On the Road to Nowhere

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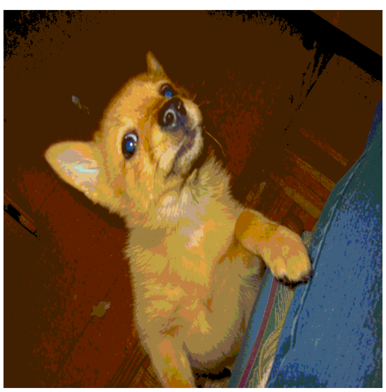
My study is in perfect order. Books are standing in alphabetical order on the shelves, the computer is sitting on the floor with its monitor on my desk. A printer is located on the top of a cabinet with copying paper next to it. Some pictures are hung on the wall, a calendar is on a table, pens and pencils orderly on their place.

The place has recently been dusted, the rug vacuumed. There are no newspapers strewn on the floor, no litter anywhere. A perfect order, indeed! Alas, it did not last: Kettu arrived, an untamed animal from the wild Finish forests.

So that when I returned the order was destroyed, the room thrashed, my beautiful study was turned into disorder in a matter of minutes



The transformation was completely destructive, it was all Kettu's fault!

But was it really? She looks so innocent (sometimes) so I would prefer to explain the change to disorder by appealing to some cosmic force...aha, the second law of thermodynamics, a favorite dogma for us amateur philosophers. One informal statement of the law says just that order is decreasing in some average sense, "average" not given a precise definition. The system becomes more and more unordered.

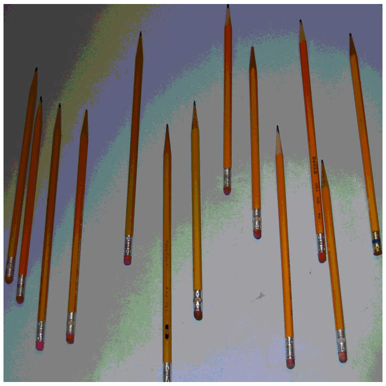
Look at the picture of my desk with pencils neatly arranged. We would certainly judge the arrangement in the figure ordered: the pencils parallel to each other, the tips aligned: a simple pattern, call it $\mathcal{P}$. How "big" is $\mathcal{P}$? Well, with n pencils, let the locations of the tips be (x_i, y_i) and let the angle of the

ith pencil relative to an x-axis bea_i,$i = 1, 2, 3, ...n$. Then we can represent the ordered pattern in terms 4 parameters x, y, h, a

$$\mathcal{P} = \{x_i = x + ih, y_i = y, a_i = a; i = 1, 2, 3...n\} \tag{1}$$

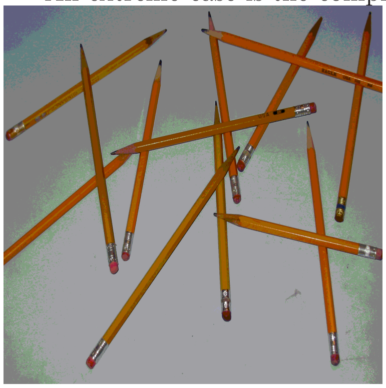
while in the unordered case we have the 3n parameters $x_i, y_i, a_i; i = 1, 2, 3, ...n)$, so that the relative dimension of $\mathcal{P}$ is $4/3n$. For a large sample of pencils the relative size is practically zero.

On the other hand the arrangement in



appears to be slightly less ordered: the pencils are still parallel but the tips seem to be unordered. Repeating the counting argument we get the relative dimension $(2n+1)/3n$ that tends to $2/3$ for a large sample of pencils.

An extreme case is the completely unordered arrangement in



with the obvious relative dimension $(3n)/3n = 1$.

It is clear that order restricts the size of the pattern, the amount of its variability is reduced. Patterns become ordered by imposing rules on the arrangement and that reduces its dimension. Since unordered patterns have the maximum relative dimension it is far from surprising that time will tend to them since almost all patterns (in terms of dimension) are unordered. Therefore, *this informal version of the second law of thermodynamics tells us practically nothing*, the statement is independent of physical knowledge.

But the second law also exists in a quantified version: the entropy of the system is an increasing function of time. There are many more or less convincing "proofs" of this statement but can one give a real proof of it at least for some dynamical systems?

Well, let us say that we describe the system by a Markov chain. The Markovian assumption can of course be questioned, systems can have longer lasting memory. But this is a question of how much information is built into the state space. As Feller said, all systems are Markovian, even God is Markovian if embedded in a sufficiently large state space. All right, let us do that. But to derive the second law under such extremely general conditions does not seem to be possible as was made clear in discussions with my learned friend Ioannis Kontoyiannis. So let us narrow down the conditions a bit and assume that the transition matrix in the Markov chain is doubly stochastic. Then the matrix can be written as a convex combination of permutation matrices that preserve entropy. Using convexity and Jensen's inequality it follows that the entropy will increase, actually be non-decreasing, as time goes on.

Hence the system develops towards nowhere in particular. The world just drifts along to unordered states. What happens in the world is just one damn thing after another.