Error Estimates of the Semi-Discrete Local Discontinuous Galerkin Method for Nonlinear Convection-Diffusion and KdV Equations

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Dedicated to Professor Ivo Babuska on the occasion of his 80th birthday

Abstract

In this paper, we provide L^2 error estimates for the semi-discrete local discontinuous Galerkin methods for nonlinear convection-diffusion equations and KdV equations with smooth solutions. The main technical difficulty is the control of the inter-element jump terms which arise because of the nonlinearity of the PDEs and the discontinuous nature of the numerical method.

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1 Introduction

In this paper, we present error estimates for the semi-discrete local discontinuous Galerkin (LDG) methods [33, 31, 15] for smooth solutions of nonlinear equations formulated by the one dimensional KdV equation

$$u_t + f(u)_x + u_{xxx} = 0, \tag{1.1}$$

the two dimensional KdV-type equation

$$u_t + f(u)_x + g(u)_y + u_{xxx} + u_{yyy} = 0, \tag{1.2}$$

the two dimensional Zakharov-Kuznetsov (ZK) equation

$$u_t + (3u^2)_x + u_{xxx} + u_{xyy} = 0, \tag{1.3}$$

and the nonlinear convection-diffusion equations

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} - \sum_{i=1}^{d}\sum_{j=1}^{d}(a_{ij}(u)u_{x_j})_{x_i} = 0, \qquad d = 1, 2 \tag{1.4}$$

where $f(u)$, $g(u)$, $f_i(u)$ and $a_{ij}(u)$ are arbitrary (smooth) nonlinear functions and $a_{ij}(u)$ are entries of a symmetric and semi-positive definite matrix. For P^k element space of piecewise polynomials of degree up to k, *a priori* error estimates in the usual L^2 norm of the form

$$\|u - u_h\| \le Ch^{k+\gamma} \tag{1.5}$$

are obtained, where u_h is the numerical LDG solution, h is the maximum mesh size of a regular triangulation, $\gamma = \frac{1}{2}$ for the one dimensional cases (1.1) and (1.4) with $d = 1$ and for the two dimensional cases (1.2), (1.3) and (1.4) with $d = 2$ for the Q^k elements (piecewise tensor product polynomials of degree k) on Cartesian meshes, and $\gamma = 0$ for the two dimensional case (1.4) with $d = 2$ for the P^k elements (piecewise polynomials of degree k) on arbitrary regular meshes. Here and below an unmarked norm $\|\cdot\|$ refers to the usual L^2 norm.

The discontinuous Galerkin (DG) method we discuss in this paper is a class of finite element methods using completely discontinuous piecewise polynomial space for the numerical solution and the test functions in the spatial variables, usually coupled with explicit and nonlinearly stable high order Runge-Kutta time discretization [28]. It was first developed for nonlinear hyperbolic conservation laws containing first derivatives by Cockburn et al. in a series of papers [13, 12, 9, 14]. For a detailed description of the method as well as its implementation and applications, we refer the readers to the lecture notes [7], the review paper [16], and the recent special issue of Journal of Scientific Computing on discontinuous Galerkin methods [17].

These DG methods were later generalized to the LDG method for solving convection diffusion equations (containing second derivatives) by Cockburn and Shu [15]. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for the compressible Navier-Stokes equations. Later, Yan and Shu developed a LDG method for a general KdV type equation containing third derivatives in [33], and they generalized the LDG method to PDEs with fourth and fifth spatial derivatives in [34]. Levy, Shu and Yan [23] developed LDG methods for solving nonlinear dispersive equations that have compactly supported traveling wave solutions, the so-called "compactons". Recently, Xu and Shu further developed the LDG method to solve a series of nonlinear wave equations [29, 30, 31, 32], which include the general KdV-Burgers type equations, the general fifth-order KdV type equations, the fully nonlinear $K(n, n, n)$ equations, the nonlinear Schrödinger equations, the Kuramoto-Sivashinsky equations, the Ito-type coupled KdV equations and the two dimensional KdV equations.

These DG and LDG methods have several attractive properties. It can be easily designed for any order of accuracy. In fact, the order of accuracy can be locally determined in each cell, thus allowing for efficient p adaptivity. It can be used on arbitrary triangulations, even those with hanging nodes, thus allowing for efficient h adaptivity. The methods have excellent parallel efficiency. It is extremely local in data communications. The evolution of the solution in each cell needs to communicate only with immediate neighbors, regardless of the order of accuracy. Finally, it has excellent provable nonlinear stability. One can typically prove a cell entropy inequality for the square entropy which leads to a strong L^2 stability, for quite general nonlinear cases, for any orders of accuracy on arbitrary triangulations in any spatial dimension, without the need for nonlinear limiters.

For smooth solutions of linear conservation laws, optimal *a priori* error estimates $O(h^{k+1})$ for the one dimensional and multi-dimensional tensor product or some other structured mesh cases, and $O(h^{k+\frac{1}{2}})$ for other cases have been given for the steady state solution or a space-time DG discretization in [22, 26, 21], with the optimality in the general situation proven in [25]. In effect, for most triangulations one could observe numerically (and prove in many cases) convergence of the order $O(h^{k+1})$ in both the L^2 and the L^∞ norms for both linear and nonlinear problems. Recently, Zhang and Shu presented *a priori* error estimates for the fully discrete Runge-Kutta DG methods with smooth solutions of scalar nonlinear conservation laws [35].

The first *a priori* error estimate for the LDG method of linear convection-diffusion equations was obtained in 1998 by Cockburn and Shu [15]. Later this analysis was extended by Cockburn and Dawson [8]. In these papers the rate of convergence obtained was in general of the order $O(h^k)$, except for some special cases when the order $O(h^{k+1})$ was obtained. Castillo [3] and Castillo et al. [4] prove the optimal rate of convergence order $O(h^{k+1})$ for the LDG method with a particular numerical flux, see also Castillo et al. [5]. Work on other discontinuous Galerkin finite element methods for convection-diffusion and for pure diffusion problems has been reviewed by Cockburn, Karniadakis and Shu [11]. In particular, we mention the numerical method of Baumann and Oden [2] and the optimal error estimates for the method as applied to nonlinear convection-diffusion equations for at least quadratic polynomials by Rivière and Wheeler [27]. Recently, Dolejšé et al. [18, 19, 20] obtained *a priori* error estimates for the discontinuous Galerkin finite element method of the convection-diffusion equations with linear diffusion terms.

There are only a few numerical works in the literature for error estimates of the LDG method for the KdV equations. In [33], *a priori* error estimates for the LDG method for linearized KdV equations in one spatial dimension were obtained. Even for such linear PDEs, there are technical difficulties to derive the L^2 *a priori* error estimates from the cell entropy inequality and approximation results, because of the possible lack of control on some of the jump terms at cell boundaries, which appear because of the discontinuous nature of the finite element space for the DG method. The remedy in [33] to handle such jump terms was via a special projection, which eliminates such troublesome jump terms in the error equation. It is more challenging to perform L^2 *a priori* error estimates for nonlinear PDEs with high order derivatives than for first order hyperbolic PDEs in [35].

In this paper, the main procedure to obtain *a priori* error estimates is the following. First, we obtain the error equation of the LDG method. Second, we estimate each term in an important *energy equality*. The main techniques we use in this paper are Taylor expansion and energy estimates employed by Zhang and Shu [35]. The paper is organized as follows. In Section 2, we introduce notations, definitions and auxiliary results used later in the paper. In Section 3, we present the LDG method for the KdV equations and present the procedure to obtain *a priori* error estimates for one and two dimensional KdV equations and the ZK equation. In Section 4, we follow the same lines as for the KdV equations to discuss the *a priori* error estimates of the LDG method for the nonlinear convection-diffusion equations in one and two dimensions. Concluding remarks are given in Section 5. Some of the more technical proofs of several lemmas are collected in the Appendix.

2 Notations, definitions and auxiliary results

In this section we introduce notations and definitions to be used later in the paper and also present some auxiliary results. We first review an important quantity measuring the relationship between the numerical flux and physical flux introduced in [35]. We then define some projections and present certain interpolation and inverse properties for the finite element spaces that will be used in the error analysis.

2.1 Basic notations

2.1.1 One-dimensional case

We denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $j = 1, \ldots, N$. The center of the cell is $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$ and the mesh size is denoted by $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max_{1 \le j \le N} h_j$ being the maximum mesh size. We assume the mesh is regular, namely the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements. We define the piecewise-polynomial space V_h as the space of polynomials of the degree up to k in each cell I_j, i.e.

$$V_h = \{v : \quad v \in P^k(I_j) \quad \text{for } x \in I_j, \quad j = 1, \ldots, N\}.$$

Note that functions in V_h are allowed to have discontinuities across element interfaces.

The solution of the numerical scheme is denoted by u_h, which belongs to the finite element space V_h. We denote by $(u_h)^+_{j+\frac{1}{2}}$ and $(u_h)^-_{j+\frac{1}{2}}$ the values of u_h at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1}, and from the left cell I_j, respectively. We use the usual notations $[u_h] = u_h^+ - u_h^-$ and $\bar{u}_h = \frac{1}{2}(u_h^+ + u_h^-)$ to denote the jump and the mean of the function u_h at each element boundary point, respectively.

2.1.2 Two-dimensional case

For a rectangular partition of $I \times J = [0, L_x] \times [0, L_y]$, we denote the mesh by $I_i \times J_j$, where $I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$ and $J_j = [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, for $i = 1, \cdots, N_x$ and $j = 1, \cdots, N_y$. The center of the element in the x-direction is $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$; the center of the element in the y-direction is $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$. The cell lengths are denoted by $h_i^x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $h_j^y = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$ with $h^x = \max_{1 \le i \le N_x} h_i^x$, $h^y = \max_{1 \le j \le N_y} h_j^x$, and $h = \max(h^x, h^y)$ being the maximum mesh size. We again assume the mesh is regular.

We define the space Z_h as the space of tensor product piecewise polynomials of degree at most k in each variable on every element, i.e.

$$Z_h = \{v : \ v \in Q^k(I_i \times J_j), \ \forall \ (x,y) \in I_i \times J_j, \ i = 1, \ldots, N_x, \ j = 1, \ldots, N_y\}, \qquad (2.1)$$

where Q^k is the space of tensor products of one-dimensional polynomials of degree up to k.

We denote by $(u_h)^+_{i+\frac{1}{2},y}$ and $(u_h)^-_{i+\frac{1}{2},y}$ the values of u_h at $(x_{i+\frac{1}{2}}, y)$, from the right cell $I_{i+1} \times J_j$ and from the left cell $I_i \times J_j$ respectively when $y \in J_j$, on all vertical edges. The notations $[u_h]_{i+\frac{1}{2},y} = (u_h)^+_{i+\frac{1}{2},y} - (u_h)^-_{i+\frac{1}{2},y}$ and $(\bar{u}_h)_{i+\frac{1}{2},y} = \frac{1}{2}\left((u_h)^+_{i+\frac{1}{2},y} + (u_h)^-_{i+\frac{1}{2},y}\right)$ denote the jump and the mean of the function u_h at $(x_{i+\frac{1}{2}}, y)$ when $y \in J_j$. Similarly, we can define $(u_h)^+_{x,j+\frac{1}{2}}$, $(u_h)^-_{x,j+\frac{1}{2}}$, $[u_h]_{x,j+\frac{1}{2}}$ and $(\bar{u}_h)_{x,j+\frac{1}{2}}$.

For an arbitrary triangulation, let $\mathbb{T}_h$ denote a tessellation of Ω with shape-regular elements K. Let Γ denote the union of the boundary faces of elements $K \in \mathbb{T}_h$, i.e. $\Gamma = \cup_{K \in \mathbb{T}_h} \partial K$, and $\Gamma_0 = \Gamma \backslash \partial\Omega$. Let $P^k(K)$ be the space of polynomials of degree at most $k \geq 0$ on $K \in \mathbb{T}_h$. We denote the finite element space by

$$W_h = \{v : \quad v \in P^k(K) \quad \text{for} \quad (x,y) \in K \quad \forall K \in \mathbb{T}_h\}. \qquad (2.2)$$

Let e be an edge shared by elements K^L and K^R. Define the normal vectors n^L and n^R on e pointing exterior to K^L and K^R, respectively. If (u_h) is a function on K^L and K^R, but possibly discontinuous across e, let $(u_h)^L$ denote $((u_h)|_{K^L})|_e$ and $(u_h)^R$ denote $((u_h)|_{K^R})|_e$. Here left and right can be individually defined on each edge e shared by two elements.

2.2 Notations for different constants

We will adopt the following convention for different constants. These constants may have a different value in each occurrence.

We will denote by C a positive constant independent of h, which may depend on the solution of the problem considered in this paper. Especially, to emphasize the nonlinearity of the flux $f(u)$ (or other nonlinear fluxes), we will denote by $C_\star$ a positive constant depending on the maximum of $|f''|$ or/and $|f'''|$. we remark that $C_\star = 0$ for a linear flux $f = cu$ with a constant c. For problems considered in this paper, the exact solution is assumed to be smooth with periodic or compactly supported boundary condition. Also, $0 \le t \le T$ for a fixed T. Therefore, the exact solution is always bounded. We follow the convention [35] to redefine the nonlinear functions $f(u)$, $g(u)$, etc. outside their ranges such that the derivatives of these nonlinear functions $f'(u)$, $f''(u)$, etc. become globally bounded functions.

2.3 A quantity related to the numerical flux

For notational convenience we would like to introduce the following numerical flux related to the discontinuous Galerkin spatial discretization. $\widehat{f}(\omega^-, \omega^+)$ is a given monotone numerical flux that depends on the two values of the function ω at the discontinuity point $x_{j+\frac{1}{2}}$, namely $\omega^{\pm}_{j+\frac{1}{2}} = \omega(x^{\pm}_{j+\frac{1}{2}})$. The numerical flux $\widehat{f}(\omega^-, \omega^+)$ satisfies the following conditions:

(a) it is locally Lipschitz continuous, so it is bounded when $\omega^{\pm}$ are in a bounded interval;

(b) it is consistent with the flux $f(\omega)$, i.e., $\widehat{f}(\omega, \omega) = f(\omega)$;

(c) it is a nondecreasing function of its first argument, and a nonincreasing function of its second argument.

In [35], Zhang and Shu introduced an important quantity to measure the difference between the numerical flux and the physical flux. For completeness, we give their definition in the following lemma.

Lemma 2.1. [35] *For any piecewise smooth function $\omega \in L^2(0,1)$, on each cell boundary point we define*

$$\alpha(\widehat{f};\omega) \equiv \alpha(\widehat{f};\omega^-,\omega^+) \triangleq \begin{cases} [\omega]^{-1}(f(\bar{\omega}) - \widehat{f}(\omega)), & \text{if}[\omega] \neq 0; \\ \frac{1}{2}|f'(\bar{\omega})|, & \text{if}[\omega] = 0 \end{cases} \tag{2.3}$$

where $\widehat{f}(\omega) \equiv \widehat{f}(\omega^-, \omega^+)$ is a monotone numerical flux consistent with the given flux f. Then $\alpha(\widehat{f};\omega)$ is non-negative and bounded for any $(\omega^-, \omega^+) \in \mathbb{R}^2$. Moreover we have

$$\frac{1}{2}|f'(\bar{\omega})| \le \alpha(\widehat{f};\omega) + C_\star|[\omega]| \tag{2.4}$$

$$-\frac{1}{8}f''(\bar{\omega})[\omega] \le \alpha(\widehat{f};\omega) + C_\star|[\omega]|^2. \tag{2.5}$$

Remark 2.1. Examples of monotone fluxes which are suitable for the discontinuous Galerkin methods can be found in, e.g., [13]. For our error estimates, we rewrite the numerical flux in a viscosity form

$$\widehat{f}(\omega^-, \omega^+) = \frac{1}{2}(f(\omega^-) + f(\omega^+) - \lambda(\omega^-, \omega^+)(\omega^+ - \omega^-)), \tag{2.6}$$

and assume the viscosity coefficient $\lambda(\omega^-, \omega^+)$ satisfies

$$\lambda(\omega^-, \omega^+) \geq \lambda_0 > 0, \quad \lambda_0 \text{ is constant.} \tag{2.7}$$

We call such flux as a "uniform dissipative flux". The well known Lax-Friedrichs flux is a uniform dissipative flux with a proper choice of λ. This property is necessary in our proof because of a lack of control for certain jump terms at cell boundaries due to the nonlinear terms and the high order derivative terms.

We would also like to use the following simplified notation. For any functions ω and ϕ, we denote

$$\alpha(\widehat{f}; \omega)[\phi]^2 = \sum_{1 \leq j \leq N} \alpha(\widehat{f}; \omega)_{j+\frac{1}{2}} [\phi]^2_{j+\frac{1}{2}}.$$

2.4 Projection and interpolation properties

2.4.1 One-dimensional case

In what follows, we will consider the standard L^2-projection of a function ω with $k+1$ continuous derivatives into space V_h, denoted by $\mathcal{P}$, i.e., for each j,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x)) v(x) dx = 0 \quad \forall v \in P^k(I_j), \tag{2.8}$$

and the special projection $\mathcal{P}^{\pm}$ into V_h, which satisfy, for each j,

$$\int_{I_j} (\mathcal{P}^+\omega(x) - \omega(x)) v(x) dx = 0 \quad \forall v \in P^{k-1}(I_j), \quad \text{and} \quad \mathcal{P}^+\omega(x^+_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}) \tag{2.9}$$

$$\int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x)) v(x) dx = 0 \quad \forall v \in P^{k-1}(I_j), \quad \text{and} \quad \mathcal{P}^-\omega(x^-_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}).$$

For the projections mentioned above, it is easy to show (c.f. [6])

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\Gamma_h} \le Ch^{k+1}, \tag{2.10}$$

where $\omega^e = \mathcal{P}\omega - \omega$ or $\omega^e = \mathcal{P}^\pm\omega - \omega$. The positive constant C, solely depending on ω, is independent of h. Γ_h denotes the set of boundary points of all elements I_j.

2.4.2 Two-dimensional case

To prove the error estimates for two-dimensional problems in Cartesian meshes, we need a suitable projection $\mathcal{P}^\pm$ similar to the one-dimensional case. We use the projections in [22, 10]. On a rectangle $I \times J = [0, L_x] \times [0, L_y]$, define

$$\mathbb{P}\omega = \mathcal{P}_x \otimes \mathcal{P}_y\omega, \qquad \mathbb{P}^\pm\omega = \mathcal{P}_x^\pm \otimes \mathcal{P}_y^\pm\omega, \tag{2.11}$$

where the subscripts indicate the application of the one-dimensional operators $\mathcal{P}$ or $\mathcal{P}^\pm$ with respect to the corresponding variable. We list some properties for the projections $\mathbb{P}^\pm$:

$$\int_{I_i}\int_{J_j} (\mathbb{P}^\pm\omega(x,y) - \omega(x,y))v(x,y)dydx = 0 \tag{2.12}$$

for any $v \in (P^{k-1}(I_i) \otimes P^k(J_j)) \cup (P^k(I_i) \otimes P^{k-1}(J_j))$. Also,

$$\int_{J_j} (\mathbb{P}^+\omega(x^+_{i-\frac{1}{2}}, y) - \omega(x_{i-\frac{1}{2}}, y))v(x^+_{i-\frac{1}{2}}, y))dy = 0 \quad \forall v \in Q^k(I_i \otimes J_j), \tag{2.13}$$

$$\int_{J_j} (\mathbb{P}^-\omega(x^-_{i+\frac{1}{2}}, y) - \omega(x_{i+\frac{1}{2}}, y))v(x^-_{i+\frac{1}{2}}, y))dy = 0 \quad \forall v \in Q^k(I_i \otimes J_j)$$

and

$$\int_{I_i} (\mathbb{P}^+\omega(x, y^+_{j-\frac{1}{2}}) - \omega(x, y_{j-\frac{1}{2}}))v(x, y^+_{j-\frac{1}{2}}))dx = 0 \quad \forall v \in Q^k(I_i \otimes J_j), \tag{2.14}$$

$$\int_{I_i} (\mathbb{P}^-\omega(x, y^-_{j+\frac{1}{2}}) - \omega(x, y_{j+\frac{1}{2}}))v(x, y^-_{j+\frac{1}{2}}))dx = 0 \quad \forall v \in Q^k(I_i \otimes J_j).$$

Similar to the one-dimensional case, there are some approximation results for the projections (2.11) in [10]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\Gamma_h} \le Ch^{k+1}, \tag{2.15}$$

where $\omega^e = \mathbb{P}\omega - \omega$ or $\omega^e = \mathbb{P}^\pm\omega - \omega$. The positive constant C, solely depending on ω, is independent of h. Γ_h denotes the set of boundary points of all elements $I_i \times J_j$.

2.5 Inverse Properties

Finally we list some inverse properties of the finite element space V_h that will be used in our error analysis. For any $\omega_h \in V_h$, there exists a positive constant C independent of ω_h and h, such that

$$\text{(i) } \|\partial_x \omega_h\| \le Ch^{-1}\|\omega_h\|, \quad \text{(ii) } \|\omega_h\|_{\Gamma_h} \le Ch^{-\frac{1}{2}}\|\omega_h\|, \quad \text{(iii) } \|\omega_h\|_\infty \le Ch^{-\frac{1}{2}}\|\omega_h\|. \tag{2.16}$$

For more details of these inverse properties, we refer to [6].

3 Error estimates of the LDG method for the KdV equations

3.1 Error analysis for the one-dimensional KdV equation

In this subsection, we formulate the LDG method for the one-dimensional KdV equation, then state the error estimate result and display the main ideas of its proof.

3.1.1 The LDG method

We present the LDG method for the following KdV equation:

$$u_t + f(u)_x + u_{xxx} = 0, \tag{3.1}$$

with an initial condition

$$u(x,0) = u_0(x) \tag{3.2}$$

and periodic boundary conditions. Here $f(u)$ is an arbitrary (smooth) nonlinear function. Notice that the assumption of periodic boundary conditions is for simplicity only and is not essential: the method can be easily designed for non-periodic boundary conditions (see e.g. [24]).

To define the LDG method, we rewrite the equation (3.1) as a first order system:

$$u_t + f(u)_x + p_x = 0, \quad p - v_x = 0, \quad v - u_x = 0. \tag{3.3}$$

Now we can use the LDG method to approximate the equations (3.3). Find u_h, p_h, $v_h \in V_h$, such that, $\forall\ \rho,\ \psi,\ \varphi \in V_h$,

$$\begin{aligned}
&\int_{I_j} (u_h)_t \rho dx - \int_{I_j} (f(u_h) + p_h)\rho_x dx + ((\widehat{f} + \widehat{p}_h)\rho^-)_{j+\frac{1}{2}} - ((\widehat{f} + \widehat{p}_h)\rho^+)_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} p_h \psi dx + \int_{I_j} v_h \psi_x dx - (\widehat{v}_h \psi^-)_{j+\frac{1}{2}} + (\widehat{v}_h \psi^+)_{j-\frac{1}{2}} = 0, \\
&\int_{I_j} v_h \varphi dx + \int_{I_j} u_h \varphi_x dx - (\widehat{u}_h \varphi^-)_{j+\frac{1}{2}} + (\widehat{u}_h \varphi^+)_{j-\frac{1}{2}} = 0.
\end{aligned} \tag{3.4}$$

The "hat" terms in (3.4) are the boundary terms that emerge from integration by parts. These are the so-called "numerical fluxes" which should be designed based on different guiding principles for different PDEs to ensure stability. For example, upwinding should be used as a guideline for odd derivatives which correspond to waves, and eventual symmetric treatment, such as an alternating choice of the fluxes for a quantity and its derivative, should be used for even derivatives. It turns out that we can take the simple choices such that

$$\widehat{f} = \widehat{f}(u_h^-, u_h^+), \quad \widehat{p}_h = p_h^+, \quad \widehat{v}_h = v_h^+, \quad \widehat{u}_h = u_h^-, \tag{3.5}$$

where we have omitted the half-integer indices $j + \frac{1}{2}$ as all quantities in (3.5) are computed at the same points (i.e. the interfaces between the cells). Here $\widehat{f}(u_h^-, u_h^+)$ is a monotone flux which has the uniform dissipation property (2.6)-(2.7). We remark that the choice for the fluxes (3.5) is not unique. In fact the crucial part is taking $\widehat{p}_h$ and $\widehat{u}_h$ from opposite sides.

3.1.2 The main result

We state the main error estimates of the semi-discrete LDG scheme (3.4). Detailed proof will be given in subsequent subsections.

Theorem 3.1. *Let u be the exact solution of the problem (3.1), which is sufficiently smooth with bounded derivatives, and assume $f \in C^3$. Let u_h be the numerical solution of the semi-discrete LDG scheme (3.4) and denote the corresponding numerical error by $e_u = u - u_h$. For regular triangulations of $I = (0, 1)$, if the finite element space V_h is the piecewise polynomials* *of degree $k \geq 1$, then for small enough h there holds the following error estimates*

$$\|u - u_h\| \leq Ch^{k+\frac{1}{2}}. \tag{3.6}$$

Remark 3.1. We could not obtain the sharp error estimates $O(h^{k+1})$ due to some extra boundary terms arising from high order derivatives. The error estimates for the linear KdV equation in [33] also has the suboptimal rate $O(h^{k+\frac{1}{2}})$. Numerical examples in [33] verify the optimal order $O(h^{k+1})$.

3.1.3 The error equation

In order to obtain the error estimate to smooth solutions for the considered semi-discrete LDG scheme (3.4), we need to first obtain the error equation.

Notice that the scheme (3.4) is also satisfied when the numerical solutions u_h, v_h, p_h are replaced by the exact solutions u, $v = u_x$, $p = u_{xx}$. We then obtain the cell error equation

$$\begin{aligned}
&\int_{I_j} (u - u_h)_t \rho dx + \int_{I_j} ((p - p_h)\psi + (v - v_h)\varphi) dx \\
&- \int_{I_j} (f(u) - f(u_h))\rho_x dx + ((f(u) - \widehat{f})\rho^-)_{j+\frac{1}{2}} - ((f(u) - \widehat{f})\rho^+)_{j-\frac{1}{2}} \\
&- \int_{I_j} (p - p_h)\rho_x dx + ((p - \widehat{p}_h)\rho^-)_{j+\frac{1}{2}} - ((p - \widehat{p}_h)\rho^+)_{j-\frac{1}{2}} \\
&+ \int_{I_j} (v - v_h)\psi_x dx - ((v - \widehat{v}_h)\psi^-)_{j+\frac{1}{2}} + ((v - \widehat{v}_h)\psi^+)_{j-\frac{1}{2}} \\
&+ \int_{I_j} (u - u_h)\varphi_x dx - ((u - \widehat{u}_h)\varphi^-)_{j+\frac{1}{2}} + ((u - \widehat{u}_h)\varphi^+)_{j-\frac{1}{2}} = 0
\end{aligned} \tag{3.7}$$

for all ρ, ψ, $\varphi \in V_h$.

Define

$$\begin{aligned}
&\mathcal{B}_j(u - u_h, p - p_h, v - v_h; \rho, \psi, \varphi) \\
&= \int_{I_j} (u - u_h)_t \rho dx + \int_{I_j} ((p - p_h)\psi + (v - v_h)\varphi) dx \\
&- \int_{I_j} (p - p_h)\rho_x dx + ((p - \widehat{p}_h)\rho^-)_{j+\frac{1}{2}} - ((p - \widehat{p}_h)\rho^+)_{j-\frac{1}{2}} \\
&+ \int_{I_j} (v - v_h)\psi_x dx - ((v - \widehat{v}_h)\psi^-)_{j+\frac{1}{2}} + ((v - \widehat{v}_h)\psi^+)_{j-\frac{1}{2}}
\end{aligned} \tag{3.8}$$

$$+\int_{I_j}(u-u_h)\varphi_x dx - ((u-\widehat{u}_h)\varphi^-)_{j+\frac{1}{2}} + ((u-\widehat{u}_h)\varphi^+)_{j-\frac{1}{2}}$$

and

$$\mathcal{H}_j(f;u,u_h;\rho) = \int_{I_j}(f(u)-f(u_h))\rho_x dx - ((f(u)-\widehat{f})\rho^-)_{j+\frac{1}{2}} + ((f(u)-\widehat{f})\rho^+)_{j-\frac{1}{2}}. \quad (3.9)$$

Summing over j, the error equation becomes

$$\sum_{j=1}^{N}\mathcal{B}_j(u-u_h,p-p_h,v-v_h;\rho,\psi,\varphi) = \sum_{j=1}^{N}\mathcal{H}_j(f;u,u_h;\rho) \quad (3.10)$$

for all ρ, ψ, $\varphi \in V_h$.

Denoting

$$w=\mathcal{P}^-u-u_h, w^e=\mathcal{P}^-u-u,\ r=\mathcal{P}p-p_h, r^e=\mathcal{P}p-p,\ z=\mathcal{P}v-v_h, z^e=\mathcal{P}v-v, \quad (3.11)$$

and taking the test functions

$$\rho=w, \quad \psi=z, \quad \varphi=-r,$$

we obtain the important *energy equality*

$$\sum_{j=1}^{N}\mathcal{B}_j(w-w^e,r-r^e,z-z^e;w,z,-r) = \sum_{j=1}^{N}\mathcal{H}_j(f;u,u_h;w). \quad (3.12)$$

3.1.4 Proof of the main result

In this subsection, we will follow the idea of [33, 35] to present the main steps in the proof of Theorem 3.1. We shall prove the theorem by analyzing each term of the energy equation (3.12).

To deal with the nonlinearity of the flux $f(u)$ we would like to make an *a priori* assumption that, for small enough h, there holds

$$\|u-u_h\| \le h. \quad (3.13)$$

For the linear flux $f(u)=cu$, this *a priori* assumption is unnecessary.

Corollary 3.2. *Suppose that the interpolation property (2.10) is satisfied, then the a priori assumption (3.13) implies that*

$$\|e_u\|_\infty \le Ch^{\frac{1}{2}} \quad \textit{and} \quad \|\mathbb{Q}u - u_h\|_\infty \le Ch^{\frac{1}{2}} \tag{3.14}$$

where $\mathbb{Q} = \mathcal{P}$ or $\mathbb{Q} = \mathcal{P}^{\pm}$ is the projection operator.

First, we consider the left hand side of the energy equation (3.12).

Lemma 3.3. *The following equation holds*

$$\begin{aligned} &\sum_{j=1}^{N} \mathcal{B}_j(w - w^e, r - r^e, z - z^e; w, z, -r) \\ &= \int_0^1 w_t w dx - \int_0^1 w_t^e w dx + \frac{1}{2}\sum_{j=1}^{N}[z]^2_{j+\frac{1}{2}} + \sum_{j=1}^{N}((r^e)^+[w])_{j+\frac{1}{2}} - \sum_{j=1}^{N}((z^e)^+[z])_{j+\frac{1}{2}}. \end{aligned} \tag{3.15}$$

Proof.

$$\mathcal{B}_j(w - w^e, r - r^e, z - z^e; w, z, -r) = \mathcal{B}_j(w, r, z; w, z, -r) - \mathcal{B}_j(w^e, r^e, z^e; w, z, -r), \tag{3.16}$$

By the same argument as that used for the cell entropy inequality [33], the first term of the right hand side in (3.16) becomes

$$\mathcal{B}_j(w, r, z; w, z, -r) = \int_{I_j} w_t w dx + \Psi_{j+\frac{1}{2}} - \Psi_{j-\frac{1}{2}} + \frac{1}{2}[z]^2_{j-\frac{1}{2}} \tag{3.17}$$

where $\Psi = r^+w^- + \frac{1}{2}(z^-)^2 - z^+z^-$.

As to the second term of the right hand side in (3.16), we have

$$\begin{aligned} &\mathcal{B}_j(w^e, r^e, z^e; w, z, -r) \\ &= \int_{I_j} w_t^e w dx + \int_{I_j} (r^e z - z^e r) dx + \int_{I_j} (-r^e w_x - w^e r_x + z^e z_x) dx \\ &\quad - ((r^e)^+[w])_{j-\frac{1}{2}} - ((w^e)^-[r])_{j-\frac{1}{2}} + ((z^e)^+[z])_{j-\frac{1}{2}} + \Phi_{j+\frac{1}{2}} - \Phi_{j-\frac{1}{2}}, \end{aligned} \tag{3.18}$$

where $\Phi = (r^e)^+w^- + (w^e)^-r^- - (z^e)^+z^-$. Because $\mathcal{P}$ is a local L^2 projection, and $\mathcal{P}^-$, even though not a local L^2 projection, does have the property that $\omega - \mathcal{P}^-\omega$ is locally orthogonal to all polynomials of degree up to $k-1$, then

$$\int_{I_j} (r^e z - z^e r) dx + \int_{I_j} (-r^e w_x - w^e r_x + z^e z_x) dx = 0.$$

Noticing the special interpolating property of the projection $\mathcal{P}^-$, we have

$$((w^e)^-[r])_{j-\frac{1}{2}} = 0.$$

The equation (3.18) becomes

$$\mathcal{B}_j(w^e, r^e, z^e; w, z, -r) = \int_{I_j} w_t^e w dx - ((r^e)^+[w])_{j-\frac{1}{2}} + ((z^e)^+[z])_{j-\frac{1}{2}} + \Phi_{j+\frac{1}{2}} - \Phi_{j-\frac{1}{2}}.$$

Combining the above equation with (3.17), summing over j and taking into account the periodic boundary condition, we can get the desired equality (3.15). □

Next, we consider the right hand side of the energy equation (3.12). We can rewrite it into the following form

$$\sum_{j=1}^{N} \mathcal{H}_j(f; u, u_h; w) = \sum_{j=1}^{N} \int_{I_j} (f(u) - f(u_h))w_x dx \tag{3.19}$$
$$+ \sum_{j=1}^{N} ((f(u) - f(\bar{u}_h))[w])_{j+\frac{1}{2}} + \sum_{j=1}^{N} ((f(\bar{u}_h) - \widehat{f})[w])_{j+\frac{1}{2}},$$

where we take into account the periodic boundary condition and we recall that the average $\bar{u}_h$ is defined by $\bar{u}_h = \frac{1}{2}(u_h^+ + u_h^-)$.

First, we give the estimate of the last term in the equation (3.19).

Lemma 3.4. *Suppose that the interpolation property (2.10) is satisfied, then we have*

$$\sum_{j=1}^{N} ((f(\bar{u}_h) - \widehat{f})[w])_{j+\frac{1}{2}} \leq -\frac{3}{4}\alpha(\widehat{f}; u_h)[w]^2 + Ch^{2k+1}. \tag{3.20}$$

Proof. Since the exact solution u of the one-dimensional KdV equation (3.1) is continuous on each boundary point, we have that

$$[u_h] = -[e_u] = [w^e - w].$$

Noticing the definition and boundedness of $\alpha(\widehat{f}; u_h)$ (see Lemma 2.1), by Young's inequality and the interpolation property (2.10), we can easily show that

$$\sum_{j=1}^{N} ((f(\bar{u}_h) - \widehat{f})[w])_{j+\frac{1}{2}} = \sum_{j=1}^{N} (\alpha(\widehat{f}; u_h)[u_h][w])_{j+\frac{1}{2}} = \sum_{j=1}^{N} (\alpha(\widehat{f}; u_h)[w^e - w][w])_{j+\frac{1}{2}}$$

$$= -\sum_{j=1}^{N}(\alpha(\widehat{f};u_h)[w]^2)_{j+\frac{1}{2}} + \sum_{j=1}^{N}(\alpha(\widehat{f};u_h)[w^e][w])_{j+\frac{1}{2}}$$
$$\leq -\frac{3}{4}\alpha(\widehat{f};u_h)[w]^2 + Ch^{2k+1}.$$

□

For the first two terms of the right hand side of the equation (3.19), the estimates are given in the lemma below. The proof of this lemma will be given in the Appendix 6.1.

Lemma 3.5. *Suppose that interpolation property (2.10) is satisfied, then we have*

$$\sum_{j=1}^{N}\int_{I_j}(f(u)-f(u_h))w_x dx + \sum_{j=1}^{N}((f(u)-f(\bar{u}_h))[w])_{j+\frac{1}{2}} \tag{3.21}$$
$$\leq \frac{1}{2}\alpha(\widehat{f};u_h)[w]^2 + (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1}.$$

Combining equations (3.20) and (3.21), we can obtain the estimate of the equation (3.19).

Corollary 3.6. *Suppose that interpolation property (2.10) is satisfied, then we have the following estimate*

$$\sum_{j=1}^{N}\mathcal{H}_j(f;u,u_h;w) \tag{3.22}$$
$$\leq -\frac{1}{4}\alpha(\widehat{f};u_h)[w]^2 + (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1}.$$

Now we are ready to get the final error estimates (3.6). Combining equations (3.12), (3.15) and (3.22), we obtain

$$\int_0^1 w_t w dx + \frac{1}{4}\alpha(\widehat{f};u_h)[w]^2 + \frac{1}{2}\sum_{j=1}^{N}[z]_{j+\frac{1}{2}}^2 + \sum_{j=1}^{N}((r^e)^+[w])_{j+\frac{1}{2}} - \sum_{j=1}^{N}((z^e)^+[z])_{j+\frac{1}{2}}$$
$$\leq \int_0^1 w_t^e w dx + (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1}.$$

Again by Young's inequality and the interpolation property (2.10), the equation becomes

$$\int_0^1 w_t w dx + \frac{1}{8}\alpha(\widehat{f};u_h)[w]^2 + \frac{1}{4}\sum_{j=1}^{N}[z]_{j+\frac{1}{2}}^2$$

$$\leq (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1},$$

where we use the uniform dissipation property of the numerical flux $\widehat{f}$. Using the results (3.14) implied by the *a priori* assumption (3.13) and the positive property of $\alpha(\widehat{f}; u_h)$, we can get the following error estimate

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx \leq C\|w\|^2 + Ch^{2k+1}.$$

Thus Theorem 3.1 follows by the triangle inequality and the interpolating property (2.10).

To complete the proof, let us verify the *a priori* assumption (3.13). In fact, the inequality (3.6) for $k \geq 1$ implies that the *a priori* assumption (3.13) is true for small enough h.

3.2 Error analysis for the two-dimensional KdV type equations

In this subsection, we follow the same line as in Subsection 3.1 to analyze error estimates of the LDG method for the two-dimensional type KdV equations in Cartesian meshes. The essential point is that we use the special projection in [10] for these two-dimensional problems in Cartesian meshes to control certain jump terms at cell boundaries.

3.2.1 The LDG method

We present the LDG method for the following problem:

$$u_t + f(u)_x + g(u)_y + u_{xxx} + u_{yyy} = 0, \tag{3.23}$$

with an initial condition

$$u(x, y, 0) = u_0(x, y) \tag{3.24}$$

and periodic boundary conditions. Here $f(u)$ and $g(u)$ are arbitrary (smooth) nonlinear functions. To define the LDG method, we rewrite the equation (3.23) as a first order system:

$$\begin{aligned} &u_t + f(u)_x + g(u)_y + p_x + q_y = 0, \\ &p - v_x = 0, \quad v - u_x = 0, \quad q - w_y = 0, \quad w - u_y = 0. \end{aligned} \tag{3.25}$$

Now we can use the LDG method to approximate the equations (3.25). Find u_h, p_h, v_h, q_h, $w_h \in Z_h$, such that, $\forall\ \rho,\ \psi,\ \varphi,\ \phi,\ \xi \in Z_h$,

$$
\begin{aligned}
&\int_{I_i}\int_{J_j}(u_h)_t\rho dxdy - \int_{I_i}\int_{J_j}(f(u_h)+p_h)\rho_x dxdy - \int_{I_i}\int_{J_j}(g(u_h)+q_h)\rho_y dxdy \\
&+\int_{J_j}((\widehat{f}+\widehat{p}_h)\rho^-)_{i+\frac{1}{2},y}dy - \int_{J_j}((\widehat{f}+\widehat{p}_h)\rho^+)_{i-\frac{1}{2},y}dy \\
&+\int_{I_i}((\widehat{g}+\widehat{q}_h)\rho^-)_{x,j+\frac{1}{2}}dx - \int_{I_i}((\widehat{g}+\widehat{q}_h)\rho^+)_{x,j-\frac{1}{2}}dx = 0, \\
&\int_{I_i}\int_{J_j} p_h\psi dxdy + \int_{I_i}\int_{J_j} v_h\psi_x dxdy - \int_{J_j}(\widehat{v}_h\psi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}(\widehat{v}_h\psi^+)_{i-\frac{1}{2},y}dy = 0, \\
&\int_{I_i}\int_{J_j} v_h\varphi dxdy + \int_{I_i}\int_{J_j} u_h\varphi_x dxdy - \int_{J_j}(\widehat{u}_h\varphi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}(\widehat{u}_h\varphi^+)_{i-\frac{1}{2},y}dy = 0, \\
&\int_{I_i}\int_{J_j} q_h\phi dxdy + \int_{I_i}\int_{J_j} w_h\phi_y dxdy - \int_{I_i}(\widehat{w}_h\phi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}(\widehat{w}_h\phi^+)_{x,j-\frac{1}{2}}dx = 0, \\
&\int_{I_i}\int_{J_j} w_h\xi dxdy + \int_{I_i}\int_{J_j} u_h\xi_y dxdy - \int_{I_i}(\widehat{u}_h\xi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}(\widehat{u}_h\xi^+)_{x,j-\frac{1}{2}}dx = 0.
\end{aligned}
\tag{3.26}
$$

The "hat" terms in (3.26) are again the numerical fluxes. It turns out that we can once more take the simple choices such that

$$
\begin{aligned}
&\widehat{f}_{i+\frac{1}{2},y} = \widehat{f}(u_h^-,u_h^+)_{i+\frac{1}{2},y},\ (\widehat{p}_h)_{i+\frac{1}{2},y} = (p_h^+)_{i+\frac{1}{2},y}, (\widehat{v}_h)_{i+\frac{1}{2},y} = (v_h^+)_{i+\frac{1}{2},y},\ (\widehat{u}_h)_{i+\frac{1}{2},y} = (u_h^-)_{i+\frac{1}{2},y}, \\
&\widehat{g}_{x,j+\frac{1}{2}} = \widehat{g}(u_h^-,u_h^+)_{x,j+\frac{1}{2}},\ (\widehat{q}_h)_{x,j+\frac{1}{2}} = (q_h^+)_{x,j+\frac{1}{2}}, (\widehat{w}_h)_{x,j+\frac{1}{2}} = (w_h^+)_{x,j+\frac{1}{2}},\ (\widehat{u}_h)_{x,j+\frac{1}{2}} = (u_h^-)_{x,j+\frac{1}{2}}.
\end{aligned}
\tag{3.27}
$$

Here $\widehat{f}(u_h^-,u_h^+)$ and $\widehat{g}(u_h^-,u_h^+)$ are monotone fluxes which have the uniform dissipation property. The algorithm is now well defined.

3.2.2 The main result

We state the main error estimates of the semi-discrete LDG scheme (3.26) for the two-dimensional problem in Cartesian meshes.

Theorem 3.7. *Let u be the exact solution of the problem (3.23), which is sufficiently smooth with bounded derivatives, and assume $f \in C^3$. Let u_h be the numerical solution of the semi-discrete LDG scheme (3.26) - (3.27) and denote the corresponding numerical error by $e_u = u - u_h$. For rectangular triangulations of $I \times J$, if the finite element space Z_h is the* *piecewise polynomials degree at most $k \geq 1$ in each variable, then for small enough h there holds the following error estimates*

$$\|u - u_h\| \leq Ch^{k+\frac{1}{2}}. \tag{3.28}$$

Remark 3.2. Notice that our proof works only for the finite element space Z_h, not for the usual k-th order polynomial space W_h. This is because the main technique is the special tensor product projection $\mathbb{P}^-$ which can eliminate certain jump terms arising from the high order derivative terms. However, numerical examples in [33, 31] do verify the optimal order of accuracy for the LDG method defined on W_h.

3.2.3 The error equation

In this subsection we give the error equation for the scheme (3.26)–(3.27). Notice that the scheme (3.26) is also satisfied when the numerical solutions u_h, v_h, p_h, w_h, q_h are replaced by the exact solutions u, $v = u_x$, $p = u_{xx}$, $w = u_y$, $q = u_{yy}$. We then obtain the cell error equation

$$\begin{aligned}
&\int_{I_i}\int_{J_j}(u-u_h)_t\rho dxdy - \int_{I_i}\int_{J_j}(f(u)-f(u_h)+p-p_h)\rho_x dxdy \\
&-\int_{I_i}\int_{J_j}(g(u)-g(u_h)+q-q_h)\rho_y dxdy + \int_{I_i}\int_{J_j}(p-p_h)\psi dxdy \\
&+\int_{I_i}\int_{J_j}(v-v_h)\varphi dxdy + \int_{I_i}\int_{J_j}(q-q_h)\phi dxdy + \int_{I_i}\int_{J_j}(w-w_h)\xi dxdy \\
&+\int_{J_j}((f(u)-\widehat{f}+p-\widehat{p}_h)\rho^-)_{i+\frac{1}{2},y}dy - \int_{J_j}((f(u)-\widehat{f}+p-\widehat{p}_h)\rho^+)_{i-\frac{1}{2},y}dy \\
&+\int_{I_i}((g(u)-\widehat{g}+q-\widehat{q}_h)\rho^-)_{x,j+\frac{1}{2}}dx - \int_{I_i}((g(u)-\widehat{g}+q-\widehat{q}_h)\rho^+)_{x,j-\frac{1}{2}}dx \\
&+\int_{I_i}\int_{J_j}(v-v_h)\psi_x dxdy - \int_{J_j}((v-\widehat{v}_h)\psi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}((v-\widehat{v}_h)\psi^+)_{i-\frac{1}{2},y}dy \\
&+\int_{I_i}\int_{J_j}(u-u_h)\varphi_x dxdy - \int_{J_j}((u-\widehat{u}_h)\varphi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}((u-\widehat{u}_h)\varphi^+)_{i-\frac{1}{2},y}dy \\
&+\int_{I_i}\int_{J_j}(w-w_h)\phi_y dxdy - \int_{I_i}((w-\widehat{w}_h)\phi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}((w-\widehat{w}_h)\phi^+)_{x,j-\frac{1}{2}}dx \\
&+\int_{I_i}\int_{J_j}(u-u_h)\xi_y dxdy - \int_{I_i}((u-\widehat{u}_h)\xi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}((u-\widehat{u}_h)\xi^+)_{x,j-\frac{1}{2}}dx = 0
\end{aligned} \tag{3.29}$$

for all $\rho,\ \psi,\ \varphi,\ \phi,\ \xi \in Z_h$. Define

$$
\begin{aligned}
&\mathcal{A}_{ij}(u-u_h, p-p_h, v-v_h, q-q_h, w-w_h; \rho, \psi, \varphi, \phi, \xi)\\
&= \int_{I_i}\int_{J_j}(u-u_h)_t\rho dxdy + \int_{I_i}\int_{J_j}(p-p_h)\psi dxdy\\
&+ \int_{I_i}\int_{J_j}(v-v_h)\varphi dxdy + \int_{I_i}\int_{J_j}(q-q_h)\phi dxdy + \int_{I_i}\int_{J_j}(w-w_h)\xi dxdy\\
&- \int_{I_i}\int_{J_j}(p-p_h)\rho_x dxdy + \int_{J_j}((p-\widehat{p}_h)\rho^-)_{i+\frac{1}{2},y}dy - \int_{J_j}((p-\widehat{p}_h)\rho^+)_{i-\frac{1}{2},y}dy\\
&- \int_{I_i}\int_{J_j}(q-q_h)\rho_y dxdy + \int_{I_i}((q-\widehat{q}_h)\rho^-)_{x,j+\frac{1}{2}}dx - \int_{I_i}((q-\widehat{q}_h)\rho^+)_{x,j-\frac{1}{2}}dx\\
&+ \int_{I_i}\int_{J_j}(v-v_h)\psi_x dxdy - \int_{J_j}((v-\widehat{v}_h)\psi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}((v-\widehat{v}_h)\psi^+)_{i-\frac{1}{2},y}dy\\
&+ \int_{I_i}\int_{J_j}(u-u_h)\varphi_x dxdy - \int_{J_j}((u-\widehat{u}_h)\varphi^-)_{i+\frac{1}{2},y}dy + \int_{J_j}((u-\widehat{u}_h)\varphi^+)_{i-\frac{1}{2},y}dy\\
&+ \int_{I_i}\int_{J_j}(w-w_h)\phi_y dxdy - \int_{I_i}((w-\widehat{w}_h)\phi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}((w-\widehat{w}_h)\phi^+)_{x,j-\frac{1}{2}}dx\\
&+ \int_{I_i}\int_{J_j}(u-u_h)\xi_y dxdy - \int_{I_i}((u-\widehat{u}_h)\xi^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}((u-\widehat{u}_h)\xi^+)_{x,j-\frac{1}{2}}dx
\end{aligned}
\tag{3.30}
$$

and

$$
\begin{aligned}
&\mathcal{G}_{ij}(f, g; u, u_h; \rho) = \int_{I_i}\int_{J_j}(f(u)-f(u_h))\rho_x dxdy + \int_{I_i}\int_{J_j}(g(u)-g(u_h))\rho_y dxdy\\
&- \int_{J_j}((f(u)-\widehat{f})\rho^-)_{i+\frac{1}{2},y}dy + \int_{J_j}((f(u)-\widehat{f})\rho^+)_{i-\frac{1}{2},y}dy\\
&- \int_{I_i}((g(u)-\widehat{g})\rho^-)_{x,j+\frac{1}{2}}dx + \int_{I_i}((g(u)-\widehat{g})\rho^+)_{x,j-\frac{1}{2}}dx.
\end{aligned}
\tag{3.31}
$$

Summing over i, j, the error equation becomes

$$
\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{A}_{ij}(u-u_h, p-p_h, v-v_h, q-q_h, w-w_h; \rho, \psi, \varphi, \phi, \xi) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{G}_{ij}(f, g; u, u_h; \rho)
\tag{3.32}
$$

for all $\rho,\ \psi,\ \varphi,\ \phi,\ \xi \in Z_h$.

Denoting

$$
\begin{aligned}
&\eta = \mathbb{P}^- u - u_h,\ \eta^e = \mathbb{P}^- u - u,\ r = \mathbb{P}p - p_h,\ r^e = \mathbb{P}p - p, z = \mathbb{P}v - v_h,\ z^e = \mathbb{P}v - v,\\
&s = \mathbb{P}q - q_h,\ s^e = \mathbb{P}q - q,\ \varsigma = \mathbb{P}w - w_h,\ \varsigma^e = \mathbb{P}w - w
\end{aligned}
\tag{3.33}
$$

and taking the test functions

$$\rho = \eta, \quad \psi = z, \quad \varphi = -r, \quad \phi = \varsigma, \quad \xi = -s,$$

we obtain the important *energy equality*

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{A}_{ij}(\eta-\eta^e, r-r^e, z-z^e, s-s^e, \varsigma-\varsigma^e; \eta, z, -r, \varsigma, -s) = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathcal{G}_{ij}(f, g; u, u_h; \eta). \tag{3.34}$$

3.2.4 Proof of the main result

In this subsection, we will follow the idea of Subsection 3.1 to present the main steps in the proof of Theorem 3.7.

To deal with the nonlinearity of the fluxes $f(u)$ and $g(u)$, we would like to make the same *a priori* assumption as in (3.13). The Corollary 3.2 is still satisfied. For linear fluxes $f(u) = c_1 u$ and $g(u) = c_2 u$ with constants c_1 and c_2, this *a priori* assumption is unnecessary.

In fact, the proof of the main results in Theorem 3.7 follows the same line as for the one-dimensional case. We just state some lemmas to give the main idea for the linear part $\mathcal{A}_{ij}$ which corresponds to the linear part $\mathcal{B}_j$ for the one-dimensional case. We will mention the difference between the proof of the one-dimensional case and the two-dimensional case, but will not repeat the details. For the nonlinear part $\mathcal{G}_{ij}$ which corresponds to the nonlinear part $\mathcal{H}_j$ for the one-dimensional case, the proof is along the same line as in the one-dimensional case and will not be repeated here.

First, we consider the left hand side of the energy equation (3.34).

Lemma 3.8. *There exist numerical entropy fluxes $\Psi_{x,j+\frac{1}{2}}$ and $\Phi_{i+\frac{1}{2},y}$ such that the following equation holds*

$$\mathcal{A}_{ij}(\eta, r, z, s, \varsigma; \eta, z, -r, \varsigma, -s) = \int_{I_i}\int_{J_j}\eta_t\eta dxdy + \frac{1}{2}\int_{J_j}[z]^2_{i-\frac{1}{2},y}dy + \frac{1}{2}\int_{I_i}[\varsigma]^2_{x,j-\frac{1}{2}}dx$$
$$+\int_{I_i}(\Psi_{x,j+\frac{1}{2}} - \Psi_{x,j-\frac{1}{2}})dx + \int_{J_j}(\Phi_{i+\frac{1}{2},y} - \Phi_{i-\frac{1}{2},y})dy. \tag{3.35}$$

The proof is by the same argument as that used for the cell entropy inequality [33].

Lemma 3.9. *There exist fluxes $\Lambda_{x,j+\frac{1}{2}}$ and $\Upsilon_{i+\frac{1}{2},y}$ such that the following equation holds*

$$\begin{aligned}
&\mathcal{A}_{ij}(\eta^e, r^e, z^e, s^e, \varsigma^e; \eta, z, -r, \varsigma, -s) \\
&= \int_{I_i}\int_{J_j} \eta_t^e \eta dxdy + \int_{I_i}\int_{J_j} (r^e z - z^e r + s^e \varsigma - \varsigma^e s) dxdy \\
&+ \int_{I_i}\int_{J_j} (-r^e \eta_x - \eta^e r_x + z^e z_x - s^e \eta_y - \eta^e s_y + \varsigma^e \varsigma_y) dxdy \\
&+ \int_{J_j} (-(r^e)^+[\eta] - (\eta^e)^-[r] + (z^e)^+[z])_{i-\frac{1}{2},y} dy + \int_{J_j} (\Lambda_{i+\frac{1}{2},y} - \Lambda_{i-\frac{1}{2},y}) dy \\
&+ \int_{I_i} (-(s^e)^+[\eta] - (\eta^e)^-[s] + (\varsigma^e)^+[\varsigma])_{x,j-\frac{1}{2}} dx + \int_{I_i} (\Upsilon_{x,j+\frac{1}{2}} - \Upsilon_{x,j-\frac{1}{2}}) dx.
\end{aligned} \tag{3.36}$$

From this lemma, we can use the property of the special projection $\mathbb{P}^-$ in (2.12)–(2.14). We have the following results similar to the one-dimensional problem

$$\int_{I_i}\int_{J_j} (-\eta^e r_x - \eta^e s_y) dxdy = 0, \quad \int_{J_j} (-(\eta^e)^-[r])_{i-\frac{1}{2},y} dy = 0, \quad \int_{I_i} (-(\eta^e)^-[s])_{x,j-\frac{1}{2}} dx = 0.$$

Notice that we can only define this kind of special projections for the Q^k elements. For other terms in (3.36), the proof is almost the same as in the one-dimensional case. Now we can get similar result to Lemma 3.3.

Lemma 3.10. *The following equation holds*

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N} \mathcal{A}_{ij}(\eta - \eta^e, r - r^e, z - z^e, s - s^e, \varsigma - \varsigma^e; \eta, z, -r, \varsigma, -s) \\
&= \int_0^1\int_0^1 \eta_t \eta dxdy - \int_0^1\int_0^1 \eta_t^e \eta dxdy + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \left(\int_{J_j} [z]^2_{i-\frac{1}{2},y} dy + \int_{I_i} [\varsigma]^2_{x,j-\frac{1}{2}} dx \right) \\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N} \left(\int_{J_j} \left(-(r^e)^+[\eta] + (z^e)^+[z]\right)_{i-\frac{1}{2},y} dy + \int_{I_i} \left(-(s^e)^+[\eta] + (\varsigma^e)^+[\varsigma]\right)_{x,j-\frac{1}{2}} dx \right).
\end{aligned} \tag{3.37}$$

We can use the same argument and technique as in Subsection 3.1.4 to proceed the remaining of the proof. The algebraic manipulations are a straightforward extension from the one-dimensional case. We therefore do not give further details.

3.2.5 The ZK equation

There is another two-dimensional generalization of the KdV equation in the following form

$$u_t + (3u^2)_x + u_{xxx} + u_{xyy} = 0, \tag{3.38}$$

which is called the Zakharov-Kuznetsov (ZK) equation. In [31], the LDG method is designed and the numerical examples are shown to illustrate the accuracy and capability of the methods. We can also use the same idea of the special projection to prove the same error estimates for the two-dimensional ZK equation (3.38) in Cartesian meshes.

To define the local discontinuous Galerkin method, we rewrite the equation (3.38) as a first order system (with $f(u) = 3u^2$):

$$
\begin{aligned}
&u_t + f(u)_x + p_x + r_y = 0, \quad p - q_x = 0, \\
&q - u_x = 0, \quad r - s_x = 0, \quad s - u_y = 0.
\end{aligned} \tag{3.39}
$$

Now we can define the local discontinuous Galerkin method to approximate the equations (3.39). Find u_h, p_h, q_h, r_h, $s_h \in Z_h$, such that, $\forall$ ρ, φ, ψ, ϕ, $\xi \in Z_h$,

$$
\begin{aligned}
&\int_{J_j}\int_{I_i}(u_h)_t\rho dxdy - \int_{J_j}\int_{I_i} f(u_h)\rho_x dxdy + \int_{J_j}((\widehat{f}\rho^-)_{i+\frac{1}{2},y} - (\widehat{f}\rho^+)_{i-\frac{1}{2},y})dy \\
&\quad - \int_{J_j}\int_{I_i} p_h\rho_x dxdy + \int_{J_j}((\widehat{p}_h\rho^-)_{i+\frac{1}{2},y} - (\widehat{p}_h\rho^+)_{i-\frac{1}{2},y})dy \\
&\quad - \int_{J_j}\int_{I_i} r_h\rho_y dxdy + \int_{I_i}((\widehat{r}_h\rho^-)_{x,j+\frac{1}{2}} - (\widehat{r}_h\rho^+)_{x,j-\frac{1}{2}})dx = 0, \\
&\int_{J_j}\int_{I_i} p_h\varphi dxdy + \int_{J_j}\int_{I_i} q_h\varphi_x dxdy - \int_{J_j}((\widehat{q}_h\varphi^-)_{i+\frac{1}{2},y} - (\widehat{q}_h\varphi^+)_{i-\frac{1}{2},y})dy = 0, \\
&\int_{J_j}\int_{I_i} q_h\psi dxdy + \int_{J_j}\int_{I_i} u_h\psi_x dxdy - \int_{J_j}((\widehat{u}_h\psi^-)_{i+\frac{1}{2},y} - (\widehat{u}_h\psi^+)_{i-\frac{1}{2},y})dy = 0, \\
&\int_{J_j}\int_{I_i} r_h\phi dxdy + \int_{J_j}\int_{I_i} s_h\phi_x dxdy - \int_{J_j}((\widehat{s}_h\phi^-)_{i+\frac{1}{2},y} - (\widehat{s}_h\phi^+)_{i-\frac{1}{2},y})dy = 0, \\
&\int_{J_j}\int_{I_i} s_h\xi dxdy + \int_{J_j}\int_{I_i} u_h\xi_y dxdy - \int_{I_i}((\bar{u}_h\xi^-)_{x,j+\frac{1}{2}} - (\bar{u}_h\xi^+)_{x,j-\frac{1}{2}})dx = 0.
\end{aligned} \tag{3.40}
$$

The "hat" or "bar" terms in (3.40) are again the numerical fluxes. It turns out that we can take the simple choices such that

$$
\begin{aligned}
&(\widehat{p}_h)_{i+\frac{1}{2},y} = (p_h)^+_{i+\frac{1}{2},y}, \quad (\widehat{u}_h)_{i+\frac{1}{2},y} = (u_h)^-_{i+\frac{1}{2},y}, \quad (\widehat{q}_h)_{i+\frac{1}{2},y} = (q_h)^+_{i+\frac{1}{2},y}, \quad (\widehat{s}_h)_{i+\frac{1}{2},y} = (s_h)^+_{i+\frac{1}{2},y}, \\
&(\bar{u}_h)_{x,j+\frac{1}{2}} = (u_h)^-_{x,j+\frac{1}{2}}, \quad (\widehat{r}_h)_{x,j+\frac{1}{2}} = (r_h)^+_{x,j+\frac{1}{2}} - \alpha_0((u_h)^+_{x,j+\frac{1}{2}} - (u_h)^-_{x,j+\frac{1}{2}}),
\end{aligned} \tag{3.41}
$$

where $\alpha_0 > 0$ is a constant. $\widehat{f}(u_h^-, u_h^+)$ is taken again as a monotone flux with the uniform dissipation property. The algorithm is now well defined. Here we modify the scheme of the

ZK equation in [31] by adding a dissipation term in the flux of $(\widehat{r}_h)_{x,j+\frac{1}{2}}$, which will give a control on the boundary terms. For the scheme (3.40)–(3.41), we can obtain the same results as in Theorem 3.7, along the same lines in the proof. The details are omitted to save space.

4 Error estimates of the LDG method for the nonlinear convection-diffusion equations

4.1 Error analysis for the one-dimensional nonlinear convection-diffusion equation

4.1.1 The LDG method

In this section, we present and analyze the LDG method for the following problem:

$$u_t + f(u)_x - (a(u)u_x)_x = 0, \tag{4.1}$$

with an initial condition

$$u(x,0) = u_0(x) \tag{4.2}$$

and periodic boundary conditions. Here $f(u)$ and $a(u) \geq 0$ are arbitrary (smooth) nonlinear functions.

To define the LDG method, we rewrite the equation (4.1) as a first order system:

$$u_t + f(u)_x - (b(u)v)_x = 0, \qquad v - B(u)_x = 0, \tag{4.3}$$

where $b(u) = \sqrt{a(u)}$ and $B(u) = \int^u b(\tau)d\tau$. Now we can use the LDG method to approximate the equations (4.3). Find u_h, $v_h \in V_h$, $\forall\ \rho$, $\psi \in V_h$, such that,

$$\begin{aligned} &\int_{I_j} (u_h)_t \rho dx - \int_{I_j} (f(u_h) - b(u_h)v_h)\rho_x dx + ((\widehat{f} - \widehat{b}(u_h)\widehat{v}_h)\rho^-)_{j+\frac{1}{2}} - ((\widehat{f} - \widehat{b}(u_h)\widehat{v}_h)\rho^+)_{j-\frac{1}{2}} = 0, \\ &\int_{I_j} v_h \psi dx + \int_{I_j} B(u_h)\psi_x dx - (\widehat{B}(u_h)\psi^-)_{j+\frac{1}{2}} + (\widehat{B}(u_h)\psi^+)_{j-\frac{1}{2}} = 0. \end{aligned} \tag{4.4}$$

The "hat" terms in (4.4) are numerical fluxes. It turns out that we can take the simple choices such that

$$\widehat{f} = \widehat{f}(u_h^-, u_h^+), \quad \widehat{b}(u_h) = \frac{B(u_h^+) - B(u_h^-)}{u_h^+ - u_h^-}, \quad \widehat{B}(u_h) = B(u_h^-), \quad \widehat{v}_h = v_h^+, \quad \widehat{u}_h = u_h^+, \tag{4.5}$$

where we have omitted the half-integer indices $j+\frac{1}{2}$ as all quantities in (4.5) are computed at the same points (i.e. the interfaces between the cells). Here $\widehat{f}(u_h^-, u_h^+)$ is monotone flux which has the uniform dissipation property. We remark that the choice for the fluxes (4.5) is not unique. In fact the crucial part is taking $\widehat{v}_h$ and $\widehat{B}(u_h)$ from opposite sides.

4.1.2 The main result

We state the main error estimates of the semi-discrete LDG scheme (4.4). Detailed proof will be given in subsequent subsections.

Theorem 4.1. *Let u be the exact solution of the problem (4.1), which is sufficiently smooth with bounded derivatives, and assume f, $a \in C^3$. Let u_h be the numerical solution of the semi-discrete LDG scheme (4.4) and denote the corresponding numerical error by $e_u = u - u_h$. For regular triangulations of $I = (0, 1)$, if the finite element space V_h is the piecewise polynomials of degree $k \geq 1$, then for small enough h there holds the following error estimates*

$$\|u - u_h\| \leq Ch^{k+\frac{1}{2}}, \tag{4.6}$$

where the constant C depends on the time t.

Remark 4.1. For the purely diffusion case, if $a(u) = cu$ (i.e. the linear heat equation) we can get the optimal $O(h^{k+1})$ order accuracy result for the numerical flux (4.5).

Remark 4.2. If we take the upwind numerical flux for the nonlinear convection term $f(u)$ which is mentioned in [35] and the linear diffusion term $a(u) = cu$, we can get the optimal error estimates $O(h^{k+1})$.

4.1.3 The error equation

In order to obtain the error estimate to smooth solutions for the considered LDG scheme (4.4), we need to first get the error equation.

Notice that the scheme (4.4) is also satisfied when the numerical solutions u_h, v_h are replaced by the exact solutions u, $v = B(u)_x$. We then obtain the cell error equation

$$\int_{I_j} (u - u_h)_t \rho dx + \int_{I_j} (v - v_h)\psi dx$$

$$
\begin{aligned}
&-\int_{I_j}(f(u)-f(u_h))\rho_x dx+((f(u)-\widehat{f})\rho^-)_{j+\frac12}-((f(u)-\widehat{f})\rho^+)_{j-\frac12} \qquad (4.7)\\
&+\int_{I_j}(b(u)v-b(u_h)v_h)\rho_x dx-((b(u)v-\widehat{b}(u_h)\widehat{v}_h)\rho^-)_{j+\frac12}+((b(u)v-\widehat{b}(u_h)\widehat{v}_h)\rho^+)_{j-\frac12}\\
&+\int_{I_j}(B(u)-B(u_h))\psi_x dx-((B(u)-\widehat{B}(u_h))\psi^-)_{j+\frac12}+((B(u)-\widehat{B}(u_h))\psi^+)_{j-\frac12}=0
\end{aligned}
$$

for all $\rho,\ \psi\in V_h$.

We define

$$
\begin{aligned}
&\mathbf{R}_j(b,B;u,v,u_h,v_h;\rho,\psi) \qquad (4.8)\\
=&\int_{I_j}(b(u)v-b(u_h)v_h)\rho_x dx-((b(u)v-\widehat{b}(u_h)\widehat{v}_h)\rho^-)_{j+\frac12}+((b(u)v-\widehat{b}(u_h)\widehat{v}_h)\rho^+)_{j-\frac12}\\
&+\int_{I_j}(B(u)-B(u_h))\psi_x dx-((B(u)-\widehat{B}(u_h))\psi^-)_{j+\frac12}+((B(u)-\widehat{B}(u_h))\psi^+)_{j-\frac12}
\end{aligned}
$$

and use the notation introduced in Section 3. Summing over j, the error equation becomes

$$
\begin{aligned}
&\sum_{j=1}^N\int_{I_j}(u-u_h)_t\rho dx+\sum_{j=1}^N\int_{I_j}(v-v_h)\psi dx \qquad (4.9)\\
=&-\sum_{j=1}^N\mathbf{R}_j(b,B;u,v,u_h,v_h;\rho,\psi)+\sum_{j=1}^N\mathcal{H}_j(f;u,u_h;\rho)
\end{aligned}
$$

for all $\rho,\ \psi\in V_h$.

Denoting

$$
w=\mathcal{P}^-u-u_h,\quad w^e=\mathcal{P}^-u-u,\quad z=\mathcal{P}^+v-v_h,\quad z^e=\mathcal{P}^+v-v \qquad (4.10)
$$

and taking the test functions

$$
\rho=w,\qquad \varphi=z,
$$

we obtain the important *energy equality*

$$
\begin{aligned}
&\sum_{j=1}^N\int_{I_j}(w-w^e)_t w dx+\sum_{j=1}^N\int_{I_j}(z-z^e)z dx\\
=&-\sum_{j=1}^N\mathbf{R}_j(b,B;u,v,u_h,v_h;w,z)+\sum_{j=1}^N\mathcal{H}_j(f;u,u_h;w). \qquad (4.11)
\end{aligned}
$$

4.1.4 Proof of the main result

In this subsection, we will follow the idea in Section 3 to present the main steps in the proof of Theorem 4.1. We shall prove the theorem by analyzing each term of the energy equation (4.11).

To deal with the nonlinearity of the flux $f(u)$, $B(u)$ and $b(u)v$, we would like to make the same *a priori* assumptions as in (3.13). The Corollary 3.2 is still satisfied. For linear fluxes $f(u) = cu$, $B(u) = u$ and $b(u)v = v$, this *a priori* assumption is unnecessary.

In fact, to present the proof of the main results in Theorem 4.1, we only need to get the estimate for the first term of the right-hand side of the equation (4.11) because the estimate for the second term of the right-hand side of the equation (4.11) has been obtained in Corollary 3.6.

We rewrite the first term of the right-hand side of the equation (4.11) into the following form

$$\begin{aligned}
&\sum_{j=1}^{N} \mathbf{R}_j(b, B; u, v, u_h, v_h; w, z) \\
&= \sum_{j=1}^{N} \int_{I_j} (b(u)v - b(u_h)v_h) w_x dx + \sum_{j=1}^{N} ((b(u)v - \widehat{b}(u_h)v_h^+)[w])_{j+\frac{1}{2}} \\
&+ \sum_{j=1}^{N} \int_{I_j} (B(u) - B(u_h)) z_x dx + \sum_{j=1}^{N} ((B(u) - B(u_h^-))[z])_{j+\frac{1}{2}},
\end{aligned} \tag{4.12}$$

where we take into account the periodic boundary condition. The estimates for the equation (4.12) are given in the lemma below. The proof of this lemma will be given in the Appendix 6.2.

Lemma 4.2. *Suppose that the interpolation property (2.10) is satisfied, then we have the following estimate*

$$\begin{aligned}
&|\sum_{j=1}^{N} \mathbf{R}_j(b, B; u, v, u_h, v_h; w, z)| \leq \sum_{j=1}^{N} (b'(u)v|w^e||[w]|)_{j+\frac{1}{2}} + \frac{1}{2}\|z\|^2 \\
&+ (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+2}.
\end{aligned} \tag{4.13}$$

Now we are ready to get the final error estimate (4.6). Combining equations (4.11), (3.22) and (4.13), we obtain

$$\begin{aligned}
&\int_0^1 w_t w dx + \frac{1}{4}\alpha(\widehat{f}; u_h)[w]^2 + \frac{1}{2}\int_0^1 z^2 dx \\
&\le \int_0^1 (w_t^e w + z_t^e z) dx + \sum_{j=1}^{N} (b'(u)v|w^e||[w]|)_{j+\frac{1}{2}} \\
&+ (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1}.
\end{aligned}$$

Again by Young's inequality and the interpolation property (2.10), the equation becomes

$$\begin{aligned}
&\int_0^1 w_t w dx + \frac{1}{8}\alpha(\widehat{f}; u_h)[w]^2 + \frac{1}{4}\int_0^1 z^2 dx \\
&\le (C + C_\star(\|w\|_\infty + h^{-1}\|e_u\|_\infty^2))\|w\|^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)h^{2k+1}.
\end{aligned}$$

Using the result (3.14) implied by the *a priori* assumption (3.13) and the positive property of $\alpha(\widehat{f}; u_h)$, we can get the following error estimate

$$\frac{1}{2}\frac{d}{dt}\int_0^1 w^2 dx \le C\|w\|^2 + Ch^{2k+1}.$$

Thus Theorem 4.1 follows by the triangle inequality and the interpolating property (2.10).

To complete the proof, let us verify the *a priori* assumption (3.13). In fact, the inequality (4.6) with $k \ge 1$ implies that the *a priori* assumption (3.13) is true for small enough h.

4.2 Error analysis for the two-dimensional nonlinear convection-diffusion equation

In this subsection, we follow the same line as in Subsection 4.1 to analyze error estimates of the LDG method for the two-dimensional nonlinear convection-diffusion equation (1.4) with $d = 2$.

4.2.1 The LDG method

We present the LDG method for the following problem [15]:

$$u_t + \sum_{i=1}^{d} f_i(u)_{x_i} - \sum_{i=1}^{d}\sum_{j=1}^{d} (a_{ij}(u)u_{x_j})_{x_i} = 0, \quad d = 2, \tag{4.14}$$

with an initial condition

$$u(x, y, 0) = u_0(x, y) \tag{4.15}$$

and periodic boundary conditions, where $f_i(u)$ and $a_{ij}(u)$ are arbitrary (smooth) nonlinear functions and $a_{ij}(u)$ are entries of a symmetric and semi-positive definite matrix. We will interchangeably use x, y to denote x_1, x_2 in two dimension. To define the LDG method, we rewrite the equation (4.14) as a first order system:

$$\begin{aligned} u_t + \sum_{i=1}^{d} f_i(u)_{x_i} - \sum_{i=1}^{d}(\sum_{l=1}^{d} b_{il}(u)q_l)_{x_i} = 0, \\ q_l - \sum_{j=1}^{d}(g_{lj}(u))_{x_j} = 0, \quad l = 1, \cdots, d. \end{aligned} \tag{4.16}$$

Since the matrix $a_{ij}(u)$ is assumed to be symmetric and semi-positive definite, there exists a symmetric matrix $b_{ij}(u)$ such that

$$a_{ij}(u) = \sum_{1 \le l \le d} b_{il}(u) b_{lj}(u) \tag{4.17}$$

and

$$g_{lj}(u) = \int^{u} b_{lj}(\tau) d\tau. \tag{4.18}$$

Now we can use the LDG method to approximate the equations (4.16). Find u_h, $(q_1)_h, \cdots$, $(q_d)_h \in W_h$, such that, $\forall\ \rho$, $\psi_1, \cdots$, $\psi_d \in W_h$,

$$\begin{aligned} &\int_K (u_h)_t \rho dx - \sum_{i=1}^{d} \int_K f_i(u_h)\rho_{x_i} dx + \sum_{i=1}^{d} \int_K (\sum_{l=1}^{d} b_{il}(u_h)(q_l)_h)\rho_{x_i} dx \\ &+ \int_{\partial K} \widehat{f}(u_h^L, u_h^R, \mathbf{n})\rho ds - \sum_{i=1}^{d} \int_{\partial K} (\sum_{l=1}^{d} \widehat{b_{il}(u_h)}\widehat{(q_l)}_h n_i)\rho ds = 0, \\ &\int_K (q_l)_h \psi_l dx + \sum_{j=1}^{d} \int_K g_{lj}(u_h)(\psi_l)_{x_j} dx - \sum_{j=1}^{d} \int_{\partial K} \widehat{g_{lj}(u_h)} n_j \psi_l ds = 0, \quad l = 1, \cdots, d. \end{aligned} \tag{4.19}$$

where $\mathbf{n} = (n_1, \cdots, n_d)$ denotes the outward unit normal to the element K at $x \in \partial K$. The "hat" terms in (4.19) are the numerical fluxes. It turns out that we can take the simple choices such that

$$\widehat{b_{il}(u_h)} = \frac{g_{il}(u_h^R) - g_{il}(u_h^L)}{u_h^R - u_h^L}, \quad \widehat{g_{lj}(u_h)} = g_{lj}(u_h^L), \quad \widehat{(q_l)}_h = (q_l)_h^R \tag{4.20}$$

and $\widehat{f}(u_h^L, u_h^R, \mathbf{n})$ is any monotone flux with uniform dissipative property which is conservative and consistent with the nonlinearity $\sum_{i=1}^{d} f_i(u)n_i$. Now the algorithm is well defined.

Remark 4.3. We can also define the LDG method for the nonlinear convection-diffusion equation (4.14) in the finite element space Z_h. The scheme is still defined by (4.19) except that the solutions and test functions are both from the space Z_h. We will also give the error estimate results for the finite element space Z_h.

4.2.2 The main result

We state the main error estimates of the semi-discrete LDG scheme (4.19) for the two-dimensional problem.

Theorem 4.3. *Let u be the exact solution of the problem (4.14), which is sufficiently smooth with bounded derivatives, and assume $f_i(u)$, $a_{ij}(u) \in C^3$. Let u_h be the numerical solution of the semi-discrete LDG scheme (4.19) - (4.20) and denote the corresponding numerical error by $e_u = u - u_h$. For arbitrary triangulations, if the finite element space W_h in each element is the piecewise polynomials degree $k \geq 1$, then for small enough h there holds the following error estimates*

$$\|u - u_h\| \leq Ch^{k+\gamma}, \tag{4.21}$$

where the constant C depends on the time t and $\gamma = 0$. For rectangular triangulations of $I \times J$, if the finite element space Z_h is the piecewise polynomials degree at most $k \geq 1$ in each variable, we have the result with $\gamma = \frac{1}{2}$ in (4.21).

Remark 4.4. In two-dimensional case, we could not get the error estimates result (4.21) with $\gamma = \frac{1}{2}$ for P^k elements because of the lack of a suitable projection $\mathcal{P}^{\pm}$ similar to the one-dimensional case to eliminate the jump terms, e.g. $\sum_{j=1}^{N}((B(u) - B(u_h^-))[z])_{j+\frac{1}{2}}$ in the equation (4.12). Otherwise, the proof for the two-dimensional P^k elements case is a straightforward extension from the one-dimensional case. We will not repeat the details here.

Remark 4.5. For the finite element space Z_h we can get the error estimates result (4.21) with $\gamma = \frac{1}{2}$. The main technique is the special tensor product projection $\mathbb{P}^{\pm}$ defined in (2.11) which can eliminate certain jump terms arising from nonlinear high order derivative terms. The proof follows the same lines as in Subsection 3.2 and Subsection 4.1.

5 Concluding remarks

In this paper, we have obtained *a priori* L^2 error estimates for the semi-discrete LDG method of two classes of nonlinear equations formulated by the KdV equations and the nonlinear convection-diffusion equations. We obtain the results for both one and two dimensional cases. It is more challenging to perform error estimates for the fully discrete LDG methods for nonlinear equations and this will be carried out in the future.

6 Appendix: Proof of several lemmas

6.1 Proof of Lemma 3.5

To complete the proof to Lemma 3.5, we would like to use the following Taylor expansions

$$\begin{aligned} & f(u) - f(u_h) && (6.1)\\ =& f'(u)w - \frac{1}{2}f''(u)w^2 - f'(u)w^e + f''(u)ww^e - \frac{1}{2}f''(u)(w^e)^2 + \frac{1}{6}f'''_u(w - w^e)^3, \\ & f(u) - f(\bar{u}_h) && (6.2)\\ =& f'(u)\bar{w} - \frac{1}{2}f''(u)\bar{w}^2 - f'(u)\bar{w}^e + f''(u)\bar{w}\bar{w}^e - \frac{1}{2}f'(u)(\bar{w}^e)^2 + \frac{1}{6}\tilde{f}'''_u(\bar{w} - \bar{w}^e)^3, \end{aligned}$$

where f'''_u and $\tilde{f}'''_u$ are the mean values. These imply the following representation

$$\begin{aligned} & \sum_{j=1}^{N}\int_{I_j}(f(u) - f(u_h))w_x dx + \sum_{j=1}^{N}((f(u) - f(\bar{u}_h))[w])_{j+\frac{1}{2}} \\ =& \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6, \end{aligned} \qquad (6.3)$$

where

$$\mathcal{T}_1 = \sum_{j=1}^{N}\int_{I_j} f'(u)ww_x dx + \sum_{j=1}^{N}(f'(u)\bar{w}[w])_{j+\frac{1}{2}},$$

$$\mathcal{T}_2 = -\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} f''(u)w^2 w_x dx + \sum_{j=1}^{N}(f''(u)\bar{w}^2[w])_{j+\frac{1}{2}}\right),$$

$$\mathcal{T}_3 = -\left(\sum_{j=1}^{N}\int_{I_j} f'(u)w^e w_x dx + \sum_{j=1}^{N}(f'(u)\bar{w}^e[w])_{j+\frac{1}{2}}\right),$$

$$\mathcal{T}_4 = \sum_{j=1}^{N}\int_{I_j} f''(u)w^e w w_x dx + \sum_{j=1}^{N}(f''(u)\bar{w}^e\bar{w}[w])_{j+\frac{1}{2}},$$

$$\mathcal{T}_5 = -\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} f''(u)(w^e)^2 w_x dx + \sum_{j=1}^{N}(f''(u)(\bar{w}^e)^2[w])_{j+\frac{1}{2}}\right),$$

$$\mathcal{T}_6 = \frac{1}{6}\left(\sum_{j=1}^{N}\int_{I_j} f'''_u(w-w^e)^3 w_x dx + \sum_{j=1}^{N}(\tilde{f}'''_u(\bar{w}-\bar{w}^e)^3[w])_{j+\frac{1}{2}}\right)$$

will be estimated separately later.

- $\mathcal{T}_1$ term.

  After a simple integration by parts, it is easy to get

$$\mathcal{T}_1 = -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(f'(u))_x w^2 dx \le C\|w\|^2. \tag{6.4}$$

- $\mathcal{T}_2$ term.

  After a simple integration by parts, it is easy to get

$$\mathcal{T}_2 = \frac{1}{6}\left(\sum_{j=1}^{N}\int_{I_j}(f''(u))_x w^3 dx + \frac{1}{4}\sum_{j=1}^{N}(f''(u)[w]^3)_{j+\frac{1}{2}}\right).$$

  By a simple Taylor expansion, there is

$$f''(u)[w] = (f''(\bar{u}_h) + f'''(u-\bar{u}_h))[w],$$

  where f''' denote a mean value of the third derivative of flux f. Combining the above equation with (2.5) of Lemma 2.1 and the interpolation property (2.10), on each boundary point $x_{j+\frac{1}{2}}$, there holds

$$f''(u)[w] \le 8\alpha(\widehat{f}, u_h) + C_\star(h + \|e_u\|_\infty^2).$$

  Thus by virtue of the inverse property (ii), we have

$$\mathcal{T}_2 \le \frac{1}{3}\alpha(\widehat{f}; u_h)[w]^2 + C_\star(C + \|w\|_\infty + h^{-1}\|e_u\|_\infty^2)\|w\|^2. \tag{6.5}$$

- $\mathcal{T}_3$ term.

  We can rewrite $\mathcal{T}_3$ into the following form

$$\mathcal{T}_3 = -\left(\sum_{j=1}^{N}\int_{I_j}(f'(u)-f'(u_j))w^e w_x dx + \sum_{j=1}^{N}\int_{I_j} f'(u_j)w^e w_x dx + \sum_{j=1}^{N}(f'(u)\bar{w}^e[w])_{j+\frac{1}{2}}\right),$$

  The second term in the above equation is zero by the definition of the projection. Because of $|f'(u)-f'(u_j)| = O(h)$ on each element I_j, then by the inverse property (i) in (2.16), together with the interpolation property (2.10), the first term in the above equation is estimated by

$$|\sum_{j=1}^{N}\int_{I_j}(f'(u)-f'(u_j))w^e w_x dx| \le C\|w^e\|\|w\| \le C\|w\|^2 + Ch^{2k+2}.$$

  Again from the conclusion (2.4) in Lemma 2.1 and the smoothness of u and f, we have

$$|f'(u_{j+\frac{1}{2}})| \le 2\alpha(\widehat{f}; u_h) + C_\star\|e_u\|_\infty.$$

  Hence by Young's inequality and the boundedness of $\alpha(\widehat{f}; u_h)$, we obtain

$$|\sum_{j=1}^{N}(f'(u)\bar{w}^e[w])_{j+\frac{1}{2}}| \le \frac{1}{6}\alpha(\widehat{f}; u_h)[w]^2 + C_\star h^{-1}\|e_u\|_\infty^2\|w\|^2 + Ch^{2k+1}.$$

  Now we can get the estimate of $\mathcal{T}_3$

$$\mathcal{T}_3 \le \frac{1}{6}\alpha(\widehat{f}; u_h)[w]^2 + (C + C_\star h^{-1}\|e_u\|_\infty^2)\|w\|^2 + Ch^{2k+1}. \tag{6.6}$$

- $\mathcal{T}_4$, $\mathcal{T}_5$ and $\mathcal{T}_6$ terms.

  Because $\mathcal{T}_4$, $\mathcal{T}_5$ and $\mathcal{T}_6$ are high order terms in Taylor expansion, it is easy to show by Young's inequality and the inverse properties (i) and (ii) in (2.16) that

$$\mathcal{T}_4 \le C_\star h^{-1}\|w^e\|_\infty\|w\|^2 \le C_\star\|w\|^2, \tag{6.7}$$

$$\mathcal{T}_5 \le C_\star h^{-1}\|w^e\|_\infty(\|w^e\| + h^{\frac{1}{2}}\|w^e\|_{\Gamma_h})\|w\| \le C_\star\|w\|^2 + C_\star h^{2k+2}, \tag{6.8}$$

$$\mathcal{T}_6 \le C_\star h^{-1}\|e_u\|_\infty^2(\|w\|^2 + Ch^{2k+2}). \tag{6.9}$$

Therefore, summing up the above estimates from equation (6.4) to (6.9), we complete the proof of Lemma 3.5.

6.2 Proof of Lemma 4.2

The proof of Lemma 4.2 is similar to the proof of Lemma 3.5. The main difference is that we take the values $\bar{u}_h = \frac{1}{2}(u_h^+ + u_h^-)$, v_h^+ and u_h^- as the reference values of the functions u_h and v_h on each boundary point.

For the nonlinear terms $b(u)v$ and $B(u)$, we use the following Taylor expansions

$$\begin{aligned} B(u) - B(u_h) =& b(u)w - \frac{1}{2}b'(u)w^2 - b(u)w^e + b'(u)ww^e - \frac{1}{2}b'(u)(w^e)^2 + \frac{1}{6}b''_u(w - w^e)^3, \\ B(u) - B(u_h^-) =& b(u)w^- - \frac{1}{2}b'(u)(w^-)^2 - b(u)(w^e)^- \\ & + b'(u)w^-(w^e)^- - \frac{1}{2}b'(u)((w^e)^-)^2 + \frac{1}{6}\tilde{b}''_u(w^- - (w^e)^-)^3, \end{aligned}$$

where b''_u and $\tilde{b}''_u$ are the mean values and here we use the property $B'(u) = b(u)$.

$$\begin{aligned} & b(u)v - b(u_h)v_h \\ =& b'(u)vw + b(u)z - \frac{1}{2}b''(u)vw^2 - b'(u)zw - b'(u)vw^e \\ & - b(u)z^e + b''(u)vww^e - \frac{1}{2}b''(u)v(w^e)^2 + b'(u)zw^e + b'(u)wz^e \\ & - b'(u)w^e z^e + \frac{1}{6}b'''_u\bar{v}(w - w^e)^3 + \frac{1}{2}b''_u(w - w^e)^2(z - z^e), \\ & b(u)v - \widehat{b}(u_h)v_h^+ \\ =& b'(u)v\bar{w} + b(u)z^+ - \frac{1}{2}b''(u)v\bar{w}^2 - b'(u)\bar{w}z^+ - \frac{1}{24}b''(u)v[w]^2 - b'(u)v\bar{w}^e \\ & - b(u)(z^e)^+ + b''(u)v\bar{w}\bar{w}^e - \frac{1}{2}b''(u)v(\bar{w}^e)^2 + b'(u)z^+\bar{w}^e + b'(u)\bar{w}(z^e)^+ \\ & - b'(u)\bar{w}^e(z^e)^+ + \frac{1}{12}b''(u)v[w][w^e] - \frac{1}{24}b''(u)v[w^e]^2 \\ & + \frac{1}{12}\tilde{b}'''_u\tilde{v}((w^+ - (w^e)^+)^2 + (w^- - (w^e)^-)^2)(\bar{w} - \bar{w}^e) \\ & + \frac{1}{6}\tilde{b}''_u((w^+ - (w^e)^+)^2 + (w^+ - (w^e)^+)(w^- - (w^e)^-) + (w^- - (w^e)^-)^2)(z^+ - (z^e)^+), \end{aligned}$$

where b''_u, $\tilde{b}''_u$, $b'''_u\bar{v}$ and $\tilde{b}'''_u\tilde{v}$ are the mean values. These imply the following representation

$$\sum_{j=1}^{N} \mathbf{R}_j(b, B; u, v, u_h, v_h; w, z) = \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4 + \mathcal{S}_5 + \mathcal{S}_6, \qquad (6.10)$$

where

$$\mathcal{S}_1 = \sum_{j=1}^{N} \int_{I_j} b'(u)vww_x dx + \sum_{j=1}^{N} (b'(u)v\bar{w}[w])_{j+\frac{1}{2}}$$

$$
\begin{aligned}
&+\sum_{j=1}^{N}\int_{I_j} b(u)(zw)_x dx+\sum_{j=1}^{N}(b(u)(z^+[w]+w^-[z]))_{j+\frac{1}{2}},\\
\mathcal{S}_2 =&-\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} b''(u)vw^2w_x dx+\sum_{j=1}^{N}(b''(u)v(\bar{w}^2+\frac{1}{12}[w]^2)[w])_{j+\frac{1}{2}}\right)\\
&-\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} b'(u)(zw^2)_x dx+\sum_{j=1}^{N}(b'(u)(2z^+\bar{w}[w]+(w^-)^2[z]))_{j+\frac{1}{2}}\right),\\
\mathcal{S}_3 =&-\sum_{j=1}^{N}\int_{I_j} b'(u)vw^e w_x dx-\sum_{j=1}^{N}(b'(u)v\bar{w}^e[w])_{j+\frac{1}{2}}\\
&-\sum_{j=1}^{N}\int_{I_j} b(u)(w^e z_x+z^e w_x)dx-\sum_{j=1}^{N}(b(u)((z^e)^+[w]+(w^e)^-[z]))_{j+\frac{1}{2}},\\
\mathcal{S}_4 =&\sum_{j=1}^{N}\int_{I_j} b''(u)vw^e ww_x dx+\sum_{j=1}^{N}(b''(u)v(\bar{w}^e\bar{w}+\frac{1}{12}[w^e][w])[w])_{j+\frac{1}{2}}\\
&+\sum_{j=1}^{N}\int_{I_j} b'(u)w^e(zw)_x dx+\sum_{j=1}^{N}(b'(u)(z^+\bar{w}^e[w]+(w^e)^-w^-[z]))_{j+\frac{1}{2}}\\
&+\sum_{j=1}^{N}\int_{I_j} b'(u)z^e ww_x dx+\sum_{j=1}^{N}(b'(u)\bar{w}(z^e)^+[w])_{j+\frac{1}{2}},\\
\mathcal{S}_5 =&-\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} b''(u)(w^e)^2w_x dx+\sum_{j=1}^{N}(b''(u)((\bar{w}^e)^2+\frac{1}{12}[w^e]^2)[w])_{j+\frac{1}{2}}\right)\\
&-\frac{1}{2}\left(\sum_{j=1}^{N}\int_{I_j} b'(u)((w^e)^2z_x+2w^e z^e w_x)dx\right.\\
&\left.+\sum_{j=1}^{N}(b'(u)(2(z^e)^+\bar{w}^e[w]+((w^e)^-)^2[z]))_{j+\frac{1}{2}}\right),\\
\mathcal{S}_6 =&\frac{1}{6}\sum_{j=1}^{N}\int_{I_j}\left(b'''_u(w-w^e)^3w_x+3b''_u(z-z^e)(w-w^e)^2w_x+b''_u(w-w^e)^3z_x\right)dx\\
&+\frac{1}{24}\sum_{j=1}^{N}\Big(4\tilde{b}''_u(w^- -(w^e)^-)^3[z]+\tilde{b}'''_u\tilde{v}((w^+-(w^e)^+)^2+(w^- -(w^e)^-)^2)(\bar{w}-\bar{w}^e)[w]\\
&+4\tilde{b}''_u((w^+-(w^e)^+)^2+(w^+-(w^e)^+)(w^- -(w^e)^-)+(w^- -(w^e)^-)^2)(z^+-(z^e)^+)[w]\Big)_{j+\frac{1}{2}}
\end{aligned}
$$

will be estimated separately later.

- $\mathcal{S}_1$ term.

After a simple integration by parts, it is easy to get

$$\mathcal{S}_1 = -\frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(b'(u)v)_x w^2 dx - \sum_{j=1}^{N}\int_{I_j}(b(u))_x zw dx \le C\|w\|^2 + \frac{1}{12}\|z\|^2. \tag{6.11}$$

- $\mathcal{S}_2$ term.

After a simple integration by parts, it is easy to get

$$\begin{aligned}\mathcal{S}_2 &= \frac{1}{6}\sum_{j=1}^{N}\int_{I_j}(b''(u)v)_x w^3 dx + \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(b'(u))_x zw^2 dx \\ &\le (C_\star\|w\|_\infty + C)\|w\|^2 + \frac{1}{12}\|z\|^2.\end{aligned} \tag{6.12}$$

- $\mathcal{S}_3$ term.

We can rewrite $\mathcal{S}_3$ into the following form

$$\begin{aligned}\mathcal{S}_3 = &-\sum_{j=1}^{N}\int_{I_j}(b'(u)v - b'(u_j)v_j)w^e w_x dx - \sum_{j=1}^{N}\int_{I_j} b'(u_j)v_j w^e w_x dx \\ &- \sum_{j=1}^{N}\int_{I_j}(b(u)-b(u_j))(w^e z_x + z^e w_x)dx - \sum_{j=1}^{N}\int_{I_j} b(u_j)(w^e z_x + z^e w_x)dx \\ &- \sum_{j=1}^{N}(b'(u)v\bar{w}^e[w])_{j+\frac{1}{2}} - \sum_{j=1}^{N}(b(u)((z^e)^+[w] + (w^e)^-[z]))_{j+\frac{1}{2}},\end{aligned}$$

The second term, the fourth term and the last term in the above equation are zero by the definition of the special projection. Because of $|b'(u)v - b'(u_j)v_j| = O(h)$ on each element I_j, then by the inverse property (i) in (2.16), together with the interpolation property (2.10), the first term in the above equation is estimated by

$$|\sum_{j=1}^{N}\int_{I_j}(b'(u)v - b'(u_j)v_j)w^e w_x dx| \le C\|w^e\|\|w\| \le C\|w\|^2 + Ch^{2k+2}.$$

By the same argument we can also get the estimate for the third term

$$|\sum_{j=1}^{N}\int_{I_j}(b(u)-b(u_j))(w^e z_x + z^e w_x)dx| \le C\|w\|^2 + \frac{1}{12}\|z\|^2 + Ch^{2k+2}.$$

Now we can get the estimate of $\mathcal{S}_3$

$$\mathcal{S}_3 \le |\sum_{j=1}^{N}(b'(u)v\bar{w}^e[w])_{j+\frac{1}{2}}| + C\|w\|^2 + \frac{1}{12}\|z\|^2 + Ch^{2k+2}. \tag{6.13}$$

- $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ terms.

  Because $\mathcal{S}_4$, $\mathcal{S}_5$ and $\mathcal{S}_6$ are high order terms in Taylor expansion, it is easy to show by Young's inequality and the inverse properties (i) and (ii) in (2.16) that

$$\begin{aligned}
\mathcal{S}_4 &\le C_\star h^{-1}\|w^e\|_\infty\|w\|^2 + Ch^{-1}(\|w^e\|_\infty + \|z^e\|_\infty)\|w\|\|z\| \\
&\le C_\star\|w\|^2 + \frac{1}{12}\|z\|^2, \qquad (6.14)\\
\mathcal{S}_5 &\le C_\star h^{-1}\|w^e\|_\infty\|w^e\|\|w\| + Ch^{-1}\|w^e\|_\infty(\|w^e\|\|z\| + \|z^e\|\|w\|) \\
&\le (C_\star + C)\|w\|^2 + \frac{1}{12}\|z\|^2 + (C_\star + C)h^{2k+2}, \qquad (6.15)\\
\mathcal{S}_6 &\le C_\star h^{-1}\|e_u\|_\infty^2(\|w\|^2 + Ch^{2k+2}) + \frac{1}{12}\|z\|^2. \qquad (6.16)
\end{aligned}$$

Therefore, summing up the above estimates from equation (6.11) to (6.16), we complete the proof of Lemma 4.2.

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