Error Estimates to Smooth Solutions of Runge-Kutta Discontinuous Galerkin Method for Symmetrizable Systems of Conservation Laws

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Abstract. In this paper we study the error estimates to sufficiently smooth solutions of symmetrizable systems of conservation laws for the Runge-Kutta discontinuous Galerkin (RKDG) method. Time discretization is the second order explicit TVD (total variation diminishing) Runge-Kutta method, and the $\mathbb{P}^k$ (piecewise polynomial) finite element is used. When $k = 1$ (piecewise linear finite element), the error estimate is obtained under the usual CFL condition $\tau \le \beta h$ for nonlinear systems in one dimension and for linear systems in multiple space dimensions. Here, h is the maximum element length, τ is the time step, and β is a positive constant independent of h and τ. Error estimates for $\mathbb{P}^k$ finite elements with $k > 1$ are obtained under a more restrictive CFL condition.

Key words: discontinuous Galerkin method, finite element method, TVD Runge-Kutta method, error estimates, symmetrizable system, conservation laws

AMS subject classification. 65M15

1 Introduction

In this paper, we continue our work in [24] and present the error estimates of the Runge-Kutta discontinuous Galerkin (RKDG) method for smooth solutions of symmetrizable systems of conservation laws

$$\mathbf{u}_{,t} + \mathbf{f}^{(i)}_{,x_i}(\mathbf{u}) = 0, \qquad (x,t) \in \Omega \times (0,T], \tag{1.1a}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0, \qquad x \in \Omega, \tag{1.1b}$$

in the spatial domain $\Omega \in \mathbb{R}^d$ and the time interval $[0,T]$. Here, $\mathbf{u}(x,t)\colon \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^m$ is the dependent solution variables, $\mathbf{f}^{(i)}(\mathbf{u})\colon \mathbb{R}^m \to \mathbb{R}^{m\times d}$, $i = 1,2,\ldots,d$, is the vector-valued flux function, and the implied summation on the index i is used in (1.1a), i.e.,

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$\mathbf{f}^{(i)}_{,x_i} = \sum_{i=1}^{d} \partial \mathbf{f}^{(i)}/\partial x_i$. We do not pay attention to boundary conditions in this paper, hence the solution is considered to be either periodic or compactly supported. For simplicity of presentation, we will only give detailed analysis for the one dimensional case where $d = 1$ and $\Omega = I = (0, 1)$; herein we drop the index i in (1.1). We will however point out similarities and differences when the analysis is generalized to multiple space dimensions. We assume in addition that each component of the flux function $\mathbf{f}(\mathbf{u})$ is smooth enough in $\mathbf{u}$, for our purpose $C^3(\mathbb{R}^m)$ will suffice. The analysis in this paper is for *smooth* solutions of (1.1). Discontinuous solutions with shocks are not considered.

The RKDG method is introduced and developed by Cockburn et al. [4, 5, 3, 2, 6] for solving nonlinear hyperbolic conservation laws, which uses discontinuous Galerkin (DG) discretization in space and combines it with an explicit total variation diminishing (TVD) Runge-Kutta time-marching algorithm [21]. This method has a good stability property, is flexible for h-p adaptivity, and has a high parallel efficiency. In recent years there has been a lot of activity in the design, analysis and application of RKDG methods. For more details, we refer to the review article [8].

Although error estimates for linear equations and for the method of lines (continuous in time) version of the RKDG method have been available for a long time, e.g. [15, 14, 7], error estimates for fully discrete RKDG method for nonlinear conservation laws with smooth solutions have been available only recently [24], in which we obtained error estimates for scalar conservations with piecewise k-th degree polynomial DG spatial discretization coupled with second order TVD Runge-Kutta time discretization. The analysis assumes the usual CFL condition $\tau \le \beta h$ for the piecewise linear k=1 case, where h is the maximum element length, τ is the time step, and β is a positive constant independent of h and τ. For the higher order $k > 1$ case, the proof has to assume a much stronger CFL condition $\tau \le \beta h^{4/3}$. In this paper, we extend these error estimates to symmetrizable systems (See Theorem 2.1).

In the symmetrization theory [20] for the first-order conservation laws, one seeks a mapping $\mathbf{u}(\mathbf{v})\colon \mathbb{R}^m \to \mathbb{R}^m$ applied to (1.1a) so that when transformed

$$\mathbf{u}_{,\mathbf{v}}\mathbf{v}_{,t} + \mathbf{f}_{,\mathbf{v}}\mathbf{v}_{,x} = 0 \tag{1.2}$$

the matrix $\mathbf{u}_{,\mathbf{v}}$ is symmetric positive definite (SPD) and the matrix $\mathbf{f}_{,\mathbf{v}} = \mathbf{f}_{,\mathbf{u}}\mathbf{u}_{,\mathbf{v}}$ is also symmetric. We further assume that each component of $\mathbf{u}_{,\mathbf{v}}$ is Lipschitz continuous with respect to the variable $\mathbf{v}$. As is well known, a conservation law system (1.1a) is symmetrizable if and only if it has a convex entropy function [10]. Well-known systems such as the Euler equations of compressible gas dynamics are symmetrizable. If $\mathbf{f}_{,\mathbf{u}}$ is already symmetric, the system (1.1a) is symmetric. It is straightforward to generalize the error estimates in [24] from the scalar case to symmetric systems. However, there are not that many physical systems which are symmetric. On the other hand, as we will see later in this paper, it is significantly more difficult to generalize the error estimates in [24] from the scalar case to symmetrizable systems.

The line of analysis in this paper follows that of [24]. The main techniques are Taylor expansions and energy analysis. In generalizing the analysis from the scalar case to systems, we need to pay attention to the suitable norm in the analysis, to a careful classification of the necessary properties for the numerical fluxes, to the complication related to the fact that derivatives (Jacobians) and second derivatives of the flux functions are matrices and super-matrices, and to a suitable generalization of the *a priori* assumption about the numerical solution. We will present a series of lemmas which mostly correspond to those in [24]. If the proofs have only minor differences, we will comment on such differences and will not repeat the details. We will concentrate our analysis on the piecewise k=1 case for the DG method, since higher order cases can be analyzed with the stronger CFL condition $\tau \leq \beta h^{4/3}$ following the same lines as that in [24], once the k=1 case is proved.

An outline of this paper is as follows. In section 2 we present, for the equation (1.1), the RKDG method and the corresponding convergence theorem with the second order TVD Runge-Kutta time discretization. In section 3 we present a proposition for an important matrix which measures the amount of numerical viscosity on each element interface, and perform some elementary analysis to the error equations. We prove the main theorem in section 4. Section 5 is an appendix in which we collect some of the technical details left out in previous sections.

2 RKDG method and the convergence theorem

We follow [6] and define the RKDG method for the problem (1.1) in one space dimension. The multi-dimensional scheme can be similarly defined, and the analysis can be carried in a similar fashion for linear systems. See Remark 4.2.

For each partition of the interval $I = (0,1)$, $\{x_{j+\frac{1}{2}}\}_{j=0}^{N}$, we set $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$ for $j = 1,\ldots,N$; we denote the quantity $\max_{1\leq j\leq N} h_j$ by h. For a given time step $\tau \equiv \tau^n$ (which could actually change from step to step, but is taken as a constant with respect to the time level n for simplicity), the solution of the scheme is denoted by $\mathbf{u}_h^n(x) = \mathbf{u}_h(x,t^n) = \mathbf{u}_h(x,n\tau)$, which belongs to the finite element space

$$\mathbb{V}_h = \{\, \mathbf{v} \in [L^1(0,1)]^m : \mathbf{v}|_{I_j} \in [\mathbb{P}^k(I_j)]^m, j = 1,\ldots,N \,\}, \tag{2.1}$$

where $\mathbb{P}^k(I_j)$ denotes the space of polynomials in I_j of degree at most k. Note that each component of a vector-valued function in $\mathbb{V}_h$ is allowed to have discontinuities across element interfaces.

In what follows, we will consider the standard L^2-projection of a vector-valued function $\mathbf{p} \in [L^2(0,1)]^m$ into the finite element space $\mathbb{V}_h$, denoted by $\mathbb{P}_h\mathbf{p}$, which is defined as the unique vector-valued function in $\mathbb{V}_h$ such that

$$\int_0^1 \mathbf{z}_h^T(x)(\mathbb{P}_h\mathbf{p}(x) - \mathbf{p}(x))\,dx = 0, \quad \forall \mathbf{z}_h \in \mathbb{V}_h. \tag{2.2}$$

where $\mathbf{z}^T$ denotes the transpose of the vector $\mathbf{z}$.

As usual, at each element interface we will denote, for a vector-valued function $\mathbf{z}$, two limiting values from different directions by $\mathbf{z}_{j+1/2}^{\pm} = \mathbf{z}(x_{j+1/2} \pm 0)$, and denote the average and jump by $\bar{\mathbf{z}} = (\mathbf{z}^+ + \mathbf{z}^-)/2$ and $[\mathbf{z}] = \mathbf{z}^+ - \mathbf{z}^-$, respectively. We also define, for any vector-valued functions $\mathbf{p}$ and $\mathbf{z}$, the following functional corresponding to the DG spatial discretization

$$\mathcal{H}_j(\mathbf{p},\mathbf{z}) = \int_{I_j} \mathbf{z}_{,x}^T \mathbf{f}(\mathbf{p})\,dx - (\mathbf{z}_{j+\frac{1}{2}}^-)^T \hat{\mathbf{h}}(\mathbf{p})_{j+\frac{1}{2}} + (\mathbf{z}_{j-\frac{1}{2}}^+)^T \hat{\mathbf{h}}(\mathbf{p})_{j-\frac{1}{2}}, \tag{2.3}$$

where $\hat{\mathbf{h}}(\mathbf{p}) \equiv \hat{\mathbf{h}}(\mathbf{p}^-, \mathbf{p}^+)$ is a given (locally) Lipschitz continuous numerical flux function consistent with the flux function $\mathbf{f}(\mathbf{p})$, that is $\hat{\mathbf{h}}(\mathbf{p}, \mathbf{p}) = \mathbf{f}(\mathbf{p})$. In this paper, we will also assume in addition that $\hat{\mathbf{h}}(\mathbf{p})$ is a generalized E-flux function, to be defined in subsection 3.3.

The approximate solution in $\mathbb{V}_h$ from time $n\tau$ to $(n+1)\tau$ given by the RKDG method with second order TVD time discretization can now be defined as follows: find successively $\mathbf{w}_h^n \equiv \mathbf{w}_h^n(x) \in \mathbb{V}_h$ and $\mathbf{u}_h^{n+1} \equiv \mathbf{u}_h^{n+1}(x) \in \mathbb{V}_h$, such that, for any $\mathbf{z}_h \equiv \mathbf{z}_h(x) \in [\mathbb{P}^k(I_j)]^m$ and $1 \le j \le N$,

$$\int_{I_j} \mathbf{z}_h^T \mathbf{w}_h^n \, dx = \int_{I_j} \mathbf{z}_h^T \mathbf{u}_h^n \, dx + \tau \mathcal{H}_j(\mathbf{u}_h^n, \mathbf{z}_h), \tag{2.4a}$$

$$\int_{I_j} \mathbf{z}_h^T \mathbf{u}_h^{n+1} \, dx = \frac{1}{2} \int_{I_j} \mathbf{z}_h^T \mathbf{u}_h^n \, dx + \frac{1}{2} \int_{I_j} \mathbf{z}_h^T \mathbf{w}_h^n \, dx + \frac{\tau}{2} \mathcal{H}_j(\mathbf{w}_h^n, \mathbf{z}_h), \tag{2.4b}$$

with the initial value $\mathbf{u}_h^0 = \mathbb{P}_h \mathbf{u}_0(x)$. This is an explicit time marching method when a local orthogonal basis is chosen for polynomials on I_j or when a small local mass matrix on I_j is inverted. Numerical results and details of this scheme can be found in [9] and [8].

We now present the main convergence theorem of the RKDG scheme (2.4). The proof will be given in the next two sections.

Theorem 2.1 *For the symmetrizable system of conservation laws (1.1), assume that the solution* $\mathbf{u}$ *and the flux function* $\mathbf{f}(\mathbf{u})$ *are sufficiently smooth with bounded derivatives. Let* $\mathbf{u}_h$ *be the numerical approximate solution of the RKDG scheme (2.4) with second order TVD Runge-Kutta time discretization and piecewise polynomial finite element space of degree* $k \ge 1$*, then for* h *small enough, the following estimate*

$$\max_{n\tau \le T} ||\mathbf{u}(t^n) - \mathbf{u}_h^n||_{L^2(0,1)} \le C(h^{k+1/2} + \tau^2), \tag{2.5}$$

holds under a suitable CFL condition $\tau \le \beta h$ *for* $k = 1$*, and under a more restrictive CFL condition* $\tau \le \beta h^{4/3}$ *for* $k > 1$*, where* C *and* β *are positive constants independent of* h *and* τ*.*

We remark that the power $h^{k+1/2}$ is optimal for general triangulations [18] for the scalar case, but is sub-optimal for the one dimensional case with scalar equations. The proof of the optimal order h^{k+1} for the scalar case requires special upwind fluxes [24], which can be done for the system case (1.1) as well in some special situations. See Remark 4.3 in section 4.

As in [24], we assume below that each component of the flux function $\mathbf{f}(\cdot)$ itself and up to its third-order derivatives are bounded in the domain $\mathbb{R}^m$. This assumption is non-essential if we consider only smooth solutions of (1.1) to a finite time T. We could achieve the desired boundedness by re-defining the flux function $\mathbf{f}(\mathbf{u})$ outside the range of the solution $\mathbf{u}$.

We denote the inverse mapping of $\mathbf{u}(\mathbf{v})$ by $\mathbf{v}(\mathbf{u})$. The symmetrizable theory provides that the Jacobians $\mathbf{u}_{,\mathbf{v}}(\mathbf{v})$ and $\mathbf{v}_{,\mathbf{u}}(\mathbf{u})$ are symmetric positive definite (SPD) and Lipschitz continuous. Similarly as above, we assume these properties hold uniformly in the domain $\mathbb{R}^m$ and the spectrum of the Jacobians are bounded.

We would also like to denote, by C, $C_\star$, M, or ε, a generic positive constant independent of n, h and τ. Herein, M and ε are used to denote constants which are independent of the solution of (1.1). $C_\star$ is used to emphasize the nonlinearity of $\mathbf{f}(\mathbf{u})$, i.e., $C_\star = 0$ for a linear flux function $\mathbf{f}(\mathbf{u}) = \mathbb{C}\mathbf{u}$. These constants may have a different value in each occurrence.

3 Error equations, energy equality and properties of the finite element spaces

We follow the idea in [24] to obtain the error estimate to sufficiently smooth solutions for the RKDG scheme (2.4). Thus we will present in this section some elementary development similar to those in [24] and we omit the detailed proof if it is similar to those in [24].

3.1 Error equations and energy equality

We first define a vector-valued function in parallel to an Euler forward time marching, namely

$$\mathbf{w}(x,t) = \mathbf{u}(x,t) + \tau \mathbf{u}_{,t}(x,t). \tag{3.1}$$

We denote the error at each stage of the RKDG scheme by $\mathbf{e}_{\mathbf{u}}^n = \mathbf{u}(t^n) - \mathbf{u}_h^n$ and $\mathbf{e}_{\mathbf{w}}^n = \mathbf{w}(t^n) - \mathbf{w}_h^n$, respectively, where for notational convenience the argument x is omitted. As is customary in finite element error analysis, we denote $\xi_{\mathbf{p}} = \mathbb{P}_h\mathbf{p} - \mathbf{p}_h$ and $\eta_{\mathbf{p}} = \mathbb{P}_h\mathbf{p} - \mathbf{p}$, where $\mathbb{P}_h$ is the local L^2-projection. Then $\mathbf{e}_{\mathbf{p}}^n = \xi_{\mathbf{p}}^n - \eta_{\mathbf{p}}^n$, where $\mathbf{p} = \mathbf{u}$ or $\mathbf{w}$.

The error equations of the RKDG scheme (2.4) can be obtained by following similar algebraic manipulations as those in [23, 24] (c.f. Lemma 4.1 in [24]). It reads

$$\int_{I_j} \mathbf{z}_h^T \xi_{\mathbf{w}}^n \, dx = \int_{I_j} \mathbf{z}_h^T \xi_{\mathbf{u}}^n \, dx + \mathcal{K}_j^n(\mathbf{z}_h), \tag{3.2a}$$

$$\int_{I_j} \mathbf{z}_h^T \xi_{\mathbf{u}}^{n+1} \, dx = \int_{I_j} \mathbf{z}_h^T \xi_{\mathbf{u}}^n \, dx + \frac{1}{2}\mathcal{K}_j^n(\mathbf{z}_h) + \frac{1}{2}\mathcal{L}_j^n(\mathbf{z}_h), \tag{3.2b}$$

for any $\mathbf{z}_h(x) \in [\mathbb{P}^k(I_j)]^m$ and $1 \le j \le N$, where

$$\mathcal{K}_j^n(\mathbf{z}_h) = \int_{I_j} \mathbf{z}_h^T(\eta_{\mathbf{w}}^n - \eta_{\mathbf{u}}^n) \, dx + \tau\mathcal{H}_j(\mathbf{u}(t^n), \mathbf{z}_h) - \tau\mathcal{H}_j(\mathbf{u}_h^n, \mathbf{z}_h), \tag{3.2c}$$

$$\mathcal{L}_j^n(\mathbf{z}_h) = \int_{I_j} \mathbf{z}_h^T(2\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{w}}^n - \eta_{\mathbf{u}}^n + 2\zeta^n) \, dx + \tau\mathcal{H}_j(\mathbf{w}(t^n), \mathbf{z}_h) - \tau\mathcal{H}_j(\mathbf{w}_h^n, \mathbf{z}_h). \tag{3.2d}$$

Here ζ^n is the truncation error in time, with the size $\mathcal{O}(\tau^3)$. We will use the short notations $\mathcal{K}^n(\mathbf{z}_h) = \sum_{1\le j\le N} \mathcal{K}_j^n(\mathbf{z}_h)$ and $\mathcal{L}^n(\mathbf{z}_h) = \sum_{1\le j\le N} \mathcal{L}_j^n(\mathbf{z}_h)$.

We will use energy estimates to analyze the error of the RKDG scheme (2.4). To this end, we define a norm which depends on the time level n, given by $||\mathbf{p}||_n = ||\mathbf{v}_{,\mathbf{u}}^{1/2}(\mathbf{u}_c^n)\mathbf{p}||$, for any vector-valued function $\mathbf{p}$, where $\mathbf{u}_c^n$ (or $\mathbf{w}_c^n$) is the piecewise constant vector-valued function which is equal to the vector $\mathbf{u}(x_j, t^n)$ (or $\mathbf{w}(x_j, t^n)$) in each element I_j. The symmetrizable theory guarantees that the $||\cdot||_n$ norm is equivalent to the usual L^2-norm $||\cdot||$.

By taking the test function $\mathbf{z}_h = \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\xi_{\mathbf{u}}^n$ in (3.2a) and $\mathbf{z}_h = \mathbf{v}_{,\mathbf{u}}(\mathbf{w}_c^n)\xi_{\mathbf{w}}^n$ in (3.2b), respectively, and adding the two equalities together, we obtain the following energy equation

$$||\xi_{\mathbf{u}}^{n+1}||_n^2 - ||\xi_{\mathbf{u}}^n||_n^2 = ||\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n||_n^2 + \mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\xi_{\mathbf{u}}^n) + \mathcal{L}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{w}_c^n)\xi_{\mathbf{w}}^n) + \mathcal{E}^n, \tag{3.3a}$$

where

$$\mathcal{E}^n = \int_I (\xi_{\mathbf{w}}^n)^T(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n) - \mathbf{v}_{,\mathbf{u}}(\mathbf{w}_c^n))(2\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{u}}^n - \xi_{\mathbf{w}}^n)\, dx. \tag{3.3b}$$

In order to get the error estimate, we shall analyze carefully each term on the right-hand side of this important energy equation (3.3) in the next section.

3.2 Properties of the finite element spaces

In this subsection we present some interpolation approximation and inverse properties of the finite element space $\mathbb{V}_h$, which consists of piecewise polynomials of degree k. The usual notations of norm and semi-norms in Sobolev spaces will be used below.

The local L^2-projection is enough to prove Theorem 2.1 for general numerical flux functions with the sub-optimal error bound $Ch^{k+1/2}$. By the standard scaling theory, it is easy to show that (c.f. [1])

$$||\eta_{\mathbf{p}}^n|| + h||\eta_{\mathbf{p}}^n||_\infty + h^{\frac{1}{2}}||\eta_{\mathbf{p}}^n||_{\Gamma_h} \leq Ch^{k+1}, \quad (\mathbf{p} = \mathbf{u}, \mathbf{w};\ \forall n : n\tau \leq T), \tag{3.4a}$$

where Γ_h is the set of boundary interfaces of all elements. Noticing the definition (3.1) of $\mathbf{w}$ and the linearity of the L^2-projection $\mathbb{P}_h$, we can conclude that

$$||\eta_{\mathbf{u}}^{n+1} - \eta_{\mathbf{u}}^n|| + ||\eta_{\mathbf{w}}^n - \eta_{\mathbf{u}}^n|| \leq Ch^{k+1}\tau, \quad \forall n : n\tau < T. \tag{3.4b}$$

In the above inequalities the positive constant C depends solely on $\mathbf{u}, \mathbf{w}$ and/or $\mathbf{u}_{,t}$, and is independent of n, h and τ.

In the following analysis we will also use some inverse properties of the finite element space $\mathbb{V}_h$. For any vector-valued function $\mathbf{z}_h \in \mathbb{V}_h$, there is a positive constant C independent of $\mathbf{z}_h$ and h, such that

$$\text{(i) } ||(\mathbf{z}_h)_{,x}|| \leq Ch^{-1}||\mathbf{z}_h||; \quad \text{(ii) } ||\mathbf{z}_h||_{\Gamma_h} \leq Ch^{-\frac{1}{2}}||\mathbf{z}_h||; \quad \text{(iii) } ||\mathbf{z}_h||_\infty \leq Ch^{-\frac{1}{2}}||\mathbf{z}_h||.$$

For more details of these inverse properties, we refer to [1].

3.3 An important matrix related to the numerical flux

In this subsection we will introduce an important matrix to measure the numerical viscosity on each element interface. The following notations will be used. If there exists an invertible matrix $\mathbb{T}$ such that $\mathbb{A} = \mathbb{T}^{-1} diag(\lambda_1, \ldots, \lambda_m)\mathbb{T}$ where $diag(\lambda_1, \ldots, \lambda_m)$ is the diagonal matrix with $\lambda_1, \cdots, \lambda_m$ on its diagonal, then we denote its absolute value by $|\mathbb{A}| = \mathbb{T}^{-1} diag(|\lambda_1|, \ldots, |\lambda_m|)\mathbb{T}$. We also denote a super-matrix function $\mathbb{G}(\mathbf{p}) = \{\mathbf{v}_{,\mathbf{u}}\mathbf{f}_{,\mathbf{u},\mathbf{u}}\mathbf{v}_{,\mathbf{u}}\}(\mathbf{p}) \in \mathbb{R}^{m\times m\times m}$, which corresponds to the second-order derivatives of the Taylor expansion.

The symmetrizable theory implies that the Jacobian $\mathbf{f}_{,\mathbf{u}}$ is similar to a symmetric matrix, because

$$\mathbf{u}_{,\mathbf{v}}^{-1/2}\mathbf{f}_{,\mathbf{u}}\mathbf{u}_{,\mathbf{v}}^{1/2} = \mathbf{u}_{,\mathbf{v}}^{-1/2}\mathbf{f}_{,\mathbf{v}}\mathbf{u}_{,\mathbf{v}}^{-1/2} \tag{3.5}$$

and the matrix on the right-hand side is symmetric. Therefore, we would like to assume in this paper that the numerical flux function $\mathbf{f}(\mathbf{u})$ under consideration satisfies

$$(\mathbf{p}^+ - \mathbf{p}^-)^T \mathbf{v}_{,\mathbf{u}}(\mathbf{p}^\theta)\{\mathbf{f}(\mathbf{p}^\pm) - \hat{\mathbf{h}}(\mathbf{p})\} \geq 0, \tag{3.6}$$

where $\mathbf{p}^\theta$ is some average of $\mathbf{p}^+$ and $\mathbf{p}^-$, i.e., $\mathbf{p}^\theta = \theta\mathbf{p}^+ + (1-\theta)\mathbf{p}^-$ for a certain $\theta : 0 \leq \theta \leq 1$. Note that the value of θ depends on the numerical fluxes and $\mathbf{p}^\pm$.

The inequality (3.6), for a symmetric system of conservation laws (in which case $\mathbf{v}_{,\mathbf{u}} \equiv \mathbf{I}$), has been considered in [13] as an E-flux (see [17] for the definition of E-fluxes for scalar conservation laws). Therefore, we refer to a numerical flux satisfying (3.6) as a generalized E-flux. It is easy to verify that the property (3.6) holds for many numerical flux functions constructed from approximate Riemann solvers, for example, Roe linearization flux function [19] with or without Harten's entropy fix [11], and the global (local) Lax-Friedrichs flux, where $\mathbf{p}^\theta$ is the so-called Roe average of $\mathbf{p}^+$ and $\mathbf{p}^-$ [19].

In the next Proposition, we would like to define an important matrix $\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})$ associated with a numerical flux function $\hat{\mathbf{h}}(\mathbf{p})$ satisfying (3.6), which measures the numerical viscosity of the flux. It is a generalization of a similar quantity in [24] for the scalar case; see also [11].

Proposition 3.1 *Assume the numerical flux $\hat{\mathbf{h}}(\mathbf{p}) \equiv \hat{\mathbf{h}}(\mathbf{p}^-, \mathbf{p}^+)$ is consistent with the vector-valued flux function $\mathbf{f}(\mathbf{p})$, and there exists an average $\mathbf{p}^\theta$ such that the generalized E-flux property (3.6) holds. Define the matrix on each element interface*

$$\mathcal{A}(\hat{\mathbf{h}};\mathbf{p}) \equiv \mathcal{A}(\hat{\mathbf{h}};\mathbf{p}^-, \mathbf{p}^+) := \begin{cases} ([\mathbf{q}]^T[\mathbf{q}])^{-1}(\mathbf{f}(\bar{\mathbf{p}}) - \hat{\mathbf{h}}(\mathbf{p}))[\mathbf{q}]^T, & \text{if } [\mathbf{p}] \neq \mathbf{0}, \\ |\mathbf{f}_{,\mathbf{v}}(\bar{\mathbf{q}})|, & \text{if } [\mathbf{p}] = \mathbf{0}. \end{cases} \tag{3.7}$$

where $\mathbf{q}^\pm = \mathbf{v}_{,\mathbf{u}}(\mathbf{p}^\theta)\mathbf{p}^\pm$. Then the spectral radius of $\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})$ is bounded, and for any vector $\mathbf{q} \in \mathbb{R}^m$ we have $[\mathbf{q}]^T\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})[\mathbf{q}] \geq 0$ and

$$\frac{1}{2}\left|[\mathbf{q}]^T\mathbf{f}_{,\mathbf{v}}(\mathbf{q}^\theta)[\mathbf{q}]\right| \leq [\mathbf{q}]^T\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})[\mathbf{q}] + C_\star||\mathbf{q}||_\infty[\mathbf{q}]^T[\mathbf{q}], \tag{3.8a}$$

$$-\frac{1}{8}\{\mathbb{G}(\mathbf{p}^\theta)\}^\iota_{\kappa,\sigma}[q_\iota][q_\kappa][q_\sigma] \leq [\mathbf{q}]^T\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})[\mathbf{q}] + C_\star||\mathbf{q}||^2_\infty[\mathbf{q}]^T[\mathbf{q}], \tag{3.8b}$$

where the implied summary on the indices ι, κ and σ is used, and $C_\star$ is a positive constant solely determined by the nonlinearity of the flux function $\mathbf{f}(\mathbf{p})$.

The proof of this proposition is straightforward by a Taylor expansion at the point $\mathbf{p}^\theta$ and the generalized E-flux property (3.6). The property (3.5) for a symmetrizable system plays an important role to ensure the positive definiteness of the matrix $\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})$. We refer to [24] for further details.

Remark 3.1. The above inequalities (3.8) also hold for other numerical flux functions which may violate the generalized E-flux property (3.6) slightly, e.g. the Harten-Hyman flux function [12] and the local Lax-Friedrichs flux function with an entropy fix (c.f. [16]). For these fluxes, the deviation to (3.6) is of the order $\mathcal{O}(||[\mathbf{p}]||^4_\infty)$, which does not affect the proof of the inequalities (3.8).

For any vector-valued function $\mathbf{p}$, we denote by $\varrho(\mathbf{p})$ the maximum of the spectral radius of $\mathcal{A}(\hat{\mathbf{h}};\mathbf{p})$ over all element interfaces. Also, if there is no confusion, we will denote

$$A(\mathbf{p}) = \sum_{1\le j\le N} [\mathbf{q}]^T_{j+\frac{1}{2}} \mathcal{A}(\hat{\mathbf{h}};\mathbf{p})_{j+\frac{1}{2}} [\mathbf{q}]_{j+\frac{1}{2}},$$

where $\mathbf{q} = \mathbf{v}_{,\mathbf{u}}(\mathbf{p}^\theta)\mathbf{p}$ and $\mathbf{p}^\theta$ is the average of $\mathbf{p}^\pm$ for (3.6) to hold.

3.4 General estimates for the operators $\mathcal{L}$ and $\mathcal{K}$

In this subsection we derive a few general inequalities to the operators $\mathcal{L}$ and $\mathcal{K}$ for any test function. It will be used in the next section, to estimate the error resulted from the second order Runge-Kutta time discretization. All estimates given in this subsection holds for finite element space $\mathbb{V}_h$ with any degree k.

By subtracting the error equation (3.2a) from (3.2b), we find out that, for any $1 \le j \le N$, there holds

$$\int_{I_j} \mathbf{z}_h^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n)\, dx = \frac{1}{2}(\mathcal{L}_j^n - \mathcal{K}_j^n)(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{z}_h), \quad \forall\, \mathbf{z}_h \in [\mathbb{P}^k(I_j)]^m, \tag{3.9}$$

thus $||\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n||_n^2 = \frac{1}{2}(\mathcal{L}^n - \mathcal{K}^n)(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n))$. It is therefore natural to start the analysis with an estimate of the difference between $\mathcal{L}^n$ and $\mathcal{K}^n$ for any test function. The key point in the analysis is to obtain a sharp bound for the error at the element interfaces.

We remark that there are only minor modifications from the scalar case [24] to the system case for this analysis. The Taylor expansions are changed from single variable to multiple variables, where the derivative $f'(u)$ becomes the Jacobian $\mathbf{f}_{,\mathbf{u}}(\mathbf{u})$, and the maximum magnitude of $|f'(u)|$ becomes the maximum spectral radius of $\mathbf{f}_{,\mathbf{u}}(\mathbf{u})$. We will therefore only present the estimates without proof and refer to [24] for more details.

Lemma 3.1 *Assume that the interpolation property (3.4) is satisfied. Given a small positive constant ε, we have for any $\mathbf{z}_h \in \mathbb{V}_h$ that*

$$\begin{aligned}(\mathcal{L}^n - \mathcal{K}^n)(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{z}_h) \le{}& \varepsilon||\mathbf{z}_h||_n^2 + M_\varepsilon \tau^2 h^{-1} \varrho(\mathbf{u}_h^n) A(\mathbf{u}_h^n) + M_\varepsilon \tau^2 h^{-1} \varrho(\mathbf{w}_h^n) A(\mathbf{w}_h^n)\\ &+ (C_\star \tau^2 h^{-2} ||\mathbf{e}_{\mathbf{u}}^n||_\infty^2 + C\tau^2)||\xi_{\mathbf{u}}^n||_n^2 + (C_\star \tau^2 h^{-2} ||\mathbf{e}_{\mathbf{w}}^n||_\infty^2 + C\tau^2)||\xi_{\mathbf{w}}^n||_n^2\\ &+ C(\Xi(n) h^{2k+2}\tau + \tau^6) - \tau \sum_{1\le j\le N} \int_{I_j} \mathbf{z}_h^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n) \mathbf{f}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}\, dx,\end{aligned}$$

where $C, C_\star$ and M_ε are positive constants independent of n, h, τ and the approximate solutions, and $\Xi(n) = 1 + C_\star h^{-1}||\mathbf{e}_{\mathbf{u}}^n||_\infty^2 + C_\star h^{-1}||\mathbf{e}_{\mathbf{w}}^n||_\infty^2$. Here, $M_\varepsilon = O(\varepsilon^{-1})$ depends on ε solely.

Similarly, we can get the following lemma to estimate $\mathcal{K}(\cdot)$ by using Taylor expansions of $\mathbf{f}(\mathbf{u})$ up to second and third order derivatives, respectively.

Lemma 3.2 *Under the assumption of Lemma 3.1, we have the following estimates for any* $\mathbf{z}_h \in \mathbb{V}_h$

$$\begin{aligned}\mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{z}_h) \le& \varepsilon||\mathbf{z}_h||_n^2 + M_\varepsilon \tau^2 h^{-1}\varrho(\mathbf{u}_h)A(\mathbf{u}_h^n) + (C + C_\star h^{-2}||\mathbf{e}_\mathbf{u}^n||_\infty^2)\tau^2||\xi_\mathbf{u}^n||_n^2 \\ &+(C + C_\star||\mathbf{e}_\mathbf{u}^n||_\infty^2)h^{2k+1}\tau - \tau\sum_{1\le j\le N}\int_{I_j}\mathbf{z}_h^T\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{f}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_\mathbf{u}^n)_{,x}\,dx,\\ \mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{z}_h) \le& \varepsilon||\mathbf{z}_h||_n^2 + C\tau^2h^{-2}||\xi_\mathbf{u}^n||_n^2 + Ch^{2k}\tau^2,\end{aligned}$$

where $C, C_\star$ *and* M_ε *are positive constants independent of* n, h, τ *and the approximate solutions, and* $M_\varepsilon = O(\varepsilon^{-1})$ *depends solely on* ε.

We take, in the second inequality of Lemma 3.2, the test function $\mathbf{z}_h = \xi_w^n - \xi_u^n$ and the positive constant ε small enough. Noticing that $||\xi_\mathbf{w}^n - \xi_\mathbf{u}^n||_n^2 = \mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_\mathbf{w}^n - \xi_\mathbf{u}^n))$ follows from the error equation (3.2a), we have the following corollary.

Corollary 3.1 *If the interpolating approximation property (3.4) and the CFL condition* $\tau = \mathcal{O}(h)$ *are satisfied, then we have*

$$||\xi_\mathbf{w}^n|| \le C||\xi_\mathbf{u}^n|| + Ch^k\tau, \quad \forall n : n\tau < T. \tag{3.10}$$

where C *is the positive constant independent of* n, h, τ *and the approximate solution.*

4 Proof of the convergence theorem

In this section we are going to prove the error estimate (2.5) of the RKDG method with finite element space of piecewise linear polynomials (k=1). The generalization to high order $k > 1$ with a more restrictive CFL condition is straightforward, along the lines of [24]. To this end, we will analyze each term on the right-hand side of the energy equation (3.3a) separately.

4.1 Estimates to each term on the right-hand side of the energy equation

Since each component of the matrix $\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n) - \mathbf{v}_{,\mathbf{u}}(\mathbf{w}_c^n)$ is of order $\mathcal{O}(\tau)$ by the smoothness assumption of the mapping $\mathbf{v}(\mathbf{u})$ and the definition (3.1) of $\mathbf{w}$, it is easy to get an estimate to the last term $\mathcal{E}^n$ in the form

$$\mathcal{E}^n \le \varepsilon||\xi_\mathbf{u}^{n+1}||_n^2\tau + C(||\xi_\mathbf{u}^n||_n^2\tau + ||\xi_\mathbf{w}^n||_n^2\tau), \tag{4.1}$$

where ε is a suitably small positive constant. Next, we will focus on estimating the last three terms on the right-hand side of the energy equation (3.3a).

Lemma 4.1 *If the interpolation approximation property (3.4) is satisfied for the linear finite element space* $\mathbb{P}^1$*, and the CFL condition* $\tau = \mathcal{O}(h)$ *is satisfied, then the following estimate holds*

$$\begin{aligned}||\xi_\mathbf{u}^{n+1} - \xi_\mathbf{w}^n||_n^2 \le& C(\Xi(n)h^3\tau + \tau^6) + \delta_1(n)A(\mathbf{u}_h^n)\tau + \delta_2(n)A(\mathbf{w}_h^n)\tau \\ &+\Big\{\frac{C_\star\tau^2}{h^2}(||\mathbf{e}_\mathbf{u}^n||_\infty^2 + ||\mathbf{e}_\mathbf{w}^n||_\infty^2) + \frac{C_\star\tau^4}{h^4}||\mathbf{e}_\mathbf{u}^n||_\infty^2 + C\tau^2\Big\}||\xi_\mathbf{u}^n||_n^2\end{aligned} \tag{4.2}$$

where C and $C_\star$ are positive constants independent of n, h, τ and the numerical solutions, and $\Xi(n)$ has been defined in Lemma 3.1. Here

$$\delta_1(n) = \frac{M_1\tau}{h}\varrho(\mathbf{u}_h^n) + \frac{M_2\tau^3}{h^3}\varrho(\mathbf{u}_h^n)\lambda^2(\mathbf{u}^n), \quad \textit{and} \quad \delta_2(n) = \frac{M_3\tau}{h}\varrho(\mathbf{w}_h^n), \tag{4.3}$$

where $\lambda(\mathbf{u}^n)$ is the maximum spectral radius of the Jacobian $\mathbf{f}_{,\mathbf{u}}(\mathbf{u}^n)$ on all element interfaces, and M_i, $i = 1, 2, 3$, are positive constants independent of the other parameters in (4.3).

Proof. We follow the framework in [24] for the scalar case and only sketch the proof below.

We take successively two test functions in both equation (3.9) and Lemma 3.1: the first one is $\mathbf{z}_h = \xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n$, and the second one is $\mathbf{z}_h = -\tau \mathbf{f}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}$. By taking a suitably small ε we obtain the following estimate

$$\begin{aligned}||\xi_{\mathbf{u}}^{n+1} - \xi_{\mathbf{w}}^n||_n^2 \le\, & C(\Xi(n)h^{2k+2}\tau + \tau^6) + M\tau^2 h^{-1}\varrho(\mathbf{w}_h^n)A(\mathbf{w}_h^n) + M\tau^2 h^{-1}\varrho(\mathbf{u}_h^n)A(\mathbf{u}_h^n) \\ & + (C_\star \tau^2 h^{-2}||\mathbf{e}_{\mathbf{u}}^n||_\infty^2 + C\tau^2)||\xi_{\mathbf{u}}^n||_n^2 + (C_\star \tau^2 h^{-2}||\mathbf{e}_{\mathbf{w}}^n||_\infty^2 + C\tau^2)||\xi_{\mathbf{w}}^n||_n^2 \\ & + M \sum_{1\le j\le N} \tau^2 \int_{I_j} (\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}^T \mathbf{v}_{,\mathbf{u}}^{1/2}(\mathbf{u}_c^n)\mathbb{S}^2(\mathbf{u}_c^n)\mathbf{v}_{,\mathbf{u}}^{1/2}(\mathbf{u}_c^n)(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}\, dx, \end{aligned} \tag{4.4}$$

under a general CFL condition $\tau = \mathcal{O}(h)$, where $\mathbb{S}(\mathbf{u}_c^n) = \mathbf{v}_{,\mathbf{u}}^{1/2}(\mathbf{u}_c^n)\mathbf{f}_{,\mathbf{v}}(\mathbf{v}(\mathbf{u}_c^n))\mathbf{v}_{,\mathbf{u}}^{1/2}(\mathbf{u}_c^n)$ is symmetric and has the same eigenvalues as $\mathbf{f}_{,\mathbf{u}}(\mathbf{u}_c^n)$ because of (3.5). Here, C and $C_\star$ are positive constants independent of n, h, τ and the approximate solutions, and M is a positive constant determined solely by the fixed constant ε in the analysis above.

We use the following fact to estimate the last term in (4.4). Since $\xi_{\mathbf{u}}^n \in \mathbb{V}_h$ is a piecewise linear vector-valued function, its derivative $(\xi_{\mathbf{u}}^n)_{,x}$ is a constant vector on each element I_j. Hence, for any vector-valued function $\mathbf{p}_h$ there holds

$$\int_{I_j} (\mathbf{p}_h - \widetilde{\mathbf{p}_h})^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\mathbf{f}_{,\mathbf{u}}(\mathbf{u}_c^n)(\xi_{\mathbf{u}}^n)_{,x}\, dx = 0, \quad (1 \le j \le N, \forall n : n\tau < T) \tag{4.5}$$

where $\widetilde{\mathbf{p}_h}$ is the average of $\mathbf{p}_h$ on each element I_j, i.e. $\int_{I_j}(\mathbf{p}_h - \widetilde{\mathbf{p}_h})\, dx = \mathbf{0}$. This property plays a key role in the following estimate. Unfortunately it holds only for piecewise linear polynomials, but not for higher order piecewise polynomials.

It is worthwhile to note, for any $\mathbf{p}_h \in \mathbb{V}_h$, that $\mathbf{p}_h - \widetilde{\mathbf{p}_h} \in \mathbb{V}_h$ and $(\mathbf{p}_h)_{,x} = (\mathbf{p}_h - \widetilde{\mathbf{p}_h})_{,x}$. The former property holds for a discontinuous finite element space, but not for general continuous finite element spaces. Set $\mathbf{p}_h = \xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n$, then the error equation (3.2a) gives

$$||\mathbf{p}_h - \widetilde{\mathbf{p}_h}||_n^2 = (\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n, \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\mathbf{p}_h - \widetilde{\mathbf{p}_h})) = \mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\mathbf{p}_h - \widetilde{\mathbf{p}_h})).$$

We take the test function $\mathbf{z}_h = \mathbf{p}_h - \widetilde{\mathbf{p}_h}$ in Lemma 3.2, then the last integral term disappears because of the equation (4.5). Let the positive constant ε be small enough, we obtain

$$||\mathbf{p}_h - \widetilde{\mathbf{p}_h}||_n^2 \le M\tau^2 h^{-1}\varrho(\mathbf{u}_h^n)A(\mathbf{u}_h^n) + (C + C_\star h^{-2}||\mathbf{e}_{\mathbf{u}}^n||_\infty^2)||\xi_{\mathbf{u}}^n||_n^2\tau^2 + (C + C_\star||\mathbf{e}_{\mathbf{u}}^n||_\infty^2)h^3\tau.$$

This gives a sharp estimate to $||(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}||$ by using $||(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}|| \le Ch^{-1}||\mathbf{p}_h - \widetilde{\mathbf{p}}||_n$, which follows from $(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x} = (\mathbf{p}_h - \widetilde{\mathbf{p}})_{,x}$, the inverse property (i) and the equivalence of the norms $||\cdot||$ and $||\cdot||_n$.

Finally, we can get the estimate (4.2) by substituting this sharp estimate of $||(\xi_{\mathbf{w}}^n - \xi_{\mathbf{u}}^n)_{,x}||$ into the inequality (4.4). It completes the proof of this lemma.

Lemma 4.2 *If the interpolation approximation property (3.4) is satisfied for the linear finite element space* $\mathbb{P}^1$*, then we have the following estimates*

$$\mathcal{K}^n(\xi_{\mathbf{u}}^n) \le \Phi(\mathbf{e}_{\mathbf{u}}^n)||\xi_{\mathbf{u}}^n||_n^2\tau - \frac{1}{2}A(\mathbf{u}_h^n)\tau + (C + C_\star||\mathbf{e}_{\mathbf{u}}^n||_\infty^2)h^3\tau. \tag{4.6a}$$

$$\mathcal{L}^n(\xi_{\mathbf{w}}^n) \le \Phi(\mathbf{e}_{\mathbf{w}}^n)||\xi_{\mathbf{w}}^n||_n^2\tau - \frac{1}{2}A(\mathbf{u}_h^n)\tau + (C + C_\star||\mathbf{e}_{\mathbf{w}}^n||_\infty^2)h^3\tau + C\tau^5. \tag{4.6b}$$

where C and $C_\star$ are positive constants independent of n, h, τ and the numerical solutions, and $\Phi(\mathbf{e}_{\mathbf{p}}^n) = C + C_\star(||\mathbf{e}_{\mathbf{p}}^n||_\infty + h^{-1}||\mathbf{e}_{\mathbf{p}}^n||_\infty^2)$ for $\mathbf{p} = \mathbf{u}$ or $\mathbf{w}$.

Proof. We will prove only (4.6a) here, since the estimate to (4.6b) is similar. Noticing the periodic or zero (compactly supported) boundary conditions, after some elementary calculations we have an equivalent form of $\mathcal{K}^n(\xi_{\mathbf{u}}^n)$. It reads

$$\begin{aligned}
\mathcal{K}^n(\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\xi_{\mathbf{u}}^n) &\equiv \sum_{j=1}^{N} \mathcal{K}_j^n(\xi_{\mathbf{u}}^n) := \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 \\
&= \sum_{1\le j\le N} \int_{I_j} (\xi_{\mathbf{u}}^n)^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)(\eta_{\mathbf{w}}^n - \eta_{\mathbf{u}}^n)\,dx \\
&+ \tau \sum_{1\le j\le N} \int_{I_j} (\xi_{\mathbf{u}}^n)_{,x}^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)\big(\mathbf{f}(\mathbf{u}^n) - \mathbf{f}(\mathbf{u}_h^n)\big)\,dx \\
&+ \tau \sum_{1\le j\le N} \Big\{ \big[\xi_{\mathbf{u}}^n\big]^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b^n)\big(\mathbf{f}(\mathbf{u}^n) - \mathbf{f}(\bar{\mathbf{u}}_h^n)\big) \Big\}_{j+\frac{1}{2}} \\
&+ \tau \sum_{1\le j\le N} \Big\{ \big[\xi_{\mathbf{u}}^n\big]^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b^n)\big(\mathbf{f}(\bar{\mathbf{u}}_h^n) - \hat{\mathbf{h}}(\mathbf{u}_h^n)\big) \Big\}_{j+\frac{1}{2}} \\
&+ \tau \sum_{1\le j\le N} \Big\{ \big((\xi_{\mathbf{u}}^{n,+})^T \mathbb{E}_b^{n,+} - (\xi_{\mathbf{u}}^{n,-})^T \mathbb{E}_b^{n,-}\big)\big(\mathbf{f}(\mathbf{u}^n) - \hat{\mathbf{h}}(\mathbf{u}_h^n)\big) \Big\}_{j+\frac{1}{2}},
\end{aligned}$$

where $\mathbb{E}_b^{n,\pm} = \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_{b\pm1/2}^n) - \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b^n)$. Here and below, we use the subscript b to stress that the evaluation is defined on at the element interfaces, for example, $(\mathbf{u}_b^n)_{j+1/2} = \mathbf{u}_{j+1/2}^n$. We will analyze each of the terms above separately, following [24].

By the interpolation approximation property (3.4) and Young's inequality, it is easy to estimate Π_1 in the form

$$\Pi_1 \le Ch^4\tau + C||\xi_{\mathbf{u}}^n||_n^2\tau. \tag{4.7a}$$

To estimate Π_2 and Π_3, we would like to consider them together, and use Taylor expansions up to third order. Here, the symmetrizable property of the system (1.1) plays an important role in the analysis. We would like to present only the result here and put aside the technical analysis, which will be given in the appendix. The final estimate reads

$$\Pi_2 + \Pi_3 \le \Phi(\mathbf{e}_{\mathbf{u}}^n)||\xi_{\mathbf{u}}^n||_n^2\tau + \frac{1}{3}A(\mathbf{u}_h^n)\tau + (C + C_\star||\mathbf{e}_{\mathbf{u}}^n||_\infty^2)h^3\tau. \tag{4.7b}$$

We can estimate the term Π_4 by using the matrix $\mathcal{A}(\hat{\mathbf{h}};\mathbf{u}_h^n)$ (see Proposition 3.1) and the following two properties: one is $[\mathbf{e}_{\mathbf{u}}^n]=-[\mathbf{u}_h^n]$ from the smoothness assumption of $\mathbf{u}^n$, and the other is $\mathcal{A}(\hat{\mathbf{h}};\mathbf{u}_h)[\mathbf{g}_h]=\mathbf{f}(\bar{\mathbf{u}}_h)-\hat{\mathbf{h}}(\mathbf{u}_h)$ from the definition of the matrix $\mathcal{A}(\hat{\mathbf{h}};\mathbf{u}_h^n)$; here, $\mathbf{g}_h^{n,\pm}=\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_h^{n,\theta})\mathbf{u}_h^{n,\pm}$. We mention that each component of $\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_h^{n,\theta})-\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b^n)$ is of the order $\mathcal{O}(||\mathbf{e}_{\mathbf{u}}^n||_\infty)$ by the smoothness assumption of the mapping $\mathbf{v}(\mathbf{u})$. Finally, by using Young's inequality and the properties of the finite element space $\mathbb{V}_h$, we have that

$$\Pi_4\leq-\frac{5}{6}A(\mathbf{u}_h^n)\tau+C_\star h^{-1}||\mathbf{e}_{\mathbf{u}}^n||_\infty^2||\xi_{\mathbf{u}}^n||_n^2+Ch^3\tau, \tag{4.7c}$$

since the spectral radius of both $\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b^n)\mathbf{v}_{,\mathbf{u}}^{-1}(\mathbf{u}_h^{n,\theta})$ and $\mathcal{A}(\hat{\mathbf{h}};\mathbf{u}_h^n)$ are bounded.

It also follows from the smoothness assumption that each component of $\mathbb{E}_b^{n,\pm}$ is of the order $\mathcal{O}(h)$. Then, the Lipschitz property of numerical flux together with the properties of the finite element space $\mathbb{V}_h$ shows that

$$\Pi_5\leq Ch^4\tau+C||\xi_{\mathbf{u}}^n||^2\tau. \tag{4.7d}$$

We can then get the estimate (4.6a) by collecting the estimates (4.7), and complete the proof of this lemma.

4.2 Proof of the convergence theorem

In this subsection we will prove the convergence Theorem 2.1 for the k=1 case by collecting all the estimates given in subsection 4.1. Moreover, we would like to use the *a priori* technique below.

To deal with the nonlinearity of the flux function $\mathbf{f}(\mathbf{u})$, we assume *a priori* that for h small enough there holds

$$||\mathbf{u}^n-\mathbf{u}_h^n||\leq h. \tag{4.8}$$

This is obviously satisfied for $n=0$ by $\mathbf{u}_h^0=\mathbb{P}_h\mathbf{u}_0(x)$ and the interpolation approximation property (3.4a). We shall later verify the correctness of (4.8), and prove that it still holds true for $n+1$ if it holds true for a given n. For a linear flux function $\mathbf{f}=\mathbb{C}\mathbf{u}$ where $\mathbb{C}$ is a constant matrix, this *a priori* assumption is unnecessary.

It follows from the *a priori* assumption (4.8) and Corollary 3.1 that $||\mathbf{w}^n-\mathbf{w}_h^n||\leq Ch$. Then the inverse property (iii) together with the assumption (4.8) implies that

$$||\mathbf{e}_{\mathbf{p}}^n||_\infty\leq Ch^{1/2},\quad ||\xi_{\mathbf{p}}^n||_\infty\leq Ch^{1/2},\quad \mathbf{p}=\mathbf{u},\mathbf{w}, \tag{4.9}$$

where the interpolation approximation property (3.4a) is used.

By combining all the results in subsection 4.1, together with Corollary 3.1 and (4.9), we can finally get, for h small enough, that

$$\begin{aligned}&||\xi_{\mathbf{u}}^{n+1}||_n^2-||\xi_{\mathbf{u}}^n||_n^2+\frac{1}{2}A(\mathbf{u}_h^n)\tau+\frac{1}{2}A(\mathbf{w}_h^n)\tau\\ \leq\,&\varepsilon||\xi_{\mathbf{u}}^{n+1}||_n^2\tau+C(||\xi_{\mathbf{u}}^n||_n^2\tau+h^3\tau+\tau^5)+\delta_1(n)A(\mathbf{u}_h^n)\tau+\delta_2(n)A(\mathbf{w}_h^n)\tau,\end{aligned} \tag{4.10}$$

under a suitable CFL condition $\tau\leq\beta h$, where β will be determined later. Here, ε is an arbitrary positive constant, and C is a positive constant independent of n,h,τ and the approximate solutions.

The number β can be determined, e.g., by both $\delta_1(n) \le 1/4$ and $\delta_2(n) \le 1/4$. We would like to mention again that those positive constants emerged in (4.3), namely M_1, M_2 and M_3, are independent of h and τ. Hence there exists a maximum positive constant r_0 also independent h and τ, such that

$$M_1 r_0 \le \frac{1}{8}, \quad M_2 r_0^3 \le \frac{1}{8}, \quad \text{and} \quad M_3 r_0 \le \frac{1}{4}.$$

Then each time step τ^n can be determined by $\tau^n \le \beta^n h$, where

$$\beta^n = r_0 \min\{\, \varrho(u_h^n)^{-1}, \varrho(w_h^n)^{-1}, (\varrho(u_h^n)\lambda^2(u^n))^{-1/3}\,\}. \tag{4.11}$$

Since the time step considered in this paper is a constant, for convenience we write the CFL condition as $\tau \le \beta h$ instead of $\tau^n \le \beta^n h$, where $\beta = \min_{\forall n: n\tau \le T} \beta^n$.

Under the usual CFL condition $\tau \le \beta h$, it follows from the inequality (4.10) when ε is suitably small, that

$$||\xi_{\mathbf{u}}^{n+1}||^2 + \sum_{0\le m\le n} A(\mathbf{u}_h^m)\tau + \sum_{0\le m\le n} A(\mathbf{w}_h^m)\tau \le C\Big(\sum_{0\le m\le n} ||\xi_{\mathbf{u}}^m||^2\tau + h^3 + \tau^4\Big),$$

where we use the equivalence of the norms $||\cdot||_n$ and $||\cdot||$ and the fact that each component of $\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^{n+1}) - \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c^n)$ is of the order $\mathcal{O}(\tau)$. Thus by Gronwall's inequality we can get the following error estimate

$$||\xi_{\mathbf{u}}^{n+1}|| \le Ch^{3/2}, \quad \forall n : n\tau \le T, \tag{4.12}$$

where C is a positive constant independent of n, h, τ and the numerical solutions. Then we can get (2.5) for k=1 easily by the triangle inequality and the interpolation approximation property (3.4a).

Finally let us verify the *a priori* assumption (4.8) before we complete the proof of Theorem 2.1 for k=1. If (4.8) is satisfied for a certain n, then it follows from (4.12) and the interpolation approximation property (3.4a) that it is also true for $n+1$. Thus the given *a priori* (4.8) is reasonable, and all of the above estimates hold for all $n : n\tau \le T$.

Remark 4.1. It is worthwhile to note that the condition (4.11) is the usual CFL condition for systems of conservation laws. By Proposition 3.1 we know that for any numerical flux function $\hat{\mathbf{h}}$ the spectral radius of $\mathcal{A}(\hat{\mathbf{h}}; \mathbf{u}_h^n)$ is bounded by a constant times the maximum of the Lipschitz constant of $\hat{\mathbf{h}}$. Fox example, for the linear flux $f = \mathbb{C}u$ the CFL number β determined by (4.11) depends solely on the spectral radius of $\mathbb{C}$. This also explains why the CFL constant β is lower bounded away from zero during a mesh refinement.

Remark 4.2. We have only carried out the error estimate and detailed proofs for the linear finite element $k = 1$ in the one dimensional case with generalized E-flux functions. The estimate (2.5) also holds for the linear flux function $f = \mathbb{C}u$ in multiple dimension, when the *a priori* assumption (4.8) is unnecessary. For higher order finite element space $k > 1$, we can prove the estimate (2.5) under a more restrictive time-space condition, e.g. $\tau = \mathcal{O}(h^{4/3})$, following the lines of analysis in [24].

Remark 4.3. For certain special numerical flux functions, we can upgrade the error estimate in Theorem 2.1 to be optimal, i.e., $\mathcal{O}(h^{k+1} + \tau^2)$.

These numerical flux functions include those constructed by the flux-vector splitting method, for example, the upwind numerical flux for a linear flux, and the Steger-Warming flux [22] for Euler equations. Their common property is that the vector-valued physical flux function is homogeneous of degree 1, i.e., $\mathbf{f}(\mathbf{u}) = \mathbf{f}_{\mathbf{u}}(\mathbf{u})\mathbf{u}$.

To obtain the optimal error estimate in this case, we would need to use a standard trick in the DG analysis as we have done in [24], which consists of two main ingredients. The first ingredient is to use the Gauss-Radau projection instead of the local L^2 projection, and the other is to use the upwind setting of the reference vector on each element interface, instead of the simple arithmetic average of two values on different sides. All the analysis is carried out in projecting to each eigenvector direction. We omit the details of the proof as they are similar to those in [24] for the scalar case.

5 Appendix

In this section we will give the detailed analysis to the inequality (4.7b). To do that, we would like to use the following Taylor expansions, with respect to the variable $\mathbf{u}$, up to third order

$$\begin{aligned}
\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_h) &= \mathbf{f}_{,\mathbf{u}}(\mathbf{u})\mathbf{e} - \frac{1}{2}\mathbf{e}^T \mathbf{f}_{,\mathbf{u},\mathbf{u}}(\mathbf{u})\mathbf{e} + \frac{1}{6}\{\mathbf{f}(\mathbf{u}^\star)\}_{,u_\iota u_\kappa u_\sigma} \mathbf{e}_\iota \mathbf{e}_\kappa \mathbf{e}_\sigma = \mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3, \\
\mathbf{f}(\mathbf{u}) - \mathbf{f}(\bar{\mathbf{u}}_h) &= \mathbf{f}_{,\mathbf{u}}(\mathbf{u}_b)\mathbf{e} - \frac{1}{2}\mathbf{e}^T \mathbf{f}_{,\mathbf{u},\mathbf{u}}(\mathbf{u}_b)\mathbf{e} + \frac{1}{6}\{\mathbf{f}(\mathbf{u}_b^\star)\}_{,u_\iota u_\kappa u_\sigma} \bar{\mathbf{e}}_\iota \bar{\mathbf{e}}_\kappa \bar{\mathbf{e}}_\sigma = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3,
\end{aligned}$$

for x inside each element and for x on each element interface, respectively, where $\mathbf{u}^\star$ and $\mathbf{u}_b^\star$ are mean values. Here and below, we suppress the subscripts n and subscripts $\mathbf{u}$ for clarity, and use the implied summation on the indices ι, κ and σ.

Thus we have $\Pi_i = \sum_{\sigma=1}^{3} \{\pi_{i\sigma}(\eta) + \pi_{i\sigma}(\mathbf{e})\}$ for $i = 2, 3$, respectively. For $\mathbf{z} = \eta$ and $\mathbf{e}$, each term in the above formulation is given as

$$\pi_{2\sigma}(\mathbf{z}) = \tau \sum_{1 \le j \le N} \int_{I_j} \mathbf{z}_{,x}^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c)\mathbf{r}_\sigma \, dx, \quad \pi_{3\sigma}(\mathbf{z}) = \tau \sum_{1 \le j \le N} \Big\{ [\mathbf{z}]^T \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b)\mathbf{s}_\sigma \Big\}_{j+\frac{1}{2}}.$$

We would like to analyze successively these terms in $\pi_{2\sigma}$ and $\pi_{3\sigma}$, for $\sigma = 1, 2, 3$, where the symmetrizable property of the system (1.1) will play an important role.

First, we remark that all the terms involving η are easy to estimate by the interpolation approximation property and Young's inequality. Here, we also use the local L^2-projection and the first inequality in Proposition 3.1. The detailed analysis is similar to those in [24], and is thus omitted. The result reads

$$\sum_{\sigma=1}^{3} \{\pi_{2\sigma}(\eta) + \pi_{3\sigma}(\eta)\} \le C(||\xi||_n^2 \tau + h^3 \tau). \tag{5.1}$$

We then focus our attention on how to estimate the remaining terms which involve $\mathbf{e}$. We have three pairs of such terms, namely, $\pi_{2i}(\mathbf{e}) + \pi_{3i}(\mathbf{e})$ for $i = 1, 2, 3$. In what follows we will estimate them separately.

Denote $\mathbb{S}(\mathbf{u}) = \mathbf{v}_{,\mathbf{u}}(\mathbf{u})\mathbf{f}_{,\mathbf{u}}(\mathbf{u})$ and $\mathbb{E}^c = \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_c) - \mathbf{v}_{,\mathbf{u}}(\mathbf{u})$. The symmetrizable theory shows that $\mathbb{S}$ is a symmetric matrix, so a simple integration by parts reveals that

$$\pi_{21}(\mathbf{e}) + \pi_{31}(\mathbf{e}) = \tau \sum_{1\le j\le N} \Big\{ \int_{I_j} \mathbf{e}_{,x}^T \mathbb{E}^c(\mathbf{u})\mathbf{f}_{,\mathbf{u}}(\mathbf{u})\mathbf{e}\,dx - \frac{1}{2}\int_{I_j} \mathbf{e}^T \partial_x \mathbb{S}(\mathbf{u})\mathbf{e}\,dx \Big\}.$$

Here $\mathbb{E}^c$ provides a factor of order $\mathcal{O}(h)$ in the first term. By using Young's inequality and the inverse properties (i), and noticing the equivalence of norms $||\cdot||$ and $||\cdot||_n$, we obtain

$$\pi_{21}(\mathbf{e}) + \pi_{31}(\mathbf{e}) \le C||\mathbf{e}||_n^2\tau + h^3\tau. \tag{5.2}$$

We have to pay more attention to the second pair of terms, namely, $\pi_{22}(\mathbf{e}) + \pi_{32}(\mathbf{e})$. This is the main obstacle in generalizing the analysis from the scalar case to symmetrizable systems. Denote $\mathbb{H}(\mathbf{u}) = \mathbf{v}_{,\mathbf{u}}^{-1}(\mathbf{u})\mathbb{E}^c$ and $\mathbf{g} = \mathbf{v}_{,\mathbf{u}}(\mathbf{u})\mathbf{e}$. An elementary manipulation shows that

$$\begin{aligned}
&\pi_{22}(\mathbf{e}) + \pi_{32}(\mathbf{e}) := \mathcal{R} + \mathcal{Q} + \mathcal{S} \\
= &-\frac{\tau}{2}\sum_{j=1}^{N} \Big[\int_{I_j} \mathbf{g}_{,x}^T \mathbf{g}^T \mathbb{G}(\mathbf{u})\mathbf{g}\,dx + \Big([\mathbf{g}]^T \bar{\mathbf{g}}^T \mathbb{G}(\mathbf{u}_b)\bar{\mathbf{g}} \Big)_{j+\frac{1}{2}} + \int_{I_j} \mathbf{g}_{,x}^T \mathbb{H}(\mathbf{u})\mathbf{g}^T \mathbb{G}(\mathbf{u})\mathbf{g}\,dx \Big].
\end{aligned}$$

In what follows we will estimate $\mathcal{R}$, $\mathcal{Q}$ and $\mathcal{S}$ separately. Below we will again use the implied summation for the indices i_1, i_2, i_3 and j (or b, since $b = j + 1/2$).

After a simple integration by parts, it is easy to get

$$\begin{aligned}
\mathcal{R} \equiv &-\frac{\tau}{2}\int_I \{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} \partial_x g_{i_1} g_{i_2} g_{i_3}\,dx \\
= &\frac{\tau}{2}\{\mathbb{G}(\mathbf{u}_b)\}_{i_1 i_2}^{i_3}[g_{i_1} g_{i_2} g_{i_3}]_b + \frac{\tau}{2}\int_I \partial_x \{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} g_{i_1} g_{i_2} g_{i_3}\,dx \\
&+ \frac{\tau}{2}\int_I \Big(\{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} g_{i_1}\partial_x g_{i_2} g_{i_3} + \{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} g_{i_1} g_{i_2} \partial_x g_{i_3} \Big)\,dx,
\end{aligned} \tag{5.3}$$

where g_ℓ means the ℓ-th component of $\mathbf{g}$. It is important to note that $\{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3}$ is invariant for all the rotations of the indices i_1, i_2 and i_3, because it has the following equivalent form

$$\{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} = \frac{\partial^2 f_{i_3}}{\partial v_{i_1}\partial v_{i_2}} + \sum_{\kappa,\gamma,\sigma,\ell} \frac{\partial f_\kappa}{\partial u_\gamma}\frac{\partial u_\gamma}{\partial v_{i_3}}\frac{\partial u_\sigma}{\partial v_{i_1}}\frac{\partial u_\ell}{\partial v_{i_2}}\frac{\partial^2 v_\kappa}{\partial u_\sigma \partial u_\ell}, \tag{5.4}$$

since $\mathbf{f}_{,\mathbf{v}}$ and $\mathbf{u}_{,\mathbf{v}}$ are symmetric matrices by the symmetrizable theory. Now, it is clear that in (5.3), the last term on right-hand side is equal to $-2\mathcal{R}$. Therefore,

$$\mathcal{R} = \frac{\tau}{6}\{\mathbb{G}(\mathbf{u}_b)\}_{i_1 i_2}^{i_3}[g_{i_1} g_{i_2} g_{i_3}]_b + \frac{\tau}{6}\int_I \partial_x \{\mathbb{G}(\mathbf{u})\}_{i_1 i_2}^{i_3} g_{i_1} g_{i_2} g_{i_3}\,dx.$$

By noticing the symmetric property of $\mathbb{G}$, we see that the jump term on each element interface has the following equivalent form

$$\begin{aligned}
\{\mathbb{G}(\mathbf{u}_b)\}_{i_1 i_2}^{i_3}[g_{i_1} g_{i_2} g_{i_3}]_b = &\{\mathbb{G}(\mathbf{u}_b)\}_{i_1 i_2}^{i_3}\Big([g_{i_1}]g_{i_2}^+ g_{i_3}^+ + g_{i_1}^-[g_{i_2}]g_{i_3}^+ + g_{i_1}^- g_{i_2}^-[g_{i_3}] \Big)_b \\
= &\{\mathbb{G}(\mathbf{u}_b)\}_{i_1 i_2}^{i_3}\Big(g_{i_1}^+ g_{i_2}^+ + \frac{1}{2}g_{i_1}^- g_{i_2}^+ + \frac{1}{2}g_{i_2}^- g_{i_1}^+ + g_{i_1}^- g_{i_2}^- \Big)_b [g_{i_3}^n]_b.
\end{aligned}$$

Hence some simple manipulations show that

$$\mathcal{R} + \mathcal{Q} = \frac{\tau}{24}\mathbb{G}^{i_3}_{i_1 i_2}(\mathbf{u}_b)[g_{i_1}]_b[g_{i_2}]_b[g_{i_3}]_b + \frac{\tau}{6}\int_I \partial_x\{\mathbb{G}(\mathbf{u})\}^{i_3}_{i_1 i_2} g_{i_1} g_{i_2} g_{i_3}\, dx.$$

We would like to point out, by the smoothness assumption of $\mathbf{u}$ and $\mathbf{f}(\mathbf{u})$, that $[\mathbf{g}] = -[\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b)\mathbf{u}_h]$ and each component of $\mathbb{G}(\mathbf{u}_b) - \mathbb{G}(\mathbf{u}_h^\theta)$ and $\mathbf{v}_{,\mathbf{u}}(\mathbf{u}_b) - \mathbf{v}_{,\mathbf{u}}(\mathbf{u}_h^\theta)$ are bounded by $C_\star||\mathbf{e}||_\infty$. Then the inequality (3.8b) of Proposition 3.1 and Young's inequality imply that

$$\mathcal{R} + \mathcal{Q} \leq \frac{1}{3}A(\mathbf{u}_h)\tau + C_\star(||\mathbf{e}||_\infty + h^{-1}||\mathbf{e}||^2_\infty)||\xi||^2_n\tau + C_\star||\mathbf{e}||^2_\infty h^3\tau. \tag{5.5}$$

Here, the interpolation approximation property (3.4a) and the inverse property (ii) are used.

It is easy to estimate the last term $\mathcal{S}$, since each component of $\mathbb{H}$ is of order $\mathcal{O}(h)$. Similarly, we can estimate easily the last pair of terms $\pi_{23}(\mathbf{e}) + \pi_{33}(\mathbf{e})$. We would like to give their estimates together. It reads

$$\mathcal{S} + \pi_{23}(\mathbf{e}) + \pi_{33}(\mathbf{e}) \leq C_\star h^{-1}||\mathbf{e}||^2_\infty(||\xi||^2\tau + h^4\tau). \tag{5.6}$$

Finally we can obtain the estimate (4.7b) by summing up the above estimates (5.1), (5.2), (5.5) and (5.6). It completes the proof of the inequality (4.7b).

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