CROSS DIFFUSION SYSTEMS

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To my Parents,

Nguyen Trong An and Nguyen Thi Huong

CROSS DIFFUSION SYSTEMS

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In this thesis we investigate regularity properties and long-time dynamics of nonnegative solutions to a class of cross diffusion (strongly coupled) parabolic systems which occur in population dynamics where the studied species are assumed to diffuse and interact with one another. One of such models which greatly interests us to obtain certain understanding is the Shigesada-Kawasaki-Teramoto (SKT) model. Briefly, our results are to address the following questions:

Global existence of classical solutions. It has been well known that one can obtain the global existence result for a general class of *regular* cross diffusion parabolic systems if he shows that the *Hölder-norms* of solutions do not blow up in finite time (see [2]). We establish the result for certain cross diffusion systems of two equations.

Existence of global attractors. These sets describe all possible long-time dynamics that the semiflow associated with the given system can produce. The results shall be proven under the assumption on the Hölder continuity of solutions. Moreover, uniform estimates of C^{μ} norms, $\mu > 1$, of solutions are also established. For quasi-linear *regular* parabolic equations, such estimates are derived by the work of Ladyzenskaja, Solonnikov, and Ural'tseva in [22] . We encompass these results for a general class of cross diffusion regular parabolic systems of m equations, $m \geq 1$.

Uniform persistence property. Mathematically speaking, the result gives the positive lower bound of solutions which *eventually* does not depend on the initial data. On the other hand, biologically speaking, it asserts that no species is completely invaded or wiped out by the others so that they coexist in time. The result is derived for the *generalized SKT model* of two competitive species.

Everywhere regularity for degenerate systems. Besides the above questions addressed for a class of *regular* cross diffusion parabolic systems, we are also concerned *everywhere regularity* of bounded weak solutions for *degenerate* ones. The general theory will be stated and then applied to a generalized porous media type SKT model in population dynamics.

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Chapter 1

INTRODUCTION

What we know is not much. What we do not know is immense.

de Laplace, Pierre-Simon.

In the most general circumstance we would like to investigate a vector function

$$u(x,t) = (u^1(x,t), ..., u^m(x,t)), \quad \forall x \in \Omega, t \geq 0,$$

to describe densities (concentrations, population densities, temperatures, charged particle densities, etc.). Here Ω is a bounded open smooth subset of $\mathbb{R}^n$, $n \geq 1$. Given any smooth subset B of Ω, the integral over B of u^i, $1 \leq i \leq m$, represents the total amount of the quantity of the component u^i within B at time t. Let $\vec{J_i}$ be a flux vector of density which controls the rate loss or increase of component u_i through ∂B, and f_i be the production/death/reaction rate for component u^i in the domain B. In general, functions $\vec{J_i}$ and f_i *may* depend on the location x, the time t, the density of u^j, as well as the vector ∇u^j, which accounts for the movement (diffusion) of u^j. Here for each $i \in \{1, 2, ..., m\}$, j may vary from 1 to m. This various dependence has driven the abundance of mathematical models. Let us briefly explain how general systems are interpreted (see [7]).

First, conservation laws assert that the rate of change of total amount of the quantity of u^i within B is equal to the negative of the flux vector $\vec{J_i}$ through ∂B and the total amount of production/death/reaction of u^i, that is,

$$\frac{d}{dt}\int_B u^i(x,t)\ dx = -\int_{\partial B} \vec{J_i}.\nu d\sigma + \int_B f_i\ dx$$

in which ν denotes the outward unit normal along B. In addition, by a view of the Gauss-Green Theorem, we deduce from the above equality

$$\int_B u^i_t(x,t)\ dx = -\int_B \mathrm{div}\vec{J_i}\ dx + \int_B f_i\ dx.$$

As the domain B is arbitrary, we can choose $B = B(y, R)$, a ball centered at any fixed y in Ω with radius $R > 0$. Dividing by $|B(y, R)|$, the Lebesgue measure of B, we derive from the above system

$$\frac{1}{|B(y,R)|}\int_{B(y,R)} (u_t^i + \mathrm{div}\vec{J}_i - f_i)\ dx = 0.$$

According to the Lebesgue theorem, letting R go to zero in the above equality gives us

$$u_t^i + \mathrm{div}\vec{J}_i = f_i, \tag{1.0.1}$$

in Ω, for any $1 \le i \le m$.

In many situations, due to the Fick's law of diffusion (Fourier's law of heat conduction, Ohm's law of electrical conduction, etc.), it is assumed that the flux vectors are of the form

$$\vec{J}_i = -a_i(x, t, u^i)\nabla u^i + b_i(x, t, u), \tag{1.0.2}$$

in which a_i's and b_i's are the real-valued known functions. a_i's account for the diffusion rate of u^i, and b_i's describe drifts in their directions. The minus sign in (1.0.2) is due to the fact that the flux is from regions of higher to lower concentration. Meanwhile, the production/death/reaction rate f_i can generally be assumed to depend on the location x, the time t, and the density of $u = (u^1, ..., u^m)$. In such cases, system (1.0.1) is the well known standard reaction diffusion system (or classical reaction diffusion, or weakly coupled) which has been very much thoroughly investigated in literature (e.g., see [24]). In many applications, reaction terms are assumed to be of competition models of Lotka-Volterra type.

However, in the above reaction diffusion models, motilities of the species are determined solely by their own characteristics in question, and therefore, hardly surprisingly, these models are not to describe many other cases of phenomena in population dynamics. For instance, by phenomenological laws, the movements of the species can be physically affected by the population pressures due to the mutual interference between the individuals, that is, the function a_i may depend also on $u = (u^1, ..., u^m)$; and in such case, we will denote $a_i(x, t, \vec{u})$ by $a_{ii}(x, t, \vec{u})$. In application, one may assume that $a_{ii}(x, t, \vec{u}) = a_{i0}(x, t) + \sum_j a_{ii}^j(x, t, u^j)$. For this reason, we refer $a_{i0}(x, t)$ as the diffusion rate of the species u_i; $a^i_{ii}(x,t,u^i)$ as the *self-diffusion* pressures, which account for the effect of the population pressures on the diffusion of its own species; and $a^j_{ii}(x,t,u^j)$, $i \neq j$, as the *density cross-diffusion* pressures, which account for the movement of u^i affected by the density of u^j.

In addition, one can assume that the motilities of the species u^j may affect the direction of the species u^i, which means the flux vector $\vec{J_i}$ also depends on ∇u^j, for some or any $j \neq i$. For convenience, the coefficient of ∇u^j, $j \neq i$, in $\vec{J_i}$ which is denoted by $a_{ij}(x,t,\vec{u})$, is the so-called *gradient (or motility) cross diffusion* pressures. Biologically speaking, the species u^i tends to be attracted by (or attacking) the species u^j (if $a_{ij}(x,t,\vec{u}) > 0$), or to be repelled (or avoiding) by one another (if $a_{ij}(x,t,\vec{u}) < 0$). Occasionally, we use *cross diffusions* to refer the gradient cross diffusion terms $a_{ij}(x,t,\vec{u})$, $j \neq i$, and *self diffusion* to refer the terms $a_{ii}(x,t,\vec{u})$.

Consequently, the flux vector $\vec{J_i}$ is reasonably specified as follows

$$\vec{J_i} = -\sum_j a_{ij}(x,t,u)\nabla u^j + b_i(x,t,u), \tag{1.0.3}$$

in which a_{ij}'s are real-valued functions (or even $n \times n$ matrices of real-valued functions), and $b_i(x,t,u)$'s are n-vector-valued functions.

The system (1.0.1) with (1.0.3) is the so-called cross diffusion (or strongly coupled) system. An interesting feature of such a system is that the motilities of the species are affected by pressures on itself population a^i_{ii} and the population and movement of the others due to the mutual interference between the individuals and the species. Finally, needless to say, the standard reaction diffusion system is a special case of cross diffusion systems as $a_{ij} \equiv 0$ with $i \neq j$.

The introduction of cross diffusion terms into classical diffusion systems allows the mathematical models to capture much more important features of many phenomena in physics, biology, ecology, and engineering sciences. For instance, the following SKT model in population dynamics is a special case of cross diffusion systems (1.0.1).

The SKT model. In paper [45] of 1979, Shigesada, Kawasaki and Teramoto proposed the following strongly coupled parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1 u - c_1 v), \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + \alpha_{21}u + \alpha_{22}v)v] + v(a_2 - b_2 u - c_2 v), \end{cases} \tag{1.0.4}$$

on a bounded smooth domain Ω in $\mathbb{R}^n$, $n \geq 1$. The Neumann boundary conditions were considered. This mathematical model describes spatial segregation of interacting species, where u and v represent the densities of two competing species. It becomes the well known Lotka-Volterra competition-diffusion system when $\alpha_{ij} = 0$, which has been thoroughly investigated. For nonzero $\alpha_{ij} \neq 0$, model (1.0.4) is a cross diffusion parabolic system and has received a lot of great attention in literature ever since its birth (e.g., see [53, 46, 27, 36, 31, 33, 37, 29, 34, 35] for recent developments) .

Main problem. Being inspired with the above mathematical model (1.0.4), we came to introduce in [29] a more general model of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla(P^u(u,v)\nabla u + P^v(u,v)\nabla v) + f(u,v), \\ \dfrac{\partial v}{\partial t} = \nabla(Q^u(u,v)\nabla u + Q^v(u,v)\nabla v) + g(u,v). \end{cases} \tag{1.0.5}$$

Apparently, the model (1.0.4) is just a special case of (1.0.5) when P^u, P^v, Q^u, Q^v are simply the partial derivatives of $P = d_1 + \alpha_{11}u^2 + \alpha_{12}uv, Q = d_2 + \alpha_{21}uv + \alpha_{22}v^2$ with respect to u, v. In many situations where we establish general theories for system (1.0.5), we shall consider P^u, P^v, Q^u, Q^v, f, g as continuous functions in u, v. As an illumination, we always confine ourself with the following case

$$\begin{aligned} P^u &= d_1 + a_{11}u + a_{12}v, \quad P^v = b_{11}u, \\ Q^v &= d_2 + a_{21}u + a_{22}v, \quad Q^u = b_{22}v, \end{aligned} \tag{1.0.6}$$

which generalizes (1.0.4) when $a_{12} = b_{11}$ and $a_{21} = b_{22}$. Here we recall that a_{12} and a_{21} are the *density cross diffusions*; and b_{11} and b_{22} are the *gradient cross diffusions*. Throughout this work, we call system (1.0.5) with (1.0.6) *generalized SKT model*.

The thesis is organized as follows.

In Chapter 2, we shall recall some standard notations, definitions of functional spaces, and some well known imbedding results that will be used throughout the thesis.

In Chapters 3 and 4, we shall establish the *Global existence* and further obtain uniform a priori estimates of solutions, which are respectively proven for a class of *triangular* cross diffusion systems (the diffusion matrix is triangular) in Chapter 3 and for *full* cross diffusion systems (the diffusion matrix is full) in Chapter 4. The fundamental theory to investigate the global existence for such strongly coupled systems was studied in [2]. There was pointed out that solutions to (1.0.5) exist globally in time if one has controls on both of their L^∞ and Hölder norms.

In particular, for *triangular* cross diffusion systems ($Q^u = 0$ in (1.0.5)), he also proved that it is sufficient to obtain the global existence if one can control the L^∞ norms of *every components* of the solution. Under certain assumptions, we shall give estimates of L^∞ norms by exploiting two different methods: L^p bootstrapping and semigroup techniques (see Section 3.1) and Lyapunov functional approach (see Section 3.3). More importantly, our estimates of L^∞ norms are *ultimately uniformly bounded* with respect to the initial data (see Definition 3.1.1). So are estimates of Hölder norms thanks to the result of [25, Theorem 6] for triangular systems (See also Theorem 3.1.3). Such a priori uniform estimates are key issues in studying long time dynamics of solutions, namely, the existence of a global attractor set. This type of the result is well known for reaction diffusion systems (e.g., see [24, 23]).

For *full* cross diffusion systems, one needs to control both the L^∞ and Hölder norms of solutions (counterexamples in [16] confirmed that they are necessary). In a recent work, Le established sufficient conditions on the parameters of (1.0.5) to obtain the global existence result (see [29]). Roughly speaking, his approach is to find a suitable function $H(u, v)$, being defined along the solution (u, v), which links the structures of the two equations in a way that he can derive certain boundedness and regularity of $H(u, v)$, as a function in (x, t). Such boundedness and regularity of H are used to study those of u and v. In Chapter 4, we shall recall his technical assumptions on the existence of function H and then give a proof of the uniform boundedness of H. This actually gives us the a priori uniform estimates of solutions to system (1.0.5) (see Theorem 4.1.2).

In Chapter 5, we prove the existence of global attractors for a general class of cross diffusion regular parabolic systems of m equations, $m \geq 1$, under the assumption on the Hölder estimates of solutions which we assure for the case of two certain equations in Chapters 3 and 4. Of course, the preceding step is to obtain a priori estimates of weak solutions that allow us to define the dynamical semiflow on $W^{1,p}$ for some $p > n$. In addition, we prove that such a priori estimates are ultimately uniform with respect to the initial data, and therefore, obtain the existence of an absorbing ball in which all the orbits eventually enter. At the same time, we actually prove the uniform compactness of the semiflow and therefore the existence of the global attractor set ([48]). Moreover, by together employing (with a minor modification) the result of Schauder estimates in [44] and using the semigroup theory developed in [8], we obtain the uniform estimates of C^ν norms, $\nu > 1$, of solutions.

In Chapter 6, we study the uniform persistence property for *regular* cross diffusion systems of the form (1.0.5) and (1.0.6). Loosely speaking, in the context of biology, this property asserts that no species is completely invaded or wiped out by the other so that they coexist in time. On the other hand, it mathematically addresses the existence of an positive equilibrium of the evolution semiflow associated with solutions of the system. We shall encompass the result under assumptions on the principal eigenvalues of linearized problems at steady states. In addition, explicit conditions on the parameters of the system are given to guarantee the positivity of such principal eigenvalues.

Besides the results for a class of *regular* cross diffusion parabolic systems, we are also concerned *everywhere regularity* of bounded weak solutions for *degenerate* ones. The last chapter is devoted to address this concern. There are a number of sophisticated technicalities in the execution. Loosely speaking, the proof relies heavily on the following main ideas, which are interesting in themselves, to employing a recent result of *partial regularity* of D. Le in [30]; to constructing a function H whose regularity can give us that of solution u (see [29]); to exploiting the technique of the auxiliary logarithmic function ω whose boundedness can imply the Hölder continuity of H (see [26, 25]); and importantly to making use of the scaled parabolic cylinders that locally transform the degenerate systems into ones that can be approximated by regular systems in suitably scaled cylinders (see [32]).

In addition, we shall find sufficient conditions which allow us to apply our general theory of everywhere regularity to a generalized porous media type SKT model in population dynamics.

Finally, the results I present in this thesis are jointly obtained by my advisor D. Le and myself. Chapters 3 and 4, where address the first question, global existence, are found in our published papers: [27, 36, 31] for triangular cross diffusion systems and [29] for full cases. The existence of a global attractor set is also addressed in [27, 36, 34] for triangular cases of two equations and in [35] where the result is obtained for a general class of cross diffusion systems. The third question about uniform persistence property is addressed in [34] for triangular cases and in [35] for full cases. Finally, the result in [32] addresses the last question, which is everywhere regularity of weak bounded solutions for a class of cross diffusion degenerate systems.

Chapter 2

PRELIMINARIES

2.1 General notations and definitions

The purpose of this section is to introduce some general notations and definitions that will be used throughout the thesis. Most of them are generally concerned with tensor notation and definitions of different spaces of functions.

Superscripts denote coordinates of points in $\mathbb{R}^n$, that is, $x = (x^1, x^2, ..., x^n) \in \mathbb{R}^n$. Also, subscripts denote differentiation with respect to x. In particular, for sufficiently smooth function u,

$$D_i u = \frac{\partial u}{\partial x^i}, \quad D_{ij} u = \frac{\partial^2 u}{\partial x^i \partial x^j}.$$

We also write ∇u (or occasionally Du) for $(D_1 u, ..., D_n u)$, Δu for $\sum_i D_{ii} u$, and $D^2 u$ for the Hessian matrix $(D_{ij} u)$. On the other hand, we write u_t (or occasionally $D_t u$) for $\dfrac{\partial u}{\partial t}$, and $D^\beta u$ denotes $D_1^{\beta_1} ... D_n^{\beta_n} u$. Here $\beta = (\beta_1, ..., \beta_2)$ with β_i are nonnegative integers. We also denote $|\beta| = \sum_i \beta_i$.

In the case of a vector-valued function $u = (u_1, ..., u_m)$, for $m \geq 1$, ∇u and Δu stand for $(\nabla u_1, ..., \nabla u_m)$ and $(\Delta u_1, ..., \Delta u_m)$, respectively. Also, denote $\sum_{i=1}^m D_i u_i$ by $\mathrm{div}(u)$.

Let $x = (x^1, ..., x^n) \in \mathbb{R}^n$ and $p = (p_{ij}) \in \mathcal{M}_{m \times n}$, a space of $m \times n$ matrices. Norms on $\mathbb{R}^n$ and $\mathcal{M}_{m \times n}$ are given by

$$|x| = \sum_{i=1}^n |x^i| \quad \text{and} \quad |p| = \sum_{j=1}^m \sum_{i=1}^n |p_{ij}|,$$

respectively.

We always use Ω to denote a bounded open connected subset of $\mathbb{R}^n$, $n \geq 1$, with boundary $\partial\Omega$. Here the boundary is smooth enough to alow us to apply the imbedding theorems.

Denote $Q = \Omega \times [0, T]$ for some $T > 0$. For a fixed point $(x, t) \in \overline{Q}$, let $B_R(x)$ denote a ball centered at x with radius R in $\mathbb{R}^n$ and $Q_R(x, t) = \Omega_R(x) \times [t - R^2, t]$ denote a parabolic cylinder. Here $\Omega_R(x) = \Omega \bigcap B_R(x)$. As far as no ambiguity can arise, we write B_R, Ω_R, Q_R instead of $B_R(x), \Omega_R(x), Q_R(x, t)$.

In addition, Ω will be assumed to be "of type A", that is, there exists a positive constant A such that for any $R > 0$ and $x_0 \in \Omega$ we have

$$\mathrm{meas}(\Omega_R(x_0)) \geq AR^n.$$

Finally we shall use the following notation

$$⨍\!\!\!\!\!⨍_A f(x,t)\, dz = \frac{1}{|A|} \iint_A f(x,t)\, dz$$

where $|A| = \mathrm{meas}(A)$ for any $A \subset Q$. In particular, if $A = Q_R$, we use

$$u_{z_0,R} = ⨍\!\!\!\!\!⨍_{Q_R} u\, dz$$

for any $z_0 \in Q_T$ and $R > 0$.

We recall the definitions of some well-known function spaces (see [10, 22]).

We shall denote by $C^k(\Omega)$, $k = 0, 1, ...$, the space of functions that have continuous derivatives up to the order k; and by $C^\infty(\Omega)$ the space of infinitely differentiable functions in Ω; and by $C^k(\overline{\Omega})$ the space of functions in $C^k(\Omega)$ whose derivatives up to the order k can be extended to continuous functions up to the boundary $\partial\Omega$; and by $C_0^k(\Omega)$ the subspace of $C^k(\overline{\Omega})$ of the functions with compact support contained in Ω.

The spaces $C^k(\overline{\Omega})$ are Banach spaces with the norm

$$\|u\|_{C^k} = \sum_{|\beta| \leq k} \sup_{x \in \Omega} |D^\beta u(x)|.$$

If $0 < \alpha < 1$, we shall denote by $C^\alpha(\Omega)$ the space of α-Hölder continuous functions in Ω, that is, continuous functions satisfying

$$[u]_\alpha := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < +\infty.$$

More generally, we shall denote $C^{k+\alpha}(\Omega)$, $k = 0, 1, ...$, the space of functions whose derivatives of order k are α-Hölder continuous in Ω. The spaces $C^{k+\alpha}(\Omega)$ are Banach spaces with the norm

$$\|u\|_{C^{k+\alpha}} = \|u\|_{C^k} + \sum_{|\beta|=k} [D^\beta u]_\alpha.$$

We shall denote by $L^p(\Omega)$, $p \geq 1$, the space of all measurable functions in Ω. $L^p(\Omega)$ is the Banach space with the norm

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p} \quad \text{and} \quad \|u\|_{\infty,\Omega} = \sup_\Omega |u|.$$

$W^{k,p}(\Omega)$ for integral k is a Banach space of all elements of $L^p(\Omega)$ such that generalized derivatives up to order k are in $L^p(\Omega)$ as well. The norm in $W^{k,p}(\Omega)$ is defined by

$$\|u\|_{k,p,\Omega} = \sum_{|\beta|\leq k} \|D^\beta u\|_{p,\Omega}.$$

$W^{l,p}(\Omega)$ for non-integral l is a Banach space of the elements of $W^{[l],p}(\Omega)$[1] with finite norm

$$\|u\|_{p,\Omega}^{(l)} = \sum_{|\beta|\leq[l]} \|D^\beta u\|_{p,\Omega} + [u]_{q,\Omega}^{(l)},$$

in which

$$[u]_{q,\Omega}^{(l)} = \sum_{|\beta|=[l]} \left(\int_\Omega \int_\Omega \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^{n+p(l-[l])}} \, dxdy\right)^{1/p}.$$

$L^{q,r}(Q_T)$ is the Banach space of all measurable functions in $Q_T = \Omega \times [0,T]$ with a finite norm

$$\|u\|_{q,r,Q_T} = \left(\int_0^T \left(\int_\Omega |u(x,t)|^q \, dx\right)^{r/q} dt\right)^{1/r}$$

in which $q, r \geq 1$. If $q = r$ then $L^{q,q}(Q_T)$ and $\|.\|_{q,q,Q_T}$ will be denoted by $L^q(Q_T)$ and $\|.\|_{q,Q_T}$, respectively.

$W_p^{2k,k}(Q_T)$ for integral k is a Banach space of the elements of $L^p(Q_T)$ that have generalized derivatives of the form $D_t^r D_x^\beta$ with any r and β satisfying the inequality $2r + |\beta| \leq 2k$. The norm is

[1] $[l]$ is the integral part of l.

defined by

$$\|u\|_{p,Q_T}^{(2k)} = \sum_{2r+|\beta|\leq 2k} \|D_t^r D_x^\beta u\|_{p,Q_T}.$$

For $0 < \alpha \leq 1$, $C^{\alpha,\alpha/2}(Q_T)$ is a Banach space of α-Hölder continuous functions in Q_T with a finite norm

$$\|u\|_{C^{\alpha,\alpha/2}} = \|u\|_{\infty,Q_T} + \sup_{z_1\neq z_2\in Q_T} \frac{|u(z_1)-u(z_2)|}{d(z_1,z_2)^\alpha},$$

where $d(z_1, z_2)$ is a parabolic distance, that is $d(z_1,z_2) = |x_1-x_2|+|t_1-t_2|^{1/2}$ for any $z_1 = (x_1,t_1), z_2 = (x_2,t_2) \in Q_T$.

We recall the definitions of the Morrey space $M^{p,\lambda}(\Omega)$, the Sobolev-Morrey space $W^{1,(p,\lambda)}$, and the Campanato space $\mathcal{L}^{p,\mu}(Q_T)$. $M^{p,\lambda}(\Omega)$ is a Banach space of elements f in $L^p(\Omega)$ with a finite norm

$$\|f\|_{M^{p,\lambda}}^p := \sup_{x\in\Omega,\rho>0} \rho^{-\lambda} \int_{B_\rho(x)} |f|^p dy < \infty.$$

$W^{1,(p,\lambda)}$ is a Banach space of elements f in $W^{1,p}(\Omega)$ with a finite norm

$$\|f\|_{W^{1,(p,\lambda)}}^p := \|f\|_{M^{p,\lambda}}^p + \|\nabla f\|_{M^{p,\lambda}}^p < \infty.$$

$\mathcal{L}^{p,\mu}(Q_T)$ is a Banach space of elements f in $L^p(Q_T)$ with a finite norm

$$\|u\|_{\mathcal{L}^{p,\mu}} := \|u\|_{p,Q_T} + [u]_{p,\mu,Q_T}$$

in which

$$[u]_{p,\mu,Q_T} := \sup_{z_0\in Q_T,R>0} R^{-\mu} \iint_{Q_R(z_0)} |u - u_{z_0,R}|^p \, dz < +\infty.$$

2.2 Auxiliary results

The purpose of this section is to present some auxiliary results used throughout this thesis. We will omit the proofs and refer the reader to books we will correspondingly specify.

Lemma 2.2.1. *(*The Uniform Gronwall Lemma - *[48, Lemma 3.1.1]) Let g,h,y be three nonnegative* *locally integrable functions on* $(t_0, +\infty)$ *such that* y' *is locally integrable on* $(t_0, +\infty)$*, and*

$$y'(t) \leq g(t)y(t) + h(t), \quad \text{for } t \geq t_0, \tag{2.2.1}$$

and the following functions in t *satisfy*

$$\int_t^{t+1} y(s)ds \leq a_1, \quad \int_t^{t+1} g(s)ds \leq a_2, \quad \int_t^{t+1} h(s)ds \leq a_3, \quad \text{for } t \geq t_0, \tag{2.2.2}$$

where a_1, a_2, a_3 *are positive constants. Then, for any* $t \geq t_0$*,*

$$y(t+1) \leq (a_1 + a_3)\exp(a_2).$$

We shall need the following useful imbedding theorems.

Lemma 2.2.2. *(*Poincaré's inequality*, [10, Theorem 3.14]) Let* $\Omega \subset \mathbb{R}^n$ *be a bounded connected open set, with Lipschitz-continuous boundary* $\partial\Omega$*. There exists a constant* $c = c(n, p, \Omega)$ *such that for every* $u \in W^{1,p}(\Omega)$

$$\int_\Omega |u - u_\Omega|^p \, dx \leq c \int_\Omega |Du|^p \, dx, \tag{2.2.3}$$

where $u_\Omega = ⨍_\Omega u \, dx$ *is the average of* u *in* Ω*.*

Lemma 2.2.3. *(*Sobolev-Poincaré's inequality*, [10, Theorem 3.15]) With the assumption of the preceding theorem, if* $p < n$*, we have*

$$\|u - u_\Omega\|_{p*} \leq c\|Du\|_p, \tag{2.2.4}$$

where $p* = \frac{np}{n-p}$*.*

Also, we have the following result

Lemma 2.2.4. *([22, II.3]) For any* $u \in W^{1,2}(Q) \bigcap L^{2,\infty}(Q)$*, we obtain*

$$\|u\|_{2\kappa,Q} \leq C \left(\|\nabla u\|_{2,Q} + \sup_\tau \|u(\bullet, \tau)\|_{2,\Omega} \right), \quad \kappa = 1 + 2/n.$$

Lemma 2.2.5. *(*Gagliardo-Nirenberg's inequality*, [13, p.37]) There exists a positive constant* $C = C(n, \Omega)$ *such that*

(a) *If $p \geq q, p \geq r, 0 \leq \theta \leq 1$, and $k - n/p \leq \theta(m - n/q) - n(1-\theta)/r$, with strict inequality if q or $r = 1$, then*

$$\|u\|_{W^{k,p}} \leq C\|u\|^{\theta}_{W^{m,q}}\|u\|^{1-\theta}_{L^r} \tag{2.2.5}$$

(b) *If $0 \leq \theta \leq 1$ and $\nu \leq \theta(m - n/q) - n(1-\theta)/r$, with strict inequality if q or $r = 1$, or if ν is an integer, then*

$$\|u\|_{C^{\nu}} \leq C\|u\|^{\theta}_{W^{m,q}}\|u\|^{1-\theta}_{L^r} \tag{2.2.6}$$

We recall the following imbedding results for functions in the Campanato space and the Sobolev-Morrey space.

Lemma 2.2.6. *([9, Proposition 1]) The spaces $\mathcal{L}^{2,n+2+2\mu}(Q_T)$ and $C^{\mu,\mu/2}(Q_T)$, $0 < \mu < 1$, are topologically and algebraically isomorphic.*

Lemma 2.2.7. *([5, Theorem 2.5]) If $\lambda < n - p$, $p \geq 1$, and $p_\lambda = \frac{p(n-\lambda)}{n-\lambda-p}$, we then have the following imbedding result*

$$W^{1,(p,\lambda)}(B) \subset M^{p_\lambda,\lambda}(B). \tag{2.2.7}$$

We will also use two following useful results by Ladyzhenskaya *et al.* [22], which are stated for scalar functions. One can easily see that they are still true for *vector-valued* functions.

Lemma 2.2.8. *([22, Lemma II.5.4]) For any function u in $W^{1,2s+2}(\Omega, \mathbb{R}^m)$ and η is a smooth function such that $\frac{\partial u}{\partial n}\eta$ or $u\eta$ vanishes on $\partial\Omega$ we have*

$$\int_{\Omega} |\nabla u|^{2s+2}\eta^2 \, dx \leq osc^2\{u,\Omega\} Cont. \int_{\Omega} (|\nabla u|^{2s-2}|D^2u|^2\eta^2 + |\nabla u|^{2s}|\nabla\eta|^2) \, dx. \tag{2.2.8}$$

Lemma 2.2.9. *([22, Lemma II.5.3]) Let $\alpha > 0$ and v be a nonnegative function such that for any ball B_R and $\Omega_R = \Omega \bigcap B_R$ the estimate*

$$\int_{\Omega_R} v(x) \, dx \leq CR^{n-2+\alpha}$$

holds. Then for any function η *from* $W_0^{1,2}(B_R)$ *the inequality*

$$\int_{\Omega_R} v(x)\eta^2 \, dx \le CR^\alpha \int_{\Omega_R} |\nabla\eta|^2 \, dx \tag{2.2.9}$$

is valid.

Chapter 3

A PRIORI ESTIMATES FOR TRIANGULAR SYSTEMS

Consider quasilinear differential operators

$$\begin{aligned}\mathcal{A}_u(u,v) &= \nabla[P(x,t,u,v)\nabla u + R(x,t,u,v)\nabla v],\\ \mathcal{A}_v(u,v) &= \nabla[Q(x,t,u,v)\nabla v],\end{aligned}$$

and the parabolic system

$$\begin{cases} u_t = \mathcal{A}_u(u,v) + f(u,v), & x\in\Omega, t>0,\\ v_t = \mathcal{A}_v(u,v) + g(u,v), & x\in\Omega, t>0,\end{cases} \tag{3.0.1}$$

with mixed boundary conditions for $x \in \partial\Omega$ and $t > 0$

$$\begin{cases} \chi(x)\left[\frac{\partial v}{\partial n}(x,t) + r(x)v(x,t)\right] + (1-\chi(x))v(x,t) = 0,\\ \bar{\chi}(x)\left[\frac{\partial u}{\partial n}(x,t) + \bar{r}(x)u(x,t)\right] + (1-\bar{\chi}(x))u(x,t) = 0,\end{cases} \tag{3.0.2}$$

where $\chi, \bar{\chi}$ are given functions on $\partial\Omega$ with values in $\{0,1\}$.

The functions $r, \bar{r}$ are given bounded nonnegative functions on $\partial\Omega$. Here, Ω is a bounded domain in $\mathbb{R}^n$ and the initial conditions are

$$v(x,0) = v^0(x), \quad u(x,0) = u^0(x), \quad x\in\Omega \tag{3.0.3}$$

for nonnegative functions v^0, u^0. In (3.0.1), P and Q represent the *self-diffusion* pressures, and R is the *cross-diffusion* pressure acting on the population u by v.

The system of form (3.0.1) is strongly coupled and of triangular form (the diffusion matrix is triangular). Such a system has recently received a lot of attention in both mathematical analysis and real life modelling. In particular, the well-known SKT model (1.0.4) (when $\alpha_{21} = 0$) in population dynamics is a special case of system (3.0.1), that is,

$$\begin{cases} u_t = \Delta[(d_1 + \alpha_{11}u + \alpha_{12}v)u] + u(a_1 - b_1u - c_1v), \\ v_t = \Delta[(d_2 + \alpha_{22}v)v] + v(a_2 - b_2u - c_2v). \end{cases} \tag{3.0.4}$$

Fundamental theory of strongly coupled systems like (3.0.1) was studied in [2]. The concept of $W^{1,p}$ weak solutions and their local existence and uniqueness results were formulated there. Roughly speaking, he showed that, for u^0, v^0 in $W^{1,p}(\Omega)$ for some $p > n$ (see [2]), there exist $\varepsilon > 0$ and a unique solution $u(t), v(t)$ in $W^{1,p}(\Omega)$ of (3.0.1) defined for $t \in (0, \varepsilon)$. In addition, one of the important issues, the global existence of solutions, was also discussed. It was pointed out that solutions to (3.0.1) exist globally in time if their L^∞ norms do not blow up.

In this chapter, we shall investigate the global existence of solutions for (3.0.1) in domains with arbitrary dimensional. Two different methods employed to obtain L^∞ estimates for weak solutions are L^p bootstrapping-semigroup techniques and Lyapunov functional approach. In addition, we highlight here that such L^∞ estimates are obtained ultimately uniformly. The ultimate uniformity of a priori estimates is one of main issues to investigate the long time dynamics of the solutions, which we shall study in later chapters.

Roughly speaking, first when differential operator $\mathcal{A}_v$ does not depend on u, we shall be able to employ L^p bootstrapping-semigroup techniques to establish the following.

> A solution (u, v) of (3.0.1) exists globally in time if $\|v(\cdot, t)\|_\infty$ and $\|u\|_{q,r,[t,t+1]\times\Omega}$ for certain numbers q, r (see (3.1.10)) do not blow up in finite time. Moreover, if these norms of the solutions are ultimately uniformly bounded then so are their Hölder norms. Therefore seen in later chapters, there is a compact global attractor, with finite Hausdörff dimension, attracting all solutions. In addition, if $\mathcal{A}_v$ is linear then the results are still proved when we replace $\|u\|_{q,r,[t,t+1]\times\Omega}$ by $\|u(\cdot, t)\|_1$.

The assumptions on the parameters defining (3.0.1) will be specified below in Section 3.1, where we consider arbitrary dimensional domains. As an application of the general results, we shall show in Section 3.2 that such assumptions are valid for the case when $n \leq 5$ and reactions are of competitive Lotka-Volterra type that is commonly hypothesized in mathematical biology contexts.

Nevertheless, the limitation of this method is to restrict the differential operator $\mathcal{A}_v$ on independence of u. Even though such settings are general enough to cover many interesting models investigated in literature (e.g., the SKT model (1.0.4) and chemotaxis systems), we are also interested in the case that $\mathcal{A}_v$ depends also on u, which basically means that the diffusion of the species (or the nutrient) v may be affected by pressures of the population (or the bacteria) u (see Chapter 1). Not surprisingly, as we shall see the L^p bootstrapping methods cannot apply to such a case. Indeed, a crucial ingredient in those techniques is an estimate of ∇v that will be used in the bootstrapping argument on the equation for u. Such an estimate, using standard results for scalar regular parabolic equations (see [22]) for the equation of v, is no longer available here. This is because of the presence of u, whose regularity is not yet known, in the diffusion term $Q(x,t,u,v)$ of the equation for v.

In order to deal with such situations, we employ the Lyapunov functional approach introduced in [29] to handle the *full* cross diffusion systems (see also Chapter 4). Roughly speaking, the method relies on the key assumptions (H.0)-(H.2) in Section 3.3 on the existence of a function $H(u,v)$, being defined along the solution (u,v), which links the structures of the two equations in a way that we can derive certain boundedness of $H(u,v)$, as a function in (x,t). Such boundedness of H will be exploited later to study that of u and v. In Section 3.5 we shall give explicit conditions on (3.0.1) that are sufficient to employ the general results.

3.1 L^p bootstrapping techniques

In this section, we will consider system (3.0.1) with the following conditions.

(P1) $P(u,v), R(u,v)$ are differentiable functions such that there exist a continuous function Φ and positive constants C, d such that

$$P(u,v) \geq d(1+u) > 0, \quad \forall u \geq 0, \tag{3.1.1}$$

$$|R(u,v)| \leq \Phi(v)u. \tag{3.1.2}$$

Moreover, the partial derivatives of P, R with respect to u, v can be majorized by some powers of u, v.

(P2) The operator $\mathcal{A}_v$ is regular linear elliptic in divergence form. That is, for some Hölder continuous functions $Q(x,t)$ and $c(x,t)$ with uniformly bounded norms

$$\mathcal{A}_v(u,v) = \nabla(Q(x,t)\nabla v) + c(x,t)v, \quad Q(x,t) \geq d > 0, \quad c(x,t) \leq 0. \tag{3.1.3}$$

We will impose the following assumption on the reaction terms.

(F) There exists a nonnegative continuous function $C(v)$ such that

$$|g(u,v)| \leq C(v)(1+u), \quad f(u,v)u^p \leq C(v)(1+u^{p+1}), \tag{3.1.4}$$

for all $u, v \geq 0$ and $p > 0$.

We will be interested only in nonnegative solutions, which are relevant in many applications. Therefore, we will assume that the solution u, v stay nonnegative if the initial data u^0, v^0 are nonnegative functions. Conditions on f, g that guarantee such positive invariance can be found in [18].

Essentially, we will establish certain a priori estimates for various spatial norms of the solutions. In order to simplify the statements of our theorems and proof, we will make use of the following terminology taken from [27].

Definition 3.1.1. Consider the initial-boundary problem (3.0.1),(3.0.2) and (3.0.3). Assume that there exists a solution (u,v) defined on a subinterval I of $\mathbb{R}_+$. Let $\mathcal{O}$ be the set of functions ω on I such that there exists a positive constant C_0, which may generally depend on the parameters of the system and the W^{1,p_0} norm of the initial value (u^0, v^0), such that

$$\omega(t) \leq C_0, \quad \forall t \in I. \tag{3.1.5}$$

Furthermore, if $I = (0,\infty)$, we say that ω is in $\mathcal{P}$ if $\omega \in \mathcal{O}$ and there exists a positive constant C_∞ that depends only on the parameters of the system but does not depend on the initial value of (u^0, v^0) such that

$$\limsup_{t\to\infty} \omega(t) \leq C_\infty. \tag{3.1.6}$$

If $\omega \in \mathcal{P}$ and $I = (0,\infty)$, we will say that ω is *ultimately uniformly bounded.*

If $\|u(\cdot,t)\|_\infty$, $\|v(\cdot,t)\|_\infty$, as functions in t, satisfy (3.1.5) the supremum norms of the solutions to (3.0.1) do not blow up in any finite time interval and are bounded by some constant that may depend on the initial conditions. This implies that the solution exists globally (see [2]). Moreover, if these norms verify (3.1.6), then they can be majorized eventually by a universal constant independent of the initial data. This property implies that there is an absorbing ball for the solution and therefore shows the existence of the global attractor if certain compactness is proven (see [11] and also Chapter 5).

Our first result is the following global existence result.

Theorem 3.1.2. *Assume (P1), (P2), and (F). Let* (u,v) *be a nonnegative solution to* (3.0.1) *with its maximal existence interval* I*. If* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_1$ *are in* $\mathcal{O}$ *then for any* $\alpha \in (0,1)$

$$\|v(\cdot,t)\|_{C^\alpha(\Omega)}, \quad \|u(\cdot,t)\|_{C^\alpha(\Omega)} \in \mathcal{O}. \tag{3.1.7}$$

If we have better bounds on the norms of the solutions then a stronger conclusion follows.

Theorem 3.1.3. *Assume (P1), (P2), and (F). Let* (u,v) *be a nonnegative solution to* (3.0.1) *with its maximal existence interval* I*. If* $\|v(\cdot,t)\|_\infty$ *and* $\|u(\cdot,t)\|_1$ *are in* $\mathcal{P}$ *then for any* $\alpha \in (0,1)$ *we have*

$$\|v(\cdot,t)\|_{C^\alpha(\Omega)}, \quad \|u(\cdot,t)\|_{C^\alpha(\Omega)} \in \mathcal{P}. \tag{3.1.8}$$

To include (3.0.4) in our study, we assume

(P2') $\mathcal{A}_v$ is a quasilinear operator given by

$$\mathcal{A}_v(u,v) = \nabla(Q(v)\nabla v) + c(x,t)v, \ Q(v) \geq d > 0, \tag{3.1.9}$$

for some differentiable function Q.

Additional a priori estimates will give the following statement.

Theorem 3.1.4. *Assume as in Theorem 3.1.2 (respectively, Theorem 3.1.3) but* $(P2)$ *is replaced by* $(P2')$*. The conclusions of Theorem 3.1.2 (respectively, Theorem 3.1.3) continue to hold if* $\|u\|_{q,r,[t,t+1]\times\Omega} =$

$\left(\int_t^{t+1}\|u(\cdot,s)\|_{q,\Omega}^r ds\right)^{1/r}$ *(as a function in t) is in $\mathcal{O}$ (respectively $\mathcal{P}$) for some q, r satisfying*

$$\frac{1}{r}+\frac{n}{2q}=1-\chi, \quad q\in\big[\frac{n}{2(1-\chi)},\infty\big], \quad r\in\big[\frac{1}{1-\chi},\infty\big] \tag{3.1.10}$$

for some $\chi\in(0,1)$.

We first consider Theorem 3.1.2 and Theorem 3.1.3. Their proofs will be based on several lemmas. Hereafter, we will use $\omega(t),\omega_1(t),\ldots$ to denote various continuous functions in $\mathcal{O}$ or $\mathcal{P}$. We first have the following fact on the component v and its spatial derivative.

In order to prove theorems, we recall some notations and the semigroup result. First, for any $t>\tau\geq 0$, we denote $Q_t=\Omega\times[0,t]$ and $Q_{\tau,t}=\Omega\times[\tau,t]$. For $r\in(1,\infty)$ and Q as one of the cylinders $Q_t, Q_{\tau,t}$, let $W_r^{2,1}(Q)$ be the Banach space of functions $u\in L^r(Q)$ having generalized derivatives $u_t, \partial_x u, \partial_{xx}u$ with finite $L^r(Q)$ norms (see [22, page 5]). For $s\geq 0$ and $r\in(1,\infty)$, we also make use of the fractional order Sobolev spaces $W_r^s(\Omega)$ (see, e.g., [1, 22] for the definition).

Let us consider the parabolic equation

$$\begin{cases} \dfrac{\partial v}{\partial t}=A(t)v+f_0(x,t), & x\in\Omega,\ t>0,\\ \dfrac{\partial v}{\partial n}(x,t)=0 & x\in\partial\Omega,\ t>0,\\ v(x,0)=v_0(x) & x\in\Omega, \end{cases} \tag{3.1.11}$$

where $A(t)$ is a uniformly regular elliptic operator of divergence form, with domain of definition $W_r^2(\Omega)$. If the coefficients of the operator $A(t)$ are uniformly Hölder continuous in a cylinder $Q_{\tau,t}$ and $(\lambda I+A(s))^{-1}$ exists for all $\lambda\geq 0$ and $s\in[\tau,t]$ then it is well known that (see, e.g., [8, Sections II.16-17]) there exists an evolution operator $U(t,s)$ for (3.1.11) such that the abstract integral version of (3.1.11) in L^r is

$$v(t)=U(t,\tau)v(\tau)+\int_\tau^t U(t,s)F(s)\,ds, \tag{3.1.12}$$

where $F(s)(x)=f_0(x,t)$. Moreover, for each $t>0$, $r>1$ and any $\beta\geq 0$, the fractional power $A^\beta(t)$, with its domain of definition $D(A_r^\beta(t))$ in $L^r(\Omega)$, of $A(t)$ is well defined ([8]). We recall the following imbeddings (see [13]).

$$D(A_r^\beta(t))\subset C^\mu(\Omega), \quad \text{for } 2\beta>\mu+n/r \tag{3.1.13}$$

and

$$D(A_r^\beta(t)) \subset W^{1,p}(\Omega), \quad \text{if } 2\beta \geq 1 - n/p + n/r. \tag{3.1.14}$$

Next, we collect some well known facts about (3.1.11).

Lemma 3.1.5. *Let $r \in (1, \infty)$. For any solution v of (3.1.11) we have*

i) For $t > \tau \geq 0$, assume that the coefficients of $A(t)$ are bounded and continuous and $f_0 \in L^r(Q_{\tau,t})$ for some $r > 3$. We have

$$\|v\|_{W_r^{2,1}(Q_{\tau,t})} \leq C(t-\tau)\left(\|f_0\|_{L^r(Q_{\tau,t})} + \|v(\cdot,\tau)\|_{W_r^{2-2/r}(\Omega)}\right) \tag{3.1.15}$$

where the constant $C(t-\tau)$ remains bounded if the length $t-\tau$ of the cylinder $Q_{\tau,t}$ is bounded and the coefficients of $A(t)$ are uniformly bounded in $Q_{\tau,t}$.

ii) Let $r > 1$ and $f(\cdot,t) \in L^r(\Omega)$. Assume that the coefficients of the operator $A(t)$ are Hölder continuous. Moreover, there exists $\delta_0 > 0$ such that $(\lambda I + A(t))^{-1}$ exists for all $\lambda \geq -\delta_0$ and all $t > 0$. For some fixed $t_0 > 0$ and any $\beta \in [0,1]$, we have

$$\|A^\beta(t_0)v(t)\|_r \leq C_\beta t^{-\beta} e^{-\delta t}\|v_0\|_r + C_\beta \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)}\|f_0(\cdot,s)\|_r ds \tag{3.1.16}$$

for some constants $\delta, C_\beta > 0$.

Proof: The proof of i) can be found in [22, Theorem 9.1, chapter IV] where Dirichlet boundary condition was considered but the result holds as well for Neumann boundary condition (see [22, page 351]). For ii), we apply $A^\gamma(t)$ to both sides of (3.1.12), take the L^r norm and then make use the inequality [8, (16.38)]. □

Going back to the solutions of (3.0.1) under the hypotheses of Theorem 3.1.2 and Theorem 3.1.3, we first have the following estimates for the component v and its spatial derivative.

Lemma 3.1.6. *There exist nonnegative functions ω_0, ω defined on the maximal interval of existence of v such that $\omega_0 \in \mathcal{P}$. For some $\delta > 0$, $r > 1$, $\beta \in (0,1)$ such that*

a. *if* $2\beta > \mu + n/r$, *we have*

$$\|v(\cdot,t)\|_{C^\mu(\Omega)} \le \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\cdot,s)\|_r ds. \tag{3.1.17}$$

b. *if* $2\beta > 1 - n/q + n/r$, *we have*

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le \omega_0(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) \|u(\cdot,s)\|_r ds. \tag{3.1.18}$$

Moreover, ω *belongs to* $\mathcal{O}$*, respectively* $\mathcal{P}$*, if* $\|v(\cdot,t)\|_\infty$ *does.*

Proof: Setting $A(t) = \nabla \cdot (Q(x,t)\nabla v + c(x,t)v) - kv$ and $\hat{f}_0(x,t) = g(u,v) + kv$ for $k > 0$ sufficiently large, we see that v satisfies (3.1.11). Since v satisfies a parabolic equation with Hölder continuous coefficients (due to (P.2)), we find that the conditions in ii) of Lemma 3.1.5 are verified. Since $\|v(\cdot,t)\|_\infty \in \mathcal{P}$, we have $\|\hat{f}_0\|_r \le \omega(t)(1+\|u(\cdot,s)\|_r)$, for some function $\omega(t) \in \mathcal{P}$. Hence, (3.1.16) of Lemma 3.1.5 gives

$$\|A_0^\beta v(t)\|_r \le C_\beta t^{-\beta} e^{-\delta t} \|v_0\|_r + C_\beta \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s)(1 + \|u(\cdot,s)\|_r) ds$$

for any fixed $t_0 > 0$. From the imbedding inequalities (3.1.13) (respectively, (3.1.14)), (3.1.17) (respectively, (3.1.18)) follows at once. □

Our starting point is the following integro-differential inequality for the L^p norm of u.

Lemma 3.1.7. *Given the conditions of Theorem 3.1.2 (respectively Theorem 3.1.3). For any* $p > \max\{n/2, 1\}$*, we set* $y(t) = \int_\Omega u^p\, dx$*. We can find* $\beta \in (0,1)$ *and positive constants* A, B, C*, and functions* $\omega_i \in \mathcal{O}$ *(respectively,* $\mathcal{P}$*) such that the following inequality holds*

$$\begin{aligned} \frac{d}{dt} y &\le -Ay^\eta + (\omega_0(t) + \|u(\cdot,t)\|_1)y + B\omega(t) \\ &\quad + Cy^\theta \Big\{\omega_1(t) + \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega_2(s) \|u(\cdot,s)\|_1^\zeta y^\vartheta(s) ds\Big\}^2. \end{aligned} \tag{3.1.19}$$

Here, $\eta = \frac{p+1}{p}$, $\theta = \frac{p-1}{p}$ *and* $\vartheta = \frac{(r-1)}{r(p-1)}$, $\zeta = \frac{(p-r)}{r(p-1)}$ *for some* $r \in (1,p)$*. Moreover,* $\eta > \theta + 2\vartheta$*.*

Proof: We assume the conditions of Theorem 3.1.3 as the proof for the other case is identical. We multiply the equation for u by u^{p-1} and integrate over Ω. Using integration by parts and noting that the boundary integrals are all nonnegative thanks to the boundary condition on u, we see that

$$\begin{aligned}\int_\Omega u^{p-1}\frac{d}{dt}u\,dx &+ \int_\Omega P(u,v)\nabla u\nabla(u^{p-1})\,dx \\ &\le \int_\Omega (-R(u,v)\nabla(u^{p-1})\nabla v + g(u,v)u^{p-1})\,dx.\end{aligned}$$

Using the conditions (3.1.1) and (3.1.2), for some positive constants $C(d,p)$, $\epsilon, C(\epsilon,d,p)$ we derive

$$\begin{aligned}\int_\Omega P(u,v)\nabla u\nabla(u^{p-1})\,dx &\ge C(d,p)\int_\Omega u^{p-1}|\nabla u|^2\,dx,\\ -\int_\Omega R(u,v)\nabla(u^{p-1})\nabla v\,dx &\le C(d,p)\int_\Omega u^{p-1}\Phi(v)\nabla u\nabla v\,dx\\ &\le \epsilon\int_\Omega u^{p-1}|\nabla u|^2\,dx + C(\epsilon,d,p)\int_\Omega u^{p-1}\Phi^2(v)|\nabla v|^2\,dx.\end{aligned}$$

From this inequality and (3.1.4), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p\,dx &+ C(d,p)\int_\Omega u^{p-1}|\nabla u|^2\,dx\\ &\le C(\epsilon,d,p)\int_\omega (u^{p-1}\Phi^2(v)|\nabla v|^2 + C(v)(u^p+1)\,dx.\end{aligned}\tag{3.1.20}$$

Furthermore, the second term on the left-hand side can be estimated as

$$\begin{aligned}\int_\Omega u^{p-1}|\nabla u|^2\,dx &= C(p)\int_\Omega |\nabla(u^{(p+1)/2})|^2\,dx\\ &\ge C\int_\Omega u^{p+1}\,dx - C\Big(\int_\Omega u^{(p+1)/2}\,dx\Big)^2\\ &\ge C\Big(\int_\Omega u^p\,dx\Big)^{\frac{p+1}{p}} - C\|u\|_1\int_\Omega u^p\,dx.\end{aligned}$$

Here, we have used the Hölder's inequality $\left(\int_\Omega u^{(p+1)/2}\ dx\right)^2 \le \|u\|_1\int_\Omega u^p\ dx.$

Next, we consider the first integral on the right of (3.1.20). By our assumption on L^∞ norm of v, $\Phi(v) \le \omega_1(t)$ for some $\omega_1 \in \mathcal{P}$. Using the Hölder inequality, we have

$$\begin{aligned}\int_\Omega u^{p-1}\Phi^2(v)|\nabla v|^2\,dx &\le \omega_1(t)\Big(\int_\Omega u^p\,dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |\nabla v|^{2p}\,dx\Big)^{1/p}\\ &= \omega_1(t)y^{\frac{p-1}{p}}\|\nabla v\|_{2p}^2.\end{aligned}$$

Since $p > \max\{n/2, 1\}$, there exists $r \in (1, p)$ such that

$$\frac{1}{n}+\frac{1}{2p} > \frac{1}{r} > \frac{1}{p}.$$

This implies $2 > 1 - n/2p + n/r$. Hence, we can find $\beta \in (0, 1)$ such that $2\beta > 1 - n/2p + n/r$. From (3.1.18), with $q = 2p > r$, we have

$$\|\nabla v\|_{2p} \le \omega_0(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r ds.$$

Applying the above estimates in (3.1.20), we derive the following inequality for $y(t)$

$$\begin{aligned}\frac{d}{dt}y + C(d,p)y^{\frac{p+1}{p}} &\le Cy^{\frac{p-1}{p}}\omega_1(t)\Big\{\omega_0(t) + \int_0^t (t-s)^{-\beta}e^{-\delta(t-s)}\omega(s)\|u(\cdot,s)\|_r ds\Big\}^2\\ &\quad + C(\omega_2(t) + \|u\|_1)y + B\omega_2(t). \qquad (3.1.21)\end{aligned}$$

Since $1 < r < p$, we can use Hölder's inequality

$$\|u\|_r \le \|u\|_1^{1-\lambda}\|u\|_p^\lambda = \|u\|_1^{1-\lambda}y^{\frac{\lambda}{p}}$$

with $\lambda = \frac{1-1/r}{1-1/p} = \frac{p(r-1)}{r(p-1)}$. Applying this in (3.1.21) and re-indexing the functions ω_i, we prove (3.1.19). The last assertion of the lemma follows from the following equivalent inequalities

$$\eta > \theta + 2\vartheta \Leftrightarrow \frac{p+1}{p} > \frac{p-1}{p} + \frac{2(r-1)}{r(p-1)} \Leftrightarrow \frac{1}{p} > \frac{(r-1)}{r(p-1)} \Leftrightarrow rp - r > pr - p \Leftrightarrow p > r.$$

This completes the proof. $\square$

Next, we will show that the L^p norm of u is in the class $\mathcal{O}$ or $\mathcal{P}$ for any $p \ge 1$.

Lemma 3.1.8. *Given the conditions of Theorem 3.1.2 (respectively Theorem 3.1.3), for any finite $p \geq 1$, there exists a function $\omega_p \in \mathcal{O}$ (respectively $\mathcal{P}$) such that*

$$\|u(\cdot, t)\|_p \leq \omega_p(t). \tag{3.1.22}$$

To prove this, we apply the following facts from [27] to the differential inequality (3.1.19).

For a function $y : \mathbb{R}^+ \to \mathbb{R}$, let us consider the inequality

$$y'(t) \leq \mathcal{F}(t, y), \quad y(0) = y_0, \quad t \in (0, \infty), \tag{3.1.23}$$

where $\mathcal{F}$ is a functional from $\mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$ into $\mathbb{R}$. The following lemma is standard and gives a global estimate for y but the estimate is still dependent on the initial data. Consider the assumptions:

F.1 Suppose that there is a function $F(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathcal{F}(t, y) \leq F(y(t), Y)$ if $y(s) \leq Y$ for all $s \in [0, t]$.

F.2 There exists a real M such that $F(Y, Y) < 0$ if $Y \geq M$.

Lemma 3.1.9. *[27, Lemma 2.17] Assume* (3.1.23), F.1, *and* F.2. *Then there exists finite M_0 such that $y(t) \leq M_0$ for all $t \geq 0$.*

The proof of this lemma is elementary, and therefore will be omitted.

Remark 3.1.10. In (F.1), the inequality $\mathcal{F}(t, y) \leq F(y(t), Y)$ is not pointwise. It requires that $y(s) \leq Y$ on the interval $s \in [0, t]$ not just that $y(t) \leq Y$. Such situation usually happens when $f(t, y)$ contains integrals of $y(t)$ over $[0, t]$.

The above constant M_0 still depends on the initial data y_0. Moreover, the function F may depend on y_0 too. Next, we consider conditions which guarantee uniform estimates for $y(t)$.

Consider the following assumptions:

(G.1) There exists a continuous function $G(y, Y) : \mathbb{R}^2 \to \mathbb{R}$ such that for τ sufficiently large, if $t > \tau$ and $y(s) \leq Y$ for every $s \in [\tau, t]$ then there exists $\tau' \geq \tau$ such that

$$\mathcal{F}(t, y) \leq G(y(t), Y) \quad \text{if } t \geq \tau' \geq \tau. \tag{3.1.24}$$

(G.2) The set $\{z : G(z, z) = 0\}$ is not empty and $z_* = \sup\{z : G(z, z) = 0\} < \infty$. Moreover, $G(M, M) < 0$ for all $M > z_*$.

(G.3) For $y, Y \geq z_*$, $G(y, Y)$ is increasing in Y and decreasing in y.

Proposition 3.1.11. *[27, Prop 2.18] Assume* (3.1.23), (G.1), (G.2)*, and* (G.3)*. If*

$$\limsup_{t\to\infty} y(t) < \infty$$

then

$$\limsup_{t\to\infty} y(t) \leq z_*.$$

Remark 3.1.12. Examples of functions F, G satisfying the conditions of the above two lemmas include

$$F(y(t), Y),\ G(y(t), Y) = -Ay^{\eta}(t) + D(y^{\gamma} + 1) + y^{\theta}(B + CY^{\vartheta})^{k}, \tag{3.1.25}$$

with positive constants $A, B, C, D, \eta, \theta, \vartheta, k$ satisfies $\eta > \theta + k\vartheta$ and $\eta > \gamma$.

Proof: [Proof of Lemma 3.1.8] Assume first the conditions of Theorem 3.1.2. From (3.1.19), we deduce the following integro-differential inequality

$$\frac{d}{dt}y \leq -Ay^{\eta} + \omega_1(t)y + B\omega_2(t) + Cy^{\theta}\{\omega_0(t) + K(t)\}^2, \tag{3.1.26}$$

where

$$K(t) := \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^{\vartheta}(s) ds$$

for some $\omega_0, \omega_1, \omega \in \mathcal{O}$ (because $\|u(\cdot, t)\|_1 \in \mathcal{O}$). We will show that Lemma 3.1.9 can be used here to assert that $y(t)$ is bounded in any finite interval. This means $\|u\|_p \in \mathcal{O}$. We define the functional

$$\mathcal{F}(t, y) = -Ay^{\eta} + \omega_1(t)y + B + Cy^{\theta}\{\omega_0(t) + K(t)\}^2. \tag{3.1.27}$$

Since $\omega_i \in \mathcal{O}$, we can find a positive constant C_ω, which may still depend on the initial data, such that $\omega_i(t) \leq C_\omega$ for all $t > 0$. Let

$$C_1 := \sup_{t>0} \int_0^t (t-s)^{-\beta} e^{-\delta(t-s)} ds \leq \int_0^\infty s^{-\beta} e^{-\delta s} ds < \infty,$$

because $\beta \in (0,1)$ and $\delta > 0$. We then set

$$F(y, Y) = -Ay^\eta + C_\omega(y + B) + Cy^\theta(C_\omega + C_\omega C_1 Y^\vartheta)^2.$$

Because $\eta > \theta + 2\vartheta$, by Lemma 3.1.7, and Remark 3.1.12, the functionals $\mathcal{F}, F$ satisfy the conditions (F.1),(F.2). Hence, Lemma 3.1.9 applies and gives

$$y(t) \leq C_0(v^0, u^0), \quad \forall t > 0. \tag{3.1.28}$$

For some constant $C_0(v^0, u^0)$ which may still depend on the initial data since F does. We have shown that $y(t) \in \mathcal{O}$.

We now seek for uniform estimates and assume the conditions of Theorem 3.1.3. From Lemma 3.1.7 we again obtain (3.1.26) with ω_i are now in $\mathcal{P}$. If a function ω belong to $\mathcal{P}$, by Definition 3.1.1, we can find $\tau_1 > 0$ such that $\omega(s) \leq \bar{C}_\infty = C_\infty + 1$ if $s > \tau_1$. We emphasize the fact that $\bar{C}_\infty$ is independent of the initial data. Let $t > \tau \geq \tau_1$ and assume that $y(s) \leq Y$ for all $s \in [\tau, t]$. Let us write

$$K(t) = \int_0^\tau (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s) ds + \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \omega(s) y^\vartheta(s) ds = J_1 + J_2.$$

By (3.1.28), there exists some constant $C(v^0, u^0)$ such that $\omega(s) y^\vartheta(s) \leq C(v^0, u^0)$ for every s. Hence, we can find $\tau' > \tau$ such that $J_1 \leq 1$ if $t > \tau'$. Thus,

$$K(t) \leq 1 + \bar{C}_\infty C_* Y^\vartheta, \quad \text{where} \quad C_* = \sup_{t>\tau, \tau>0} \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} ds < \infty.$$

Therefore, for $t > \tau'$ we have $f(t, y) \le G(y(t), Y)$ with

$$G(y(t), Y) = -Ay^{\eta}(t) + \bar{C}_{\infty}(y + B) + y^{\theta}(\bar{C}_{\infty} + 1 + \bar{C}_{\infty}C_* Y^{\vartheta})^2. \tag{3.1.29}$$

We see that G is independent of the initial data and satisfies (G1)-(G3) as $\eta > \theta + 2\vartheta$ (see Remark 3.1.12). Therefore, Proposition 3.1.11 applies here to complete the proof. □

We conclude this section by giving the following proofs.

Proof: [Proofs of Theorems 3.1.2 and 3.1.3] We first apply i) of Lemma 3.1.5 to the equation for v in (3.0.1). Since $\|u(\cdot, t)\|_p \in \mathcal{P}$ for any p large, we see that $f(u, v) \in L^q(Q_{\tau,t})$ for any $q > 1$. In fact, with $\tau = t - 1$, $\|f(u, v)\|_{L^q(Q_{\tau,t})}$, as a function in t, is in the class $\mathcal{P}$. Hence,

$$\|v\|_{W_q^{2,1}(Q_{\tau,t})} \le C\left(\|f(u, v)\|_{L^q(Q_{\tau,t})} + \|v(\cdot, \tau)\|_{W_q^{2-2/q}(\Omega)}\right). \tag{3.1.30}$$

Choosing $\beta \in (0, 1)$ (close to 1) and r sufficiently large such that $2\beta > 2 - 1/q + n/r$, Lemma 3.1.6 states that the norm of $v(\cdot, t)$ in $C^{2-1/q}(\Omega)$, and therefore $W_q^{2-2/q}(\Omega)$, is in the class $\mathcal{P}$ for any $q > 1$. We then conclude that $\|v\|_{W_q^{2,1}(Q_{\tau,t})} \in \mathcal{P}$ for any $q > 1$. So,

$$\int_{t-1}^{t} \int_{\Omega} \left(|\frac{\partial v}{\partial t}(x, s)|^q + |\Delta v(x, s)|^q \right) dx\, ds \le \omega(t), \quad \forall t \in I \tag{3.1.31}$$

for some $\omega \in \mathcal{P}$. We now write the equation for u as follows

$$\frac{\partial u}{\partial t} = \operatorname{div}(A(x, t)\nabla u) + B(x, t)\nabla u + \hat{F}(x, t),$$

where $A(x, t) = P(u, v)$, $B = R_u \nabla v$ and $\hat{F}(x, t) = g(u, v)R(u, v)\Delta v + R_v|\nabla v|^2$. Using (3.1.31), we easily see that $b(x, t)$ and $\hat{F}(x, t)$ belong to $L^{q,q}$ for any q large. Standard regularity theories for quasilinear parabolic equations (see [26]) can be applied here to conclude that $u(x, t)$ is in class $C^{\alpha,\alpha/2}$ for some $\alpha > 0$. □

Proof: [Proof of Theorem 3.1.4] The proof is exactly the same as that of Theorem 3.1.3 if we can regard $\mathcal{A}_v$ as a *linear* regular elliptic operator with Hölder continuous coefficients (whose norms are also ultimately uniformly bounded) so that Lemma 3.1.6 is applicable. To this end, we need only to show that $Q(v(x,t))$, as a function in (x,t), is Hölder continuous. Since we assume that $\|v(\cdot,t)\|_\infty \in \mathcal{P}$ and (3.1.4) holds, the assumption of the theorem implies that $\|g(u,v)\|_{q,r,[t,t+1]\times\Omega} \in \mathcal{P}$. The range of q, r in (3.1.10) and well known regularity theory for quasilinear parabolic equations (see [22, Chap.5, Theorem 1.1] or [26]) assert that there is $\alpha > 0$ such that $v \in C^{\alpha,\alpha/2}(\Omega \times (0,\infty))$ with uniformly bounded norm. So is $Q(v(x,t))$. In fact, by [9], we also have that $\nabla v \in C^{\alpha,\alpha/2}(\Omega\times(0,\infty))$.

□

3.2 The competitive Lotka-Volterra reaction terms

In this section we show that the hypotheses of Theorems 3.1.3 and 3.1.4 are verified for (3.0.1) if the reaction terms are of Lotka-Volterra type used in (3.0.4), that is,

$$f(u,v) = u(a_1 - b_1 u - c_1 v), \quad g(u,v) = v(a_2 - b_2 u - c_2 v), \tag{3.2.1}$$

where a_i, b_i, c_i's are given constants. The main result of this section is the following.

Theorem 3.2.1. *Assume that $\mathcal{A}_u$ satisfies $(P1)$, $\mathcal{A}_v$ is of the form* (3.1.9)*, $n \le 5$, and that $b_1, b_2, c_2 > 0$. The assumption on the dimension will be omitted if $\mathcal{A}_v$ is a linear operator of the form in (3.1.3).*

For any given $p_0 > n$ and any given nonnegative initial data u^0, v^0 in

$$X = \{(u,v) \in W^{1,p_0}(\Omega) \times W^{1,p_0}(\Omega) : u(x), v(x) \ge 0, \quad \forall x \in \Omega\}.$$

Then weak solutions (u,v) to (3.0.1) with (3.2.1) *are classical and exist globally. Furthermore, for any $\alpha \in (0,1)$ we have*

$$\|v(\cdot,t)\|_{C^\alpha(\Omega)}, \quad \|u(\cdot,t)\|_{C^\alpha(\Omega)} \in \mathcal{P}. \tag{3.2.2}$$

For given nonnegative initial data $u^0, v^0 \in X$, it is standard to show that the solution stays nonnegative (see [18]). Clearly, the functions f, g satisfy the condition (F). Thus, the above theorem is a consequence of Theorems 3.1.3 and 3.1.4 if we can show that the norms $\|v(\bullet,t)\|_\infty$ and $\|u\|_{q,r,[t,t+1]\times\Omega} = \left(\int_t^{t+1} \|u(\cdot,s)\|_{q,\Omega}^r ds\right)^{1/r}$ (respectively, $\|u(\bullet,t)\|_1$) belong to $\mathcal{P}$ for some q, r satisfying $1/r + n/2q \in (0,1)$. These will be done in several steps.

First of all, since $b_2, c_2 > 0$, using invariant principle for scalar parabolic equation or test the equation of v by $(v-k)_+$ for some k large we easily derive

Lemma 3.2.2. $\|v(\cdot,t)\|_\infty \in \mathcal{P}$.

The followings are devoted to obtain estimates of $\|u\|_{q,r}$.

Lemma 3.2.3. *For the component u we have*

$$\|u(\cdot,t)\|_1 \in \mathcal{P}, \tag{3.2.3}$$

$$\int_t^{t+1}\int_\Omega u^2 dx \in \mathcal{P}. \tag{3.2.4}$$

Proof: Integrating the equation for u over Ω. Using the Robin boundary condition and the fact that $u, v \geq 0$ we can drop the boundary integrals result in the integration by parts to obtain

$$\frac{d}{dt}\int_\Omega u dx \leq \int_\Omega f(u,v)\, dx \leq c\int_\Omega u\, dx - b_1\int_\Omega u^2\, dx \tag{3.2.5}$$

this implies (here $c = a_1 + |c_1|\|v\|_\infty$)

$$\frac{d}{dt}\int_\Omega u\, dx \leq c\int_\Omega u\, dx - b_1(\int_\Omega u\, dx)^2 \tag{3.2.6}$$

It is easy to see that (3.2.6) gives (3.2.3) (see also Proposition 3.1.11). Integrating (3.2.5) from t to $t+1$ and using (3.2.3), we get (3.2.4). □

We need to estimate the norms of ∇v and v_t.

Lemma 3.2.4. *We assert that*

$$\|\nabla v(\cdot,t)\|_2 \in \mathcal{P}, \tag{3.2.7}$$

$$\int_t^{t+1}\int_\Omega v_t^2(x,s) dx\, ds \in \mathcal{P}. \tag{3.2.8}$$

Proof: First of all, using the boundary condition for v, we notice that

$$\begin{aligned}
\int_\Omega \nabla(Q\nabla v)Qv_t\, dx &= -\int_\Omega Q\nabla v(Q_v\nabla v v_t + Q\nabla(v_t))dx + \int_{\partial\Omega} Q\frac{\partial v}{\partial n}Qv_t\, d\sigma \\
&= -\frac{1}{2}\int_\Omega \frac{d}{dt}(Q^2|\nabla v|^2)dx - \int_{\partial\Omega_1} r(x)Qvv_t\, d\sigma
\end{aligned}$$

where $\partial\Omega_1$ is a subset of $\partial\Omega$ on which the Robin or Neumann conditions are given, that is, $\chi = 1$ in (3.0.2). Therefore, by multiplying the equation for v by Qv_t, we get

$$\int_\Omega Qv_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega Q^2|\nabla v|^2 dx \le \int_\Omega f(u,v)Qv_t\, dx - \frac{d}{dt}\int_{\partial\Omega_1} r(x)\hat{Q}(v)\, d\sigma,$$

where $\hat{Q}(v) = \int_0^v Q(s)s\; ds$. The above then gives

$$\int_\Omega Qv_t^2 dx + \frac{d}{dt}\int_\Omega Q^2|\nabla v|^2 dx \le \int_\Omega f^2(u,v)Q\, dx - \frac{d}{dt}\int_{\partial\Omega_1} r(x)\hat{Q}(v)\, d\sigma. \tag{3.2.9}$$

On the other hand, let $\bar{Q}(v) = \int_0^v Q(s)ds$ and multiply the equation for v by $\bar{Q}(v)$ to obtain

$$\int_\Omega \bar{Q}v_t dx = -\int_\Omega Q^2|\nabla v|^2 dx - \int_{\partial\Omega_1} rv\bar{Q}\, d\sigma + \int_\Omega f(u,v)\bar{Q}(v)\, dx.$$

But

$$\int_\Omega Qv_t^2 dx \ge -2\int_\Omega \bar{Q}v_t\, dx - \int_\Omega \frac{\bar{Q}^2}{Q}\, dx$$

by Young inequality. We now set

$$y(t) = \int_\Omega Q^2|\nabla v|^2 dx + \int_{\partial\Omega_1} r(x)\hat{Q}(v)\, d\sigma$$

and add $2\int_{\partial\Omega_1} r\hat{Q}\, d\sigma$ to both sides of (3.2.9). Using the above inequalities, we easily obtain

$$y'(t) + 2y(t) \le \int_\Omega [f^2Q + \frac{\bar{Q}^2}{Q} + 2f\bar{Q}]dx - 2\int_{\partial\Omega_1} rv\bar{Q}\, d\sigma + 2\int_{\partial\Omega_1} r\hat{Q}\, d\sigma.$$

From the assumption $f(u,v) \le C(v)(1+u)$ and (3.2.4) we see that the above implies $y(t) \in \mathcal{P}$. But v, and therefore $\int_{\partial\Omega_1} r\hat{Q}\, d\sigma$ and $\int_{\partial\Omega_1} rv\bar{Q}\, d\sigma$, belongs to $\mathcal{P}$. We conclude that $\int_\Omega Q^2|\nabla v|^2 dx \in \mathcal{P}$. This and (3.1.3) give (3.2.7).

Finally, we can integrate (3.2.9) and use (3.2.7), (3.1.3) to obtain (3.2.8). □

For $n = 3$, we note that the assumptions of Theorem 3.1.4 immediately follow from this lemma if we take $q = 2 > n/2$ and $r = \infty$ in (3.1.10). However, we will present a unified proof for all $n \le 5$ below. We will also employ the variance of the Gronwall inequality (see Lemma 2.2.1)

Lemma 3.2.5. *For any $q \le 2^* = 2n/(n-2)$, we have*

$$\int_t^{t+1} \|\nabla v(\cdot, s)\|_q^2 ds \in \mathcal{P}. \tag{3.2.10}$$

Proof: By standard Sobolev embedding theorem [1, Theorem 5.4], we have

$$\|\nabla v\|_{2^*}^2 \le \frac{1}{d^2}\Big(\int_\Omega |Q\nabla v|^{2^*}\, dx\Big)^{2/2^*} \le C\int_\Omega (|Q\nabla v|^2 + |\nabla(Q\nabla v)|^2)\, dx \tag{3.2.11}$$

From the equation for v and the condition on f, we have

$$|\nabla(Q\nabla v)|^2 \le |f(u,v)|^2 + |v_t|^2 \le \omega(t)(u^2+1) + |v_t|^2.$$

This and (3.2.11) imply

$$\|\nabla v\|_{2^*}^2 \le C\omega_1(t)\int_\Omega (|\nabla v|^2 + |u|^2 + |v_t|^2)\, dx.$$

We then integrate the above inequality over $[t, t+1]$ and make use of Lemma 3.2.3 to get (3.2.10) for $q = 2^*$. Finally, if $q < 2^*$, we have $\|\nabla v\|_q \le C\|\nabla v\|_{2^*}$ (due to Hölder's inequality and the fact that Ω is bounded) for some constant C and complete the proof. □

Multiplying the equation for u by u^{2p-1} ($p > 1/2$) and using the boundary condition, we derive

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^{2p}\, dx \quad &+ \quad \frac{2p-1}{p}\int_\Omega P|\nabla u^p|^2\, dx \\ &\le \quad C(p)\int_\Omega |R\nabla u^{2p-1}\nabla v|\, dx + \omega(t)(\int_\Omega (u^{2p}+1)\, dx.\end{aligned} \tag{3.2.12}$$

Using the conditions on P, R and Young's inequality, we have

$$\int_\Omega P|\nabla u^p|^2\, dx \ge d(\int_\Omega u|\nabla u^p|\, dx + \int_\Omega |\nabla u^p|\, dx),$$

$$\begin{aligned}\int_\Omega |R\nabla u^{2p-1}\nabla v|\,dx &\leq \omega(t)\int_\Omega |u^p\nabla u^p\nabla v|\,dx\\ &\leq \epsilon\int_\Omega u|\nabla u^p|^2\,dx + C(\epsilon)\omega(t)\int_\Omega u^{2p-1}|\nabla v|^2\,dx.\end{aligned}$$

for any $\epsilon > 0$. Moreover,

$$\int_\Omega u^{2p-1}|\nabla v|^2\,dx \leq \Big(\int_\Omega u^{2p}\,dx\Big)^{1-1/2p}\|\nabla v\|_{4p}^2 \leq \Big(\int_\Omega u^{2p}\,dx + 1\Big)\|\nabla v\|_{4p}^2$$

By choosing appropriately small ϵ, we derive from (3.2.12) and the above inequalities the following key inequality

$$\frac{d}{dt}y(t) + C_p\int_\omega (1+u)|\nabla u^p|^2\,dx \leq g(t)y(t) + h(t), \tag{3.2.13}$$

where $y(t) = \int_\Omega u^{2p}\,dx$, $g(t) = \|\nabla v\|_{4p}^2 + \omega(t) + C(p)$, $h(t) = \omega(t) + C(p)$ for some $\omega \in \mathcal{P}$ and $C_p, C(p) > 0$.

We then have the following lemma.

Lemma 3.2.6. *For $\lambda = \min\{n/(n-2), 2\}$, we have $\|u(\cdot,t)\|_\lambda \in \mathcal{P}$.*

Proof: We choose p in (3.2.13) such that $2p = \lambda$. Firstly, $h(t)$ in (3.2.13) satisfies (2.2.2). On the other hand, as $4p = 2\lambda \leq 2^*$ we see that $\|\nabla v(\cdot,t)\|_{4p}^2 \in \mathcal{P}$ by Lemma 3.2.5. Thus, $g(t)$ in (3.2.13) also verifies (2.2.2). Thanks to (3.2.4) and because $\lambda \leq 2$, we see that $y(t) = \int_\Omega u^\lambda\,dx$ verifies the assumption of Lemma 2.2.1. This gives our lemma. □

We conclude this section with the following proof.

Proof: [Proof of Theorem 3.2.1] Thanks to Lemma 3.2.3, we need only to verify the last assumption on $\|u\|_{q,r}$ of the theorem. Let $p = \lambda/2$ and $l = \frac{\lambda+1}{2}$ in (3.2.13), and $U = u^l$. We integrate (3.2.13) over $[t, t+1]$ and use the above lemma to get

$$\int_t^{t+1}\int_\Omega \|\nabla U|^2\,dx\,ds = \big(1+\frac{1}{2p}\big)^2\int_t^{t+1}\int_\Omega u|\nabla u^p|^2\,dx\,ds \in \mathcal{P}. \tag{3.2.14}$$

The function $W = U - \int_\Omega U\,dx$ has zero average and we can use the Gagliardo-Nirenberg inequality to get

$$\|W\|_{2^*,\Omega} \leq C\|\nabla W\|_{2,\Omega} \Rightarrow \|U\|_{2^*,\Omega} \leq C(\|\nabla U\|_{2,\Omega} + \|U\|_{1,\Omega}).$$

For $r = 2l$, $q = l2^*$, we derive

$$\int_t^{t+1} \|u\|_{q,\Omega}^r \, ds = \int_t^{t+1} \|U\|_{2^*,\Omega}^2 \, ds \le C\Big(\int_t^{t+1} \|\nabla U\|_{2,\Omega}^2 \, ds + \sup_{[t,t+1]} \|U\|_{1,\Omega}^2 \Big).$$

As $l \le \lambda$, $\|U(\cdot,t)\|_{1,\Omega} = \|u(\cdot,t)\|_{l,\Omega}^l \in \mathcal{P}$ (see Lemma 3.2.6). Thus, (3.2.14) and the above show that $\|u\|_{q,r,[t,t+1]\times\Omega} \in \mathcal{P}$, with r, q satisfying

$$1 - \chi := \frac{1}{r} + \frac{n}{2q} = \frac{1}{l}(\frac{1}{2} + \frac{n}{22^*}) = \frac{n}{4l}$$

Set $A := q - \frac{n}{2(1-\chi)} = q - 2l$, $B := r - \frac{1}{1-\chi} = 2l - \frac{4l}{n}$. To see that q, r satisfy the condition (3.1.10) of Theorem 3.1.4, we show that $\chi \in (0, 1)$ and $A, B \ge 0$. Computing the values of χ, A, B for $n = 3, 4, 5$ gives:

$$n = 3: \quad \chi = 1/2, \quad A = 6, \quad B = 1.$$

$$n = 4: \quad \chi = 1/3, \quad A = 3, \quad B = 3/2.$$

$$n = 5: \quad \chi = 1/16, \quad A = 16/9, \quad B = 8/5.$$

The assumptions of Theorem 3.1.4 are fulfilled and our proof is complete (we should also remark that $\chi = -1/5 < 0$ if $n = 6$). □

3.3 Lyapunov functional method

In the rest of this chapter, we will employ the Lyapunov functional method, which is introduced in [29] to deal with full regular system (see also next chapter), to establish a priori estimates of solutions to system (3.0.1), that is,

$$\begin{cases} u_t = \mathcal{A}_u(u, v) + F(u, v), & x \in \Omega, t > 0, \\ v_t = \mathcal{A}_v(u, v) + G(u, v), & x \in \Omega, t > 0. \end{cases} \tag{3.3.1}$$

Here we assume that

(P) $\mathcal{A}_u = \nabla(P(u, v)\nabla u + R(u, v)\nabla v), \mathcal{A}_v = \nabla(Q(u, v)\nabla v)$ in which functions P, Q, R are differentiable functions in (u, v). Moreover, P, Q are positive for nonnegative u, v, and u_0, v_0 are nonnegative on Ω.

(F) Reaction terms F, G are continuous functions in (u, v) such that $F(0, v) = G(u, 0) = 0$ and

$$F(u, v) \text{ and } G(u, v) \text{ are negative if either } u \text{ or } v \text{ is sufficiently large.} \tag{3.3.2}$$

Firstly, for the sake of simplicity, we consider here systems with homogeneous Neumann boundary conditions, and leave the mixed boundary case of the form (3.0.2) to Remark 3.3.6. The nonnegativity of the solutions is easy to establish. Indeed, by testing the equations of u, v respectively by their negative parts u_-, v_- and using elementary differential inequalities, one can prove that u_-, v_- are zero for all t (see also [28]). This shows that u, v stay nonnegative for all $t \in (0, \infty)$.

By multiplying the equation of v in (3.3.1) with $(v - K_v)_+$ and using the assumption on G we easily prove the following.

Lemma 3.3.1. *Assume (P) and (F). Then there exists a constant $K_v > 0$, which may depend on the initial data v_0, such that $v(x, t) \leq K_v$ for all $(x, t) \in \Omega_T$.*

Under a stronger assumption on G, we shall obtain uniformity of the bound with respect to the initial data. However, for convenience, we impose here an assumption on F as well, that is,

(F') Reaction terms F, G are continuous functions in (u, v) and there exist positive constants $\alpha, M_0 > 0$ such that $F(0, v) = G(u, 0) = 0$ and

$$F(u, v) \leq -\alpha u, \quad G(u, v) \leq -\alpha v, \quad \text{ if either } u \geq M_0 \text{ or } v \geq M_0. \tag{3.3.3}$$

We shall prove the following.

Lemma 3.3.2. *Assume (P) and (F'). Then there exists a constant K_v that is independent of the initial data such that*

$$\limsup_{t \to \infty} \|v(\bullet, t)\|_\infty \leq K_v. \tag{3.3.4}$$

In the study of the boundedness of u, we consider the following subset of $\mathbb{R}^2$

$$\Gamma = \{(u,v) \,:\, u > 0, \quad 0 < v < K_v\}, \tag{3.3.5}$$

and the following assumptions.

(H.0) There exist a C^2 function $H(u,v)$ defined on a neighborhood Γ_0 of Γ and a constant K_0 such that $(H_u F + H_v G)(H - K)_+ \leq 0$ for every $(u,v) \in \Gamma_0$ and $K \geq K_0$.

(H.1) There exists $\lambda_1 > 0$ such that

$$[H_u(P\nabla u + R\nabla v) + H_v Q\nabla v]\nabla H \geq \lambda_1 |\nabla H|^2, \tag{3.3.6}$$

$$(P\nabla u + R\nabla v)\nabla H_u + Q\nabla v \nabla H_v \geq 0, \tag{3.3.7}$$

for every $(u,v) \in \Gamma_K := \Gamma \bigcap \{(u,v) : H(u,v) \geq K\}$, $K \geq K_0$, with K_0 given in (H.0).

(H.2) If $u \to \infty$ in $\mathbb{R}^2$ then $H(u,v) \to \infty$.

Here, we write $H_u = \frac{\partial}{\partial u}H(u,v)$, $H_{uu} = \frac{\partial^2}{\partial u \partial u}H(u,v)$, $\nabla H = \nabla_x H(\vec{u}(x))$, and so on. Furthermore, w_+ denotes the nonnegative part $\sup\{w, 0\}$ of a function w. Our first main result on the boundedness of weak solutions is the following.

Theorem 3.3.3. *The conditions (P), (F), (H.0), (H.1), and (H.2) imply that u, v are bounded.*

To obtain uniform estimates we need to assume further that

(H.0') There are constants $C_1, K_0 > 0$ such that $(H_u F + H_v G) \leq -C_1 H$ for every $(u,v) \in \Gamma_0$ such that $H \geq K_0$.

Our second main result on the uniform boundedness of u, v reads

Theorem 3.3.4. *Assume as in Theorem 3.3.3 with replacing (F) and (H.0) by (F') and (H.0'), respectively. Then u, v are ultimately uniformly bounded, that is, there exist positive constants K_u, K_v independent of the initial data u_0, v_0 such that*

$$\limsup_{t\to\infty} \|u(\bullet, t)\|_\infty \leq K_u, \quad \limsup_{t\to\infty} \|v(\bullet, t))\|_\infty \leq K_v. \tag{3.3.8}$$

Thanks to Lemmas 3.3.1 and 3.3.2, we need to prove the boundedness of u in order to complete Theorems 3.3.3 and 3.3.4.

Proof: *[Proof of Theorem 3.3.3]* Firstly, for nonnegative $\phi \in W^{1,2}(\Omega)$, we can test the equations of u, v respectively by $H_u\phi$ and $H_v\phi$, add the results, and use (3.3.7) to get

$$\int_\Omega \frac{\partial H}{\partial t}\phi\, dx + \int_\Omega (H_u(P\nabla u + R\nabla v) + H_v Q\nabla v)\nabla\phi\, dx \le C\int_\Omega (H_u F + H_v G)\phi\, dx. \tag{3.3.9}$$

Here, we have used the homogeneous Neumann boundary conditions so that the boundary integrals, which appear in the integration by parts, are all zero.

We set $H_0 = \sup_{x\in\Omega} H(u_0(x), v_0(x))$, which is finite because u_0, v_0 are bounded on Ω. Let $K \ge \max\{K_0, H_0\}$ and ϕ be $(H-K)_+$ in (3.3.9). Integrate the result in t and use (H.0), (3.3.6) to obtain

$$\int_\Omega (H-K)_+^2\, dx\,|_0^t + \lambda_1\int_0^t\int_{H\ge K} |\nabla H|^2 dxds \le 0. \tag{3.3.10}$$

Since $(H-K)_+ = 0$ when $t=0$ (as $K \ge H_0$), the above shows that $(H-K)_+ = 0$ for all t. We conclude that $H \le K$ on Ω. Condition (H.2) basically says that boundedness of u comes from that of $H(u,v)$. Thus, u, v are bounded by some constant depending on K_0 and the initial data u_0, v_0.

□

We now turn to prove Theorem 3.3.4. Firstly in order to prove the uniform boundedness of the component v (Lemma 3.3.2), we need the following standard Moser's iteration technique.

Lemma 3.3.5. *For $T_1 > T > T_0$, let V be a function on $\Omega\times[T_0, T_1]$ such that*

$$\sup_{\tau\in I_t}\int_{\Omega\times\tau} V^q dx + \iint_{Q_t} |\nabla V^{q/2}|^2\, dz \le C\frac{q^\nu}{t-s}\iint_{Q_s} V^q\, dz, \tag{3.3.11}$$

for all $q \ge m$, $T_0 < s < t < T$ and some $\nu \ge 0$. Here $I_t := [t, T_1], Q_t := \Omega\times I_t$, for any $t \in [T_0, T_1]$. Then there exists a positive constant C_0 depending on $T - T_0$ such that

$$\sup_{Q_T} V(x,t) \le C_0\left(⨌_{Q_{T_0}} V^m\, dz\right)^{1/m}. \tag{3.3.12}$$

Proof: Applying Lemma 2.2.4 with $U = V^{q/2}$, we obtain from (3.3.11)

$$\left(\iint_{Q_t} (V^q)^\kappa \, dz\right)^{1/\kappa} \le \frac{Cq^\nu}{s-t} \iint_{Q_s} V^q \, dz, \tag{3.3.13}$$

which $\kappa = 1 + 2/n$.

For $i = 0, 1, \ldots$, set $s_i = T - (T - T_0)2^{-i}$, $Q_i = Q_{s_i}$, and $q_i = m\kappa^i$. Using $s = s_i, t = s_{i+1}$ in (3.3.13), we obtain

$$\left(\iint_{Q_{i+1}} V^{m\kappa^{i+1}} \, dz\right)^{1/\kappa} \le \frac{Cm^\nu (\kappa^\nu)^i 2^{-i-1}}{T - T_0} \iint_{Q_i} V^{m\kappa^i} \, dz.$$

Dividing both sides by $|Q_i||Q_{i+1}|$ and using the fact that $Q_T \subset Q_i \subset Q_{T_0}$ for any i, we easily get

$$\left(⨌_{Q_{i+1}} V^{m\kappa^{i+1}} \, dz\right)^{1/\kappa} \le C_1 (\kappa^\nu)^i 2^{-i} ⨌_{Q_i} V^{m\kappa^i} \, dz$$

which $C_1 = Cm^\nu 2^{-1} \frac{|Q_{T_0}|}{|Q_T|} \frac{1}{T - T_0}$.

Hence,

$$\left(⨌_{Q_{i+1}} V^{m\kappa^{i+1}} \, dz\right)^{1/(m\kappa^{i+1})} \le C_2^{\kappa^{-i}} C_3^{i\kappa^{-i}} \left(⨌_{Q_i} V^{m\kappa^i} \, dz\right)^{1/m\kappa^i},$$

with $C_2 = C_1^{1/m}$ and $C_3 = (\kappa^\nu/2)^{1/m}$. Iterating the above gives

$$\left(⨌_{Q_{k+1}} V^{m\kappa^{k+1}} \, dz\right)^{1/(m\kappa^{k+1})} \le C_2^{\sum_{i=0}^k \kappa^{-i}} C_3^{\sum_{i=0}^k i\kappa^{-i}} \left(⨌_{Q_{d_0, T_0}} V^m \, dz\right)^{1/m}.$$

Since the series in the exponents converge, we can let k tend to ∞ to obtain (3.3.12). $\square$

Proof: *(Proof of Lemma 3.3.2)* Let $T > 1$ and $s < t$ be two numbers in $(T - 1, T)$. We consider a C^1 function $\eta\ :\ (0,\infty) \longrightarrow [0,1]$ that satisfies

$$\eta(\tau) = \begin{cases} 0 & \text{if} \quad \tau < s, \\ \\ 1 & \text{if} \quad \tau > s \end{cases} \qquad \text{and} \quad |\eta'| \leq \frac{1}{t-s}. \tag{3.3.14}$$

For T sufficiently large and any $q \geq 1$, we test the equation of v by $V^{q-1}\eta$, with $V = (v - K_0)_+$. Here K_0 is in (F). Integration by parts gives

$$\iint_Q \frac{1}{q}\frac{\partial V^q}{\partial t}\eta\ dz + d(q-1)\iint_Q V^{q-2}|\nabla V|^2\ dz = \iint_Q GV^{q-1}\eta\ dz \leq 0. \tag{3.3.15}$$

By (3.3.14), this implies

$$\iint_Q \frac{\partial(V^q\eta)}{\partial t}\ dz + \lambda\iint_Q |\nabla V^{q/2}|^2\ dz \leq Cq\iint_Q V^q|\eta_t|\ dz \leq \frac{Cq}{t-s}\iint_Q V^q\ dz.$$

We then apply Lemma 3.3.5 to assert that

$$\sup_{Q_T} V(x,t) \leq C_0\left(\int_{T-1}^{T+1}\int_\Omega V^2\ dx\right)^{1/2}. \tag{3.3.16}$$

On the other hand, we test the equation of v by V. We easily obtain

$$Y' \leq -CY, \quad \text{with} \quad Y(t) = \int_\Omega V^2(x,t)\ dx.$$

This shows that $\limsup_{t\to\infty} Y(t)$ is bounded by some constant independent of $Y(0)$ or u_0, u_0. Hence, this fact and (3.3.16) prove (3.3.4). □

Proof: *[Proof of Theorem 3.3.4]* The proof of uniform boundedness of H is exactly the same as that of v. Indeed, we first still have (3.3.9) in the proof of Theorem 3.3.3 which is due only to assumption (H.1). Therefore, replacing ϕ by $V^{q-1}\eta$ with $V = (H - K_1)_+$ and η defined by (3.3.14), and using (H.1), we obtain

$$\iint_Q \frac{1}{q}\frac{\partial V^q}{\partial t}\eta\ dz + \lambda_1(q-1)\iint_Q V^{q-2}|\nabla V|^2\ dz = \iint_Q (H_uF + H_vG)V^{q-1}\eta\ dz.$$

From this, using (H.0'), we obtain (3.3.15) and therefore (3.3.16). Finally, with noting that $H_uF + H_vG \le -C_1H$ (also due to (H.0')), we have the uniform bound of L^2-norm of H. This and (3.3.16) give the uniform boundedness of H. The proof is complete due to (H.2). $\square$

Remark 3.3.6. We remark that our proof goes unchanged if the mixed boundary conditions

$$\begin{cases} \chi(x)\left[\frac{\partial v}{\partial n}(x,t) + r(x)v(x,t)\right] + (1-\chi(x))v(x,t) = 0, \\ \bar{\chi}(x)\left[\frac{\partial u}{\partial n}(x,t) + \bar{r}(x)u(x,t)\right] + (1-\bar{\chi}(x))u(x,t) = 0, \end{cases}$$

are considered. Here $\chi, \bar{\chi}$ are given functions on $\partial\Omega$ with values in $\{0,1\}$, and functions $r, \bar{r}$ are given bounded nonnegative functions on $\partial\Omega$. In fact, the only difference in our calculation is that there would be boundary integrals, resulting from the integration by parts, to appear on the right hand sides of (3.3.10) and (3.3.15). However, by using the fact that $u, v, H_u, H_v, r, \bar{r}$ are nonnegative on the boundary and choosing K_0, H_0 sufficiently large but independent of u, v, we can see easily that these boundary integrals are negative. Thus, (3.3.10) and (3.3.15) are still valid and our argument can continue as before.

3.4 The existence of the Lyapunov functional H

We now see that the assumption on the existence of a function H, satisfying (H.1), is crucial for our main results in the previous section. Obviously, it is not clear whether this function ever exists. In this section we will find sufficient conditions on the structure of (3.3.1) such that we can find such H.

Clearly, the conditions (3.3.6),(3.3.7) are satisfied if the following quadratics (in $U, V \in \mathbb{R}^n$) are positive definite.

$$A_1 := (P-\lambda)H_u^2U^2 + [RH_uH_v + (Q-\lambda)H_v^2]V^2 + [RH_u^2 + (Q+P-2\lambda)H_uH_v]UV, \tag{3.4.1}$$

$$A_2 := PH_{uu}U^2 + (RH_{uv} + QH_{vv})V^2 + [RH_{uu} + (P+Q)H_{uv}]UV. \tag{3.4.2}$$

A_1 is positive definite if the coefficients of U^2, V^2 are nonnegative and its discriminant Θ_1 is nonpositive. However, a simple calculation shows that

$$\Theta_1 = (PH_uH_v - RH_u^2 - QH_uH_v)^2 = H_u^2((P-Q)H_v - RH_u)^2.$$

This suggests that we will require H to fulfill $(P-Q)H_v = RH_u$. In other words, we will consider the following equations

$$f(u,v) = (P-Q)/R, \tag{3.4.3}$$

$$H_u = f(u,v)H_v. \tag{3.4.4}$$

Lemma 3.4.1. *Assume that (3.4.4) holds. There exists $\lambda > 0$ such that A_1 is positive definite.*

Proof: By (3.4.3) and (3.4.4), the coefficients of U^2, V^2 in A_1 are respectively $H_u^2(P-\lambda)$ and $H_v^2(Rf + Q - \lambda) = H_v^2(P-\lambda)$. They are nonnegative if we choose $\lambda = \inf_\Gamma P$. □

To verify the positivity of A_2 in (3.4.2), we consider its discriminant Θ_2. Easy computation shows that

$$\Theta_2 := (RH_{uu} + PH_{uv} + QH_{uv})^2 - 4PH_{uu}\left(RH_{uv} + QH_{vv}\right).$$

Differentiating $H_u = fH_v$, we get $H_{uu} = f_uH_v + fH_{uv}$ and $H_{uv} = f_vH_v + fH_{vv}$. Substitute these into Θ_2 and simplify to obtain

$$\Theta_2 := \alpha_1 H_{vv}^2 + \alpha_2 H_{vv}H_v + \alpha_3 H_v^2. \tag{3.4.5}$$

Using (3.4.3), we easily see that $\alpha_1 = 0$. Similarly, we have

$$\begin{aligned} \alpha_2 &= 2\left(R\left(f_u + ff_v\right) + Pf_v + Qf_v\right)\left(Rf^2 + Pf + Qf\right) \\ &\quad -4P[\left(f_u + ff_v\right)\left(Rf + Q\right) + Rf^2 f_v] \\ &= 4(-PRf^2 f_v + P^2 f_v f - PQf_u) \\ &= -4PQ(f_u - f_v f). \end{aligned}$$

$$\begin{aligned}
\alpha_3 &= (R(f_u + ff_v) + Pf_v + Qf_v)^2 - 4P(f_u + ff_v)Rf_v \\
&= (f_u + ff_v)^2R^2 + (P+Q)^2f_v^2 + 2Rf_v[(f_u + ff_v)(Q-P)] \\
&= (f_u + ff_v)^2R^2 + (P-Q)^2f_v^2 + 2Rf_v(f_u + ff_v)(Q-P) + 4PQf_v^2 \\
&= [(f_u + ff_v)R + f_v(Q-P)]^2 + 4PQf_v^2 \\
&= R^2f_u^2 + 4PQf_v^2.
\end{aligned}$$

Let g be a solution to (3.4.4)and G be any differentiable function on $\mathbb{R}$. We observe that $H(u,v) = G(g(u,v))$ is also a solution to (3.4.4). We will make the following main assumptions of this section.

(H.3) Assume that there exists a connected neighborhood Γ_K^0 of Γ_K such that g belong to $C^2(\Gamma_K^0)$. Moreover,

$$g_v \neq 0, \text{ and } \alpha_2 = -4PQ(f_u - f_v f) \neq 0, \quad \forall (u,v) \in \Gamma_K^0. \tag{3.4.6}$$

(H.4) The quantities $g_{vv}/g_v^2 + \alpha_3/(\alpha_2 g_v)$, $\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$ are bounded on Γ_K. Here, we denoted

$$\delta_{12} = P[f^2g_{vv} + (f_u + ff_v)g_v], \quad \delta_{21} = Pg_{vv} + Rf_vg_v,$$

and $\delta_{11} = \delta_{22} = Pfg_v^2$.

The existence of H is then given by

Theorem 3.4.2. *Assume (H.3), (H.4) and let $H(u,v) = \exp(\mu g(u,v))$. There exists μ such that (H.1) holds.*

Proof: Thanks to Lemma 3.4.1 and the choice of g, we need only to check the positivity of A_2. We first show that $\Theta_2 < 0$ on Γ_K for suitable choice of μ. Let $G(x) = \exp(\mu x)$. As $H_v = G'g_v$, $H_{vv} = (G''g_v^2 + G'g_{vv})$, and $G''/G' = \mu$, we have

$$\Theta_2 = H_{vv}H_v\alpha_2 + H_v^2\alpha_3 = (G')^2g_v^3\alpha_2\left[\mu + (\frac{g_{vv}}{g_v^2} + \frac{\alpha_3}{\alpha_2 g_v})\right].$$

Thanks to our assumption (3.4.6) and because Γ_K^0 is connected, the coefficient of μ never vanishes on Γ_K. That is, either $g_v^3\alpha_2 < 0$ or $g_v^3\alpha_2 > 0$ on Γ_K. Because $g_{vv}/g_v^2 + \alpha_3/(\alpha_2 g_v)$ is bounded on Γ_K and $g_v, G' \neq 0$, the above shows that $\Theta_2 < 0$ on Γ_K for suitable choice of μ with $|\mu|$ being sufficiently large.

Finally, we show that the coefficients of U^2, V^2 in A_2 are positive. It suffices to show that the following quantities $\delta_1 = PH_{uu}$ and $\delta_2 = (RH_{uv} + QH_{vv})$ are strictly positive on Γ_K. Similar calculation as before yields

$$\delta_1 = P[f^2 H_{vv} + (f_u H_v + f f_v H_v)] = \exp(\mu g) f \delta_{11} \left[\mu \frac{\delta_{12}}{f\delta_{11}} + \mu^2\right],$$

$$\delta_2 = (Rf + Q)\, H_{vv} + R f_v H_v = \exp(\mu g) \frac{\delta_{22}}{f} \left[\mu \frac{f\delta_{21}}{\delta_{22}} + \mu^2\right],$$

where δ_{ij} are defined as in (H.4). Since the coefficients of μ, $\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$, are bounded on Γ_K , and $f\delta_{11}$, δ_{22}/f are positive, we can choose $|\mu|$ large to have that $\delta_1, \delta_2 > 0$ on Γ_K. □

3.5 The Shigesada-Kawasaki-Teramoto model

This section is to illustrate the validity of our results obtained in the preceding sections. We consider functions P, Q, R in (3.3.1) as follows

$$P(u,v) = d_1 + a_{11}u + a_{12}v, \quad R(u,v) = b_{11}u, \quad Q(u,v) = d_2 + a_{21}u + a_{22}v. \tag{3.5.1}$$

In this form, system (3.3.1) is a generalized version of the SKT model (3.0.4) when $a_{12} = b_{11}$ and $a_{21} = 0$. Our first result is the following global existence result.

Theorem 3.5.1. *Assume (F), (3.5.1) and that $a_{ij} \geq 0, d_i, a_{11}, b_{11} > 0$, $i,j = 1,2$. In addition, suppose that*

$$a_{11} \neq a_{21},\ a_{22} > a_{12}, \ \text{ and } a_{22} \neq a_{12} + b_{11}. \tag{3.5.2}$$

Then weak solutions to (3.3.1) with nonnegative initial data are classical and exist globally.

Furthermore, for any $\alpha \in (0,1)$ we have

$$\|v(\cdot,t)\|_{C^\alpha(\Omega)}, \quad \|u(\cdot,t)\|_{C^\alpha(\Omega)} \in \mathcal{O}. \tag{3.5.3}$$

With an additional condition on the parameters of the system, we have the following uniform estimates.

Theorem 3.5.2. *Assume as in Theorem 3.5.1 and further that $a_{11} > a_{21}$. Then for any $\alpha \in (0,1)$ we have*

$$\|v(\cdot,t)\|_{C^\alpha(\Omega)}, \quad \|u(\cdot,t)\|_{C^\alpha(\Omega)} \in \mathcal{P}. \tag{3.5.4}$$

Clearly, because the system is triangular, the estimates in L^∞ norms are sufficient to obtain estimates (3.5.3) and (3.5.4) (e.g., see proofs of Theorems 3.1.2 and 3.1.3 or [25, Theorem 6]). In addition, the boundedness of v was proven in Lemmas 3.3.1 (respectively, 3.3.2) where only assumptions (P) and (F) (respectively, (F')) are required so that we will only concern ourselves with the boundedness of u here. We apply Theorem 3.3.3 to establish Theorem 3.5.1.

By Lemma 3.3.1, we can take Γ_0 to be the strip $\{(u,v)|u > 0, 0 < v < K_v\}$. We also see that h of (3.4.3) is given by

$$h(u,v) = \frac{d + au - bv}{u},$$

where

$$d = \frac{d_1 - d_2}{b_{11}}, \quad a = \frac{a_{11} - a_{21}}{b_{11}}, \quad b = \frac{a_{22} - a_{12}}{b_{11}}.$$

Our assumption (3.5.2) simply means $a \neq 0$, $b \neq 1$ and $b > 0$. Moreover, the equation (3.4.4) can be solved by methods of characteristics (see [7]). In fact, it is elementary to see that the general solution of (3.4.4) is given by

$$g(u,v) = L\left(\frac{u^b}{d(b-1) + abu - b(b-1)v}\right),$$

where L can be any C^1 function on $\mathbb{R}$.

Since $a^2 b > 0$ and $F(u,v), G(u,v) \leq 0$ if u is large, we can find $K_1 > 0$ such that if $u \geq K_1$

then $F(u,v), G(u,v) \leq 0$ and $a[d(b-1)+abu-b(b-1)K_v] > 1$. We define

$$\Gamma_1 := \{(u,v) \in \Gamma_0 \,|\, u \geq K_1\}$$

and

$$\hat{g}(u,v) = (b-1)\log\left(\frac{u^b}{a[d(b-1)+abu-b(b-1)v]}\right), \quad (u,v) \in \Gamma_1. \tag{3.5.5}$$

Put $G_0 = \sup\{\hat{g}(u,v) \,|\, u = K_1,\, 0 < v \leq K_v\}$. Let $g(u,v)$ be a C^1 extension of $\hat{g}(u,v)$ on Γ_0 that satisfies $\sup_{\Gamma_0 \backslash \Gamma_1} g(u,v) \leq G_0 + 1$. We then set $G_1 := G_0 + 2$. Obviously, we have

$$g(u,v) \geq G_1 \Rightarrow (u,v) \in \Gamma_1 \Rightarrow u \geq K_1. \tag{3.5.6}$$

We study the function g on Γ_1. Firstly, we compute and find

$$g_v = \frac{b(b-1)^2}{d(b-1)+abu-b(b-1)v}, \quad g_{vv} = \frac{g_v^2}{b-1}, \tag{3.5.7}$$

$$f_u = \frac{bv-d}{u^2}, \quad f_v = -\frac{b}{u}. \tag{3.5.8}$$

We then prove the following lemmas.

Lemma 3.5.3. *For $(u,v) \in \Gamma_1$, we have $g_v\alpha_2 < 0$ and $\alpha_3/(\alpha_2 g_v)$ is bounded.*

Proof: By (3.5.8), we have

$$f_u - ff_v = \frac{d(b-1)+abu-b(b-1)v}{u^2} \neq 0, \quad \forall (u,v) \in \Gamma_1.$$

Thus, by (3.5.7), $g_v\alpha_2 = -4PQb(b-1)^2/u^2 < 0$ on Γ_1. On the other hand, we write

$$\frac{\alpha_3}{\alpha_2 g_v} = \frac{R^2 f_u^2}{\alpha_2 g_v} - \frac{f_v^2}{(f_u - ff_v)g_v},$$

which can be simplified to

$$-\frac{(bv-d)^2 b_{11}^2}{4b(b-1)^2 PQ} - \frac{b}{(b-1)^2}.$$

The above quantity is bounded on Γ_1 since $P \geq d_1$, $Q \geq d_2$ and v is bounded. The proof of this lemma is complete. $\square$

Lemma 3.5.4. *$\delta_{12}/(f\delta_{11})$ and $f\delta_{21}/\delta_{22}$ are bounded on Γ_1.*

Proof: We have

$$\frac{\delta_{12}}{f\delta_{11}} = \frac{g_{vv}}{g_v^2} + \frac{f_u + ff_v}{f^2 g_v} = \frac{1}{b-1} + \frac{f_u + ff_v}{f^2 g_v}.$$

The last fraction is

$$-\frac{(d(1+b) + abu - b(b+1)v)(d(b-1) + abu - b(b-1)v)}{(d + au - bv)^2 b(b-1)^2},$$

which is bounded because v is bounded on Γ_1 and the powers of u in the numerator and denominator are equal (so that the fraction is bounded when u is large).

Next, we have

$$\frac{f\delta_{21}}{\delta_{22}} = \frac{g_{vv}}{g_v^2} + \frac{Rf_v}{Pg_v} = \frac{1}{b-1} - \frac{b_{11}(d(b-1) + abu - b(b-1)v)}{(d_1 + a_{11}u + a_{12}v)(b-1)^2}.$$

The last fraction is bounded on Γ_1 by the same reason as before. $\square$

We have shown that the conditions (H.3) and (H.4) are satisfied on the set Γ_1. In particular, because $g_v\alpha_2 < 0$, we see that the factor μ in the proof of Theorem 3.4.2 can be chosen to be positive and sufficiently large. Fix such a constant μ, we then define $H(u,v) = \exp(\mu g(u,v))$. Let $K_0 = \exp(\mu G_1)$. We see that $H(u,v) \geq K_0 \Rightarrow g(u,v) \geq G_1$. Therefore, thanks to (3.5.6), we have

$$\Gamma_{K_0} = \{(u,v) \in \Gamma_0 \,|\, H(u,v) \geq K_0\} \subset \Gamma_1. \tag{3.5.9}$$

The definition of Γ_1, Theorem 3.4.2 and the above lemmas show that (H.0) and (H.1) are verified on Γ_1. By (3.5.9), they also hold on Γ_{K_0}. It is easy to see that $g(u,v) \sim \log(u^{(b-1)^2})$ when u is large so that $H(u,v) \sim u^{\mu(b-1)^2}$. Since $\mu > 0$ and $H(u,v)$ is bounded on $\Gamma_0 \setminus \Gamma_1$ (by $\exp(\mu G_1)$) we easily see that $H(u,v) \to \infty$ iff $u \to \infty$. Hence (H.2) also holds. Theorem 3.3.3 asserts that u is bounded. Our proof of Theorem 3.5.1 is complete.

Finally, to finish the proof of Theorem 3.5.2, we just need to verify (H.0'). Indeed, when u is large, we have

$$g_v = \frac{b(b-1)^2}{d(b-1) + abu - b(b-1)v} > 0.$$

Hence, H_u, H_v are positive and $H_u F + H_v G$ can be estimated from above by

$$\alpha(H_u u + H_v v) = -\alpha\mu H g_v(fu + v) = -\alpha\mu H \frac{b(b-1)^2(d + au - (b-1)v)}{d(b-1) + abu - b(b-1)v}.$$

It is easy to see that the last fraction is bounded from below by $(b-1)^2$. This gives (H.0'). The proof is complete.

Notes and Remarks

The L^p bootstrapping techniques presented in Section 3.1 rely largely on a paper of Le in 2002 where he proved the result under the assumption that the domain is of dimension 2 ([27]). The results are also in our paper [36]. When this work was completed, we learned that Choi, Lui and Yamada ([4]) were also able to prove global existence results for the SKT model (3.0.4) (also when $n \leq 5$). However, their method was pure PDE and did not provide time independent estimates so that they could only assert that the solutions exist globally. Meanwhile, not only does our method, using PDE and semigroup techniques, apply to more general systems and gives stronger conclusions; but it also requires a much weaker assumption in some cases to obtain the *uniform a priori estimates* which are key (sufficient) issues to investigate the long-time dynamics of solutions (see Chapter 5). In particular, we only need L^1 estimates of u if the second equation is semi-linear.

The Lyapunov functional approach was also employed in [46, 47, 52, 53] to address the question of the global existence for full cross diffusion systems. However, these authors must assume certain special structure conditions on their systems and also the domain Ω to be of dimension at most 2 due to their use of Sobolev imbedding inequalities. Our result in Section 3.3 is obtained without any requirement on the dimension of considered domains. Furthermore, the general theory can be applied to a general class of triangular cross diffusion systems (with conditions (P) and (F), of course) rather than the SKT model as long as one guarantees the existence of function H satisfying conditions (H.0)-(H.2), or (H.3) and (H.4). The result can be also found in our published papers [31, 33].

Chapter 4

A PRIORI ESTIMATES FOR FULL SYSTEMS

The purpose of this chapter is to recall the global existence result of Le [29] and go further to establish uniform a priori estimates for solutions of a class of strongly coupled parabolic systems.

4.1 Main result

Consider the following strongly coupled parabolic system

$$\begin{cases} u_t &= \nabla(P^u(u,v)\nabla u + P^v(u,v)\nabla v) + F(u,v), \\ v_t &= \nabla(Q^u(u,v)\nabla u + Q^v(u,v)\nabla v) + G(u,v), \end{cases} \tag{4.1.1}$$

on a bounded domain Ω in $\mathbb{R}^n$. Here, the initial conditions are described by $u(x,0) = u_0$ and $v(x,0) = v_0$, $x \in \Omega$.

Concerning the boundary conditions, for the sake of simplicity, we consider here homogeneous Neumann boundary conditions. In fact, by a view of Remark 3.3.6, our main results could apply to the following mixed conditions

$$\begin{cases} \chi(x)\left[\frac{\partial v}{\partial n}(x,t) + r(x)v(x,t)\right] + (1-\chi(x))v(x,t) = 0, \\ \bar{\chi}(x)\left[\frac{\partial u}{\partial n}(x,t) + \bar{r}(x)u(x,t)\right] + (1-\bar{\chi}(x))u(x,t) = 0, \end{cases} \tag{4.1.2}$$

where $\chi, \bar{\chi}$ are given functions on $\partial\Omega$ with values in $\{0,1\}$, and functions $r, \bar{r}$ are given bounded nonnegative function on $\partial\Omega$.

The above system arises in many applications and has recently received a lot of attention in both mathematical analysis and real life modelling. As we already mentioned in Chapter 1, it is also referred to by the cross diffusion system names as P^u, Q^v model the *self diffusion* of the components u, v and P^v, Q^u represent the *cross diffusion* that result from the influence of one component on the other. If P^v, Q^u are zero, (4.1.1) is the well studied weakly coupled parabolic system (or classical reaction diffusion system). The introduction of cross diffusion terms into classical diffusion systems allows the mathematical models to capture much more important features of many phenomena in physics, biology, ecology, and engineering sciences. Of course, the presence of these terms caused enormous difficulties in the mathematical treatment due to the strong coupling in the diffusion terms.

Most recently, under certain assumptions on the parameters of the system, Le proved in [29] that weak solutions of (4.1.1) are bounded and everywhere regularity. Therefore they are classical and exist globally. In particular, he also gave explicit conditions on the parameters of the system (4.1.1), which includes the SKT model (1.0.4) (when $a_{12} = b_{11}, a_{21} = b_{22}$), with

$$\begin{aligned} P^u &= d_1 + a_{11}u + a_{12}v, \quad P^v = b_{11}u, \\ Q^v &= d_2 + a_{21}u + a_{22}v, \quad Q^u = b_{22}v. \end{aligned} \tag{4.1.3}$$

In this chapter, we shall go further to obtain the uniform a priori estimates for weak solutions of (4.1.1) with (4.1.3). Such estimates of solutions will be used to prove the existence of global attractors (see Chapter 5). First we would recall Le's result of global existence whose proof is in [29].

Theorem 4.1.1. *[29, Theorem 1.2] Assume that $a_{ij} \geq 0, d_i, b_{11}, b_{22} > 0, i, j = 1, 2,$ and $F(0, v) = G(u, 0) = 0$ for all u, v. In addition suppose that*

$$F(u,v) \text{ and } G(u,v) \text{ are negative if either } u \text{ or } v \text{ are sufficiently large.} \tag{4.1.4}$$

and

$$a_{11} - a_{21} > b_{22}, \quad a_{22} - a_{12} > b_{11}. \tag{4.1.5}$$

Then weak solutions, with nonnegative initial data, to (4.1.1) with (4.1.3) are classical and exist globally. Furthermore, the Hölder norms of solutions depend only on the bound of their L^∞ norms.

We shall give the proof of our following main result.

Theorem 4.1.2. *Assume as in Theorem 4.1.1 and that there exist positive constants K_0 and α such that if either $u \geq K_0$ or $v \geq K_0$, then*

$$F(u,v) \leq -\alpha u, \quad G(u,v) \leq -\alpha v. \tag{4.1.6}$$

Then for any nonnegative solution (u, v) to (4.1.1) and any $\alpha \in (0,1)$, there exists a $C_\infty(\alpha) > 0$ independent of initial data such that

$$\limsup_{t\to\infty} \|u(\bullet, t)\|_{C^\alpha} + \limsup_{t\to\infty} \|v(\bullet, t)\|_{C^\alpha} \le C_\infty(\alpha). \tag{4.1.7}$$

In population dynamics terms, condition (4.1.5) means that self diffusion rates are stronger than cross diffusion ones. Obviously, our assumption (4.1.6) is satisfied if the reactions F, G are the well known Lotka-Volterra competitive reaction type of the form

$$F(u, v) = u(a_1 - b_1 u - c_1 v), \quad G(u, v) = v(a_2 - b_2 u - c_2 v). \tag{4.1.8}$$

We also remark that the condition $F(0, v) = G(u, 0) = 0$ and maximum principles imply that the solutions stay positive if their initial data are nonnegative.

4.2 Proof of main result

Clearly, in order to obtain uniform estimate (4.1.7), thanks to Theorem 4.1.1, it suffices to show that the L^∞ norms of the solution are ultimately uniformly bounded. That is, we need only find a positive constant C_∞ independent of the initial data such that

$$\limsup_{t\to\infty} \|u(\bullet, t)\|_\infty + \limsup_{t\to\infty} \|v(\bullet, t))\|_\infty \le C_\infty. \tag{4.2.1}$$

The proof of this fact will largely base on the analysis in [29, Section 4.2] where Le proved the existence of a C^2 function $H(u, v)$ defined on $\mathbb{R}^2_+$ such that the below conditions are satisfied.

(H.0) $H(u, v) = \exp(\mu g(u, v))$ for some sufficiently large $\mu > 0$ (depending only on the parameters of the system) and g is a solution of $g_u = f(u, v) g_v$, with $f(u, v)$ being the positive solution of

$$F(f) := -P^v f^2 + (P^u - Q^v) f + Q^u = 0. \tag{4.2.2}$$

(H.1) There exists a constant K_1 such that $(H_u F + H_v G)(H - K)_+ \le 0$ for any $(u, v) \in \Gamma_0$ and $K \ge K_1$.

(H.2) Let $\mathcal{P} = P^u \nabla u + P^v \nabla v$ and $\mathcal{Q} = Q^u \nabla u + Q^v \nabla v$. There exists $\lambda > 0$ such that

$$H_u \mathcal{P} + H_v \mathcal{Q} \geq \lambda |\nabla H|^2, \tag{4.2.3}$$

$$\mathcal{P} \nabla H_u + \mathcal{Q} \nabla H_v \geq 0, \tag{4.2.4}$$

for any $(u, v) \in F \bigcap \{(u, v) \ : \ H(u, v) \geq K_1\}$, with K_1 given in $(H.1)$.

(H.2) If $(u, v) \to \infty$ in $\mathbb{R}^2$, then $H(u, v) \to \infty$.

Under our additional assumption (4.1.6), we observe that function H has the following property.

Lemma 4.2.1. *There exists a positive constant C such that*

$$H_u F + H_v G \leq -CH \tag{4.2.5}$$

if either $u \geq K_0$ or $v \geq K_0$, with K_0 being given in (4.1.6).

Proof: Without loss of generality, we can assume that $d_1 \leq d_2$. Substituting $f = \frac{a_{11} - a_{21}}{b_{11}} > 0$ in the quadratic on the left hand side of (4.2.2) and simplifying the result, we get

$$-\frac{[(a_{11} - a_{21})(a_{22} - a_{12}) - b_{11} b_{22}]v + (d_2 - d_1)(a_{11} - a_{21})}{b_{11}}.$$

By (4.1.5) and the fact that $d_1 \leq d_2$, the above is negative. Since leading coefficient $-P^v$ is negative and $f(u, v)$ is the positive root, we must have that $f(u, v)$ is bounded by $(a_{11} - a_{21})/b_{11}$ for all $u, v \geq 0$. This and [29, Lemma 4.3] imply that there exists a positive constant C such that

$$g_v \geq \frac{C}{f(u, v)} \geq \frac{b_{11} C}{a_{11} - a_{21}}.$$

Now, from $H = \exp(\mu g)$ and assumption (4.1.6) on the reaction terms F and G, we easily obtain

$$H_u F + H_v G = \mu H g_v (f F(u, v) + G(u, v)) \leq -\mu \alpha H g_v (f u + v) \leq -C_1 H$$

if either $u \geq K_0$ or $v \geq K_0$. The proof of this lemma is complete. $\square$

We are now ready to give the proof of Theorem 4.1.2

Proof of Theorem 4.1.2: For any positive test function ϕ, we test the equations of u, v respectively with $H_u\phi, H_v\phi$ and add the results. By using (4.2.4), we easily obtain

$$\int_\Omega H_t\phi \, dx + \int_\Omega [H_u\mathcal{P} + H_v\mathcal{Q}] \nabla\phi \, dx \leq \int_\Omega (H_uF + H_vG)\phi \, dx. \tag{4.2.6}$$

Now follow exactly proof of Theorem 3.3.4, we obtain the ultimately uniform boundedness of H. Therefore $(H.2)$ gives that of u and v, that is, estimate (4.2.1). By our earlier discussion, this completes our proof of Theorem 4.1.1. □

Notes and Remarks

As we mentioned, the presence of cross diffusion terms caused enormous difficulties in the mathematical treatment. It is not surprising that many classical methods, which were developed successfully for regular reaction diffusion systems, could not be extended to handle (4.1.1) (even for the SKT model). Not much work had been done before.

Technically, the boundedness of weak solutions to the SKT system was studied in [19, 43] by using invariance principles. Of course, this method required severe restrictions on the initial data of the solutions. L^p estimates and Lyapunov functional approachs were used in [46, 47, 52, 53] to attack this question. However, not only that these authors must assume that the systems are of certain special form but also their use of Sobolev imbedding inequalities forced the domain Ω to be of dimension at most 2.

On the other hand, the question of whether bounded weak solutions are Hölder continuous also presents great difficulties, and little progress has been made in the last twenty years. Partial regularity results were obtained by Giaquinta and Struwe in [9] for a general class of systems. Everywhere regularity results for bounded solutions were proven only in very few situations assuming additional structure conditions. Among these are triangular systems (see [2, 27, 25]) or strongly coupled systems of special form (see [25, 51]). In [25], we assumed rather restrictive structural conditions that prevent the application of our results to many important models. In fact, the SKT model (1.0.4) does not satisfy the structures studied in [25, 51]. Under the assumption that the domain Ω is of two dimensional, the authors of [17] studied regularity of certain full systems. Very recently, Le proposed a general assumption, namely, (H.0)-(H.2) on the parameters of the systems to address the global existence issues (see [29]). In addition, in [35] we went further to establish the ultimate uniformity of a priori estimates, which is Theorem 4.1.2 presented here.

Chapter 5

GLOBAL ATTRACTORS AND ESTIMATES FOR GRADIENTS

In the preceding chapters, we have obtained the global existence result and the uniform a priori estimates of solutions to cross diffusion parabolic systems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla[(d_1 + a_{11}u + a_{12}v)\nabla u + b_{11}u\nabla v] + F(u,v), \\ \dfrac{\partial v}{\partial t} = \nabla[b_{22}v\nabla u + (d_2 + a_{21}u + a_{22}v)\nabla v] + G(u,v), \end{cases} \tag{5.0.1}$$

which is supplied with the Neumann or Robin type boundary conditions

$$\frac{\partial u}{\partial n} + r_1(x)u = 0, \qquad \frac{\partial v}{\partial n} + r_2(x)v = 0, \tag{5.0.2}$$

on the boundary $\partial\Omega$ of a bounded domain Ω in $\mathbb{R}^n$. Here r_1, r_2 are given nonnegative functions on $\partial\Omega$. The initial conditions are described by $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, $x \in \Omega$. Here $u_0, v_0 \in W^{1,p}(\Omega)$ for some $p > n$.

Our main results in this chapter are to obtain the existence of a global attractor and uniform estimates of gradients of solutions. In order to state these results, let us first recall some definitions in the dynamical system theory. Let (X, d) be a metric space and Φ be a semiflow on X. That is, $(x,t) \mapsto \Phi_t(x)$ is continuous, $\Phi_0 = id_X$, and $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for $s, t \geq 0$. A subset A of X is said to be **positively invariant** for Φ if $\Phi_t(A) \subset A$ for all $t \geq 0$, and **invariant** if $\Phi_t(A) = A$ for all $t \geq 0$.

A subset $A \subset X$ is said to be an **attractor** for Φ if A is nonempty, compact, invariant, and there exists some open neighborhood U of A in X such that $\lim_{t\to\infty} d(\Phi_t(u), A) = 0$ for all $u \in U$. Here, $d(x, A)$ is the usual Hausdorff distance from x to the set A. If A is an attractor which attracts every point in X, A is called **global attractor**. There are two other names, universal or maximal attractor, used in literature which have the same meaning as global attractor.

It is shown that a dynamical system possessing a global attractor is observable in the sense that all the orbits converge towards the set, or in another word, the set describes all possible long-time dynamics that the solution semiflow associated with the system can produce. For background information and references, the reader is referred to the book of Temam [48].

We obtain the following result whose proof is given in Section 5.2.

Theorem 5.0.1. *Assume that* $a_{ij}, b_{22} \geq 0, d_i, b_{11} > 0, i, j = 1, 2,$ *and*

$$a_{11} - a_{21} > b_{22}, \quad a_{22} - a_{12} > b_{11}. \tag{5.0.3}$$

In addition, there exist positive constants K_0 *and* K_1 *such that if either* $u \geq K_0$ *or* $v \geq K_0$*, then*

$$F(u, v) \leq -K_1 u, \quad G(u, v) \leq -K_1 v. \tag{5.0.4}$$

Then (5.0.1) and (5.0.2) define a dynamical system on $W^{1,p}_+(\Omega, \mathbb{R}^2)$*, the positive cone of* $W^{1,p}(\Omega, \mathbb{R}^2)$*, for some* $p > n$*. And this dynamical system possesses a global attractor set.*

Furthermore, let (u, v) *be a nonnegative solution to (5.0.1). Then there exist* $\nu > 1$ *and* $C_\infty > 0$ *independent of initial data such that*

$$\limsup_{t\to\infty} \|u(\bullet, t)\|_{C^\nu(\Omega)} + \limsup_{t\to\infty} \|v(\bullet, t)\|_{C^\nu(\Omega)} \leq C_\infty. \tag{5.0.5}$$

In population dynamics terms, condition (5.0.3) means that self diffusion rates are stronger than cross diffusion ones. In fact, these assumptions are only needed in the preceding chapters to establish that weak solutions are bounded and Hölder continuous and their Hölder norms are uniformly bounded in time (see Chapters 3 and 4). Moreover, for sake of generalization, estimate (5.0.5) will be in fact proven for solutions of a more general class of cross diffusion systems (of m equations) rather than those to (5.0.1).

Let us consider the following nonlinear parabolic systems of m equations ($m \geq 2$) given by

$$u_t = \mathrm{div}(a(u)\nabla u) + f(u, \nabla u), \tag{5.0.6}$$

in a domain $Q = \Omega \times (0, T) \subset \mathbb{R}^{N+1}$, with Ω being an open subset of $\mathbb{R}^N$, $N \geq 1$. The *vector valued* functions $u = (u^1, ..., u^m)$, $f = (f^1, ..., f^m)$ take values in $\mathbb{R}^m$, $m \geq 1$. ∇u denotes $(\nabla u^1, ..., \nabla u^m)$ in which ∇u^i is the spatial gradient of u^i, that is, $\nabla u^i = (D_\alpha u^i)_{\alpha=1}^n$. Here, $a(u) = (a_{ij}(u))$ is a $m \times m$ matrix. See also Remark 5.2.2 for the case $a(u)$ is a symmetric tensor, that is, $a_{ij} = (a^{ij}_{\alpha\beta})$ is an $n \times n$

matrix and $a^{ij}_{\alpha\beta} = a^{ji}_{\beta\alpha}$.

We need the following assumption on parameters of the system: there exist a positive constant λ and a continuous function $C(|u|)$ such that for any $\xi \in \mathbb{R}^m$

$$|f(u,\xi)| + |f_u(u,\xi)| \le C(|u|)(1+|\xi|^2), \quad |f_\xi(u,\xi)| \le C(|u|)(1+|\xi|), \tag{5.0.7}$$

$$\lambda|\xi|^2 \le a_{ij}(u)\xi_i\xi_j \le C(|u|)|\xi|^2. \tag{5.0.8}$$

For the sake of simplicity, we will deal with the Neumann conditions $\dfrac{\partial u}{\partial n} = 0$ in the proof below, and leave the Robin case to Remark 5.2.3.

Later we shall define our semi-flow $\Phi_t(u_0) = (u(\bullet,t))$ for all $t \ge 0$ and any u_0 in $W^{1,p}_+(\Omega, \mathbb{R}^m) = \{u \in W^{1,p}(\Omega,\mathbb{R}^m) \,:\, u(x) \ge 0 \,\forall x \in \Omega\}$. Here $(u(\bullet,t))$ be the solution to (5.0.6). Our goal in this chapter is to show that the semi-flow Φ_t is well defined and possesses a global attractor in $W^{1,p}_+(\Omega,\mathbb{R}^m)$. Precisely, we obtain the following.

Theorem 5.0.2. *Let $u = (u^i)$ be a nonnegative solution of (5.0.6). Suppose that there exists a positive constant $C_\infty(\alpha)$ independent of initial data such that*

$$\limsup_{t\to\infty} \|u_i(\bullet,t)\|_{C^\alpha(\Omega)} \le C_\infty(\alpha) \tag{5.0.9}$$

for all $\alpha \in (0,1)$ and $i = 1, ..., m$.

Then there exist $\nu > 1$ and a positive constant C_∞ independent of the initial data such that

$$\limsup_{t\to\infty} \|u^i(\bullet,t)\|_{C^\nu(\Omega)} \le C_\infty. \tag{5.0.10}$$

Moreover, for $p > n \ge 2$, let K be a closed bounded subset in $W^{1,p}(\Omega,\mathbb{R}^m)$. We consider solutions u with their initial data $u_0 \in K$. Then the image of K under solution flow $K_t := \{u(\bullet,t) : u_0 \in K\}$ is a compact subset of $W^{1,p}(\Omega,\mathbb{R}^m)$.

5.1 L^p estimates of gradients

Our main results in this section are the following estimates for higher order norms of solutions. We first establish uniform estimates in $W^{1,p}$ norms of solutions to prove the existence of an absorbing ball in the $W^{1,p}(\Omega, \mathbb{R}^m)$ space. This is a crucial step of proving the existence of the global attractor set.

Theorem 5.1.1. *Assume as in Theorem 5.0.2. Then there exists a positive constant $C_\infty(p)$ independent of the initial data such that*

$$\limsup_{t\to\infty} \|u^i(\bullet, t)\|_{W^{1,p}(\Omega)} \leq C_\infty(p) \tag{5.1.1}$$

for any $p > 1$ and $i = 1, ..., m$.

The proof of these theorems will be based on several lemmas. The main idea to prove the above theorems is to use the imbedding results for Morrey's spaces. We recall the definitions of the Morrey space $M^{p,\lambda}(\Omega)$ and the Sobolev-Morrey space $W^{1,(p,\lambda)}$. Let $B_R(x)$ denote a cube centered at x with radius R in $\mathbb{R}^n$.

We say that $f \in M^{p,\lambda}(\Omega)$ if $f \in L^p(\Omega)$ and

$$\|f\|^p_{M^{p,\lambda}} := \sup_{x\in\Omega, \rho>0} \rho^{-\lambda} \int_{B_\rho(x)} |f|^p dy < \infty.$$

Also, f is said to be in Sobolev-Morrey space $W^{1,(p,\lambda)}$ if $f \in W^{1,p}(\Omega)$ and

$$\|f\|^p_{W^{1,(p,\lambda)}} := \|f\|^p_{M^{p,\lambda}} + \|\nabla f\|^p_{M^{p,\lambda}} < \infty.$$

If $\lambda < n-p$, $p \geq 1$, and $p_\lambda = \frac{p(n-\lambda)}{n-\lambda-p}$, we then have the following imbedding result (see Theorem 2.5 in [5])

$$W^{1,(p,\lambda)}(B) \subset M^{p_\lambda,\lambda}(B). \tag{5.1.2}$$

We then proceed by proving some estimates for the Morrey norms of the gradients of the solutions. In the sequel, the temporal variable t is always assumed to be sufficiently large such that

$$\|u(.,t)\|_{C^\alpha} \le C_\infty(\alpha),\ \forall \alpha \in (0,1) \text{ and } t \ge T, \tag{5.1.3}$$

where T may depend on the initial data.

From now on, let us fix a point $(x,t) \in \overline{\Omega} \times (T,\infty)$. As far as no ambiguity can arise, we write $B_R = B_R(x)$, $\Omega_R = \Omega \bigcap B_R$, and $Q_R = \Omega_R \times [t-R^2,t]$. In the proofs, we will always use $\xi(x,t)$ as a cut off function between $B_R \times [t-R^2,t]$ and $B_{2R} \times [t-4R^2,t]$, that is, ξ is a smooth function that $\xi = 1$ in $B_R \times [t-R^2,t]$ and $\xi = 0$ outside $B_{2R} \times [t-4R^2,t]$.

We first have the following technical lemmas.

Lemma 5.1.2. *For any sufficiently small $R > 0$, solution u of (5.0.6) satisifes*

$$\int_{\Omega_{2R}} |D^2 u|^2 \xi^2 \ dx \le C \int_{\Omega_{2R}} \left[|u_t|^2 \xi^2 + (|\nabla u|^2 + 1)(|\nabla \xi|^2 + \xi^2) \right] \ dx. \tag{5.1.4}$$

Proof: We first consider the points x on the boundary $\partial\Omega$. As $\partial\Omega$ is smooth, we can locally flatten the boundary and assume that $\partial\Omega$ is the plane $\{x_n = 0\}$. Therefore u solves a new system of the form

$$u_t^i = D_\alpha(a'^{ij}_{\alpha\beta} D_\beta u^j) + f'(u, Du) \tag{5.1.5}$$

with the Neumann boundary condition, $D_n u^i = 0$. This new system still satisfies the same conditions as (5.0.7) and (5.0.8) (e.g., see [7, pp. 320-322]). In particular, we have the ellipticity condition, that is, for any $p \in \mathbb{R}^{m\times n}$,

$$\sum_{ij\alpha\beta} a'^{ij}_{\alpha\beta} p^i_\alpha p^j_\beta \ge \lambda |p|^2 = \lambda \sum_{i\alpha} |p^i_\alpha|^2. \tag{5.1.6}$$

We test the equations (5.1.5) of u^i by $\sum_{s=1}^{n-1} D_s(D_s u^i \xi^2)$. Here $D_s = D_{x_s}$. Using integration by parts, we obtain

$$
\begin{aligned}
\int_{\Omega_{2R}} (u_t^i - f')D_s(D_s u^i \xi^2)\ dx &= -\int_{\Omega_{2R}} (a'^{ij}_{\alpha\beta} D_\beta u^j) D_\alpha D_s(D_s u^i \xi^2)\ dx \\
&= \int_{\Omega_{2R}} D_s(a'^{ij}_{\alpha\beta} D_\beta u^j) D_\alpha (D_s u^i \xi^2)\ dx \\
&\quad - \int_{\partial\Omega_{2R}} a'^{ij}_{\alpha\beta} D_\beta u^j D_\alpha (D_s u^i \xi^2) \cos(n, x_s)\ d\sigma
\end{aligned}
$$

Since $\cos(n, x_s) = \delta_s^n = 0$ for any $s \neq n$, the above boundary integral is zero. In addition, by using Young's inequality and the ellipticity condition, with $p_\alpha^i = D_{s\alpha} u^i$ in (5.1.6), it is not difficult to deduce the inequality

$$
\sum_{s=1}^{n-1} \sum_{\alpha,i=1}^{n} \int_{\Omega_{2R}} |D_{s\alpha} u^i|^2 \xi^2\ dx \leq C \int_{\Omega_{2R}} \left[|u_t|^2 \xi^2 + |\nabla u|^4 \xi^2 + (|\nabla u|^2 + 1)(|\nabla \xi|^2 + \xi^2) \right]\ dx.
$$

On the other hand, by solving equations of u with respect to the derivative $D_{nn} u^i$, we get

$$
|D_{nn} u|^2 \leq C \left(|u_t|^2 + \sum_{s=1}^{n-1} \sum_{k,i=1}^{n} |D_{ks} u^i|^2 + |\nabla u|^4 + 1 \right).
$$

Combining the last two inequalities, we derive

$$
\int_{\Omega_{2R}} |D^2 u|^2 \xi^2\ dx \leq C \int_{\Omega_{2R}} \left[|u_t|^2 \xi^2 + |\nabla u|^4 \xi^2 + (|\nabla u|^2 + 1)(|\nabla \xi|^2 + \xi^2) \right]\ dx.
$$

Using Lemma 2.2.8 with $s = 1$, we then have

$$
\iint_{Q_{2R}} |\nabla u|^4 \xi^2\ dz \leq C R^{2\alpha} \int_{\Omega_{2R}} \left(|D^2 u|^2 \xi^2 + |\nabla u|^2 |\nabla \xi|^2 \right)\ dx. \tag{5.1.7}
$$

Therefore, the estimates of the lemma for points on the boundary follows at once from two last inequalities with sufficiently small R. In order to obtain the interior estimates, we test the equations by $\sum_{s=1}^{n} D_s(D_s u^i \xi^2)$ and use integration by parts and the conditions (5.0.7) and (5.0.8). The lemma follows at once.

□

Lemma 5.1.3. *For sufficiently small $R > 0$, we have the following estimate*

$$\int_{\Omega_R} |\nabla u|^2 \, dx + \iint_{Q_R} [|u_t|^2 + |D^2 u|^2] \, dz \leq C R^{n-2+2\alpha}.$$

In the proof below, we will need two useful Lemmas 2.2.8 and 2.2.9 by Ladyzhenskaya *et al.* [22].

Proof: Rewrite (5.0.6) as follows

$$u_t = a(u)\Delta u + (a_{u_i} \nabla u_i)\nabla u + f(u, \nabla u) \tag{5.1.8}$$

and test this by $-\Delta u \xi^2$. Integration by parts gives

$$\iint_{Q_{2R}} u_t \Delta u \xi^2 \, dz = -\frac{1}{2}\iint_{Q_{2R}} \frac{\partial(|\nabla u|^2 \xi^2)}{\partial t} \, dz + \iint_{Q_{2R}} \left[|\nabla u|^2 \xi \xi_t - u_t \nabla u \xi \nabla \xi\right] \, dz.$$

Note that we have used $\xi \dfrac{\partial u}{\partial n} = 0$ on ∂Q_{2R} that is due to the choice of ξ and the Neumann condition of u. Therefore the boundary integrals resulting in the integration by parts are all zero.

Since $a(u)\Delta u \Delta u \geq \lambda |\Delta u|^2$ (see (5.0.8)), we obtain

$$\begin{aligned} \int_{\Omega_R} |\nabla u(x,t)|^2 \, dx + \iint_{Q_{2R}} |\Delta u|^2 \xi^2 \, dz &\leq C \iint_{Q_{2R}} |\nabla u|^2 (\xi |\xi_t| + \xi^2 + \xi^2 |\Delta u|) \, dz \\ &+ C \iint_{Q_{2R}} \left[|u_t||\nabla u| \xi |\nabla \xi| + |f||\Delta u| \xi^2\right] \, dz. \end{aligned}$$

By Young's inequality and the facts that $|\xi_t|, |\nabla \xi|^2 \leq C/R^2$, we derive

$$\begin{aligned} \int_{\Omega_R} |\nabla u(x,t)|^2 \, dx &+ \iint_{Q_{2R}} |\Delta u|^2 \xi^2 \, dz \leq \epsilon \iint_{Q_{2R}} |u_t|^2 \xi^2 \, dz \\ &+ C \iint_{Q_{2R}} \left(|\nabla u|^4 \xi^2 + \frac{1}{R^2} |\nabla u|^2 \right) \, dz + C R^{n+2}. \end{aligned} \tag{5.1.9}$$

From (5.1.8), we get

$$\iint_{Q_{2R}} |u_t|^2 \xi^2 \, dz \leq \iint_{Q_{2R}} \left(|\Delta u|^2 + |\nabla u|^4 + |\nabla u|^2 + 1\right) \xi^2 \, dz.$$

Together with Lemma 5.1.2, we obtain

$$\iint_{Q_{2R}} |D^2u|^2\xi^2 \, dz \leq \iint_{Q_{2R}} \left[(|\Delta u|^2 + |\nabla u|^4)\xi^2 + (|\nabla u|^2+1)(|\nabla \xi|^2 + \xi^2)\right] \, dz.$$

We then choose ϵ sufficiently small in (5.1.9) to derive that

$$\int_{\Omega_R} |\nabla u|^2 \, dx + \iint_{Q_R} (|u_t|^2 + |D^2u|^2) \, dz \leq C\iint_{Q_{2R}} [|\nabla u|^4\xi^2 + \frac{1}{R^2}|\nabla u|^2] \, dz + CR^{n+2}.$$

By using the estimate (5.1.7) with R sufficiently small and again noting that $|\nabla\xi| \leq CR^{-1}$, we obtain from the last inequality

$$\int_{\Omega_R} |\nabla u|^2 \, dx + \iint_{Q_R} (|u_t|^2 + |D^2u|^2) \, dz \leq \frac{C}{R^2}\iint_{Q_{2R}} |\nabla u|^2 \, dz + CR^{n+2}. \tag{5.1.10}$$

On the other hand, by testing (5.0.6) with $(u-u_R)\xi^2$, which u_R is the average of u over Q_R, one can easily get

$$\iint_{Q_{2R}} |\nabla u|^2 \, dz \leq CR^{n+2\alpha}.$$

This and (5.1.10) complete the proof of this lemma. □

The following lemma shows that ∇u is uniformly bounded in $W^{1,(2,n-4+2\alpha)}(\Omega_R)$ so that imbedding (5.1.2) can be employed.

Lemma 5.1.4. *For $R > 0$ sufficiently small, we have the following estimates*

$$\int_{\Omega_R} (u_t^2 + |D^2u|^2) \, dx \leq CR^{n-4+2\alpha}. \tag{5.1.11}$$

Proof: We now test (5.0.6) with $-(u_t\xi^2)_t$. Integration by parts in t gives

$$\begin{aligned} -\frac{1}{2}\frac{\partial}{\partial t}\iint_{Q_{2R}} u_t^2\xi^2 \, dz \quad &+ \quad \iint_{Q_{2R}} u_t^2\xi\xi_t \, dz + \iint_{Q_{2R}} (a(u)\nabla u)_t\nabla(u_t\xi^2) \, dz \\ &= \quad -\iint_{Q_{2R}} f_t(u,\nabla u)u_t\xi^2 \, dz. \end{aligned} \tag{5.1.12}$$

We again note that the boundary integrals resulting in the integration by parts are all zero. We consider the term

$$(a(u)\nabla u)_t\nabla(u_t\xi^2) = (a(u)\nabla u_t + a_u(u)u_t\nabla u)(\nabla u_t\xi^2 + 2u_t\xi\nabla\xi).$$

Using assumptions (5.0.7), (5.0.8), and Young's inequality, we have the following inequalities: $a(u)\nabla u_t\nabla u_t \geq \lambda|\nabla u_t|^2$, and

$$\begin{aligned}
|u_t\nabla u_t\xi\nabla\xi| &\leq \epsilon|\nabla u_t|^2\xi^2 + C(\varepsilon)u_t^2|\nabla\xi|^2, \\
|u_t\nabla u\nabla u_t\xi^2| &\leq \epsilon|\nabla u_t|^2\xi^2 + C(\varepsilon)u_t^2|\nabla u|^2\xi^2, \\
|u_t^2\nabla u\xi\nabla\xi| &\leq u_t^2|\nabla u|^2\xi^2 + u_t^2|\nabla\xi|^2, \\
|f_t(u,\nabla u)u_t\xi^2| &\leq \epsilon|\nabla u_t|^2\xi^2 + C(\varepsilon)u_t^2|\nabla u|^2\xi^2 + C(\varepsilon)u_t^2\xi^2.
\end{aligned}$$

These inequalities and (5.1.12) yield

$$\int_{\Omega_R} |u_t|^2\, dx + \iint_{Q_{2R}} |\nabla u_t|^2\xi^2\, dz \leq C\iint_{Q_{2R}} |u_t|^2\left(|\nabla u|^2\xi^2 + \xi^2 + |\nabla\xi|^2 + |\xi_t|\right)\, dz. \tag{5.1.13}$$

As we have shown in Lemma 5.1.3, $\int_{\Omega_R} |\nabla u|^2\, dx \leq cR^{n-2+\alpha}$. This allows us to apply Lemma 2.2.9, with the function v being $|\nabla u|^2$, to derive

$$\iint_{Q_{2R}} |\nabla u|^2u_t^2\xi^2\, dz \leq cR^{2\alpha}\iint_{Q_{2R}} [|\nabla u_t|^2\xi^2 + u_t^2|\nabla\xi|^2]\, dz.$$

Hence, for R sufficiently small, we obtain from the above and (5.1.13) that

$$\int_{\Omega_R} |u_t|^2\, dx + \iint_{Q_R} |\nabla u_t|^2\, dz \leq C\iint_{Q_{2R}} |u_t|^2\left(\xi^2 + |\nabla\xi|^2 + |\xi_t|\right)\, dz. \tag{5.1.14}$$

Applying Lemma 5.1.3 and using the fact that $|\xi_t|, |\nabla\xi|^2 \leq CR^{-2}$, we obtain the desired inequality u_t. Finally, employ Lemma 5.1.2 and then again Lemma 5.1.3 to obtain the estimate for D^2u. We conclude the proof. □

We are now ready to give

Proof of Theorem 5.1.1: Thanks to the estimate (5.1.11), we can assert that ∇u is in $W^{1,(2,\lambda)}(\Omega_R)$, $\lambda = n - 4 + 2\alpha$, and furthermore its norms are uniformly bounded. Therefore, by the imbedding (5.1.2) and the fact that $M^{2_\lambda,\lambda} \subset L^{2_\lambda}$, we have $\|\nabla u(\bullet, t)\|_{L^{2_\lambda}(\Omega)}$ with $2_\lambda = \frac{2(4-2\alpha)}{2-2\alpha}$ bounded by some constant C. Since α is arbitrarily chosen in $(0, 1)$, 2_λ can be as large as we wish. This proves that there exists a positive constant $C_\infty(p)$ such that $\|u(\bullet, t)\|_{W^{1,p}(\Omega)} \leq C_\infty(p)$, for any $p > 1$ and $t \geq T$. T is in (5.1.3). The proof of Theorem 5.1.1 is complete. $\square$

5.2 Proofs

In this section we shall give the proof of Theorem 5.0.2 and therefore that of Theorem 5.0.1 thanks to the uniform estimate (5.0.9) in the preceding chapters. To this end we will need the following Schauder estimate by Schlag in [44].

Lemma 5.2.1. *Let $u \in C^{2,1}(Q_T)$ be a solution of (5.0.6). Then, for $1 < q < \infty$, there exists a constant $C(q, T)$ such that*

$$\|D^2 u\|_{L^q(Q_T)} \leq C(q, T) \left[\|f\|_{L^q(Q_T)} + \|u\|_{L^q(Q_T)}\right], \tag{5.2.1}$$

where $Q_T = \Omega \times [0, T]$.

In fact, this result was proven in [44] under the assumption that a is a symmetric tensor, that is, $a = (a_{ij}^{\alpha\beta})$ with $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$. In our case, a is a matrix $a = (a_{ij})$ and it is not necessary symmetric. However, the above estimate is still in force as we will discuss the necessary modifications in the argument of [44] at the end of this section after the proof of our main theorem.

Proof of Theorem 5.0.2: For each i, we rewrite each equation for u_i as follows

$$u_t^i = \Delta u_i + F_i$$

where $F_i = \sum_{i,j}(a_{ij}(u) - \delta_{ij})\Delta u_j + a_u(u)\nabla u \bullet \nabla u + f(u, \nabla u)$, where δ_{ij} is the Kronecker delta. We now apply $ii)$ of [27, Lemma 2.5] here to obtain

$$\|u(\bullet, t)\|_{C^\nu(\Omega)} \leq C\|u(\bullet, \tau)\|_{L^r(\Omega)} + C_\beta \int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \|F(\bullet, s)\|_{L^r(\Omega)} ds$$

for any $t > T+1$, $\tau = t-1$ and $\beta \in (0,1)$ satisfying $2\beta > \nu + n/r$, and for some fixed constants $C, \delta, C_\beta > 0$. By Hölder's inequality, we have

$$\int_\tau^t (t-s)^{-\beta} e^{-\delta(t-s)} \|F(\bullet, s)\|_{L^r(\Omega)} ds \leq \|F\|_{L^r(Q_{\tau,t})} \left[\int_\tau^t (t-s)^{-\beta r'} e^{-\delta(t-s)r'} ds\right]^{1/r'}.$$

Here $r' = \frac{r}{r-1}$. The last integral is bounded by a constant $C(\beta, r, \delta)$ as long as $\beta r' \in (0,1)$ or r is sufficiently large. On the other hand, since $\|u(\bullet, t)\|_{L^\infty(\Omega)}$ is uniformly bounded for large t, $|F(\bullet, t)| \leq C(|\Delta u| + |\nabla u|^2)$. This, (5.1.1) (with $p = 2r$) and Schauder estimate (5.2.1) imply that there is a constant C_r such that

$$\|F\|_{L^r(Q_{\tau,t})} \leq C_r, \quad \forall t > T.$$

Putting these facts together, we now choose r sufficiently large and $\beta < 1$ such that $\nu > 1$. We then see that $\|u_i(\bullet, t)\|_{C^\nu(\Omega)}$ is uniformly bounded for large t. This proves (5.0.10).

Concerning the compactness, let $p > n \geq 2$ be given and K be a bounded subset in $\bigotimes_{i=1}^m W^{1,p}(\Omega)$. We consider solutions u with their initial data $u_0 \in K$. Estimate (5.0.10) shows that K_t is a bounded subset of $\bigotimes_{i=1}^m C^\nu(\Omega)$. By using the well-known compact imbedding $C^\nu(\Omega) \subset W^{1,p}(\Omega)$, K_t is a compact subset of $\bigotimes_{i=1}^m W^{1,p}(\Omega)$. The proof of this theorem is complete. □

Remark 5.2.2. Our results here are still true for the case a is a symmetric tensor. Indeed, one may realize that the only place needed to reprove is Lemma 5.1.3. To this end, we test the equations of u^i by $u_t^i \xi^2$. Symmetry of the tensor a and integration by parts give

$$\begin{aligned}
\iint_{Q_{2R}} f(u, \nabla u) u_t \xi^2 \, dz &= \iint_{Q_{2R}} [u_t^i - D_\alpha(a_{\alpha\beta}^{ij} D_\beta u^j)] u_t^i \xi^2 \, dz \\
&= \iint_{Q_{2R}} [|u_t|^2 \xi^2 + a_{\alpha\beta}^{ij} D_\beta u^j D_\alpha u_t^i \xi^2 + 2a_{\alpha\beta}^{ij} D_\beta u^j u_t^i \xi D_\alpha \xi] \, dz \\
&= \iint_{Q2R} [|u_t|^2 \xi^2 + \frac{\partial}{\partial t}\left(a_{\alpha\beta}^{ij} D_\beta u^j D_\alpha u^i \xi^2/2\right) + 2a_{\alpha\beta}^{ij} D_\beta u^j u_t^i \xi D_\alpha \xi \\
&- (a_{\alpha\beta}^{ij})_t D_\beta u^j D_\alpha u^i \xi^2/2 - a_{\alpha\beta}^{ij} D_\beta u^j u^i \xi \xi_t] dz
\end{aligned}$$

From this, by using the ellipticity condition $a_{\alpha\beta}^{ij} D_\beta u^j D_\alpha u^i \geq \lambda |Du|^2$ and Young's inequality, it is not difficult to deduce

$$\int_{\Omega_R} |\nabla u|^2 \, dx + \iint_{Q_{2R}} |u_t|^2 \xi^2 \, dz \le C \iint_{Q_{2R}} [|\nabla u|^4 \xi^2 + (|\nabla u|^2 + 1)(|\nabla \xi|^2 + |\xi_t| + \xi^2)] \, dz.$$

Thanks to Lemmas 5.1.2 and 2.2.8, we obtain from this

$$\int_{\Omega_R} |\nabla u|^2 \, dx + \iint_{Q_R} (|u_t|^2 + |D^2 u|^2) \, dz \le C \iint_{Q_{2R}} (|\nabla u|^2 + 1)(|\nabla \xi|^2 + |\xi_t| + \xi^2) \, dz.$$

Lemma 5.1.3 follows from this at once. The proof for the case a is the tensor is complete. $\square$

Remark 5.2.3. The case of Robin boundary conditions can be reduced to the Neumann one by a simple change of variables. First of all, since our proof is based on the local estimates of Lemmas 5.1.3 and 5.1.4, we need only to study these inequalities near the boundary. As $\partial\Omega$ is smooth, we can locally flatten the boundary and assume that $\partial\Omega$ is the plane $\{x_n = 0\}$. Furthermore, we can take $\Omega_R = \{(x', x_n) \,:\, x_n > 0, \, |(x', x_n)| < R\}$. The boundary conditions become

$$\frac{\partial u_i}{\partial x_n} + \widetilde{r}_i(x') u_i = 0.$$

We then introduce $U(x', x_n) = (U_1(x', x_n), ..., U_m(x', x_n))$ with

$$U_i(x', x_n) = \exp(x_n \widetilde{r}_i(x')) u_i(x', x_n)$$

Obviously, U satisfies the Neumann boundary condition on $x_n = 0$. Simple calculations also show that U verifies a system similar to that for u, and conditions (Q.1), (Q.2) are still valid. In fact, there will be some extra terms occurring in the divergence parts of the equations for U, but these terms can be handled by a simple use of Young's inequality so that our proof is still in force. Thus Theorem 5.0.2 applies to U, and the estimates for u then follow.

Finally, we conclude this section by a brief discussion of Lemma 5.2.1. A careful reading of [44] reveals that the only place where the symmetry of $a(u)$ is needed is the proof of [44, Lemma 1].

This lemma concerns the estimates for solutions to homogeneous systems with constant coefficients

$$v_t^i - A_{ij}\Delta v^j = 0 \quad \text{in } Q_R \tag{5.2.2}$$

which $v = 0$ on $\partial B_R^+ \bigcap \{x_n > 0\} \times [-R^2, 0]$ and on $B_R^+ \times \{-R^2\}$ and $\dfrac{\partial v}{\partial n} = 0$ on $B_R^+ \bigcap \{x_n = 0\} \times [-R^2, 0]$.

The lemma is stated as follows.

Lemma 5.2.4. *Let $0 < r \leq R$. Then any smooth solution v of (5.2.2) satisfies*

a.

$$\iint_{Q_{r/2}} |v_t|^2 \, dz \leq Cr^{-2} \iint_{Q_r} |\nabla v|^2 \, dz. \tag{5.2.3}$$

b. *for $k = 1, 2, 3, ...$*

$$\iint_{Q_{r/2}} |\nabla^k v|^2 \, dz \leq C_k r^{-2k} \iint_{Q_r} |v|^2 \, dz. \tag{5.2.4}$$

c. *for any $0 < \rho < r \leq R$,*

$$⨏\!\!\!⨏_{Q_\rho} |v|^2 \, dz \leq C \left(\frac{\rho}{r}\right)^2 ⨏\!\!\!⨏_{Q_r} |v|^2 \, dz. \tag{5.2.5}$$

Thus, Lemma 5.2.1 is proven if we can relax the symmetry assumption in this lemma.

Proof: Let $v = (v^i)$ be a solution of (5.2.2), that is,

$$\iint_{Q_{2R}} v_t^i \phi + A_{ij} \nabla v^j \nabla \phi \, dz = 0, \tag{5.2.6}$$

where $\phi \in C^1(Q_R)$ such that $\phi = 0$ on $\partial B_R^+ \bigcap \{x_n > 0\} \times [-R^2, 0]$ and on $B_R^+ \times \{-R^2\}$.

Let η be a cut-off function in Q_r such that $\eta = 1$ in $Q_{r/2}$, $\eta(., -r^2) = 0$, and η vanishes on $\partial B_r \bigcap \{x_n > 0\} \times [-r^2, 0]$.

By squaring equations (5.2.2) and summing up the results, we have

$$\iint_{Q_r} |v_t|^2 \eta \, dz \leq C \iint_{Q_r} |\Delta v|^2 \eta \, dz. \tag{5.2.7}$$

Now, by choosing $\phi = \Delta v^i \eta^2$ in (5.2.6), one can easily see that

$$\iint_{Q_r} v_t^i \Delta v^i \eta^2 \, dz - \iint_{Q_r} A_{ij} \Delta v^j \Delta v^i \eta^2 \, dz = 0.$$

Thanks to the ellipticity and integrations by parts, we obtain

$$\begin{aligned}
\lambda \iint_{Q_r} |\Delta v|^2 \eta^2 \, dz &\leq \iint_{Q_r} v_t^i \Delta v^i \eta^2 \, dz \\
&= -\frac{1}{2} \frac{\partial}{\partial t} \iint_{Q_r} |\nabla v^i|^2 \eta^2 \, dz - \iint_{Q_r} (\nabla v^i v_t^i \nabla \eta - |\nabla v^i|^2 \eta_t) \eta \, dz \\
&\leq \epsilon \iint_{Q_r} |v_t|^2 \eta \, dz + C \iint_{Q_r} (|\eta_t| + |\nabla \eta|^2) |\nabla v|^2 \, dz.
\end{aligned}$$

Using this, (5.2.7), and the fact that $|\eta_t|, |\nabla \eta|^2 \leq Cr^{-2}$, we easily get (5.2.3).

In order to prove (5.2.4) for $k = 1$, we choose $\phi = v^i \eta^2$ in (5.2.2). It is now standard to see that

$$\iint_{Q_{r/2}} |\nabla v|^2 \, dz \leq Cr^{-2} \iint_{Q_r} |v|^2 \, dz.$$

From this point on, we can follow [44] to complete the proof. □

Notes and Remarks

It is well known that the existence of the global attractor for reaction diffusion systems was very much investigated (e.g., see [24]). However, to the best of our knowledge, there has been not much work on such dynamics or long time behavior of solutions to the cross diffusion systems. Some works in this direction are due to Redlinger in [42] for certain triangular systems and in some years later to Le and others in [27, 20, 36] for more general triangular systems. Theorems 5.0.1 and 5.0.2 which now also appear in [35] are therefore new in literature for full cross diffusion systems (and even for triangular ones, see [34]). More importantly, these theorems give uniform estimates in Hölder norms of gradients. It should be noticed that the fact the Hölder continuity of Du follows from that of solutions u to (5.0.6) was established in [9]. However, no such estimates of Hölder norms of gradients were established in there as well as in recent developments. As far as we are aware of, such estimates first exist in literature due to [35].

It is also worth noticing that even although our theory was stated for a very general class of cross diffusion systems, it relies presumedly on the *everywhere Hölder continuity* of weak solutions to (5.0.6), which has been very long standing in question. Certainly, the best regularity one can obtain for solutions of such systems is *partial regularity* (see [9]). Moreover, counterexamples in [16] show that it is hopeless in general to expect that if we do not add more conditions on structure of the systems. Therefore, finding certain conditions to obtain the everywhere regularity becomes an interesting problem. Chapters 3.0.1 and 4 are affirmative examples in this direction.

Finally, we would like to highlight that the establishment of the global attractor result and uniform estimates in C^1 of solutions for the systems allows us to apply powerful results on uniform persistence (permanence), developed by Hale-Waltman [12], Thieme [49, 50], and Hirsch-Smith-Zhao [15]. The employment will be in detail discussed in the following chapter. Once uniform persistence is obtained, the existence of a positive equilibrium solution of the modeling system, representing the survival of the competitive population, then follows at once (e.g., see [41]).

Chapter 6

PERSISTENCE PROPERTY

The goal of this chapter is to study uniform persistence property of positive solutions of the generalized SKT model. We seek sufficient conditions in terms of principal eigenvalues of linearized problems for the existence of positive threshold levels below which time dependent solutions will never be for large t. Our proof mainly bases on a persistence result in [15] for general dynamical systems defined on metric spaces.

6.1 Main result

Owing to the preceding chapters, we have known the regularity as well as ultimately uniform estimates in C^1-norms of solutions to the cross diffusion system of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla[(d_1 + a_{11}u + a_{12}v)\nabla u + b_{11}u\nabla v] + u(a_1 - b_1 u - c_1 v), \\ \dfrac{\partial v}{\partial t} = \nabla[b_{22}v\nabla u + (d_2 + a_{21}u + a_{22}v)\nabla v] + v(a_2 - b_2 u - c_2 v), \end{cases} \tag{6.1.1}$$

which is supplied by the Robin boundary condition $\dfrac{\partial u}{\partial n} + r_1 u = \dfrac{\partial v}{\partial n} + r_2 v = 0$ on $\partial\Omega$, for some nonnegative continuous functions r_1, r_2.

Also due to Theorem 5.0.1, the existence of the global attractor for the above system was established. That is, the corresponding dynamical system possesses a set that attracts all bounded sets or towards which all the orbits converge. However, it is possible that the global attractor may contain trivial or semitrivial steady states. Regarding to biological implications, such solutions describe the ultimate wash out of populations. This possibility may be the case since each species in population dynamics is assumed to be capable of attachment to one another ($c_1, b_2 > 0$). Therefore we are interested in seeking sufficient conditions to protect against the wash out or to guarantee the coexistence.

Coexistence problems for cross diffusion systems were also extensively studied (see [21] and the reference therein). However, whether these coexistence states are observable, that is their stability, is still yet to be determined. This question remains widely open even for the simpler Lotka-Volterra counterpart. Coexistence in the sense of uniform persistence would then be more appropriate and realistic. Strictly speaking, by uniform persistence, we take the sense that we seek sufficient conditions for the existence of $\eta > 0$, independent of the nonnegative initial data (u_0, v_0), such that

$$\|u(\bullet, t)\|_{C^1}, \|v(\bullet, t)\|_{C^1} \geq \eta$$

for all large t, say $t \geq T$, where T may depend on the initial data, provided that densities of both species are initially present, that is, both u_0 and u_0 do not vanish identically. In biological terms, uniform persistence means that no species will be either wiped out or completely invaded by others so that they coexist in time. For background information about persistence properties and references we refer to [49, 50].

We attain our main results by taking advantage of the theory on uniform persistence for general dynamical systems defined on metric spaces, developed by Hirsch, Smith, and Zhao [15]. The results will be stated under assumptions on the positivity of the principal eigenvalues of linearized problems at the wash out steady states. That is, these steady states are assumed to be unstable (or repelling) in their complementary directions. Once uniform persistence is obtained, the existence of a positive equilibrium solution of the modeling system, representing the survival of the competitive population, then follows at once (e.g., see [41]).

Let u_*, v_* be the unique positive solutions (see [3]) to

$$0 = \nabla(P^u(u_*, 0)\nabla u_*) + f(u_*, 0), \qquad 0 = \nabla(Q^v(0, v_*)\nabla v_*) + g(0, v_*),$$

and the same Robin boundary condition as that of u, v. Here and throughout this chapter, for simplicity, we set

$$\begin{aligned} P^u &= d_1 + a_{11}u + a_{12}v, \quad P^v = b_{11}u, \\ Q^v &= d_2 + a_{21}u + a_{22}v, \quad Q^u = b_{22}v, \end{aligned}$$

and

$$f(u,v) = u(b_1 - c_{11}u - c_{12}v), \quad g(u,v) = v(b_2 - c_{21}u - c_{22}v).$$

We consider the eigenvalue problems

$$\lambda\psi = d_1\Delta\psi + a_1\psi, \quad \text{and} \quad \lambda\phi = d_2\Delta\phi + a_2\phi, \tag{6.1.2}$$

$$\lambda\psi = \nabla[P^u(0,v_*)\nabla\psi + \partial_u P^v(0,v_*)\psi\nabla v_*] + \partial_u f(0,v_*)\psi, \tag{6.1.3}$$

$$\lambda\phi = \nabla[Q^v(u_*,0)\nabla\phi + \partial_v Q^u(u_*,0)\phi\nabla u_*] + \partial_v g(u_*,0)\phi \tag{6.1.4}$$

with the boundary conditions $\dfrac{\partial\psi}{\partial n} + r_1\psi = \dfrac{\partial\phi}{\partial n} + r_2\phi = 0$.

Our persistence result reads as follows.

Theorem 6.1.1. *Assume that $a_{ij}, b_{22} \geq 0, d_i, b_{11} > 0, i,j = 1,2$, and*

$$a_{11} - a_{21} > b_{22}, \quad a_{22} - a_{12} > b_{11}. \tag{6.1.5}$$

Furthermore, suppose that the principal eigenvalues of (6.1.2), (6.1.3) and (6.1.4) are positive. If Robin boundary conditions are considered, we also assume further that the two quantities $a_{12} - b_{11}$ and $a_{21} - b_{22}$ are positive and sufficiently small.

Then system (6.1.1) is uniformly persistent. That is, there exists $\eta > 0$ such that any its solution (u,v), whose initial data $u_0, v_0 \in W^{1,p}(\Omega)$ are positive, satisfies

$$\liminf_{t\to\infty} \|u(\bullet,t)\|_{C^1(\Omega)} \geq \eta, \quad \liminf_{t\to\infty} \|v(\bullet,t)\|_{C^1(\Omega)} \geq \eta. \tag{6.1.6}$$

The conditions (6.1.5) are due to Theorem 5.0.1, where they are required to obtain the existence of the global attractor for the dynamical system associated with (6.1.1) (More precisely, they are only needed to establish the uniform a priori estimates of solutions in Theorems 3.5.2 and 4.1.2). As mentioned earlier, the positivity of the principal eigenvalues in theorem means that the trivial steady state $(0,0)$ is repelling in the $(u,0),(0,v)$ directions, and the semitrivial steady states $(u_*,0)$, $(0,v_*)$ are unstable in their complementary directions. At the end of this chapter (see Lemma 6.2.6 and

Lemma 6.2.7), we also present explicit sufficient conditions on the parameters of (6.1.1) that guarantee this positivity.

6.2 Proof of the main result

Let us first recall some definitions in the dynamical system theory. Let (X, d) be a metric space and Φ be a semiflow on X. For a nonempty invariant set M, the set $W^s(M) := \{x \in X : \lim_{t\to\infty} d(\Phi_t(x), M) = 0\}$ is called the **stable** set of M. Here, $d(x, A)$ is the usual Hausdorff distance from x to the set A. A nonempty invariant subset M of X is said to be **isolated** if it is the maximal invariant set in some neighborhood of itself.

Let A and B be two isolated invariant sets. A is said to be **chained** to B, denoted by $A \to B$, if there exists a globally defined trajectory $\Phi_t(x)$, $t \in (-\infty, \infty)$, through some $x \notin A \bigcup B$ whose range has compact closure such that the omega limit set $\omega(x) \subset B$ and the alpha limit set $\alpha(x) \subset A$. A finite sequence $\{M_1, M_2, ..., M_k\}$ of isolated invariant sets is called a **chain** if $M_1 \to M_2 \to ... \to M_k$. The chain is called a **cycle** if $M_k = M_1$.

Let $X_0 \subset X$ be an open set and $\partial X_0 = X \setminus X_0$. Assume that X_0 is positively invariant. Let $p(x) = d(x, \partial X_0)$, the distance from x to ∂X_0. Φ is said to be **uniformly persistent** with respect to $(X_0, \partial X_0, p)$ if there exists $\eta > 0$ such that

$$\liminf_{t\to\infty} p(\Phi_t(x)) \geq \eta$$

for all $x \in X_0$.

The following uniform persistence result is established in [15].

Theorem 6.2.1. *(Theorem 4.3 in [15]) Assume that*

(C1) Φ *has a global attractor* A*;*

(C2) *There exists a finite sequence* $M = \{M_1, ..., M_k\}$ *of pairwise disjoint, compact and isolated invariant sets in* ∂X_0 *with the following properties:*

(m.1) $\bigcup_{x\in\partial X_0} \omega(x) \subset \bigcup_{i=1}^{k} M_i$,

(m.2) *no set of M forms a cycle in ∂X_0,*

(m.3) *M_i is isolated in X,*

(m.4) *$W^s(M_i)\bigcap X_0 = \emptyset$ for each $i = 1, ..., k$.*

Then there exists $\delta > 0$ such that for any $x \in X_0$, the following inequality holds

$$\inf_{y\in\omega(x)} d(y, \partial X_0) > \delta.$$

We will apply this theorem to prove our main result, Theorem 6.1.1. Denote $X = C^1_+(\Omega) \times C^1_+(\Omega)$ and its positive cone $X_0 = \{(u, v) \in X \ : \ u > 0 \text{ and } v > 0\}$. Then (X, d), with $d(x, y) = \|x - y\|_{C^1(\Omega)}$, is a complete metric space. The boundary of X_0 consists of $I_u := \{(u, 0) \ : \ u \geq 0\}$ and $I_v := \{(0, v) \ : \ v \geq 0\}$. Thanks to Theorem 5.0.1, we can define the semiflow on X as follows: for any initial data (u_0, v_0) in X, define $\Phi_t(u_0, v_0) = (u(\bullet, t), v(\bullet, t))$ for all $t \geq 0$. Estimate (5.0.5) also gives that Φ is a compact semiflow and possesses a global attractor in X. A simple application of maximum principles for scalar parabolic equations shows that X_0, I_u, I_v are positively invariant under Φ (see [38]). Therefore, (C.1) is verified.

Next, we consider the condition (C.2). It is clear that the "boundary" parts $u = 0$ or $v = 0$ of X_0 are also invariant with respect to Φ. On these boundaries, the dynamics of (6.1.1) is reduced to those of the following scalar parabolic equations.

$$u_t = \nabla(P^u(u, 0)\nabla u) + f(u, 0), \quad u(0) > 0, \tag{6.2.1}$$

$$v_t = \nabla(Q^v(0, v)\nabla v) + g(0, v), \quad v(0) > 0. \tag{6.2.2}$$

Investigating the dynamics of these equations leads us to the following steady state equations

$$\nabla(P^u(u_*, 0)\nabla u_*) + f(u_*, 0) = 0, \quad \nabla(Q^v(0, v_*)\nabla v_*) + g(0, v_*) = 0,$$

together with the Robin boundary conditions. If the principal eigenvalues of (6.1.2) are positive, the above equations admit unique solutions, which are denoted respectively by u_* and v_*. Furthermore, the solutions $u(x, t), v(x, t)$ of (6.2.1), (6.2.2) converge to u_*, v_*, respectively, in the C^1 norm as t

tends to infinity. Meanwhile, the trivial solution 0 is an unstable steady state for both equations. These claims are obtained by following closely the proof of [3, Corollary 2.4], where the Dirichlet boundary condition was assumed.

Therefore, the sets $M_0 = (0,0)$, $M_1 = (u_*, 0)$, and $M_2 = (0, v_*)$ are pairwise disjoint, compact and isolated invariant sets in ∂X_0 with respect to Φ. Moreover, no set of $\{M_i\}$ can form a cycle in ∂X_0; and $\bigcup_{x\in\partial X_0} \omega(x) \subset \bigcup_{i=0}^{2} M_i$. We thus show that the conditions (m.1) and (m.2) are satisfied.

Checking (m.3) and (m.4) requires much more effort. The role of the parameters r_1, r_2 will play an important role here. Let us assume that the system (6.1.1) satisfies the Robin boundary condition with $r_1, r_2 \neq 0$. The Neumann case is simpler, and will be discussed later in Remark 6.2.5.

We discuss first the property (m.4) at M_0. We will show below that the instability of M_0 is determined by the principal eigenvalue λ of (see (6.1.2))

$$\begin{cases} \lambda\phi = d_2\Delta\phi + a_2\phi, \\ \dfrac{\partial\phi}{\partial n} + r_2\phi = 0. \end{cases} \tag{6.2.3}$$

Proposition 6.2.2. *Assume that the principal eigenvalue λ of (6.2.3) is positive. There exists $\eta_0 > 0$ such that for any solution (u, v) of (6.1.1) with $(u_0, v_0) \in X_0$, we have*

$$\limsup_{t\to\infty} \|(u(.,t), v(.,t))\|_X \geq \eta_0.$$

Proof: Let ϕ be the positive eigenfunction associated with the principal eigenvalue λ of (6.2.3). By testing the equation of v by ϕ and (6.2.3) by v, we subtract the results to get

$$\begin{aligned} \frac{d}{dt}\int_\Omega v\phi\, dx &= \lambda\int_\Omega v\phi\, dx - \int_{\partial\Omega} (Q_0 r_2 + Q_v^u r_1 u)v\phi\, d\sigma \\ &+ \int_\Omega [-Q_0\nabla v\nabla\phi - Q^u\nabla u\nabla\phi + (g - a_2 v)\phi]\, dx. \end{aligned} \tag{6.2.4}$$

Here, we denoted $Q_0 = Q^v - d_2 = a_{21}u + a_{22}v$. Integration by parts yields

$$-\int_\Omega Q_0\nabla v\nabla\phi\, dx = \int_\Omega v\nabla(Q_0\nabla\phi)\, dx + \int_{\partial\Omega} r_2 Q_0 v\phi\, d\sigma.$$

Putting this in (6.2.4), we infer

$$\frac{d}{dt}\int_{\Omega} v\phi\ dx = \lambda\int_{\Omega} v\phi\ dx + \int_{\Omega} v\phi\frac{\nabla(Q_0\nabla\phi) - Q_v^u\nabla u\nabla\phi}{\phi}\ dx - \int_{\Omega}(b_2u + c_2v)v\phi\ dx.$$

Now, suppose that our claim was false. For any $\eta > 0$, there would be a solution (u, v) such that $\|(u(.,t), v(.,t))\|_X \leq \eta$ when t is large. This implies that the quantities $\frac{|\nabla(Q_0\nabla\phi)|}{\phi}$, $\frac{|Q_v^u\nabla u\nabla\phi|}{\phi}$, and $(b_2u + c_2v)$ can be very small. Thus, if η is sufficiently small, then the above equation yields

$$\frac{d}{dt}\int_{\Omega} v\phi\ dx \geq \frac{\lambda}{2}\int_{\Omega} v\phi\ dx.$$

This shows that, as $t \to \infty$, $\displaystyle\int_{\Omega} v(.,t)\phi\ dx$ goes to infinity, contradicting the fact that $\|(u,v)\|_X$ is bounded. Our proof is complete. □

Next, we study M_1 and M_2. Our main assumption for (m.3) and (m.4) to hold is the instability of M_1, M_2 in their complement v, u directions, respectively. To this end, we consider the linearization of the system (6.1.1) at a general steady state point (u, v).

$$\begin{cases} \lambda\psi = \nabla\left[(P_u^u\psi + P_v^u\phi)\nabla u + P^u\nabla\psi + P_u^v\psi\nabla v + P^v\nabla\phi\right] + f_u\psi + f_v\phi, \\ \lambda\phi = \nabla[Q_v^u\phi\nabla u + Q^u\nabla\psi + (Q_u^v\psi + Q_v^v\phi)\nabla v + Q^v\nabla\phi] + g_u\psi + g_v\phi. \end{cases} \tag{6.2.5}$$

Here ψ (respectively, ϕ) satisfies the same boundary condition as that of u (respectively, v).

Putting $(u, v) = (0, v_*)$ and $(\psi, \phi) = (\psi, 0)$ in (6.2.5), the instability of $M_2 = (0, v_*)$ in the direction u is determined by the sign of the principal eigenvalue of the following system.

$$\lambda\psi = \nabla(P^u(0, v_*)\nabla\psi + P_u^v\psi\nabla v_*) + f_u(0, v_*)\psi, \tag{6.2.6}$$

with v_* being the solution of

$$0 = \nabla(Q^v(0, v_*)\nabla v_*) + g(0, v_*). \tag{6.2.7}$$

We shall establish the following repelling property of $(0, v_*)$.

Proposition 6.2.3. *Suppose that the principal eigenvalue λ of (6.2.6) is positive. If $P_v^u - P_u^v = a_{12} - b_{11}$ is positive and sufficiently small, then there exists $\eta_0 > 0$ such that for any solution (u, v)*

of (6.1.1) with $(u_0, v_0) \in X_0$*, we have*

$$\limsup_{t\to\infty} \|(u(.,t), v(.,t)) - (0, v_*)\|_X \geq \eta_0.$$

Similarly, putting $(u, v) = (u_*, 0)$ and $(\psi, \phi) = (0, \phi)$ in (6.2.5), the instability of $M_1 = (u_*, 0)$ in the direction v is determined by the sign of the principal eigenvalue of the following system.

$$\lambda\phi = \nabla(Q^u_v \phi \nabla u_* + Q^v(u_*, 0)\nabla\phi) + g_v(u_*, 0)\phi, \tag{6.2.8}$$

with u_* being the solution of

$$0 = \nabla(P^u(u_*, 0)\nabla u_*) + f(u_*, 0). \tag{6.2.9}$$

We shall also establish the following repelling property of $(u_*, 0)$.

Proposition 6.2.4. *Suppose that the principal eigenvalue* λ *of (6.2.8) is positive. If* $Q^v_u - Q^u_v = a_{21} - b_{22}$ *is positive and sufficiently small, then there exists* $\eta_0 > 0$ *such that for any solution* (u, v) *of (6.1.1) with* $(u_0, v_0) \in X_0$*, we have*

$$\limsup_{t\to\infty} \|(u(.,t), v(.,t)) - (u_*, 0)\|_X \geq \eta_0.$$

An immediate consequence of these propositions is that $W^s(M_i) \bigcap X_0 = \emptyset$, $i = 0, 1, 2$ respectively. Otherwise, by the definition of $W^s(M_i)$, there exists $(u_0, v_0) \in X_0$ such that $d((u(t), v(t)), M_i) \to 0$ as $t \to \infty$, a contradiction to the above corresponding propositions.

Moreover, we also see that M_i is isolated in X. Indeed, consider a neighborhood of M_i in X_0, $V = \{(u, v) \in X_0 \ : \ d((u, v), M_i) < \eta_0/2\}$. For any $(u_0, v_0) \in X_0 \bigcap V$, the above proposition shows that $(u(t), v(t))$ will inevitably exits V. This means M_i is maximal in V and isolated in X.

We now give the proof of Proposition 6.2.3. The proof of Proposition 6.2.4 is obviously the same and we will omit it.

Proof: *(Proof of Proposition 6.2.3).* The proof is by contradiction. Assume that for any $\eta > 0$ there exist a solution (u, v) of (6.1.1) and $T > 0$ such that

$$\|u(.,t)\|_{C^1(\Omega)}, \|v(.,t) - v_*\|_{C^1(\Omega)} < \eta \tag{6.2.10}$$

for all $t > T$. Hereafter, we always consider $t > T$.

We denote $P_0 = P^u(0, v_*)$ and recall (6.2.5):

$$\lambda\psi = \nabla(P_0\nabla\psi + P^v_u\psi\nabla v_*) + f_u(0, v_*)\psi.$$

Set $\overline{P}(u, v) = \int_0^u P^u(s, v)ds$. We note that $\nabla\overline{P}(u, v) = P^u\nabla u + P^u_v u\nabla v$. Testing the above equation with $\overline{P}$, we obtain

$$\begin{aligned}\lambda\int_\Omega \psi\overline{P}(u, v)\ dx &= -\int_\Omega P_0P^u\nabla\psi\nabla u\ dx - \int_\Omega P_0P^u_v u\nabla\psi\nabla v\ dx \\ &- \int_\Omega P^v_u\psi\nabla v_*(P^u\nabla u + P^u_v u\nabla v)\ dx + \int_\Omega f_u((0, v_*))\psi\overline{P}\ dx \\ &+ \int_{\partial\Omega} (P_0\frac{\partial\psi}{\partial n} + P^v_u\psi\frac{\partial v_*}{\partial n})\overline{P}\ d\sigma.\end{aligned} \tag{6.2.11}$$

Similarly, we test the equation of u in (6.1.1) with $P_0\psi$ ($\nabla(P_0\psi) = P^u_v\psi\nabla v_* + P_0\nabla\psi$), and get

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega P_0u\psi\ dx &= -\int_\Omega P^uP_0\nabla u\nabla\psi\ dx - \int_\Omega P^uP^u_v\psi\nabla u\nabla v_*\ dx \\ &- \int_\Omega P^v\nabla v(P_0\nabla\psi + P^u_v\psi\nabla v_*)\ dx + \int_\Omega fP_0\psi\ dx \\ &+ \int_{\partial\Omega} (P^u\frac{\partial u}{\partial n} + P^v\frac{\partial v}{\partial n})P_0\psi\ d\sigma.\end{aligned} \tag{6.2.12}$$

From (6.2.11), (6.2.12) and the boundary condition, we find

$$\begin{aligned}\frac{\partial}{\partial t}\int_\Omega P_0u\psi\ dx &= \lambda\int_\Omega \psi\overline{P}\ dx + (P^u_v - P^v_u)\int_\Omega [P_0u\nabla v\nabla\psi - P^u\psi\nabla u\nabla v_*]\ dx \\ &+ \int_\Omega (fP_0 - f_u(0, v_*)\overline{P})\psi\ dx + I_\partial,\end{aligned} \tag{6.2.13}$$

where $I_\partial = \displaystyle\int_{\partial\Omega} (\overline{P} - P^u u)r_1\psi P_0 + (\overline{P}v_* - uvP_0)P^v_u r_2\psi\ d\sigma.$

Next, we shall show that the integrals on the right of (6.2.13) are either nonnegative or controlled by the first integral. From the definition of the parameters, we have

$$\begin{aligned}\overline{P}\psi &= (d_1+\alpha_{12}v)u\psi+\frac{\alpha_{11}u^2}{2}\psi \geq P_0u\psi+P_v^u(v-v_*)u\psi,\\ (fP_0-f_u(0,v_*)\overline{P})\psi &= (c_1(v_*-v)-b_1u)P_0u\psi+f_u(\alpha_{12}(v_*-v)-\frac{\alpha_{11}u}{2})u\psi.\end{aligned}$$

Hence, if η in (6.2.10) is sufficiently small, the above gives

$$\int_\Omega \overline{P}\psi\ dx \geq \frac{3}{4}\int_\Omega P_0u\psi\ dx, \quad \left|\int_\Omega (fP_0-f_u(0,v_*)\overline{P})\psi\ dx\right| \leq \frac{\lambda}{4}\int_\Omega P_0u\psi\ dx.$$

On the other hand, integrate by parts to get

$$\begin{aligned}-\int_\Omega P\psi\nabla u\nabla v_*\ dx &= \int_\Omega u\nabla(P^u\psi\nabla v_*)\ dx-\int_{\partial\Omega} uP^u\psi\frac{\partial v_*}{\partial n}\ d\sigma\\ &= \int_\Omega u\psi\frac{\nabla(P^u\psi\nabla v_*)}{\psi}\ dx+\int_{\partial\Omega} uP^u\psi r_2v_*\ d\sigma.\end{aligned}$$

Thanks to (6.2.10) and the fact that $\psi>0$ on $\bar{\Omega}$, the quantities $|\nabla v||\frac{\nabla\psi}{\psi}|$, $\frac{\nabla(P^u\psi\nabla v_*)}{P_0\psi}$ are bounded. Thus, if $P_v^u-P_u^v$ is positive and sufficiently small, then

$$(P_v^u-P_u^v)\int_\Omega [P_0u\nabla v\nabla\psi-P^u\psi\nabla u\nabla v_*]\ dx \geq -\frac{\lambda}{4}\int_\Omega P_0u\psi\ dx+(P_v^u-P_u^v)\int_{\partial\Omega} P^ur_2v_*u\psi\ d\sigma.$$

Putting these facts in (6.2.13), we derive

$$\frac{\partial}{\partial t}\int_\Omega P_0u\psi\ dx \geq \frac{\lambda}{4}\int_\Omega P_0u\psi\ dx+I_\partial+(P_v^u-P_u^v)\int_{\partial\Omega} uP^u\psi r_2v_*\ d\sigma.$$

Finally, we study the boundary integrals. Straightforward calculations show

$$I_\partial=\int_{\partial\Omega}[-\frac{\alpha_{11}}{2}ur_1P_0+(d_1(v_*-v)+\frac{\alpha_{11}}{2}uv_*)P_u^vr_2]u\psi\ d\sigma.$$

If η in (6.2.10) is small, then it is clear that the quantity in the brackets can be very small. Thus, I_∂ can be controlled by the positive boundary integral $(P_v^u-P_u^v)\int_{\partial\Omega} P^ur_2v_*u\psi\ d\sigma$.

Therefore

$$\frac{\partial}{\partial t}\int_{\Omega} P_0 u\psi\ dx \geq \frac{\lambda}{4}\int_{\Omega} P_0 u\psi\ dx. \tag{6.2.14}$$

As $\lambda > 0$, this shows that $\displaystyle\int_{\Omega} P^u(0, v_*)u\psi\ dx$ goes to infinity as t does. This contradicts (6.2.10) and completes this proof. □

Remark 6.2.5. If the boundary conditions are of Neumann type, then u_*, v_*, ψ, ϕ in the above proofs are just constant functions and our calculations will be much simpler. In fact, it is easy to see that the smallness condition for $P^u_v - P^v_u$ (respectively $Q^v_u - Q^u_v$) in Proposition 6.2.3 (respectively Proposition 6.2.4) is no longer needed.

Next, we will present explicit and simple criteria on the parameters of (6.1.1) for the positivity of the principal eigenvalues of (6.2.6), (6.2.8).

Lemma 6.2.6. *Assume that either $r_1 = r_2 \equiv 0$ and $a_1/a_2 > c_1/c_2$, or $r_1, r_2 \neq 0$ and*

$$\frac{a_1}{a_2} > \max\left\{\frac{c_1}{c_2}, \frac{2a_{12}}{a_{22}}\right\}, \tag{6.2.15}$$

and

a) *$a_{12} > b_{11}$ and $d_1 a_{22} \geq 2d_2 b_{11}$;*

b) *$\sup_{\partial\Omega}(r_1(x) - r_2(x))_+$ and $(a_2 d_1 - a_1 d_2)_+$ are sufficiently small.*

Then λ in (6.2.6) is positive.

Proof: Set $P_0 = P^u(0, v_*)$, $Q_0 = Q^v(0, v_*)$. We test (6.2.6) with $\overline{Q} = \int_0^{v_*} Q^v(0, s)ds$ and test (6.2.7) with $P^u(0, v_*)\psi$. Together, we get

$$\begin{aligned}
\lambda\int_{\Omega}\psi\overline{Q}\ dx &= -\int_{\Omega} P_0 Q_0 \nabla\psi\nabla v_*\ dx - \int_{\Omega} P^v_u Q_0 \psi|\nabla v_*|^2\ dx \\
&+ \int_{\partial\Omega}(P_0\frac{\partial\psi}{\partial n} + P^v_u\psi\frac{\partial v_*}{\partial n})\overline{Q}\ d\sigma + \int_{\Omega} f_u\psi\overline{Q}\ dx \\
&= (P^u_v - P^v_u)\int_{\Omega} Q_0\psi|\nabla v_*|^2\ dx + \int_{\Omega}(f_u\overline{Q} - g_0 P_0)\psi\ dx \\
&+ \int_{\partial\Omega}(P_0\frac{\partial\psi}{\partial n} + P^v_u\psi\frac{\partial v_*}{\partial n})\overline{Q}\ d\sigma - \int_{\partial\Omega} Q_0 P_0\psi\frac{\partial v_*}{\partial n}\ d\sigma.
\end{aligned}$$

We need only show that the right hand side is positive. Since $P^u_v = a_{12} > b_{11} = P^v_u$, the first term on the right is nonnegative. For the second integral, we note that

$$f_u\overline{Q} - g_0P_0 = v_*[(a_1 - c_1v_*)(d_2 + \frac{a_{22}}{2}v_*) - (a_2 - c_2v_*)(d_1 + a_{12}v_*)].$$

We study the quantity in the brackets by considering the quadratic

$$\begin{aligned} F(X) &= (a_1 - c_1X)(d_2 + \frac{a_{22}}{2}X) - (a_2 - c_2X)(d_1 + a_{12}X) \\ &= (c_2a_{12} - \frac{1}{2}c_1a_{22})X^2 + (\frac{1}{2}a_1a_{22} - a_2a_{12} + c_2d_1 - c_1d_2)X + a_1d_2 - a_2d_1. \end{aligned}$$

First of all, by a simple use of maximum principles, we can easily show that $0 < v_*(x) \le a_2/c_2$ for all $x \in \overline{\Omega}$. Let $\mu = \inf_\Omega v_*(x) > 0$.

We will show that $F(v_*) > 0$. Firstly, due to (6.2.15),

$$F(0) = a_1d_2 - a_2d_1 \text{ and } F(a_2/c_2) = (a_1 - \frac{a_2c_1}{c_2})(d_2 + \frac{a_2a_{22}}{2c_2}) > 0.$$

Consider the case when the coefficient of X^2 in $F(X)$ is negative. If $F(0) \ge 0$ then $F(v_*) > 0$ because $0 < \mu \le v_*(x) \le a_2/c_2$. If $F(0) < 0$, then $F(X) = 0$ has two positive roots X_1, X_2 with $X_2 > a_2/c_2$. Hence, if $|F(0)|$ is sufficiently small then $\mu > X_1$ and therefore $F(v_*) > 0$.

Otherwise, by (6.2.15), we have $F(v_*) \ge (\frac{1}{2}a_1a_{22} - a_2a_{12} + c_2d_1 - c_1d_2)v_* + F(0)$. If $(c_2d_1 - c_1d_2) \ge 0$, the last quantity is obviously positive when either $F(0) \ge 0$ or $F(0) < 0$ but $|F(0)|$ is small. Or else, because $v_* \le a_2/c_2$ we have

$$F(v_*) \ge (c_2d_1 - c_1d_2)\frac{a_2}{c_2} + a_1d_2 - a_2d_1 = a_2d_2(\frac{a_1}{a_2} - \frac{c_1}{c_2}) > 0.$$

In all cases, $F(v_*) > 0$. Thus, the second integral is also positive. It remains to consider the boundary integrals. In view of the boundary condition, they are

$$\int_{\partial\Omega} (r_2 - r_1)P_0\overline{Q}\psi \, d\sigma + \int_{\partial\Omega} r_2\psi v_*^2 \left(\frac{a_{22}}{2}d_1 - b_{11}d_2 + \frac{a_{22}}{2}(P_v - R_u)v_*\right) \, d\sigma$$

The last integrand is positive due to the first condition in b). Therefore the above sum is nonnegative if either $r_2 \geq r_1$ or $r_1 - r_2 > 0$ but sufficiently small. Therefore, under the stated assumptions in the lemma, λ is positive. $\square$

Similarly, we have the following result.

Lemma 6.2.7. *Assume that either* $r_1 = r_2 \equiv 0$ *and* $b_1/b_2 > a_1/a_2$*, or* $r_1, r_2 \neq 0$ *and*

$$\frac{a_1}{a_2} < \min\left\{\frac{b_1}{b_2}, \frac{a_{11}}{2a_{21}}\right\},$$

and

a) $a_{21} > b_{22}$ *and* $d_2 a_{11} \geq 2d_1 b_{22}$*;*

b) $\sup_{\partial\Omega}(r_2(x) - r_1(x))_+$ *and* $(a_1 d_2 - a_2 d_1)_+$ *are sufficiently small.*

Then λ *in (6.2.8) is positive.*

We conclude this paper by giving the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. It is clear that the stated conditions (P.1) or (P.2) satisfy those of our propositions and lemmas of this section. The theorem then follows from Theorem 6.2.1. $\square$

Notes and Remarks

It is worth noticing that when homogeneous Neumann boundary conditions are assumed, the conditions in Lemma 6.2.6 and Lemma 6.2.7 read

$$\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2},$$

which is already well known for the Lotka-Volterra counterparts (see [3, 14] and the references therein). It is not quite surprising to see that the cross diffusion parameters (a_{ij}, b_{11}, b_{22}) do not manifest in this case as the semitrivial steady states u_*, v_* are being just constants. The situation will be more interesting when we consider the Robin boundary conditions. In this case, the semitrivial steady states are non-constants; and the cross diffusion (or gradient) effects will play an essential role as seen in Lemma 6.2.6 and Lemma 6.2.7.

Finally, we would like to remark that the uniform persistence property in this chapter is established in the C^1 norm instead of the usual L^∞ norm widely used in literature of Lotka-Volterra systems. This is in part due to the setting of the phase space $W^{1,p}$ for strongly coupled parabolic systems (see [2]). So, our persistence result does not rule out the possibility that solutions might form spikes at some points but approach zero almost everywhere as $t \to \infty$. That type of behavior can be seen in some models for chemotaxis, which also involve a form of strong coupling, so it may be that the results presented here are optimal. However, it is naturally to ask if it is impossible for one species can survive in the sense that its density is going to be almost negligible (that is, the L^∞ norm goes to zero) while oscillating wildly to maintain the positivity of its C^1 norm. The answer to this question is still under investigation.

Chapter 7

DEGENERATE PARABOLIC SYSTEMS

In this chapter, a class of strongly coupled degenerate parabolic systems is considered. We shall give sufficient conditions to guarantee that bounded weak solutions are Hölder continuous everywhere. The general theory will be applied to a porous media type Shigesada-Kawasaki-Teramoto model in population dynamics.

Let us consider the following nonlinear parabolic systems of m equations ($m \geq 2$) given by

$$u_t = \mathrm{div}(a(x,t,u)\nabla u) + f(x,t,u), \tag{7.0.1}$$

in a domain $Q = \Omega \times (0,T) \subset \mathbb{R}^{N+1}$, with Ω being an open subset of $\mathbb{R}^N$, $N \geq 1$. The *vector valued* functions u, f take values in $\mathbb{R}^m$, $m \geq 1$. ∇u denotes the spatial derivative of u. Here, $a(x,t,u) = (A_{ij}^{\alpha\beta})$ is a tensor in $\mathrm{Hom}(\mathbb{R}^{nm}, \mathbb{R}^{nm})$.

A weak solution u to (7.0.1) is a function $u \in W_2^{1,0}(Q, \mathbb{R}^m)$ such that

$$\iint_Q [-u\phi_t + a(x,t,u)\nabla u \nabla \phi]\ dz = \iint_Q f(x,t,u)\phi\ dz$$

for all $\phi \in C_c^1(Q, \mathbb{R}^m)$. Here, we write $dz = dxdt$.

In a recent work [30], we investigated the question of partial regularity of (7.0.1) having the following structure conditions.

(A.1) There exists a C^1 map $g : \mathbb{R}^m \to \mathbb{R}^m$, with $\Phi(u) = \nabla_u g(u)$, such that for some positive constants $\lambda, \Lambda > 0$ there hold

$$a(u)\nabla u \cdot \nabla u \geq \lambda |\nabla g(u)|^2, \quad |a(u)\nabla u| \leq \Lambda |\Phi(u)||\nabla g(u)|.$$

(A.2) (Degeneracy condition) $\Phi(0) = 0$. There exist positive constants C_1, C_2 such that

$$C_1(|\Phi(u)| + |\Phi(v)|)|u - v| \leq |g(u) - g(v)| \leq C_2(|\Phi(u)| + |\Phi(v)|)|u - v|.$$

(A.3) (Comparability condition) For any $\beta \in (0,1)$, there exist constants $C_1(\beta), C_2(\beta)$ such that if $u, v \in \mathbb{R}^m$ and $\beta|u| \le |v| \le |u|$, then $C_1(\beta)|\Phi(u)| \le |\Phi(v)| \le C_2(\beta)|\Phi(u)|$.

(A.4) (Continuity condition) $\Phi(u)$ is invertible for $u \neq 0$. The map $a(u)\Phi(u)^{-1}$ is continuous on $\mathbb{R}^m \backslash \{0\}$. Moreover, there exists a monotone nondecreasing concave function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$, ω is continuous at 0, and

$$|a(v)\Phi(v)^{-1} - a(u)\Phi(u)^{-1}| \le (|\Phi(u)| + |\Phi(v)|)\omega(|u-v|^2), \tag{7.0.2}$$

$$|\Phi(u) - \Phi(v)| \le (|\Phi(u)| + |\Phi(v)|)\omega(|u-v|^2) \tag{7.0.3}$$

for all $u, v \in \mathbb{R}^m$.

Introducing the so called *A-heat approximation method*, we were able to extend the partial regularity results in [9] to the degenerate system (7.0.1). The main result of [30] is the following characterization of the regular sets of bounded weak solutions.

Theorem 7.0.1. *([30]) Let u be a bounded weak solution to (7.0.1) satisfying (A.1)-(A.4). Set*

$$Reg(u) = \{(x,t) \in \Omega \times (0,T) : \ u \text{ is Hölder continuous in a neighborhood of } (x,t)\}$$

and $Sing(u) = \Omega \times (0,T) \backslash Reg(u)$. Then $Sing(u) \subseteq \Sigma_1 \bigcup \Sigma_2$, where

$$\Sigma_1 = \{(x,t) \in \Omega \times (0,T) : \liminf_{R \to 0} |(u)_{Q_R(x,t)}| = 0\},$$

$$\Sigma_2 = \{(x,t) \in \Omega \times (0,T) : \liminf_{R \to 0} ⨏\!\!⨏_{Q_R} |u - (u)_{Q_R(x,t)}|^2 \, dz > 0\}.$$

Here, for each $R > 0$, $Q_R(x,t) = B_R(x) \times (t - R^2, t)$ and $(u)_{Q_R(x,t)} = ⨏\!\!⨏_{Q_R(x,t)} u \, dz$. Moreover, $H^n(\Sigma_2) = 0$, where H^n is the n-dimensional Hausdorff measure.

Obviously, whether bounded weak solutions are Hölder continuous everywhere, that is Sing$(u) = \emptyset$, is an important question and still remains open. There are no previous results concerning everywhere regularity for general systems of the form (7.0.1). The results and methods in aforementioned works [17, 27, 51] for regular systems cannot apply here. New techniques and additional structure conditions will be needed. This will be the main goal of this chapter.

We begin our chapter, in Section 7.2, by considering systems like (7.0.1) of m equations ($m \geq 2$) and giving sufficient conditions (in addition to (A.1)-(A.4)) that guarantee everywhere regularity of bounded weak solutions. Roughly speaking, our method relies on the key assumption on the existence of a function $H(u)$. This function links the structures of the equations in a way that we can derive certain regularity of $H(u)$, which is regarded as a function in (x, t). Such regularity of $H(u)$ will be exploited later to study that of u. This technique was first introduced by us in [27] to handle the regular cases. Here, we make use of the scaled parabolic cylinders in order to reflect the degeneracy $\Phi(u)$. This idea was originally introduced in [6] to deal with scalar p-Laplacian equations. However, the case of degenerate systems needs much more sophisticated techniques. Another difficulty arises as the L^2 estimate for ∇u, derived by Giaquinta and Struwe in [9, page 443] for regular cases, is no longer available here to obtain the smallness of the average of the deviation $|u - (u)_{Q_R}|^2$ on Q_R. Direct estimates of these quantities must be rediscovered. In addition, we must also show that the system is *averagely* not too degenerate in certain scaled cylinder so that the component Σ_1 of the singular set is empty.

We demonstrate our general theory by considering a degenerate Shigesada-Kawasaki-Teramoto (SKT) model arising in population dynamics. Here, we incorporate the porous media type diffusion into the well studied regular (SKT) systems. We will give sufficient conditions on the parameters of this system such that a function H can be found; and the results of Section 7.2 are applicable. The existence of a function H for general regular (SKT) systems was studied in [29]. Our degenerate system (SKT) obviously necessitates a different H, but some calculations in [29] are reusable here. The new choice of H in this work also greatly simplifies many complicated calculations in [29].

We would like to remark that we assume no presence of ∇u in the lower order term f in (7.0.1) for the sake of simplicity. In fact, in [30] and this present work, we could allow f to depend on ∇u, and to have growth like $\varepsilon|\Phi(u)|^2|\nabla u|^2$ for sufficiently small $\varepsilon > 0$. The proof for this case is similar, with an exception of some minor technical modifications.

The chapter is organized as follows. In Section 7.1, we introduce our notations, hypotheses and main theorems. We study the general system (7.0.1) in Section 7.2. Section 7.3 is devoted to the degenerate (SKT) system and concludes this chapter.

7.1 Main results

Throughout this chapter, Ω is a bounded domain in $\mathbb{R}^N$. For a scalar function $h(x,t)$, with $(x,t) \in \mathbb{R}^{N+1}$, its spatial (resp. temporal) derivative with respect to the x (resp. t)variable is denoted by ∇h (resp. $\partial h/\partial t$ or h_t). If $u = (u_1, \ldots, u_m)$ is a vector valued function, then $\nabla u = (\nabla u_1, \ldots, \nabla u_m)$. If H is a function in u, then $H_u = \nabla_u H = (\partial_{u_1} H, \ldots, \partial_{u_m} H)$.

For a given set $X \subset \mathbb{R}^n$ we denote by $|X|$ its n dimensional Lebesgue measure. We write $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$, the ball centered at x_0 with radius R. For a measurable bounded X, we denote the average of a given measurable function h over X by $h_X = \frac{1}{|X|}\int_X h(x)dx$.

In our proofs, $C, C_1, \ldots$ will denote various constants whose values change from line to line but are independent of the solutions in question. For $a, b \geq 0$, we also write $a \sim b$ if there are positive constants C_1, C_2 such that $C_1 a \leq b \leq C_2 a$.

In the sequel, we first consider a bounded weak solution u to (7.0.1) on $\Omega \times (0,T)$ and the following conditions.

(H.1) There exists a C^2 real function $H(u)$ defined on a neighborhood of the range of the solution u. Moreover, for some $\gamma \geq 2$ and $|u|$ small, we have $H(u) \sim |u|^\gamma$.

(H.2) There are positive constants λ_1, λ_2, and λ_3 such that

$$H_u^T a(u) \nabla u \nabla H \geq \lambda_1 |\Phi(u)|^2 |\nabla H|^2,$$

$$\nabla H_u^T a(u) \nabla u \geq \lambda_2 |u|^{\gamma-2} |\nabla g(u)|^2,$$

$$|H_u^T a(u) \nabla u| \leq \lambda_3 |\Phi(u)|^2 |\nabla H|.$$

We are now in a position to state our first theorem on the everywhere regularity.

Theorem 7.1.1. *Given the conditions (A.1)-(A.4) and (H.1)-(H.2), bounded weak solutions to (7.0.1) are Hölder continuous on* $\Omega \times (0,T)$.

To illustrate this general result, we then study a system of the form

$$\begin{cases} u_t = \nabla(P^u \nabla u + P^v \nabla v) + F(u,v), \\ v_t = \nabla(Q^u \nabla u + Q^v \nabla v) + G(u,v). \end{cases} \tag{7.1.1}$$

with the following degenerate structure, for some $m > 0$,

$$P^u = a_{11}u^m + a_{12}v^m, \qquad P^v = b_{11}u^m + b_{12}v^m,$$
$$Q^v = a_{21}u^m + a_{22}v^m, \qquad Q^u = b_{21}u^m + b_{22}v^m.$$

In this form, (7.1.1) is a generalized version of the well known Shigesada-Kawasaki-Teramoto model in population dynamics (see [45]). By allowing the presence of powers of u, v and dropping random diffusion terms, we take into account porous media type diffusion effects. The system becomes degenerate and has not been ever discussed in existing literature.

In applications, u, v represent population densities of the species under investigation, and thus only positive solutions are of interest. Our second result deals with the regularity of these positive weak solutions.

Theorem 7.1.2. *Assume that $m \geq 1$ and the following conditions on the coefficients of (7.1.1):*

$$\alpha := a_{11} - a_{21} > 0, \quad \beta := a_{22} - a_{12} > 0, \tag{7.1.2}$$

and

$$\alpha^2 b_{12} + \alpha\beta b_{11} + b_{11}b_{12}b_{21} \geq b_{22}b_{11}^2, \quad \beta^2 b_{21} + \alpha\beta b_{22} + b_{22}b_{12}b_{21} \geq b_{11}b_{22}^2. \tag{7.1.3}$$

If (u, v) is a positive bounded weak solution to (7.1.1), then (u, v) is Hölder continuous everywhere.

7.2 The general case

We give the proof of Theorem 7.1.1 in this section. For the sake of simplicity, we will assume throughout that $f(x, t, u) \equiv 0$. The presence of this term would cause no major difficulties.

By translation, we will assume that $(x_0, t_0) = (0, 0)$. Fix an $\epsilon \in (0, 2)$ and sufficiently small $R_0 > 0$. We consider the cylinder

$$Q(2R_0, R_0^{2-\epsilon}) = B_{2R_0}(0) \times [-R_0^{2-\epsilon}, 0] \subseteq \Omega_T.$$

Given $\rho > 0$, we will determine the positive constants θ and $\delta \in (0,1)$ and construct the following sequences:

$$R_n = \frac{R_0}{\theta^n}, \ \mu_0 = \sup_{Q(2R_0, R_0^{2-\epsilon})} H(u(x,t)), \ \mu_{n+1} = \max\{\delta\mu_n, \theta R_n^\epsilon\},$$

$$\Phi_{\mu_n} = \sup\{|\Phi(u)| \, : \, H(u) \le \mu_n\}, \ Q_n = B_{R_n}(0) \times [-\Phi_{\mu_n}^{-2} R_n^2, 0].$$

We also define the following function on Q_n:

$$w(x,t) := \log\left(\frac{\mu_n}{N(u)}\right), \text{ with } N(u) = \frac{1}{\rho}(\mu_n - H(u)).$$

For each n, let $Q_n^0 = \{(x,t) \in Q_n \, : \, w(x,t)_+ = 0\}$. We consider the following two alternatives.

(A) For all integers n, we have

$$|Q_n^0| > \rho |Q_n|. \tag{7.2.1}$$

(B) For some integer n, we have

$$|Q_n^0| \le \rho |Q_n|. \tag{7.2.2}$$

Let us briefly explain how Theorem 7.1.1 follows from these two alternatives.

Given any $\varepsilon > 0$, we will show that $\rho = \rho(\varepsilon) > 0$ can be chosen such that if (7.2.2) holds for some (fixed) n, then there are fixed constants $\mu, \beta > 0$ such that

$$\sup_{Q_R} |u| \le \mu, \quad ⨏⨏_{Q_R} |u - u_{Q_R}|^2 \, dz \le \varepsilon\mu^2 \text{ and } |u_{Q_R}| \ge \beta\mu, \quad R = R_n/2. \tag{7.2.3}$$

The Hölder continuity of u then follows immediately from (7.2.3) and Theorem 7.0.1.

Otherwise, for such ρ, we suppose that (7.2.1) holds for all integers n. We will show that the followings are true for all integers n.

$$H(u(x,t)) \le \mu_n \quad \forall (x,t) \in Q_n, \tag{7.2.4}$$

$$Q_{n+1} \subseteq Q_n. \tag{7.2.5}$$

Arguing as in the proof of [22, Lemma 5.8], we can see that the sequence $\{\mu_n\}$ satisfies $\mu_n \le C(R_n/R_0)^\alpha$ for some $\alpha > 0$ and some constant C depending only on θ, R_0, μ_0. Due to (7.2.4), $H(u(x,t))$ is Hölder continuous. The assumption (H.1) then gives the Hölder continuity of $u(x,t)$.

Remark 7.2.1. If $H(u) \ge \sigma\mu_n$ for some $\sigma > 0$, then there is constant $C = C(\sigma) > 0$ such that $|\Phi(u)| \ge C\Phi_{\mu_n}$. Indeed, let $\Phi_{\mu_n} = |\Phi(u_0)|$ for some u_0 such that $H(u_0) \le \mu_n$. (H.1) implies that $|u|^\gamma \ge C_1(\sigma)\mu_n$ and $|u_0|^\gamma \le C_2\mu_n$. Hence, $|u_0| \le C_3(\sigma)|u|$ for some $C_3(\sigma)$. This and (A.3) give $\Phi_{\mu_n} \le C(\sigma)|\Phi(u)|$.

Alternative (A): First of all, by scaling and assuming that $\Phi_{\mu_0} \ge CR_0^\epsilon$, we can make $Q_0 \subseteq Q(2R_0, R_0^{2-\epsilon})$ so that (7.2.4) and (7.2.5) are verified for $n = 0$. Moreover, we can also assume that $\mu_n \le 1$ for all n.

Assume that (7.2.4) holds for some integer n. Let $R = R_n/4$ and η be a function with compact support in $Q_R = B_R \times [-\Phi_{\mu_n}^{-2}R^2, 0]$.

We test the equation of u_i by $H_{u_i}\eta/N$ and add the results to get

$$\int_\Omega \frac{\partial w}{\partial t}\eta\, dx + \int_\Omega \left[\frac{H_u^T a(u)\nabla u}{N}\nabla\eta + \frac{\nabla H_u^T a(u)\nabla u}{N}\eta + \frac{H_u^T a(u)\nabla u \nabla H}{N^2}\eta\right] dx = 0. \tag{7.2.6}$$

If $\eta \ge 0$, then (H.2) and the above imply

$$\int_\Omega \frac{\partial w}{\partial t}\eta\, dx + \int_\Omega \frac{H_u^T a(u)\nabla u}{N}\nabla\eta\, dx \le 0. \tag{7.2.7}$$

We first show that $\|w\|_{\infty,Q_R}$ can be estimated in terms of $\|w\|_{2,Q_{2R}}$. By (H.2), we have

$$\left|\frac{H_u^T a(u)\nabla u}{N}\right| \leq \lambda_3|\Phi(u)|^2|\nabla w|, \quad \frac{H_u^T a(u)\nabla u}{N}\nabla w = \frac{H_u^T a(u)\nabla u \nabla H}{N^2} \geq \lambda_1|\Phi(u)|^2|\nabla w|^2.$$

We then see that the assumptions of [26, Lemma 3.3] are satisfied. Moreover, on the set $w^+ > 0$ we have $H > (1-\rho)\mu_n$. Remark 7.2.1 asserts that $|\Phi(u)| \geq C\Phi_{\mu_n}$ on the set $w^+ > 0$. Furthermore, since $H(u(x,t)) \leq \mu_n$ on Q_n by (7.2.4), we have $|\Phi(u(x,t)| \leq \Phi_{\mu_n}$ on Q_n. Hence, the comparability property (3.12) of [26, Lemma 3.4] is verified too. The iteration argument of [26, Lemma 3.5] then gives a constant C independent of R and Φ such that

$$\sup_{B_R\times[-\Phi_{\mu_n}^{-2}R^2,0]} w \leq C\Big(1 + \frac{1}{|Q_R|}\iint_{Q_R} (w_+)^2 \, dz\Big). \tag{7.2.8}$$

Next, we replace η by η^2 in (7.2.6) and use (H.2) to get

$$\int_\Omega \frac{\partial w}{\partial t}\eta^2 \, dx + \int_\Omega |\Phi(u)|^2|\nabla w|^2\eta^2 \, dx \leq C\int_\Omega (|\Phi(u)|^2|\nabla w|\eta|\nabla\eta| \, dx.$$

Having established that $|\Phi(u)| \sim \Phi_{\mu_n}$ on the set $w^+ > 0$ and $meas(\{w^+ = 0\}) = meas(Q_n^0) \geq \rho|Q_n|$ by (7.2.1), we can follow the proof of [26, Lemma 3.6] to show that $\frac{1}{|Q_R|}\iint_{Q_R} (w_+)^2 \, dz$ can be bounded by a constant independent of R and Φ. By (7.2.8), $\sup_{B_R\times[-\Phi_{\mu_n}^{-2}R^2,0]} w$ is also bounded by a constant, denoted by $\ln(C)$, independent of R and Φ. From the definition of w, we easily get

$$H(u(x,t)) \leq \delta\mu_n, \quad \forall (x,t) \in Q_R, \tag{7.2.9}$$

with $\delta = \frac{C-\rho}{C} < 1$ and depends only on ρ.

We now show that (7.2.5) is verified by a suitable choice of θ.

To proceed, we claim that there is a constant $C_0 = C(\delta)$ such that $\Phi_{\mu_n} \leq C_0\Phi_{\mu_{n+1}}$. Indeed, let u_1 be such that $\Phi_{\mu_n} = |\Phi(u_1)|$ and $H(u_1) \leq \mu_n$. Since $\mu_n \leq \mu_{n+1}/\delta$, we have $|u_1|^\gamma \leq C_1(\delta)\mu_{n+1}$. Hence, for some $C_2(\delta)$, we have $u_2 = C_2(\delta)u_1$ satisfying $H(u_2) \leq \mu_{n+1}$. This gives that $|\Phi(u_1)| \leq C_3(\delta)|\Phi(u_2)| \leq C_0(\delta)\Phi_{\mu_{n+1}}$, due to (A.3). Our claim then follows.

We then determine θ such that $Q_{n+1} \subseteq Q_R$. This is to say, $R_{n+1} \leq R = R_n/4$ and $\Phi_{\mu_{n+1}}^{-2}R_{n+1}^2 \leq \Phi_{\mu_n}^{-2}R^2$. To this end, we need $\theta \geq 4$ and $\Phi_{\mu_n} \leq \Phi_{\mu_{n+1}}\theta/4$. We then choose $\theta =$

$\max\{4, 4C_0(\delta)\}$.

Therefore, $Q_{n+1} \subseteq Q_R \subseteq Q_n$ and (7.2.9) holds on Q_{n+1}. This shows that (7.2.4) continues to hold for $n+1$. By induction, we conclude that (7.2.4) and (7.2.5) hold for all integers n. Our proof is complete in this case.

Alternative B: We now have (7.2.2) for some n. Denote $R = R_n/4$ and

$$Q_{4R} = Q_n, \quad Q_R = B_R \times [-\Phi_{\mu_n}^{-2} R^2, 0].$$

It is easy to see that (7.2.2) yields

$$|Q_n^0| = |\{(x,t) \in Q_{4R} \,:\, H \le (1-\rho)\mu_n\}| < \rho |Q_{4R}|. \tag{7.2.10}$$

We first derive L^p estimates for $|\nabla g|$. Test (7.0.1) with $H_u \eta$ to get

$$\int_\Omega \frac{\partial H}{\partial t}\eta \, dx + \int_\Omega (\nabla H_u)^T a(u) \nabla u \eta \, dx + \int_\Omega H_u^T a(u) \nabla \eta \, dx = 0. \tag{7.2.11}$$

Let $H_k^+ = (H(u(x,t)) - k)^+$. Replacing η in (7.2.11) by $H_k^+ \eta^2$, we easily obtain

$$\begin{aligned} \iint_{\Omega_T} \frac{\partial (H_k^+ \eta)^2}{\partial t} \, dz \quad &+ \quad \iint_{\Omega_T} [H_u^T a(u) \nabla u \nabla H_k^+ \eta^2 + \nabla H_u^T a(u) \nabla u H_k^+ \eta^2] \, dz \\ &\le \quad \iint_{\Omega_T} [H_u^T a(u) \nabla u H_k^+ \eta \nabla \eta + (H_k^+)^2 \eta \eta_t] \, dz. \end{aligned}$$

By (H.2), this implies

$$\iint_{\Omega_T} [\lambda_1 |\Phi(u)|^2 |\nabla H|^2 \eta^2 + \lambda_2 |u|^{\gamma-2} |\nabla g(u)|^2 H_k^+ \eta^2] \, dz \le \iint_{\Omega_T} [|\Phi(u)|^2 |H_k^+|^2 |\nabla \eta|^2 + |H_k^+|^2 |\eta_t|] \, dz.$$

We now take η to be a cut-off function with respect to the scaled cylinders Q_R, Q_{4R}. We have that $|\nabla \eta| \le \frac{1}{R}$ and $|\eta_t| \le \frac{\Phi_{\mu_n}^2}{R^2}$.

We then take $k = (1-2\rho)\mu_n$ and note that $H_k^+ \le 2\rho\mu_n$ on Q_{4R}. Moreover, because $H(u(x,t)) \le \mu_n$ on Q_{4R}, we have $|\Phi(u(x,t))| \le \Phi_{\mu_n}$. Using the fact that $|Q_R| \sim \Phi_{\mu_n}^{-2} R^{N+2}$, we obtain

$$\iint_{Q_R} |u|^{\gamma-2}|\nabla g(u)|^2 H_k^+ \; dz \leq C(\rho\mu_n)^2 R^N.$$

Let $A_0 := \{(x,t) \in Q_R | H \geq (1-\rho)\mu_n\}$. Then $(H-k)_+ \geq \rho\mu_n$ on A_0. So,

$$\iint_{A_0} |u|^{\gamma-2}|\nabla g(u)|^2 \; dz \leq C\rho\mu_n R^N. \tag{7.2.12}$$

Since $H(u) \sim |u|^\gamma$ ($\gamma \geq 2$) and $\rho < 1/2$, the following is valid on A_0

$$|u|^{\gamma-2} \sim H^{(\gamma-2)/\gamma} \geq ((1-\rho)\mu_n)^{(\gamma-2)/\gamma} \geq C\mu_n^{(\gamma-2)/\gamma}$$

Therefore, (7.2.12) implies

$$\iint_{A_0} |\nabla g(u)|^2 \; dz \leq C\rho\mu_n^{2/\gamma} R^n. \tag{7.2.13}$$

In addition, we can find C such that if $\mu = (C\mu_n)^{1/\gamma}$ then

$$\sup_{Q_R} |u(x)| \leq \mu \text{ and } \mu_n \leq C\mu^\gamma. \tag{7.2.14}$$

It means we obtain

$$\iint_{A_0} |\nabla g(u)|^2 \; dz \leq C\rho\mu^2 R^n. \tag{7.2.15}$$

By testing (7.0.1) with $u\eta$, it is standard to show that

$$\iint_{Q_R} |\nabla g(u)|^2 \; dz \leq C\mu^2 R^n. \tag{7.2.16}$$

For any subset A of Q_R, Hölder's inequality gives

$$\iint_A |\nabla \vec{u}|^q \; dz \leq \left(\iint_A |\nabla \vec{u}|^2 \; dz\right)^{\frac{q}{2}} |A|^{1-\frac{q}{2}}. \tag{7.2.17}$$

Taking $q = \frac{2n}{n+1} < 2$, $A = A_0$ and using (7.2.12), we obtain

$$\iint_{A_0} |\nabla g(u)|^q \; dz \le C(\rho\mu^2 R^n)^{\frac{n}{n+1}} \Phi_{\mu_n}^{\frac{-2}{n+1}} R^{\frac{n+2}{n+1}} = C(\rho\mu^2)^{\frac{n}{n+1}} \Phi_{\mu_n}^{\frac{-2}{n+1}} R^{n+\frac{2}{n+1}}.$$

Similarly, we take $A = Q_R \setminus A_0$ in (7.2.17). Using (7.2.16) and also the fact that $|A| \le \rho|Q_R|$, we have

$$\iint_{Q_R\setminus A_0} |\nabla g(u)|^q \; dz \le C(\mu^2 R^n)^{\frac{n}{n+1}} (\rho\Phi_{\mu_n}^{-2} R^{n+2})^{\frac{1}{n+1}} = C\rho^{\frac{1}{n+1}} \Phi_{\mu_n}^{\frac{-2}{n+1}} \mu^{\frac{2n}{n+1}} R^{n+\frac{2}{n+1}}.$$

The above inequalities give us the following estimate for $|\nabla g|$:

$$\iint_{Q_R} |\nabla g(u)|^q \; dz \le C(\rho^{\frac{n}{n+1}} + \rho^{\frac{1}{n+1}})\Phi_{\mu_n}^{\frac{-2}{n+1}} \mu^{\frac{2n}{n+1}} R^{n+\frac{2}{n+1}}. \tag{7.2.18}$$

We now try to estimate the deviation $|u - u_{Q_R}|$. We recall the following inequality ([Ladyzenskaja, (2.10), p.45]), with $r = 1, p = 2$ and $m = 2n/n+1$), for functions u with $u_\Omega = 0$.

$$\int_\Omega u^2 \; dx \le C \int_\Omega |\nabla u|^{\frac{n}{n+1}} \; dx \left(\int_\Omega |u| \; dx \right)^{\frac{2}{n+1}}$$

Let $V(t)$ be a vector such that $g(V(t)) = g_{B_R}(u)$. The above yields

$$\begin{aligned} \iint_{Q_R} |g(u) - g(V(t))|^2 \; dz &\le C \iint_{Q_R} |\nabla g(u)|^q \; dz \sup_t \left(\int_{B_R(t)} |g(u) - g_{B_R}(u)| \; dx \right)^{\frac{2}{n+1}} \\ &\le C[\rho^{\frac{n}{n+1}} + \rho^{\frac{1}{n+1}}] \mu^{\frac{2n}{n+1}} \Phi_{\mu_n}^{\frac{-2}{n+1}} R^{n+\frac{2}{n+1}} (\Phi_{\mu_n}\mu)^{\frac{2}{n+1}} R^{\frac{2n}{n+1}} \end{aligned}$$

Hence,

$$\iint_{Q_R} |g(u) - g(V(t))|^2 \; dz \le C[\rho^{\frac{n}{n+1}} + \rho^{\frac{1}{n+1}}]\mu^2 R^{n+2}.$$

As $|g(u) - g(V(t))| \ge C(|\Phi(u)|^{1/2} + |\Phi(V(t))|^{1/2})|u - V(t)|$ and $H(u) \ge (1-\rho)\mu_n$ on A_0, Lemma 7.2.1 showed that $|\Phi(u)| \sim \Phi_{\mu_n}$ on the set A_0. Thus,

$$\Phi_{\mu_n}^2 \iint_{A_0} |u - V(t)|^2 \, dz \le C[\rho^{\frac{n}{n+1}} + \rho^{\frac{1}{n+1}}]\mu^2 R^{n+2} = \varepsilon(\rho)\mu^2 R^{n+2},$$

with $\varepsilon(\rho) = C[\rho^{\frac{n}{n+1}} + \rho^{\frac{1}{n+1}}]$.

Because $\int_{B_r} |u - u_{B_R}|^2 \, dx \le C \int_{B_r} |u - V(t)|^2 \, dx$, we have

$$\begin{aligned}
\Phi_{\mu_n}^2 \iint_{Q_R} |u - u_{B_R}|^2 \, dz &\le \Phi_{\mu_n}^2 \iint_{A_0} |u - V(t)|^2 \, dz + \Phi_{\mu_n}^2 \iint_{Q_R \setminus A_0} |u - V(t)|^2 \, dz \\
&\le \varepsilon(\rho)\mu^2 R^{n+2} + \Phi_{\mu_n}^2 \mu^2 \rho |Q_R|
\end{aligned}$$

This gives $\Phi_{\mu_n}^2 \iint_{Q_R} |u - u_{B_R}|^2 \, dz \le (\varepsilon(\rho) + \rho)\mu^2 R^{n+2}$.

On the other hand, for $G = \iint_{Q_R} |u_{B_R} - u_{Q_R}|^2 \, dz$, we have

$$\begin{aligned}
G &\le |Q_R| \sup_{t \in I_R} \left| \fint_{B_R} u(x,t) \, dx - \frac{1}{|I_R|} \int_{I_R} \fint_{B_R} u(x,s) \, dx ds \right|^2 \\
&\le |Q_R||B_R|^{-2} \sup_{t,s \in I_R} \left| \int_{B_R} [u(x,t) - u(x,s)] \, dx \right|^2
\end{aligned}$$

From the equation, for $\sigma \in (0,1)$, we have

$$\begin{aligned}
\sup_{t,s \in I_R} \left| \int_{B_R} [u(x,t) - u(x,s)] \, dx \right| &\le \int_{B_{(\sigma+1)R} \setminus B_R} |u(x,t) - u(x,s)| \, dx \\
&\quad + \iint_{Q_{(\sigma+1)R}} |a(u)\nabla u \nabla \eta| \, dz \\
&\le C\sigma\mu R^n + \frac{\Phi_{\mu_n}}{\sigma R} \iint_{Q_{2R}} |\nabla g(u)| \, dz
\end{aligned}$$

Using the inequality $\int_A u \le \sqrt{\int_A u^2 |A|}$ we argue the same way as in (7.2.18) to get

$$\begin{aligned}
\iint_{Q_{2R}} |\nabla g(u)| \, dz &\le \sqrt{\rho\mu^2 R^n \Phi_{\mu_n}^{-2} R^{n+2}} + \sqrt{(C\mu^2 R^n)(\rho \Phi_{\mu_n}^{-2} R^{n+2})} \\
&= C\sqrt{\rho \Phi_{\mu_n}^{-2}} \mu R^{n+1}
\end{aligned}$$

Therefore, by choosing $\sigma = \rho^{1/4}$, we have

$$\sup_{t,s\in I_R}\left|\int_{B_R} [u(x,t) - u(x,s)]\ dx\right| \leq C\left(\sigma + \frac{\sqrt{\rho}}{\sigma}\right)\mu R^n \leq C\rho^{1/4}\mu R^n$$

Thus,

$$G \leq C\rho^{1/2}\mu^2|Q_R|$$

Hence,

$$⨌_{Q_R} |u - u_{Q_R}|^2\ dz \leq C(\varepsilon(\rho) + \rho)\mu^2 + C\rho^{1/2}\mu^2 \leq o(\rho)\mu^2$$

Given any $\varepsilon > 0$. We can choose ρ such that

$$⨌_{Q_R} |u - u_{Q_R}|^2\ dz \leq \varepsilon\mu^2.$$

We show that u_{Q_R} is not small. We have

$$\iint_{Q_R} u^2\ dz \leq (\varepsilon\mu^2 + |u_{Q_R}|^2)|Q_R|.$$

Because $|H| \geq (1-\rho)\mu_n$ implies $|u|^\gamma \geq C\mu^\gamma$. Therefore $|u| \geq C\mu$ on A_0 ($|A_0| \geq (1-\rho)|Q_R|$, we have

$$\iint_{Q_R} u^2\ dz \geq \iint_{A_0} u^2\ dz \geq C^2\mu^2(1-\rho)|Q_R|.$$

By choosing ρ, ε small, we see that $|u_{Q_R}| \geq \beta\mu$ for some constant $\beta > 0$. This gives (7.2.3).

7.3 The degenerate (SKT) system

We prove Theorem 7.1.2 in this section. Let us recall the system

$$\begin{cases} u_t = \nabla(P^u\nabla u + P^v\nabla v) + F(u,v), \\ v_t = \nabla(Q^u\nabla u + Q^v\nabla v) + G(u,v). \end{cases} \tag{7.3.1}$$

with the following degenerate structure, for some $m > 0$,

$$P^u = a_{11}u^m + a_{12}v^m, \qquad P^v = b_{11}u^m + b_{12}v^m,$$
$$Q^v = a_{21}u^m + a_{22}v^m, \qquad Q^u = b_{21}u^m + b_{22}v^m.$$

We also recall the following conditions stated in Theorem 7.1.2.

(P) Assume $m \geq 1$, $\alpha := a_{11} - a_{21} > 0$, and $\beta := a_{22} - a_{12} > 0$. Moreover

$$\alpha^2 b_{12} + \alpha\beta b_{11} + b_{11}b_{12}b_{21} \geq b_{22}b_{11}^2, \quad \beta^2 b_{21} + \alpha\beta b_{22} + b_{22}b_{12}b_{21} \geq b_{11}b_{22}^2. \tag{7.3.2}$$

Let (u, v) be a positive solution to (7.3.1). Our goal is to find a suitable function H that satisfies the conditions $(H.1), (H.2)$ such that Theorem 7.1.1 can apply here.

To begin, we set $\det a(u, v) = P^u Q^v - P^v Q^u$ and

$$\Phi^2(u, v) = \frac{P^u + Q^v + \sqrt{(P^u + Q^v)^2 - \det(a)}}{2}$$

We also take $g_u(u, v) = \Phi(u, v)I_2$, where I_2 is the 2×2 identity matrix. Thanks to (P), it is easy to see that (A.1)-(A.4) are satisfied here.

We first observe that the condition (H.2) is verified if we can find a function H that satisfies the following conditions.

$$H_u^T a(\vec{u})\nabla\vec{u}\nabla H \geq \overline{\lambda}_1|\nabla H|^2, \tag{7.3.3}$$

$$\nabla H_u^T a(\vec{u})\nabla\vec{u} \geq \overline{\lambda}_2|\nabla\vec{u}|^2, \tag{7.3.4}$$

$$|H_u^T a(\vec{u})\nabla\vec{u}| \leq \overline{\lambda}_3|\nabla H|, \tag{7.3.5}$$

where $\overline{\lambda}_1 = \lambda_1\Phi^2(u, v), \overline{\lambda}_2 = \lambda_2|\vec{u}|^{\mu-2}\Phi^2(u, v), \overline{\lambda}_3 = \lambda_3\Phi^2(u, v)$, with λ_i being positive constants.

These conditions amount to the positivity of the following quadratics in $U, V \in \mathbb{R}^N$:

$$
\begin{aligned}
A_1 &= \left((P_uH_u + Q_uH_v)\,H_u - \overline{\lambda}_1 {H_u}^2\right) U^2 + \left((P_vH_u + Q_vH_v)\,H_v - \overline{\lambda}_1 {H_v}^2\right) V^2 \\
&\quad + \left((P_vH_u + Q_vH_v)\,H_u + (P_uH_u + Q_uH_v)\,H_v - 2\overline{\lambda}_1 H_vH_u\right) VU,
\end{aligned} \tag{7.3.6}
$$

$$
\begin{aligned}
A_2 &= \left(Q_uH_{uv} + P_uH_{uu} - \overline{\lambda}_2\right) U^2 + \left(P_vH_{uv} - \overline{\lambda}_2 + Q_vH_{vv}\right) V^2 \\
&\quad + (P_vH_{uu} + P_uH_{uv} + Q_vH_{uv} + Q_uH_{vv})\,VU,
\end{aligned} \tag{7.3.7}
$$

and

$$
\begin{aligned}
A_3 &= \left(\overline{\lambda}_3 {H_u}^2 - (P_uH_u + Q_uH_v)^2\right) U^2 + \left(\overline{\lambda}_3 {H_v}^2 - (P_vH_u + Q_vH_v)^2\right) V^2 \\
&\quad + \left(2\overline{\lambda}_3 H_vH_u - 2\,(P_vH_u + Q_vH_v)\,(P_uH_u + Q_uH_v)\right) VU.
\end{aligned} \tag{7.3.8}
$$

Following [29], the discriminants of A_1, A_3 will be nonpositive if the following first order equation is satisfied.

$$
H_u = f(u,v)H_v, \tag{7.3.9}
$$

where f is the solution to

$$
-P_vf^2 + (P_u - Q_v)f + Q_u = 0. \tag{7.3.10}
$$

Because $P_vQ_u > 0$, (7.3.10) has two solutions f_1, f_2 with $f_1f_2 < 0$. In what follows, we denote by $f = f(u,v)$ the positive solution of (7.3.10).

We first have the following simple lemma.

Lemma 7.3.1. *Assume that H satisfies (7.3.9). There exist positive numbers λ_1, λ_3 such that A_1, A_3 are positive definite.*

Proof: Following the proof of [29, Lemma 3.2], we need only choose λ_1, λ_3 such that the coefficients of U^2, V^2 in A_1, A_3 can be positive. By (7.3.10) and (7.3.9), these coefficients in A_1 can be written as

$$
H_v^2 f^2 (P_vf + Q_v - \overline{\lambda}_1), \quad H_v^2 (P_vf + Q_v - \overline{\lambda}_1).
$$

Similarly, for A_3, they are

$$H_u^2(\overline{\lambda}_3 - (P_v f + Q_v)^2), \quad H_v^2(\overline{\lambda}_3 - (P_v f + Q_v)^2).$$

Since $f > 0$ and $\Phi^2 = P^v f + Q^v$, we can take $\lambda_1 = \frac{1}{2}$ and $\lambda_3 = 2\sup_\Gamma(P_v f + Q_v)$, which is finite positive number because of the boundedness of f (see below) and the fact that u, v are bounded.

□

Thus, we are left with the positivity of A_2. The rest of this section will be devoted to finding H that solves (7.3.9) and makes A_2 positive definite. This is also the crucial step in proving Theorem 7.1.2.

To proceed, we pick a solution g of the first order equation

$$g_u - f(u,v)g_v = 0, \tag{7.3.11}$$

and let G be any C^2 differentiable function on $\mathbb{R}$. Notice that $H(u,v) = G(g(u,v))$ is also a solution to (7.3.9).

We shall follow the calculations of [29]. To verify the positivity of A_2, we consider its discriminant Θ. Owing to $H_u = fH_v$, an easy computation shows that

$$\Theta = -4\overline{\lambda}_2^{\,2} + 4\Theta_1\overline{\lambda}_2 + \Theta_2$$

Where

$$\begin{aligned}
\Theta_1 &= \beta_2 H_{vv} + \beta_3 H_v = \delta_1 + \delta_2 \\
\Theta_2 &= -\alpha_2 H_{vv} H_v + \alpha_3 H_v^2 = -H_v^2\left(\alpha_2 \frac{H_{vv}}{H_v} - \alpha_3\right)
\end{aligned}$$

with (see [29])

$$
\begin{aligned}
\alpha_2 &= 4\left(P^u Q^v - Q^u P^v\right)\left(f_u - f f_v\right), \\
\alpha_3 &= \left[(f_u + f f_v)P^v - f_v(P^u - Q^v)\right]^2 + 4\left(P^u\, Q^v - Q^u\, P^v\right) f_v^2 \\
&= \left[f_u P^v + \frac{f_v Q^u}{f}\right]^2 + 4\det(a) f_v^2, \\
\beta_2 &= \Phi^2(f^2+1), \quad \beta_3 = \Phi^2 f f_v + P^u f_u + P^v f_v.
\end{aligned}
$$

Meanwhile, the coefficients of U^2 and V^2 in A_2 are $\delta_1 - \overline{\lambda}_2$ and $\delta_2 - \overline{\lambda}_2$, respectively, with

$$
\delta_1 = \Phi^2 f^2 H_{vv} + \left(\Phi^2 f f_v + P^u\, f_u\right) H_v, \quad \delta_2 = \Phi^2 H_{vv} + P^v\, f_v\, H_v.
$$

We first study on f, f_u, f_v. We obtain the following result.

Lemma 7.3.2. *Assume (P). We have*

i) *There exist two positive constants C_1, C_2 such that $C_1 \le f(u,v) \le C_2$.*

ii) $f_u u + f_v v = 0$ *and* $f_u > 0$ *and* $f_v < 0$.

Proof: We recall the formula of f,

$$
f = \frac{P^u - Q^v + \sigma(u,v)}{2P^v} = \frac{\alpha u^m - \beta v^m + \sigma(u,v)}{2(b_{11}u^m + b_{12}v^m)}
$$

Therefore, it is easy to see that

$$
f_u = u^{m-1}\frac{-b_{11}f^2 + \alpha f + b_{21}}{\sigma(u,v)}, \quad f_v = v^{m-1}\frac{-b_{12}f^2 - \beta f + b_{22}}{\sigma(u,v)}.
$$

Here $\sigma(u,v) = \sqrt{(P_u - Q_v)^2 + 4P_v Q_u}$.

Let f_1 (f_2) be the positive solution of $f_v = 0$ ($f_u = 0$, respectively). We easily get

$$
f_1 = \frac{2b_{22}}{\beta + \sqrt{\beta^2 + 4\,b_{12}b_{22}}}, \quad f_2 = \frac{\alpha + \sqrt{\alpha^2 + 4\,b_{11}b_{21}}}{2b_{11}}
$$

We would recall that f is the positive solution of

$$F(f) := -(b_{11}u^m + b_{12}v^m)f^2 + (\alpha u^m - \beta v^m)f + b_{21}u^m + b_{22}v^m = 0.$$

By simple calculations, we get $F(f_1)$ is

$$\frac{2u^m\left(-2\,{b_{22}}^2 b_{11} + b_{22}\alpha\,\beta + b_{22}\alpha\,\sqrt{\beta^2+4\,b_{12}b_{22}} + b_{21}\beta^2 + b_{21}\beta\,\sqrt{\beta^2+4\,b_{12}b_{22}} + 2\,b_{12}b_{22}b_{21}\right)}{\left(\beta+\sqrt{\beta^2+4\,b_{12}b_{22}}\right)^2}$$

and, similarly, $F(f_2)$ is exactly equal to

$$\frac{v^m\left(b_{12}\alpha^2 + b_{12}\alpha\,\sqrt{\alpha^2+4\,b_{11}b_{21}} + 2\,b_{12}b_{11}b_{21} + b_{11}\beta\,\alpha + b_{11}\beta\,\sqrt{\alpha^2+4\,b_{11}b_{21}} - 2\,b_{22}{b_{11}}^2\right)}{-2{b_{11}}^2}$$

Our condition (P) simply makes $F(f_1) > 0$ and $F(f_2) < 0$. Since there is a unique positive solution f of $F(f) = 0$, we immediately get $f_u > 0$, $f_v < 0$, and $f_1 < f < f_2$. This proves i). Straight calculation shows ii). $\square$

Our next step is to determine solution g of

$$g_u = f g_v, \tag{7.3.12}$$

which can be solved by characteristic methods. From [7, pp. 97-99], we know that $\vec{x}(t) = (u(t), v(t))$, $z(t) = g(\vec{x}(t))$ and $\vec{p}(t) = (p_u(t), p_v(t)) = \nabla g(\vec{x}(t))$ solve the following system:

$$\begin{aligned} \vec{x}'(t) &= (1, -f), \\ \vec{p}'(t) &= (f_u p_v, f_v p_v), \\ z'(t) &= p_u - f p_v = 0. \end{aligned} \tag{7.3.13}$$

We choose the initial data for $x, \vec{p}$ on the line $\Upsilon = \{(u, v) : u = v > 0\}$, which is noncharacteristic, to be

$$\vec{x}(0) = (u, u); \quad p_u(0) = f(u, u), \quad p_v(0) = 1. \tag{7.3.14}$$

A smooth solution g of (7.3.12) can be found by setting g to be constant along each flow line

$\vec{x}(t)$. In fact, we will define g on the line Υ by

$$g(u,v) = \int_0^u f(s,s)ds + v.$$

The following lemma provides useful properties of the solution g of the above system.

Lemma 7.3.3. *The followings hold for (7.3.13) and (7.3.14).*

i) *g is defined on the first quadrant $\{(u,v) : u,v > 0\}$.*

ii) *There exist $C_1, C_2 > 0$ such that $C_1 \le g_v \le C_2$.*

iii) *There are positive constants C_1, C_2 such that $C_1(fu+v) \le g(u,v) \le C_2(fu+v)$.*

iv) $g_{vv} = -g_v \frac{f_v}{f}$.

Proof: i) Because f is bounded by Lemma 7.3.2, $\vec{x}(t)$ exists for all $t \in \mathbb{R}$. It is trivial to show that the flow lines cross every point in the first quadrant so that g is well defined on this set.

ii) Consider a characteristic curve emanating from a point (u_0, v_0) on Υ. From the first equation of (7.3.13) and the fact that f is bounded, we easily see that there is a constant C such that $-u_0 \le t \le Cv_0$ for $u(t), v(t)$ to be positive. From the equation for $\vec{p}$ in (7.3.13), we have $p_v(t) = \exp(\int_0^t f_v(u(s), v(s))ds$. We will estimate the last integral.

Consider the case $t \ge 0$. We have $u(t) \ge u_0$. The proof of ii) of Lemma 7.3.2 reveals that

$$|f_v(u(t), v(t))| \le \frac{C_1 v^{\alpha-1}}{\bar{\sigma}(u,v)} \le \frac{C_1 v^{\alpha-1}}{\sqrt{4b_{11}b_{22}u^\alpha v^\alpha}} \le \frac{C_2 v^{\alpha/2-1}}{u_0^{\alpha/2}}.$$

Thus,

$$\int_0^t |f_v| ds = \int_0^t |f_v| \frac{dv(s)}{-f} \le C_3 \int_0^{Cv_0} \frac{v^{\alpha/2-1}}{u_0^{\alpha/2}} dv \le C_4.$$

For $t < 0$, we use the fact that $f_v v = -f_u u$ to get

$$\int_0^t |f_v| ds \le \int_0^{|t|} \left| \frac{f_u u}{v} \right| ds \le C_5 \int_0^{u_0} \frac{u^{\alpha/2}}{v_0^{\alpha/2+1}} du \le C_6.$$

In both cases, we find $g_v = \exp(\int_0^t f_v(u(s), v(s))ds$ is bounded from above and below by positive constants, and conclude the proof of ii).

iii) Using the fact that f is homogeneous, we have

$$\begin{aligned} g(u,v) &= \int_0^1 g'(tu,tv)dt = \int_0^1 g_u(tu,tv)u + g_v(tu,tv)vdt \\ &= \int_0^1 (f(tu,tv)u+v)g_v(tu,tv)dt = (fu+v)\int_0^1 g_v(tu,tv)dt. \end{aligned}$$

This and ii) give the assertion.

iv) From the fact that $g_v = \exp(\int_0^t f_v(u(s),v(s))ds$, we obtain

$$g_{vv} = \frac{\partial}{\partial t} g_v \frac{\partial t}{\partial v} = \exp(\int_0^t f_v(u(s),v(s))ds) f_v(u(t),v(t))\frac{1}{-f} = -g_v \frac{f_v}{f}.$$

This concludes our proof of the lemma. □

Theorem 7.3.4. *Assume $m \geq 1$. There exists a μ sufficiently large such that $H(u,v) = (g(u,v))^\mu$ satisfies $(H.2)$.*

Proof: First, we shall show that $\Theta_2 < 0$ with sufficiently large μ. In deed, we write

$$\begin{aligned} \Theta_2 &= -H_v^2\left(\alpha_2 \frac{H_{vv}}{H_v} - \alpha_3\right) = -H_v^2\left(\alpha_2 g^{-1} g_v(\mu-1) - \alpha_2\frac{f_v}{f} - \alpha_3\right) \\ &= -H_v^2 \alpha_2 g^{-1} g_v \left(\mu - 1 - \frac{f_v g}{f g_v} - \frac{\alpha_3 g}{\alpha_2 g_v}\right) \end{aligned}$$

Since $\alpha_2 = \det(a)(f_u - ff_v)$ and $f_u > 0, f_v < 0$ due to Lemma 7.3.2, we have $H_v^2\alpha_2 g^{-1} g_v > 0$. Therefore it suffices for the negativity of Θ_2 to show that $\frac{f_v g}{f g_v}$ and $\frac{\alpha_3 g}{\alpha_2 g_v}$ are bounded.

Indeed, we observe

$$\sigma(u,v) \sim u^m + v^m, \quad \det(a) \sim \sigma^2, \quad f_u \sim \frac{u^{m-1}}{\sigma(u,v)}, \quad f_v \sim \frac{v^{m-1}}{\sigma(u,v)}$$

Thus thanks to Young's inequality $(m \geq 1)$, we obtain

$$\left|\frac{gf_v}{g_v}\right| + \left|\frac{gf_u}{g_v}\right| \leq \frac{(u+v)(u^{m-1}+v^{m-1})}{\sigma(u,v)} \leq C$$

We recall

$$\frac{\alpha_3 g}{\alpha_2 g_v} = \frac{g[ff_uP^v + f_vQ^u]^2}{\det(a)g_vf^2(f_u - ff_v)} + \frac{gf_v^2}{g_v(f_u - ff_v)}.$$

Since $f_u - ff_v > -ff_v > 0$ and $f_u - ff_v > f_u > 0$, we have

$$\frac{gf_v^2}{g_v(f_u - ff_v)} \leq \left|\frac{gf_v}{fg_v}\right| \leq C$$

and

$$\frac{g[ff_uP^v + f_vQ^u]^2}{\det(a)g_vf^2(f_u - ff_v)} \leq \frac{(P^v)^2 + (Q^u)^2}{\det(a)} \left(\left|\frac{gf_u}{g_v}\right| + \left|\frac{gf_v}{f^3g_v}\right|\right) \leq C.$$

Therefore $\Theta_2 < 0$ by choosing a sufficiently large μ. We shall next show that

$$\Theta = -4\overline{\lambda}_2^{\,2} + 4\Theta_1\overline{\lambda}_2 + \Theta_2 < 0$$

Here $\Theta_1 = \beta_2 H_{vv} + \beta_3 H_v$ and $\Theta_2 = -\alpha_2 H_{vv}H_v + \alpha_3 H_v^2$. By a view of the above proof of $\Theta_2 < 0$, it is obviously enough to show that $\frac{\Theta_1\overline{\lambda}_2}{\alpha_3 H_v^2}$ is bounded. We have

$$\frac{\Theta_1\overline{\lambda}_2}{\alpha_3 H_v^2} = \frac{\lambda_2\beta_2 H_{vv}|\vec{u}|^{\mu-2}\Phi^2}{\alpha_3 H_v^2} + \frac{\lambda_2\beta_3|u|^{\mu-2}\Phi^2}{\alpha_2 H_v}. \tag{7.3.15}$$

We recall that $g \sim u + v, \Phi^2 \sim u^m + v^m, H_{vv} \sim |\vec{u}|^{\mu-2}, H_v \sim g|\vec{u}|^{\mu-2}, \alpha_2 \sim g[u^{m-1} + v^{m-1}]^2$,

$$\alpha_3 = \left[f_uP^v + \frac{f_vQ^u}{f}\right]^2 + 4\det(a)f_v^2 \geq \left[f_uP^v + \frac{f_vQ^u}{f}\right]^2 \sim \left[u^{m-1} + v^{m-1}\right]^2,$$

and

$$\beta_2 = \Phi^2(f^2 + 1) \sim \Phi^2, \quad \beta_3 = \Phi^2 ff_v + P^uf_u + P^vf_v \sim u^{m-1} + v^{m-1}.$$

Put these estimates into (7.3.15) and note that λ_2 can be as small as we wish, we easily see that $\frac{\Theta_1\overline{\lambda}_2}{\alpha_3 H_v^2}$ are bounded by a constant independent of μ. This proves the existence of μ such that $\Theta < 0$.

Finally, we need to show that the coefficients of U^2 and V^2 in A_2 are positive, that is, the following quantities are positive.

$$\begin{aligned}
\delta_1 - \overline{\lambda}_2 &= -\overline{\lambda}_2 + \Phi^2 f^2 H_{vv} + \left(\Phi^2 f f_v + P^u fu\right) H_v \\
&= \Phi^2 f^2 g^{-1} H_v \left[-\frac{g\overline{\lambda}_2}{\Phi^2 f^2 H_v} + \frac{g(\Phi^2 f f_v + P^u f_u)}{\Phi^2 f^2} + \frac{g H_{vv}}{H_v} \right], \\
\delta_2 - \overline{\lambda}_2 &= -\overline{\lambda}_2 + P^v f_v H_v + \Phi^2 H_{vv} = \Phi^2 g^{-1} H_v \left[-\frac{g\overline{\lambda}_2}{\Phi^2 H_v} + \frac{g P^v f_v}{\Phi^2} + \frac{g H_{vv}}{H_v} \right].
\end{aligned}$$

Note that $g\frac{H_{vv}}{H_v}$ can be very large as μ is sufficiently large, while the other terms are bounded due to the same arguments as above. Therefore $\delta_1 - \overline{\lambda}_2, \delta_2 - \overline{\lambda}_2$ are positive and so is A_2 thanks to the negativity of Θ. The proof is complete. □

Notes and Remarks

The regularity of bounded solutions to cross diffusion (strongly coupled) systems is a long-standing problem. For systems with *regular* diffusion part $a(x, t, u)$, *partial* regularity results were established by Giaquinta and Struwe in [9]. However, the question of whether bounded weak solutions are Hölder continuous *everywhere* was only answered in very few situations under either a severe restriction on the dimension N of the domain Ω, $N \leq 2$, as in [17], or special structural conditions on $a(x, t, u)$ for arbitrary N (see [29, 51]).

To the best of our knowledge, the best regularity one can obtain for cross diffusion systems like (7.0.1), which has certain degeneracy in the tensor a, is the *partial regularity* established by Le in [30]. Important examples of (7.0.1) include cross diffusion systems modelling phenomena in porous media. In contrast to the single equation case (see [26]), one cannot expect in general that bounded weak solutions of (7.0.1) will be Hölder continuous everywhere. Our result here thus are new in literature, and this is the content of our paper [32].

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