A local-structure-preserving local discontinuous Galerkin method for the Laplace equation

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Abstract

In this paper, we present a local-structure-preserving local discontinuous Galerkin (LDG) method for the Laplace equation. The method is based on the standard LDG formulation and uses piecewise harmonic polynomials, which satisfy the partial differential equation (PDE) exactly inside each element, as the approximating solutions for the primitive variable u, leading to a significant reduction of the degrees of freedom for the final system and hence the computational cost, without sacrificing the convergence quality of the solutions. An *a priori* error estimate in the energy norm is established. Numerical experiments are performed to verify optimal convergence rates of the local-structure-preserving LDG method in the energy norm and in the L^2-norm, as well as to compare it with the standard LDG method to demonstrate comparable errors of the two methods even though the new local-structure-preserving LDG method is computational less expensive.

Keywords: discontinuous Galerkin method, Laplace equation, local-structure-preserving

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1 Introduction

It is well known that by allowing discontinuities in the finite element solution spaces, discontinuous Galerkin (DG) methods [4, 6], or the local discontinuous Galerkin (LDG) methods for partial differential equations (PDEs) containing higher than first spatial derivatives [5, 12, 13, 11], have more degrees of freedom compared with the traditional finite element methods. This is often considered as a drawback of the DG (LDG) methods. However, these "extra" degrees of freedom may provide algorithm developers opportunities to design stable and accurate schemes by properly choosing the inter-element treatment (also called numerical fluxes). This issue has been pursued by many authors regarding different problems, see for example the review paper [6] and the special issue of the Journal of Scientific Computing on discontinuous Galerkin methods [7].

In this paper, we study a local-structure-preserving LDG method for solving the Laplace equation as a model equation, to explore another issue related to these "extra" degrees of freedom. That is, the discontinuities of the solution spaces also provide flexibility for us to choose the local solution spaces in each element, which is definitely not easy for traditional continuous finite element methods.

Our local-structure-preserving LDG method for the Laplace equation is based on the standard LDG method for elliptic equations [1]. The distinctive feature of the method is the use of harmonic polynomials (polynomials which satisfy $\triangle u = 0$) to approximate the primitive variable u in each element. In other words, the PDE is satisfied exactly in each element by the numerical solution. As a byproduct, the number of degrees of freedom for the final system is significantly reduced, therefore less computational cost is expected compared with the standard LDG method. Meanwhile, the approximation properties of the chosen spaces can guarantee no loss of the accuracy. This local-structure-preserving idea has been successfully studied before for several types of equations: the Maxwell equations [3] and the MHD equations [9] in which the solution space is locally divergence-free, and the Hamilton-Jacobi equations [10] in which the solution space is locally curl-free.

Following the approach in [1], we prove that if harmonic polynomials of degree at most k are used for the primitive variable u, with properly chosen spaces for the auxiliary variable $\mathbf{q} = \nabla u$, the numerical solutions will give an optimal k-th order approximation in the energy norm (the error of $\mathbf{q}$ in L^2-norm plus terms involving the jumps of u across element interfaces) for sufficiently smooth solutions. This estimate is confirmed by numerical experiments. We also perform numerical experiments to demonstrate that the optimal $(k+1)$-th order accuracy is achieved for u in the L^2-norm, however the proof of this cannot be obtained by the standard application of the duality argument due to the choice of the solution spaces. This is left as an open issue to be addressed later.

Notice that the spirit of the LDG method, for which the auxiliary variable $\mathbf{q} \approx \nabla u$ can be locally eliminated, ensures us a final system with the size depending only on the dimension of the space for u [1]. A simple derivation shows that the dimension of our proposed space for u depends on the polynomial degree k *linearly*, whereas the dimension of the standard choice of the LDG solution space depends on k *quadratically*. At the same time, one can see numerically that this local-structure-preserving LDG method gives comparable results as the ones by the standard LDG method, and the condition number of the final linear system is in the same order as the one by the standard LDG method. Therefore with the reduced degrees of freedom, less computational cost is needed using this proposed method in order to achieve the same accuracy of approximation, while the structure of the solution is maintained exactly in each element.

The paper is organized as follows. In Section 2, we introduce the local-structure-preserving LDG method for the Laplace equation. The well-posedness of the method is established in the same section. Section 3 contains an *a priori* error estimate in the energy norm, which is confirmed by numerical experiments in Section 4. Also in Section 4, we include the numerical convergence results of the solution u in the L^2-norm, a comparison between the results of the local-structure-preserving LDG method and the ones of the standard LDG method, and a brief study on the choice of the solution space $\mathbf{M}_h$ for $\mathbf{q}_h$. We end in Section 5 with concluding remarks and remarks for future work.

2 Local-structure-preserving LDG method

The following model problem is considered

$$\begin{aligned} -\triangle u &= 0 \ \text{ in } \Omega \\ u &= g_D \ \text{ on } \Gamma_D \\ \frac{\partial u}{\partial n} &= \mathbf{g}_N \cdot \mathbf{n} \ \text{ on } \Gamma_N \end{aligned} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with the outward unit normal $\mathbf{n}$ to its boundary $\Gamma = \Gamma_N \cup \Gamma_D$. We assume $|\Gamma_D|_{\mathbb{R}^{d-1}} > 0$ for simplicity.

The standard LDG method for solving (2.1) would start with a triangulation $\mathcal{T}_h$ of the domain Ω with the element being denoted by K, the edge by e, and the meshsize by $h = \max_{K\in\mathcal{T}_h}(\text{diameter of } K)$. We further denote by $\mathcal{E}_i$ the union of all interior edges, by $\mathcal{E}_D$ the union of the edges on Γ_D, and by $\mathcal{E}_N$ the union of the edges on Γ_N. By introducing the auxiliary variable $\mathbf{q} = \nabla u$, (2.1) can be rewritten as

$$\mathbf{q} = \nabla u \ \text{ in } \Omega \tag{2.2}$$

$$-\nabla \cdot \mathbf{q} = 0 \ \text{ in } \Omega \tag{2.3}$$

$$u = g_D \ \text{ on } \Gamma_D \tag{2.4}$$

$$\frac{\partial u}{\partial n} = \mathbf{g}_N \cdot \mathbf{n} \ \text{ on } \Gamma_N \tag{2.5}$$

Following the usual definition of the LDG method for elliptic equations, e.g. [1], we obtain the LDG method for (2.2)-(2.5): find $(u_h, \mathbf{q}_h) \in V_h \times \mathbf{M}_h$ such that $\forall K \in \mathcal{T}_h$,

$$\int_K \mathbf{q}_h \cdot \mathbf{r} dx = -\int_K u_h \nabla \cdot \mathbf{r} dx + \int_{\partial K} \hat{u}_h \mathbf{r} \cdot \mathbf{n} ds, \tag{2.6}$$

$$\int_K \mathbf{q}_h \cdot \nabla v dx = \int_{\partial K} v \hat{\mathbf{q}}_h \cdot \mathbf{n} ds. \tag{2.7}$$

There are two things to be specified to finalize the method. One is the numerical fluxes: $\hat{u}_h$ and $\hat{\mathbf{q}}_h$, which are defined by

$$\hat{\mathbf{q}} = \{\{\mathbf{q}\}\} - C_{11}[\![u]\!] - \mathbf{C}_{12}[\![\mathbf{q}]\!] \tag{2.8}$$

$$\hat{u} = \{\{u\}\} + \mathbf{C}_{12} \cdot [\![u]\!] \tag{2.9}$$

for any interior edge e, and by

$$\hat{\mathbf{q}} := \begin{cases} \mathbf{q}^+ - C_{11}(u^+\mathbf{n}^+ + g_D\mathbf{n}^-) & \text{on } \Gamma_D \\ \mathbf{g}_N & \text{on } \Gamma_N \end{cases} \tag{2.10}$$

$$\hat{u} := \begin{cases} g_D & \text{on } \Gamma_D \\ u^+ & \text{on } \Gamma_N \end{cases} \tag{2.11}$$

for the boundary edges. Here the averages $\{\{\cdot\}\}$ and the jumps $[\![\cdot]\!]$ are defined as follows: at any point $x \in e \in \mathcal{E}_i$,

$$\{\{u\}\} := (u^+ + u^-)/2, \qquad \{\{\mathbf{q}\}\} := (\mathbf{q}^+ + \mathbf{q}^-)/2$$

$$[\![u]\!] := (u^+\mathbf{n}^+ + u^-\mathbf{n}^-)/2, \qquad [\![\mathbf{q}]\!] := (\mathbf{q}^+ \cdot \mathbf{n}^+ + \mathbf{q}^- \cdot \mathbf{n}^-)/2,$$

with the following meanings of the notations: suppose $e = K^+ \cap K^-$, let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the outer unit normals of K^+ and K^- along e respectively, and $(u^+, \mathbf{q}^+)$ (respectively $(u^-, \mathbf{q}^-)$) be the trace of the piecewise smooth function $(u, \mathbf{q})$ inside K^+ (respectively K^-) along e. For a boundary edge $e \subset K$, we denote $K^+ = K$ and $\mathbf{n}^+$ is the outer unit normal along $\partial\Omega$. The parameters C_{11} and C_{12} in general depend on $x \in e$, and we will comment on the choice of these parameters later.

The second thing to be specified to complete the scheme is to choose the solution spaces V_h and $\mathbf{M}_h$. The standard choice of these spaces is [1]

$$\bar{V}_h := \bar{V}_h^k = \{u \in L^2(\Omega) : u|_K \in P^k(K), \forall K \in \mathcal{T}_h\},$$

$$\bar{\mathbf{M}}_h := \bar{\mathbf{M}}_h^k = \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \forall K \in \mathcal{T}_h\},$$

where $P^k(K)$ denotes the space of polynomials in K of degree at most k. In this paper, we would like to propose a different choice of the solution spaces

$$V_h := V_h^k = \{u \in L^2(\Omega) : u|_K \in P^k(K), \triangle u|_K = 0, \forall K \in \mathcal{T}_h\},$$

Table 2.1: The dimensions of the local-structure-preserving space $V_h^k|_K$ and the standard choice of the space $\bar{V}_h^k|_K$ in each element.

k	1	2	3	4	5	6	7
$\|V_h^k\|_K\|$	3	5	7	9	11	13	15
$\|\bar{V}_h^k\|_K\|$	3	6	10	15	21	28	36

$$\mathbf{M}_h := \mathbf{M}_h^k = \{\mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \nabla \cdot \mathbf{q}|_K = 0, \forall K \in \mathcal{T}_h\}.$$

Notice the piecewise harmonic polynomials are used as the approximation for u. It is easy to write a local basis of $V_h^k|_K$. For example, take $K = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \in \mathbb{R}^2$, one choice of the basis of $V_h^k|_K$ could be:

$k = 1 :$ 1, x, y

$k = 2 :$ add $x^2 - y^2$, xy

$k = 3 :$ add $x^3 - 3xy^2$, $y^3 - 3x^2y$

$k = 4 :$ add $x^4 - 6x^2y^2 + y^4$, $x^3y - xy^3$

Remark 2.1.

- *Using the piecewise harmonic polynomials ($\triangle u = 0$) for u_h, the PDE is satisfied exactly in each element by the approximating solutions.*

- *Due to the choice of the numerical fluxes (2.8)-(2.11), the function $\mathbf{q}_h$ can be locally solved in terms of u_h from (2.6) and eliminated from the equations, hence the size of the final system to be solved depends on the size of the solution space for u_h only. This is the reason that the method is called a* local *discontinuous Galerkin method in [5].*

- *Notice that the dimension of the local-structure-preserving space $V_h^k|_K$ for u_h is $2k+1$, which depends on k linearly, whereas the dimension of the standard choice of the space $\bar{V}_h^k|_K$ for u_h is $(k+2)(k+1)/2$, which depends on k quadratically. In other words, using the proposed space will significantly reduce the degrees of freedom of the final system especially for large k. In Table 2.1, we display the dimension of these two spaces for some values of the polynomial degree k.*

- *There are two conditions on the choice of V_h and $\mathbf{M}_h$. One is*

$$\nabla V_h \subset \mathbf{M}_h \tag{2.12}$$

which ensures the well-posedness of the scheme (see Lemma 2.2); the other is

$$\nabla \cdot \mathbf{M}_h \subset V_h \tag{2.13}$$

which is used in our error estimate analysis (see Theorem 3.1), yet it does not seem to be a necessary condition for accuracy according to our numerical evidence (see Tables 4.7-4.8 in Section 4). Both the proposed space $V_h \times \mathbf{M}_h$ and the standard choice of the space $\bar{V}_h \times \bar{\mathbf{M}}_h$ satisfy these conditions. An alternative choice for $\mathbf{M}_h$ could be

$$\mathbf{M}'_h = \mathbf{M}'^k_h = \left\{ \mathbf{q} \in [L^2(\Omega)]^d : \mathbf{q}|_K \in [P^k(K)]^d, \triangle \nabla \cdot \mathbf{q}|_K = 0, \forall K \in \mathcal{T}_h \right\}.$$

- *The coefficient C_{11} need to be positive for the well-posedness of the scheme (Lemma 2.2). C_{12} is taken to be $O(1)$ throughout this paper. In the numerical experiments, we take the parameter C_{12} to be along the direction of the (pre-chosen) normal vector of each edge with the modulus $1/2$. By doing so, if $C_{11} = 0$ in (2.8)-(2.11), the numerical fluxes will become the "alternating fluxes" $(\hat{u}, \hat{\mathbf{q}}) = (u^+, \mathbf{q}^-)$, or $= (u^-, \mathbf{q}^+)$, which simplify significantly the computation and make the effective stencil narrower when the auxiliary variable $\mathbf{q}$ is eliminated locally. Such "alternating fluxes" have been used extensively in time-dependent problems [5, 12, 13, 11] with great success.*

By summing up over all $K \in \mathcal{T}_h$, the method can be written as: find $(u_h, \mathbf{q}_h) \in V_h \times \mathbf{M}_h$, such that

$$\begin{aligned} a(\mathbf{q}_h, \mathbf{r}) + b(u_h, \mathbf{r}) &= F(\mathbf{r}) \quad \forall\ \mathbf{r} \in \mathbf{M}_h && (2.14) \\ -b(v, \mathbf{q}_h) + c(u_h, v) &= G(v) \quad \forall\ v \in V_h && (2.15) \end{aligned}$$

where

$$a(\mathbf{q},\mathbf{r}) = \int_\Omega \mathbf{q}\cdot\mathbf{r}dx \tag{2.16}$$

$$b(u,\mathbf{r}) = \sum_K \int_K u\nabla\cdot\mathbf{r}dx - \int_{\mathcal{E}_i}(\{\{u\}\} + \mathbf{C}_{12}\cdot[[u]])[[\mathbf{r}]]ds - \int_{\mathcal{E}_N} u\mathbf{r}\cdot\mathbf{n}ds \tag{2.17}$$

$$c(u,v) = \int_{\mathcal{E}_i} C_{11}[[u]]\cdot[[v]]ds + \int_{\mathcal{E}_D} C_{11}uvds \tag{2.18}$$

$$F(\mathbf{r}) = \int_{\mathcal{E}_D} g_D\mathbf{r}\cdot\mathbf{n}dx, \qquad G(v) = \int_{\mathcal{E}_D} C_{11}g_Dvds + \int_{\mathcal{E}_N} v\mathbf{g}_\mathbf{N}\cdot\mathbf{n}ds \tag{2.19}$$

A more compact way of writing the scheme is to find $(u_h,\mathbf{q}_h)\in V_h\times\mathbf{M}_h$, which satisfies

$$\mathcal{A}(\mathbf{q}_h,u_h;\mathbf{r},v) = \mathcal{F}(\mathbf{r},v) \qquad \forall\ (v,\mathbf{r})\in V_h\times\mathbf{M}_h \tag{2.20}$$

where

$$\mathcal{A}(\mathbf{q},u;\mathbf{r},v) = a(\mathbf{q},\mathbf{r}) + b(u,\mathbf{r}) - b(v,\mathbf{q}) + c(u,v), \qquad \mathcal{F}(\mathbf{r},v) = F(\mathbf{r}) + G(v).$$

Lemma 2.2 (Well-posedness). *The LDG method defined by (2.20) with $C_{11}>0$ provides a unique approximation solution $(u_h,\mathbf{q}_h)\in V_h\times\mathbf{M}_h$.*

Proof. The same argument as that in the Proposition 2.1 in [1] can be used here directly, since the condition (2.12) is the only property of V_h and $\mathbf{M}_h$ needed in the proof. □

3 An *a priori* error estimate in the energy norm

In this section, we present an *a priori* error estimate in the energy norm for the scheme (2.20). We follow a similar argument as that in [1]. Since what we emphasize in this paper is the local-structure-preserving idea, we assume the full elliptic regularity in this section for simplicity. That is, the exact solution $(u,\mathbf{q})\in V\times\mathbf{M}$, where

$$V := \{u\in H^{s+2}(\Omega) : \triangle u = 0 \text{ in } \Omega\} \qquad \mathbf{M} := \{\mathbf{q}\in[H^{s+1}(\Omega)]^d : \nabla\cdot\mathbf{q} = 0 \text{ in } \Omega\},\ \ s\geq 0.$$

The general cases can be obtained by more delicate analysis.

Theorem 3.1 (Error estimate in the energy norm)**.** *Let* $(u, \mathbf{q}) \in V \times \mathbf{M}$ *be the exact solution of (2.1) and* $(u_h, \mathbf{q}_h) \in V_h^k \times \mathbf{M}_h^k$ *be the approximate local-structure-preserving LDG solution of (2.20). Then we have*

$$|(\mathbf{q} - \mathbf{q}_h, u - u_h)|_{\mathcal{A}} \leq Ch^P \|u\|_{s+2}, \tag{3.1}$$

where

$$|(\mathbf{q}, u)|_{\mathcal{A}}^2 = \left(\|\mathbf{q}\|_0^2 + \int_{\mathcal{E}_i} C_{11} [\![u]\!]^2 ds + \int_{\mathcal{E}_D} C_{11} u^2 ds \right)^{1/2} \tag{3.2}$$

$$P = \begin{cases} \min\{s + \frac{1}{2}, k\} & C_{11} \sim O(1) \\ \min\{s + 1, k\} & C_{11} \sim O(h^{-1}) \end{cases} \tag{3.3}$$

There are mainly two ingredients in the proof. One is the Galerkin orthogonality from the consistency of the scheme:

$$\mathcal{A}(\mathbf{e_q}, e_u; \mathbf{r}, v) = 0, \quad \forall\ (\mathbf{r}, v) \in \mathbf{M}_h \times V_h \tag{3.4}$$

where $(\mathbf{e_q}, e_u) = (\mathbf{q} - \mathbf{q}_h, u - u_h)$. The other is the approximation property of the solution spaces V_h and $\mathbf{M}_h$.

Lemma 3.2 (Approximation property of V_h and $\mathbf{M}_h$)**.** *Let* $(w, \mathbf{r}) \in (V \times \mathbf{M})|_K$*, and let* Π *be the* L^2 *projection from* $V|_K$ *to* $V_h^k|_K$*,* $\mathbf{\Pi}$ *be the* L^2 *projection from* $\mathbf{M}|_K$ *to* $\mathbf{M}_h^k|_K$*, then*

$$||w - \Pi w||_{0,K} \leq Ch_K^{\min\{s+1,k\}+1} ||w||_{s+2,K}, \qquad ||w - \Pi w||_{0,\partial K} \leq Ch_K^{\min\{s+1,k\}+\frac{1}{2}} ||w||_{s+2,K}.$$

$$||\mathbf{r} - \mathbf{\Pi r}||_{0,K} \leq Ch_K^{\min\{s,k\}+1} ||\mathbf{r}||_{s+1,K}, \qquad ||\mathbf{r} - \mathbf{\Pi r}||_{0,\partial K} \leq Ch_K^{\min\{s,k\}+\frac{1}{2}} ||\mathbf{r}||_{s+1,K}$$

with h_K *as the diameter of* K*.*

Proof. These approximation results can be obtained by the application of the Bramble-Hilbert lemma (see [2]) and the standard scaling argument. We refer to [8] for more details.

□

The following inverse inequality will also be needed in the proof (see [2]):

Lemma 3.3 (Inverse inequality)**.** *For* $w \in P^k(K)$*, we have*

$$||w||_{0,\partial K} \leq Ch_K^{1/2} ||w||_{0,K}.$$

Proof of Theorem 3.1. Without loss of generality, we take $C_{11}(x) = c_{11}$ in the proof for the simplicity of the notation, where c_{11} is a constant which might depend on the meshsize h. By denoting $(\xi_u, \xi_{\mathbf{q}}) = (u - \Pi u, \mathbf{q} - \boldsymbol{\Pi}\mathbf{q})$, we have

$$\begin{aligned} |(\mathbf{e}_{\mathbf{q}}, e_u)|_{\mathcal{A}} =& |\left((\xi_{\mathbf{q}}, \xi_u) + (\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u)\right)|_{\mathcal{A}} \\ \le& |(\xi_{\mathbf{q}}, \xi_u)|_{\mathcal{A}} + |(\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u)|_{\mathcal{A}} = \mathbf{I} \quad + \quad \mathbf{II}. \end{aligned} \tag{3.5}$$

We will then estimate the terms $\mathbf{I}$ and $\mathbf{II}$ separately.

$$\begin{aligned} \mathbf{I}^2 =& |(\xi_{\mathbf{q}}, \xi_u)|_{\mathcal{A}}^2 = \mathcal{A}(\xi_{\mathbf{q}}, \xi_u; \xi_{\mathbf{q}}, \xi_u) \\ =& a(\xi_{\mathbf{q}}, \xi_{\mathbf{q}}) + c(\xi_u, \xi_u) \\ \le& \sum_K \left(||\xi_{\mathbf{q}}||_{0,K}^2 + 2c_{11}||\xi_u||_{0,\partial K}^2 \right) \\ \le& C\sum_K \left(h_K^{2\min\{s,k\}+2}||\mathbf{q}||_{s+1,K}^2 + c_{11}h_K^{2\min\{s+1,k\}+1}||u||_{s+2,K}^2 \right) \end{aligned} \tag{3.6}$$

The last inequality is from the approximation results in Lemma 3.2.

$$\begin{aligned} \mathbf{II}^2 =& \mathcal{A}(\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u; \boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u) \\ =& \mathcal{A}(-\xi_{\mathbf{q}}, -\xi_u; \boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u) \qquad \text{(Galerkin orthogonality)} \\ =& \mathcal{A}(-\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \Pi e_u; \xi_{\mathbf{q}}, -\xi_u) \qquad \text{(definition of } \mathcal{A}\text{)} \\ =& a(-\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}, \xi_{\mathbf{q}}) + b(\Pi e_u, \xi_{\mathbf{q}}) - b(-\xi_u, -\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}) + c(\Pi e_u, -\xi_u) \\ =& -\sum_K \left(\int_K \boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}} \cdot \xi_{\mathbf{q}} + \int_K \nabla(\Pi e_u) \cdot \xi_{\mathbf{q}} + \int_K \xi_u \nabla \cdot \boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}} dx \right) + \mathbf{III} \\ =& \mathbf{III} \end{aligned}$$

The fourth equality needs another form of $b(\cdot,\cdot)$

$$b(u, \mathbf{r}) = -\sum_{K \in \mathcal{T}_h} \nabla u \cdot \mathbf{r} dx + \int_{\mathcal{E}_i} (\{\{\mathbf{r}\}\} - \mathbf{C_{12}}[[\mathbf{r}]]) \cdot [[u]] ds + \int_{\mathcal{E}_D} u\mathbf{r} \cdot \mathbf{n} ds.$$

The last equality holds since Π and $\boldsymbol{\Pi}$ are L^2 projections onto V_h and $\mathbf{M}_h$ respectively, and we have the inclusion relations $\nabla V_h \subset \mathbf{M}_h$ and $\nabla \cdot \mathbf{M}_h \subset V_h$. Finally,

$$\begin{aligned} \mathbf{III} =& \int_{\mathcal{E}_i} (\{\{\xi_{\mathbf{q}}\}\} - C_{12}[[\xi_{\mathbf{q}}]]) \cdot [[\Pi e_u]] ds + \int_{\mathcal{E}_D} \Pi e_u \xi_{\mathbf{q}} \cdot \mathbf{n} ds + \int_{\mathcal{E}_i} (\{\{\xi_u\}\} + C_{12}[[\xi_u]])[[\boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}}]] \\ &+ \int_{\mathcal{E}_N} \xi_u \boldsymbol{\Pi}\mathbf{e}_{\mathbf{q}} \cdot \mathbf{n} ds - \int_{\mathcal{E}_i} C_{11}[[\Pi e_u]][[\xi_u]] ds - \int_{\mathcal{E}_D} C_{11} \Pi e_u \xi_u ds \end{aligned}$$

By the Cauchy-Schwartz inequality and the inverse inequality in Lemma 3.3

$$\mathbf{III} \leq C\ \mathbf{II}\ \Lambda(\xi_{\mathbf{q}}, \xi_u)$$

where

$$\begin{aligned}\Lambda(\mathbf{q}, u)^2 = &\int_{\mathcal{E}_D} \left(\frac{1}{c_{11}}(\mathbf{q}\cdot\mathbf{n})^2 + C_{11}u^2\right) ds + \int_{\mathcal{E}_N} \frac{u^2}{h_K} ds \\ &+ \int_{\mathcal{E}_i} \left(\frac{1}{c_{11}}|\{\{\mathbf{q}\}\} - \mathbf{C_{12}}[[\mathbf{q}]]|^2 + \frac{1}{h^K}(\{\{u\}\} + \mathbf{C}_{12}\cdot[[u]])^2 + c_{11}[[u]]^2\right) ds\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbf{II} \leq &C\ \Lambda(\xi_{\mathbf{q}}, \xi_u) \\ \leq &C\left\{\sum_K \left(\frac{1}{c_{11}} h_K^{2\min\{s,k\}+1}||\mathbf{q}||^2_{s+1,K} + (c_{11} + \frac{1}{h_K})h_K^{2\min\{s+1,k\}+1}||u||^2_{s+2,K}\right)\right\}^{1/2} \qquad (3.7)\end{aligned}$$

In the last step, we again use the approximation results in Lemma 3.2. Now combining (3.5)-(3.7), we obtain the error estimate in the energy norm (3.1)-(3.3). ☐

Remark 3.4. *Unfortunately, a direct application of the standard duality argument [1] can not provide the error estimate of* u *in the* L^2*-norm due to the choice of the solution space. This issue will be addressed elsewhere. Numerically, however, we observe the optimal convergence rate for* u *in the* L^2*-norm, see Tables 4.1-4.2 in Section 4.*

4 Numerical results

In this section, we include two numerical examples in the two dimensional case. One is the smooth example with the exact solution $u(x,y) = e^{-x}\cos(y)$ on $[0,1]\times[0,1]$. The other is the singular example with the exact solution $u(x,y) = r^{\alpha}\sin(\alpha\theta)$ on $[0,1]\times[0,1]$, where (r,θ) is the polar coordinate and $\alpha = 4/3$. The Dirichlet boundary conditions are considered in both cases with empty Neumann boundary. We have also computed the singular example with $\alpha = 2/3$ and obtained similar results, which are not included here to save space. The computation is performed on uniform rectangular meshes, with the parameter C_{12} to be along the direction of the (pre-chosen) normal vector of each edge with the modulus 1/2 (see Remark 2.1). Diagonally pre-conditioned BiConjugate Gradient iterative method is used to solve the final linear system, the stopping criterion is such that the relative residual error is less than 10^{-13}.

The following notations are used in Tables 4.1-4.8: LSP means the local-structure-preserving LDG method; ST means the standard LDG method; Mix means the local-structure-preserving LDG method with $\bar{\mathbf{M}}_h$ as the solution space for $\mathbf{q}_h$.

4.1 Validation of the error estimate with different C_{11}

We first check the convergence rate of our local-structure-preserving LDG method for smooth and singular solutions with different stabilizing parameter C_{11}, see Tables 4.1 and 4.2. For both $C_{11} = 10 = O(1)$ and $C_{11} = 1/h = O(1/h)$, the theoretical convergence is confirmed. The errors for both cases are in the same magnitude. Moreover, $C_{11} = O(1)$ seems to give better convergence rates for $\mathbf{q}_h$ in the L^2-norm and for $(u_h, \mathbf{q}_h)$ in the energy norm than $C_{11} = O(1/h)$ does, and the convergence rates are also higher than the theoretically expected ones. Although we have not provided an error estimate for u in the L^2-norm theoretically, we observe numerically the $(k+1)$-th order convergence rate for smooth solutions which are optimal.

4.2 Comparison of the local-structure-preserving LDG method and the standard LDG method.

When we introduce the local-structure-preserving LDG method, we claim that by doing so, the size of the final linear system is greatly reduced due to the choice of the solution spaces. In this subsection, we show that this theoretically less expensive method will produce comparable results as the standard LDG methods for both the smooth example and the singular example with different choices of the stabilizing parameter C_{11}, see Tables 4.3-4.6. Therefore we conclude that we have a less expensive method to implement which can produce approximating solutions with the same accuracy.

Table 4.1: Local-structure-preserving LDG solutions with different parameter C_{11} for the smooth example $u(x, y) = e^{-x}\cos(y)$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^1-LSP with $C_{11} = 10$						
h_0	5.33E-04	-	6.61E-03	-	3.15E-02	-
$h_0/2$	1.41E-04	1.92	2.07E-03	1.67	6.64E-03	1.44
$h_0/4$	3.64E-05	1.95	6.28E-04	1.72	2.40E-03	1.47
$h_0/8$	9.30E-06	1.97	1.89E-04	1.74	8.57E-04	1.48
$h_0/16$	2.35E-06	1.98	5.63E-05	1.74	3.04E-04	1.49
$h_0/32$	5.91E-07	1.99	1.67E-05	1.75	1.08E-04	1.50
P^1-LSP with $C_{11} = 1/h$						
h_0	5.33E-04	-	6.61E-03	-	1.80E-02	-
$h_0/2$	1.34E-04	1.99	3.11E-03	1.08	9.04E-03	1.00
$h_0/4$	3.36E-05	1.99	1.50E-03	1.05	4.52E-03	1.00
$h_0/8$	8.42E-06	2.00	7.34E-04	1.03	2.26E-03	1.00
$h_0/16$	2.11E-06	2.00	3.63E-04	1.02	1.13E-03	1.00
$h_0/32$	5.27E-07	2.00	1.80E-04	1.01	5.66E-04	1.00
P^2-LSP with $C_{11} = 10$						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.93E-06	2.64	5.16E-05	2.49	9.55E-05	2.30
$h_0/4$	2.92E-07	2.73	8.08E-06	2.67	1.85E-05	2.37
$h_0/8$	4.11E-08	2.83	1.15E-06	2.81	3.46E-06	2.42
$h_0/16$	5.48E-09	2.90	1.56E-07	2.89	6.31E-07	2.46
$h_0/32$	7.10E-10	2.95	2.04E-08	2.93	1.13E-07	2.48
P^2-LSP with $C_{11} = 1/h$						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.53E-06	2.97	7.51E-05	1.95	1.20E-04	1.97
$h_0/4$	1.93E-07	2.99	1.91E-05	1.97	3.04E-05	1.98
$h_0/8$	2.43E-08	2.99	4.82E-06	1.99	7.65E-06	1.99
$h_0/16$	3.04E-09	2.99	1.21E-06	1.99	1.92E-06	2.00
$h_0/32$	3.81E-10	3.00	3.03E-07	2.00	4.80E-07	2.00
P^3-LSP with $C_{11} = 10$						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.74E-08	3.73	5.33E-07	3.41	8.10E-07	3.26
$h_0/4$	1.26E-09	3.79	4.78E-08	3.48	8.11E-08	3.32
$h_0/8$	8.71E-11	3.85	4.44E-09	3.43	7.97E-09	3.35
$h_0/16$	5.57E-12	3.97	4.50E-10	3.30	7.85E-10	3.34
P^3-LSP with $C_{11} = 1/h$						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.46E-08	3.98	7.21E-07	2.97	9.85E-07	2.98
$h_0/4$	9.15E-10	3.99	9.10E-08	2.99	1.24E-07	2.99
$h_0/8$	5.74E-11	3.99	1.14E-08	2.99	1.55E-08	3.00
$h_0/16$	4.11E-12	3.80	1.43E-09	3.00	1.94E-09	3.00

Table 4.2: Local-structure-preserving LDG solutions with different parameter C_{11} for the singular example $u(x,y) = r^\alpha \sin(\alpha\theta)$, where (r,θ) is the polar coordinate and $\alpha = 4/3$. $h_0 = 0.1$.

meshsize h	$\lVert e_u \rVert_0$	rate	$\lVert \mathbf{e_q} \rVert_0$	rate	$\lVert(\mathbf{e_q}, \mathbf{e_u})\rVert_{\mathcal{A}}$	rate
P^1-LSP with $C_{11} = 10$						
h_0	5.72E-04	-	8.36E-03	-	2.46E-02	-
$h_0/2$	1.47E-04	1.96	3.21E-03	1.38	9.13E-03	1.43
$h_0/4$	3.78E-05	1.97	1.24E-03	1.37	3.35E-03	1.45
$h_0/8$	9.62E-06	1.97	4.81E-04	1.36	1.22E-03	1.46
$h_0/16$	2.45E-06	1.98	1.88E-04	1.35	4.43E-04	1.46
$h_0/32$	6.23E-07	1.97	7.44E-05	1.34	1.62E-04	1.46
P^1-LSP with $C_{11} = 1/h$						
h_0	5.72E-04	-	8.36E-03	-	2.46E-02	-
$h_0/2$	1.46E-04	1.97	3.54E-03	1.24	1.25E-02	0.98
$h_0/4$	3.70E-05	1.98	1.53E-03	1.21	6.29E-03	0.99
$h_0/8$	9.35E-06	1.98	6.77E-04	1.18	3.16E-03	0.99
$h_0/16$	2.36E-06	1.99	3.08E-04	1.14	1.59E-03	0.99
$h_0/32$	5.92E-07	1.99	1.43E-04	1.10	7.96E-04	1.00
P^2-LSP with $C_{11} = 10$						
h_0	5.30E-05	-	2.35E-03	-	2.83E-03	-
$h_0/2$	1.13E-05	2.23	9.03E-04	1.38	1.02E-03	1.48
$h_0/4$	2.36E-06	2.26	3.52E-04	1.36	3.78E-04	1.43
$h_0/8$	4.84E-07	2.28	1.38E-04	1.35	1.44E-04	1.39
$h_0/16$	9.81E-08	2.30	5.47E-05	1.34	5.58E-05	1.37
$h_0/32$	1.97E-08	2.32	2.16E-05	1.34	2.19E-05	1.35
P^2-LSP with $C_{11} = 1/h$						
h_0	5.30E-05	-	2.35E-03	-	2.83E-03	-
$h_0/2$	1.06E-05	2.33	9.32E-04	1.33	1.12E-03	1.33
$h_0/4$	2.10E-06	2.33	3.70E-04	1.33	4.46E-04	1.33
$h_0/8$	4.17E-07	2.33	1.47E-04	1.33	1.77E-04	1.33
$h_0/16$	8.27E-08	2.33	5.83E-05	1.33	7.03E-05	1.33
$h_0/32$	1.64E-08	2.33	2.31E-05	1.33	2.79E-05	1.33
P^3-LSP with $C_{11} = 10$						
h_0	1.98E-05	-	1.62E-03	-	1.73E-03	-
$h_0/2$	4.23E-06	2.23	6.34E-04	1.36	6.58E-04	1.39
$h_0/4$	9.01E-07	2.23	2.48E-04	1.35	2.54E-04	1.37
$h_0/8$	1.89E-07	2.25	9.78E-05	1.35	9.90E-05	1.36
$h_0/16$	3.91E-08	2.27	3.86E-05	1.34	3.89E-05	1.35
P^3-LSP with $C_{11} = 1/h$						
h_0	1.98E-05	-	1.62E-03	-	1.73E-03	-
$h_0/2$	3.93E-06	2.33	6.44E-04	1.33	6.87E-04	1.33
$h_0/4$	7.80E-07	2.33	2.56E-04	1.33	2.72E-04	1.33
$h_0/8$	1.54E-07	2.33	1.01E-04	1.33	1.08E-04	1.33
$h_0/16$	3.07E-08	2.33	4.03E-05	1.33	4.29E-05	1.33

Table 4.3: Local-structure-preserving LDG solutions and the standard LDG solutions for the smooth example $u(x, y) = e^{-x}\cos(y)$. $C_{11} = 10$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^1-LSP						
h_0	5.33E-04	-	6.61E-03	-	3.15E-02	-
$h_0/2$	1.41E-04	1.92	2.07E-03	1.67	6.64E-03	1.44
$h_0/4$	3.64E-05	1.95	6.28E-04	1.72	2.40E-03	1.47
$h_0/8$	9.30E-06	1.97	1.89E-04	1.74	8.57E-04	1.48
$h_0/16$	2.35E-06	1.98	5.63E-05	1.74	3.04E-04	1.49
$h_0/32$	5.91E-07	1.99	1.67E-05	1.75	1.08E-04	1.50
P^1-ST						
h_0	5.33E-04	-	6.78E-03	-	1.81E-02	-
$h_0/2$	1.40E-04	1.93	2.16E-03	1.65	6.66E-03	1.44
$h_0/4$	3.62E-05	1.95	6.79E-04	1.67	2.41E-03	1.47
$h_0/8$	9.21E-06	1.97	2.18E-04	1.64	8.62E-04	1.48
$h_0/16$	2.32E-06	1.99	7.26E-05	1.59	3.06E-04	1.49
$h_0/32$	5.84E-07	1.99	2.48E-05	1.55	1.09E-04	1.50
P^2-LSP						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.93E-06	2.64	5.16E-05	2.49	9.55E-05	2.30
$h_0/4$	2.92E-07	2.73	8.08E-06	2.67	1.85E-05	2.37
$h_0/8$	4.11E-08	2.83	1.15E-06	2.81	3.46E-06	2.42
$h_0/16$	5.48E-09	2.90	1.56E-07	2.89	6.31E-07	2.46
$h_0/32$	7.10E-10	2.95	2.04E-08	2.93	1.13E-07	2.48
P^2-ST						
h_0	8.55E-06	-	4.70E-04	-	5.52E-04	-
$h_0/2$	1.08E-06	2.98	1.21E-04	1.95	1.32E-04	2.06
$h_0/4$	1.36E-07	2.99	3.11E-05	1.97	3.24E-05	2.03
$h_0/8$	1.70E-08	3.00	7.86E-06	1.98	8.03E-06	2.01
$h_0/16$	2.13E-09	3.00	1.98E-06	2.00	2.00E-06	2.00
$h_0/32$	2.67E-10	3.00	4.96E-07	2.00	4.98E-07	2.00
P^3-LSP						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.74E-08	3.73	5.33E-07	3.41	8.10E-07	3.26
$h_0/4$	1.26E-09	3.79	4.78E-08	3.48	8.11E-08	3.32
$h_0/8$	8.71E-11	3.85	4.44E-09	3.43	7.97E-09	3.35
$h_0/16$	5.57E-12	3.97	4.50E-10	3.30	7.85E-10	3.34
P^3-ST						
h_0	1.32E-07	-	6.52E-06	-	7.57E-06	-
$h_0/2$	8.61E-09	3.94	8.24E-07	2.98	8.96E-07	3.08
$h_0/4$	5.51E-10	3.97	1.03E-07	2.99	1.08E-07	3.05
$h_0/8$	3.50E-11	3.98	1.30E-08	3.00	1.33E-08	3.03
$h_0/16$	3.17E-12	3.46	1.63E-09	3.00	1.65E-09	3.01

Table 4.4: Local-structure-preserving LDG solutions and the standard LDG solutions for the smooth example $u(x,y) = e^{-x}\cos(y)$. $C_{11} = 1/h$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^1-LSP						
h_0	5.33E-04	-	6.61E-03	-	1.80E-02	-
$h_0/2$	1.34E-04	1.99	3.11E-03	1.08	9.04E-03	1.00
$h_0/4$	3.36E-05	1.99	1.50E-03	1.05	4.52E-03	1.00
$h_0/8$	8.42E-06	2.00	7.34E-04	1.03	2.26E-03	1.00
$h_0/16$	2.11E-06	2.00	3.63E-04	1.02	1.13E-03	1.00
$h_0/32$	5.27E-07	2.00	1.80E-04	1.01	5.66E-04	1.00
P^1-ST						
h_0	5.33E-04	-	6.78E-03	-	1.81E-02	-
$h_0/2$	1.34E-04	1.99	3.16E-03	1.10	9.05E-03	1.00
$h_0/4$	3.36E-05	2.00	1.51E-03	1.06	4.52E-03	1.00
$h_0/8$	8.43E-06	2.00	7.37E-04	1.04	2.26E-03	1.00
$h_0/16$	2.11E-06	2.00	3.64E-04	1.02	1.13E-03	1.00
$h_0/32$	5.27E-07	2.00	1.81E-04	1.02	5.66E-04	1.00
P^2-LSP						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.53E-06	2.97	7.51E-05	1.95	1.20E-04	1.97
$h_0/4$	1.93E-07	2.99	1.91E-05	1.97	3.04E-05	1.98
$h_0/8$	2.43E-08	2.99	4.82E-06	1.99	7.65E-06	1.99
$h_0/16$	3.04E-09	2.99	1.21E-06	1.99	1.92E-06	2.00
$h_0/32$	3.81E-10	3.00	3.03E-07	2.00	4.80E-07	2.00
P^2-ST						
h_0	8.55E-06	-	4.70E-04	-	5.52E-04	-
$h_0/2$	1.06E-06	3.02	1.23E-04	1.93	1.42E-04	1.96
$h_0/4$	1.31E-07	3.01	3.14E-05	1.97	3.60E-05	1.98
$h_0/8$	1.64E-08	3.00	7.95E-06	1.98	9.06E-06	1.99
$h_0/16$	2.04E-09	3.00	2.00E-06	1.99	2.27E-06	2.00
$h_0/32$	2.56E-10	3.00	5.01E-07	2.00	5.69E-07	2.00
P^3-LSP						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.46E-08	3.98	7.21E-07	2.97	9.85E-07	2.98
$h_0/4$	9.15E-10	3.99	9.10E-08	2.99	1.24E-07	2.99
$h_0/8$	5.74E-11	3.99	1.14E-08	2.99	1.55E-08	3.00
$h_0/16$	4.11E-12	3.80	1.43E-09	3.00	1.94E-09	3.00
P^3-ST						
h_0	1.31E-07	-	6.52E-06	-	7.57E-06	-
$h_0/2$	8.30E-09	3.99	8.39E-07	2.96	9.64E-07	2.97
$h_0/4$	5.21E-10	4.00	1.06E-07	2.98	1.22E-07	2.99
$h_0/8$	3.28E-11	3.99	1.34E-08	2.99	1.53E-08	2.99
$h_0/16$	8.45E-12	1.96	1.68E-09	2.99	1.91E-09	3.00

Table 4.5: Local-structure-preserving LDG solutions and the standard LDG solutions for the singular example $u(x, y) = r^\alpha \sin(\alpha\theta)$, where (r, θ) is the polar coordinate and $\alpha = 4/3$. $C_{11} = 10$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^1-LSP						
h_0	5.72E-04	-	8.36E-03	-	2.46E-02	-
$h_0/2$	1.47E-04	1.96	3.21E-03	1.38	9.13E-03	1.43
$h_0/4$	3.78E-05	1.97	1.24E-03	1.37	3.35E-03	1.45
$h_0/8$	9.62E-06	1.97	4.81E-04	1.36	1.22E-03	1.46
$h_0/16$	2.45E-06	1.98	1.88E-04	1.35	4.43E-04	1.46
$h_0/32$	6.23E-07	1.97	7.44E-05	1.34	1.62E-04	1.46
P^1-ST						
h_0	5.73E-04	-	8.46E-03	-	2.47E-02	-
$h_0/2$	1.47E-04	1.96	3.28E-03	1.37	9.15E-03	1.43
$h_0/4$	3.74E-05	1.97	1.28E-03	1.35	3.36E-03	1.45
$h_0/8$	9.47E-06	1.98	5.06E-04	1.34	1.23E-03	1.45
$h_0/16$	2.38E-06	1.99	2.01E-04	1.33	4.47E-04	1.46
$h_0/32$	5.98E-07	1.99	8.00E-05	1.33	1.63E-04	1.46
P^2-LSP						
h_0	5.30E-05	-	2.35E-03	-	2.83E-03	-
$h_0/2$	1.06E-05	2.33	9.32E-04	1.33	1.12E-03	1.33
$h_0/4$	2.10E-06	2.33	3.70E-04	1.33	4.46E-04	1.33
$h_0/8$	4.17E-07	2.33	1.47E-04	1.33	1.77E-04	1.33
$h_0/16$	8.27E-08	2.33	5.83E-05	1.33	7.03E-05	1.33
$h_0/32$	1.64E-08	2.33	2.31E-05	1.33	2.79E-05	1.33
P^2-ST						
h_0	4.57E-05	-	2.46E-03	-	2.89E-03	-
$h_0/2$	9.10E-06	2.33	9.76E-04	1.33	1.15E-03	1.33
$h_0/4$	1.81E-06	2.33	3.88E-04	1.33	4.56E-04	1.33
$h_0/8$	3.59E-07	2.33	1.54E-04	1.33	1.81E-04	1.33
$h_0/16$	7.12E-08	2.33	6.11E-05	1.33	7.18E-05	1.33
$h_0/32$	1.41E-08	2.33	2.42E-05	1.33	2.85E-05	1.33
P^3-LSP						
h_0	1.98E-05	-	1.62E-03	-	1.73E-03	-
$h_0/2$	4.23E-06	2.23	6.34E-04	1.36	6.58E-04	1.39
$h_0/4$	9.01E-07	2.23	2.48E-04	1.35	2.54E-04	1.37
$h_0/8$	1.89E-07	2.25	9.78E-05	1.35	9.90E-05	1.36
$h_0/16$	3.91E-08	2.27	3.86E-05	1.34	3.89E-05	1.35
P^3-ST						
h_0	1.40E-05	-	1.60E-03	-	1.67E-03	-
$h_0/2$	2.77E-06	2.33	6.35E-04	1.33	6.64E-04	1.33
$h_0/4$	5.50E-07	2.33	2.52E-04	1.33	2.64E-04	1.33
$h_0/8$	1.09E-07	2.33	1.00E-04	1.33	1.05E-04	1.33
$h_0/16$	2.16E-08	2.33	3.97E-05	1.33	4.15E-05	1.33

Table 4.6: Local-structure-preserving LDG solutions and the standard LDG solutions for the singular example $u(x,y) = r^{\alpha}\sin(\alpha\theta)$, where (r,θ) is the polar coordinate and $\alpha = 4/3$. $C_{11} = 1/h$. $h_0 = 0.1$.

meshsize h	$\lVert e_u \rVert_0$	rate	$\lVert \mathbf{e_q} \rVert_0$	rate	$\lVert(\mathbf{e_q}, \mathbf{e_u})\rVert_{\mathcal{A}}$	rate
P^1-LSP						
h_0	5.72E-04	-	8.36E-03	-	2.46E-02	-
$h_0/2$	1.46E-04	1.97	3.54E-03	1.24	1.25E-02	0.98
$h_0/4$	3.70E-05	1.98	1.53E-03	1.21	6.29E-03	0.99
$h_0/8$	9.35E-06	1.98	6.77E-04	1.18	3.16E-03	0.99
$h_0/16$	2.36E-06	1.99	3.08E-04	1.14	1.59E-03	0.99
$h_0/32$	5.92E-07	1.99	1.43E-04	1.10	7.96E-04	1.00
P^1-ST						
h_0	5.73E-04	-	8.46E-03	-	2.47E-02	-
$h_0/2$	1.46E-04	1.97	3.58E-03	1.21	1.25E-02	0.98
$h_0/4$	3.71E-05	1.98	1.55E-03	1.21	6.30E-03	0.99
$h_0/8$	9.37E-06	1.98	6.84E-04	1.18	3.17E-03	0.99
$h_0/16$	2.36E-06	1.99	3.10E-04	1.14	1.59E-03	0.99
$h_0/32$	5.93E-07	1.99	1.44E-04	1.11	7.96E-04	1.00
P^2-LSP						
h_0	5.30E-05	-	2.35E-03	-	2.83E-03	-
$h_0/2$	1.06E-05	2.33	9.32E-04	1.33	1.12E-03	1.33
$h_0/4$	2.10E-06	2.33	3.70E-04	1.33	4.46E-04	1.33
$h_0/8$	4.17E-07	2.33	1.47E-04	1.33	1.77E-04	1.33
$h_0/16$	8.27E-08	2.33	5.83E-05	1.33	7.03E-05	1.33
$h_0/32$	1.64E-08	2.33	2.31E-05	1.33	2.79E-05	1.33
P^2-ST						
h_0	4.57E-05	-	2.46E-03	-	2.89E-03	-
$h_0/2$	9.36E-06	2.29	9.58E-04	1.36	1.06E-03	1.45
$h_0/4$	1.90E-06	2.30	3.77E-04	1.34	3.98E-04	1.41
$h_0/8$	3.80E-07	2.32	1.49E-04	1.34	1.54E-04	1.37
$h_0/16$	7.59E-08	2.32	5.91E-05	1.34	6.00E-05	1.36
$h_0/32$	1.51E-08	2.33	2.35E-05	1.33	2.36E-05	1.34
P^3-LSP						
h_0	1.98E-05	-	1.62E-03	-	1.73E-03	-
$h_0/2$	3.93E-06	2.33	6.44E-04	1.33	6.87E-04	1.33
$h_0/4$	7.80E-07	2.33	2.56E-04	1.33	2.72E-04	1.33
$h_0/8$	1.54E-07	2.33	1.01E-04	1.33	1.08E-04	1.33
$h_0/16$	3.07E-08	2.33	4.03E-05	1.33	4.29E-05	1.33
P^3-ST						
h_0	1.40E-05	-	1.60E-03	-	1.67E-03	-
$h_0/2$	2.81E-06	2.31	6.33E-04	1.34	6.49E-04	1.37
$h_0/4$	5.64E-07	2.32	2.51E-04	1.34	2.54E-04	1.35
$h_0/8$	1.12E-07	2.33	9.94E-05	1.33	1.00E-04	1.34
$h_0/16$	2.24E-08	2.33	3.95E-05	1.33	3.96E-05	1.34

4.3 The local-structure-preserving LDG method with $\mathbf{M}_h$ or $\bar{\mathbf{M}}_h$ as the space for $\mathbf{q}_h$

Our error analysis relies on the condition (2.13) for the choice of the solution spaces. In this subsection, we use $V_h \times \mathbf{M}_h$ and $V_h \times \bar{\mathbf{M}}_h$ as the solution spaces in the local-structure-preserving LDG method (2.20). Similar convergence rates and comparable numerical results are observed, see Tables 4.7-4.8. Therefore, the condition (2.13) does not seem to be necessary for convergence. Moreover, in actual implementation, once the approximation properties in Lemma 3.2 are satisfied, there is some flexibility in choosing the space for the auxiliary variable $\mathbf{q}_h$. Notice that the P^1 results are not included in the tables, as they are actually the same as the P^1 results in Tables 4.3 and 4.4.

Remark 4.1. *The condition number of the final linear system of the local-structure-preserving LDG method for the Laplace equation is in the same order as the one from the standard LDG method for the same equation. This is observed through our numerical experiments where the iteration numbers for convergence are comparable for these two cases.*

5 Concluding remarks

By taking advantage of the flexibility of solution spaces in the discontinuous Galerkin and the local discontinuous Galerkin methods, we have developed a local-structure-preserving local discontinuous Galerkin method for the Laplace equation. By using this method, the equation is satisfied exactly within each element by the approximating solutions, while there is no sacrifice of the convergence quality of these approximations with a much smaller solution space and hence much reduced computational cost. Future work will include an L^2 error estimate, and a generalization of the method to more general PDEs such as the Poisson equation and other elliptic equations. The full elliptic regularity is assumed in the analysis in this paper for simplicity. More general cases can be analyzed with more delicate details.

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Table 4.7: Local-structure-preserving LDG solutions using $\mathbf{M}_h$ (LSP) or $\bar{\mathbf{M}}_h$ (Mix) as the space for $\mathbf{q}_h$. $C_{11} = 10$ for the smooth example $u(x, y) = e^{-x}\cos(y)$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^2-LSP						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.93E-06	2.64	5.16E-05	2.49	9.55E-05	2.30
$h_0/4$	2.92E-07	2.73	8.08E-06	2.67	1.85E-05	2.37
$h_0/8$	4.11E-08	2.83	1.15E-06	2.81	3.46E-06	2.42
$h_0/16$	5.48E-09	2.90	1.56E-07	2.89	6.31E-07	2.46
$h_0/32$	7.10E-10	2.95	2.04E-08	2.93	1.13E-07	2.48
P^2-Mix						
h_0	8.58E-06	-	4.73E-04	-	5.55E-04	-
$h_0/2$	1.09E-06	2.98	1.23E-04	1.95	1.33E-04	2.06
$h_0/4$	1.37E-07	2.99	3.13E-05	1.97	3.26E-05	2.03
$h_0/8$	1.71E-08	3.00	7.90E-06	1.98	8.07E-06	2.01
$h_0/16$	2.14E-09	3.00	1.99E-06	1.99	2.01E-06	2.01
$h_0/32$	2.68E-10	3.00	4.98E-07	2.00	5.01E-07	2.00
P^3-LSP						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.74E-08	3.73	5.33E-07	3.41	8.10E-07	3.26
$h_0/4$	1.26E-09	3.79	4.78E-08	3.48	8.11E-08	3.32
$h_0/8$	8.71E-11	3.85	4.44E-09	3.43	7.97E-09	3.35
$h_0/16$	5.57E-12	3.97	4.50E-10	3.30	7.85E-10	3.34
P^3-Mix						
h_0	1.74E-07	-	8.31E-06	-	9.11E-06	-
$h_0/2$	1.11E-08	3.96	1.05E-06	2.99	1.10E-06	3.05
$h_0/4$	7.06E-10	3.98	1.32E-07	2.99	1.35E-07	3.03
$h_0/8$	4.48E-11	3.98	1.65E-08	3.00	1.67E-08	3.01
$h_0/16$	5.19E-12	3.11	2.06E-09	3.00	2.08E-09	3.01

Table 4.8: Local-structure-preserving LDG solutions using $\mathbf{M}_h$ (LSP) or $\bar{\mathbf{M}}_h$ (Mix) as the space for $\mathbf{q}_h$. $C_{11} = 1/h$ for the smooth example $u(x,y) = e^{-x}\cos(y)$. $h_0 = 0.1$.

meshsize h	$\|e_u\|_0$	rate	$\|\mathbf{e_q}\|_0$	rate	$\|(\mathbf{e_q}, \mathbf{e_u})\|_{\mathcal{A}}$	rate
P^2-LSP						
h_0	1.20E-05	-	2.89E-04	-	4.71E-04	-
$h_0/2$	1.53E-06	2.97	7.51E-05	1.95	1.20E-04	1.97
$h_0/4$	1.93E-07	2.99	1.91E-05	1.97	3.04E-05	1.98
$h_0/8$	2.43E-08	2.99	4.82E-06	1.99	7.65E-06	1.99
$h_0/16$	3.04E-09	2.99	1.21E-06	1.99	1.92E-06	2.00
$h_0/32$	3.81E-10	3.00	3.03E-07	2.00	4.80E-07	2.00
P^2-Mix						
h_0	8.58E-06	-	4.73E-04	-	5.55E-04	-
$h_0/2$	1.06E-06	3.01	1.24E-04	1.93	1.43E-04	1.96
$h_0/4$	1.32E-07	3.01	3.16E-05	1.97	3.61E-05	1.98
$h_0/8$	1.65E-08	3.00	7.99E-06	1.99	9.09E-06	1.99
$h_0/16$	2.05E-09	3.00	2.01E-06	1.99	2.28E-06	2.00
$h_0/32$	2.57E-10	3.00	5.03E-07	2.00	5.71E-07	2.00
P^3-LSP						
h_0	2.31E-07	-	5.65E-06	-	7.78E-06	-
$h_0/2$	1.46E-08	3.98	7.21E-07	2.97	9.85E-07	2.98
$h_0/4$	9.15E-10	3.99	9.10E-08	2.99	1.24E-07	2.99
$h_0/8$	5.74E-11	3.99	1.14E-08	2.99	1.55E-08	3.00
$h_0/16$	4.11E-12	3.80	1.43E-09	3.00	1.94E-09	3.00
P^3-Mix						
h_0	1.74E-07	-	8.30E-06	-	9.11E-06	-
$h_0/2$	1.09E-08	3.99	1.06E-06	2.97	1.15E-06	2.98
$h_0/4$	6.82E-10	4.00	1.33E-07	2.99	1.45E-07	2.99
$h_0/8$	4.30E-11	3.99	1.67E-08	2.99	1.81E-08	3.00
$h_0/16$	4.97E-12	3.11	2.09E-09	3.00	2.27E-09	3.00

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