$

with the boundary condition

$$\psi(0) = \psi(1) = 0. \quad (37)$$

From the stability of the Babuška-Zlámal method, we obtain

$$C_s \| \|u_I - u_h\| \|^2 \leq B_h(u_I - u, u_I - u_h) + B_h(u - u_h, u_I - u_h) =: T_1 + T_2. \quad (38)$$

Since  $u_I$  is the continuous interpolant of  $u$  in  $S^k$ , we can use the continuity of  $u - u_I$  to estimate  $T_1$ :

$$T_1 \leq C \| \|u_I - u\| \| \|u_I - u_h\| \leq C(\Delta x)^k \| \|u_I - u_h\| \| \|f\|_{H^{k-1}(0,1)}. \quad (39)$$

We also have

$$\begin{aligned} T_2 &= \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) \{u_x\}_{j+\frac{1}{2}} [u_I - u_h]_{j+\frac{1}{2}} \\ &\leq C(\Delta x)^k \| \|u_I - u_h\| \| \|u\|_{H^2(0,1)} \leq C(\Delta x)^k \| \|u_I - u_h\| \| \|f\|_{L^2(0,1)} \end{aligned} \quad (40)$$

where the second inequality comes from the regularity of the elliptic PDE (10) and the first inequality is estimated by using the auxiliary inequality

$$\begin{aligned} \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) \{u_x\}_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} &= \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) (\Delta x)^{k+1/2} \{u_x\}_{j+\frac{1}{2}} [v]_{j+\frac{1}{2}} (\Delta x)^{-k-1/2} \\ &\leq C \| \|v\| \| \left( \sum_{j=0}^N (\Delta x)^{2k+1} \{u_x\}_{j+\frac{1}{2}}^2 \right)^{1/2} \leq C(\Delta x)^k \| \|v\| \| \|u\|_{H^2(0,1)} \end{aligned} \quad (41)$$

with the last step following from the trace inequality which is proven in [1].

Substituting (39) and (40) back into (38), we get

$$\|u_I - u_h\| \leq C(\Delta x)^k \|f\|_{H^{k-1}(0,1)},$$

and

$$\|u - u_h\| \leq C(\Delta x)^k \|f\|_{H^{k-1}(0,1)}.$$

by the triangle inequality.

For the  $L^2$ -error estimate, we have

$$\|u - u_h\|_{L^2(0,1)}^2 \leq B_h(u - u_h, \psi) - \sum_{j=0}^N a^\varepsilon(x_{j+\frac{1}{2}}) \{\psi_x\}_{j+\frac{1}{2}} [u - u_h]_{j+\frac{1}{2}} =: T_3 + T_4. \quad (42)$$

Let  $\psi_I$  be the continuous interpolant of  $\psi$  in  $S^k$ , then  $B_h(u, \psi_I) = \int_0^1 f \psi_I dx$ . Therefore,

$$T_3 = B_h(u - u_h, \psi - \psi_I) \leq C \|u - u_h\| \|\psi - \psi_I\| \leq C \Delta x \|u - u_h\| \|u - u_h\|_{L^2(0,1)}. \quad (43)$$

The term  $T_4$  can be estimated by using the auxiliary inequality (41)

$$T_4 \leq C(\Delta x)^k \|u - u_h\| \|\psi\|_{H^2(0,1)} \leq C(\Delta x)^k \|u - u_h\| \|u - u_h\|_{L^2(0,1)}. \quad (44)$$

Substituting (43) and (44) back into (42), we get

$$\|u - u_h\|_{L^2(0,1)} \leq C(\Delta x)^{k+1} \|f\|_{H^{k-1}(0,1)}$$

which finishes the proof.  $\square$

## 6. A numerical example

In this section, we present a numerical example of using the Babuška-Zlámal DG method based on the proposed multi-scale approximation spaces for solving the elliptic multi-scale problem (10) with

$$f = x, \quad a = \frac{1}{2 + x + \sin(\frac{2\pi x}{\varepsilon})} \quad (45)$$

with  $\varepsilon = 0.1$  and  $\varepsilon = 0.01$  respectively. The numerical results are shown in Table II. We can clearly observe that the solutions are numerically well resolved and the expected  $(k + 1)$ -th order of accuracy is achieved without any resonance effects when the mesh size  $\Delta x$  changes from larger than  $\varepsilon$  to smaller than  $\varepsilon$ . Our numerical implementation does show an amplification of the round-off errors to around the level at  $10^{-7}$  in double precision, probably due to the large condition number of the mass matrix. We will explore more accurate implementation of the method in the future.

## 7. Concluding remarks

DG methods based on multi-scale approximation spaces for solving a class of second order elliptic PDEs with multi-scale solutions are studied in this paper. The basis functions of the

Table II.  $L^2$  and  $L^\infty$ -errors for the Babuška-Zlámal DG method based on multi-scale approximation spaces.  $N$  uniform cells.

$S^1$ space								
$N$	$\varepsilon = 0.1$				$\varepsilon = 0.01$			
	$L^2$ -error	order	$L^\infty$ -error	order	$L^2$ -error	order	$L^\infty$ -error	order
10	9.36E-04		2.45E-03		7.06E-04		1.72E-03	
20	2.37E-04	1.98	7.64E-04	1.68	1.72E-04	2.00	5.24E-04	1.71
40	6.12E-05	1.95	2.27E-04	1.75	4.40E-05	1.97	1.47E-04	1.90
80	1.57E-05	1.96	6.05E-05	1.91	1.13E-05	1.97	4.20E-05	1.84
160	3.96E-06	1.99	1.54E-05	1.97	2.59E-06	2.06	9.55E-06	2.08
320	9.92E-07	2.00	3.88E-06	1.99	7.13E-07	1.93	3.28E-06	1.64
640	2.48E-07	2.00	9.74E-07	1.99	1.85E-07	1.94	8.67E-07	1.94
1280	1.17E-07	1.08	5.79E-07	0.75	1.19E-07	0.45	6.81E-07	1.68
$S^2$ space								
$N$	$\varepsilon = 0.1$				$\varepsilon = 0.01$			
	$L^2$ -error	order	$L^\infty$ -error	order	$L^2$ -error	order	$L^\infty$ -error	order
5	3.20E-04		5.47E-04		3.20E-04		5.53E-04	
10	2.08E-05	3.94	4.31E-05	3.67	2.08E-05	3.94	4.12E-05	3.75
20	2.00E-06	3.38	4.29E-06	3.33	1.98E-06	3.39	3.81E-06	3.43
40	2.35E-07	3.09	4.91E-07	3.13	2.33E-07	3.09	4.45E-07	3.10
80	3.76E-08	2.64	1.09E-07	2.17	7.27E-08	1.68	1.93E-07	1.21

multi-scale approximation spaces are constructed from the differential operators to capture the micro-scale structure information of solutions so that the solutions of multi-scale PDEs can be well approximated. Both theoretical proofs and numerical experiments show that, compared to the usual piecewise polynomial spaces, more accurate results are obtained for coarse meshes. No resonance effects appear, that is, the rate of convergence is independent of  $\varepsilon$ .

From a practical point of view, we have only considered the one dimensional problem and a special class of two dimensional problems for which the multi-scale basis functions can be explicitly constructed. This renders the DG scheme very efficient, with a cost comparable to that for the regular DG scheme with polynomial basis functions. For general multi-dimensional problems, the multi-scale basis functions may not be available explicitly and must also be computed numerically [11], which would increase the computational cost of the multi-scale DG method tremendously.

In future work we will generalize both the multi-scale DG method and its analysis to broader classes of multi-dimensional problems with higher order of accuracy. We will also explore applications of the multi-scale DG method in fluid dynamics and semi-conductor device simulations.

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