ASYMPTOTIC STABILITY OF NONCHARACTERISTIC VISCOUS BOUNDARY LAYERS

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To my wife,

Thanh Thi-Hai Tran

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Abstract

In this dissertation we give a rigorous mathematical analysis of the asymptotic stability of arbitrary–amplitude noncharacteristic viscous boundary layers in dimensions $d \geq 1$, motivated by physical applications such as compressible gas dynamics and magnetohydrodynamics (MHD) equations. Briefly, the dissertation addresses the following problems.

One–dimensional stability. Our first result concerns the stability of one–dimensional boundary layers for a class of symmetrizable hyperbolic-parabolic systems. We obtain the results by following the approach of detailed derivation of pointwise Green function bounds; more specifically, we build on the works of C. Mascia and K. Zumbrun in their treating the shock cases for the hyperbolic-parabolic systems and of S. Yarahmadian and K. Zumbrun in their treating the boundary layers for the strictly parabolic systems.

Multi–dimensional stability. Our second result concerns the long-time stability of multi-dimensional boundary layers of a general class of systems. Under the so–called uniform Evans stability condition, we prove the stability of the layers in dimensions $d \geq 2$, following a modified version of the approach of K. Zumbrun in treating the multi-dimensional shock cases, involving estimates between various L^p spaces.

Multi–dimensional stability for systems with variable multiplicities. Our third result is to extend the existing stability results to certain MHD layers for which the constant multiplicity assumption used in previous analyses fails to hold. In addition, the removal of a technical assumption is done. We encompass the extension by employing the recent work of O. Guès, G. Métivier, M. Williams, and K. Zumbrun in the construction of Kreiss' symmetrizers.

Spectral stability of isentropic Navier–Stokes layers. Our final result concerns the verification of the uniform Evans stability condition. By making use of the Evans-function framework of K. Zumbrun and others, we verify numerically the stability of large-amplitude compressive, or "shock-like", boundary layers of the isentropic Navier–Stokes equations.

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Chapter 1

INTRODUCTION

1.1 Boundary layers

The notion of boundary layers appears in many applications, for instance in gas dynamics, magnetohydrodynamics (MHD), and fluid mechanics; see, for example, physical discussions in Schlichting and Gersten [53] and Braslow [4]. See also discussions in the Introduction of S. Yarahmadian's doctoral thesis [55]. For the mathematical analysis of the boundary layer theory, see a very interesting book of Métivier [40] and the references therein.

Specifically, we consider a boundary layer, or stationary solution,

$$\tilde{U} = \bar{U}(x_1), \quad \lim_{x_1 \to +\infty} \bar{U}(x_1) = U_+, \quad \bar{U}(0) = \bar{U}_0 \tag{1.1}$$

of a hyperbolic–parabolic system of conservation laws on the quarter-space

$$\tilde{U}_t + \sum_j F^j(\tilde{U})_{x_j} = \sum_{jk} (B^{jk}(\tilde{U})\tilde{U}_{x_k})_{x_j}, \quad x \in \mathbb{R}^d_+ = \{x_1 > 0\}, \quad t > 0, \tag{1.2}$$

$\tilde{U} = (\tilde{u}, \tilde{v})^{tr} \in \mathbb{R}^{n-r} \times \mathbb{R}^r$, $F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n \times n}$, with initial data $\tilde{U}(x,0) = \tilde{U}_0(x)$ and Dirichlet type boundary conditions specified later.

In this dissertation, we restrict our studies to boundary layers assuming that the layer solution is *noncharacteristic*, that is, the matrix dF^1_{11} in the hyperbolic equations of $\tilde{u}$ is either strictly positive (inflow case) or strictly negative (outflow case). Roughly speaking, the noncharacteristicity limits the signals to be transmitted into or out of but not along the boundary. In the context of gas dynamics or MHD, this corresponds to the situation of a porous boundary with prescribed inflow or outflow conditions accomplished by suction or blowing, a scenario that has been suggested as a means to reduce drag along an airfoil by stabilizing laminar flow; see Example 1.2.1 below.

A fundamental question connected to the physical motivations from aerodynamics is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small (initial and boundary) perturbation of $\bar{U}$ remains close to $\bar{U}$, or converges time-asymptotically to $\bar{U}$, under the evolution of (1.2). That is the question we address here.

1.2 Physical examples.

Example 1.2.1. The main example we have in mind consists of *laminar solutions* $(\rho, u, e)(x_1, t)$ of the compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u^t u) + \nabla p = \varepsilon\mu\Delta u + \varepsilon(\mu + \eta)\nabla \mathrm{div} u \\ \partial_t(\rho E) + \mathrm{div}\big((\rho E + p)u\big) = \varepsilon\kappa\Delta T + \varepsilon\mu \mathrm{div}\big((u\cdot\nabla)u\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon(\mu+\eta)\nabla(u\cdot \mathrm{div} u), \end{cases} \tag{1.3}$$

$x \in \mathbb{R}^d$, on a half-space $x_1 > 0$, where ρ denotes density, $u \in \mathbb{R}^d$ velocity, e specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho, e)$ pressure, $T = T(\rho, e)$ temperature, $\mu > 0$ and $|\eta| \leq \mu$ first and second coefficients of viscosity, $\kappa > 0$ the coefficient of heat conduction, and $\varepsilon > 0$ (typically small) the reciprocal of the Reynolds number, with no-slip *suction-type* boundary conditions on the velocity,

$$u_j(0, x_2, \dots, x_d) = 0,\ j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) < 0,$$

and prescribed temperature, $T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x})$. Under the standard assumptions p_ρ, $T_e > 0$, this can be seen to satisfy all of the hypotheses that we shall make in Chapters 2 and 3 with the *outflow* boundary condition; indeed these are satisfied also under much weaker van der Waals gas assumptions [37, 59, 9, 19, 18]. In particular, boundary-layer solutions are of noncharacteristic type, scaling as $(\rho, u, e) = (\bar{\rho}, \bar{u}, \bar{e})(x_1/\varepsilon)$, with layer thickness $\sim \varepsilon$ as compared to the $\sim \sqrt{\varepsilon}$ thickness of the characteristic type found for an impermeable boundary.

This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow; see [53, 4]. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds [4].[1] Possible mechanisms for this reduction are smaller thickness $\sim \varepsilon << \sqrt{\varepsilon}$ of noncharacteristic boundary layers as compared to characteristic type, and greater stability, delaying the transition from laminar to turbulent flow. In particular, stability properties appear to be quite important for the understanding of this phenomenon. For further discussion, including the related issues of matched asymptotic expansion, multi-dimensional effects, and more general boundary configurations, see [19].

Example 1.2.2. Alternatively, we may consider the compressible Navier–Stokes equations (1.3) with *blowing-type* boundary conditions

$$u_j(0, x_2, \dots, x_d) = 0,\ j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) > 0,$$

and prescribed temperature and pressure

$$T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x}), \quad p(0, x_2, \dots, x_d) = p_{wall}(\tilde{x})$$

(equivalently, prescribed temperature and density). Under the standard assumptions $p_\rho, T_e > 0$ on the equation of state (alternatively, van der Waals gas assumptions), this again can be seen to satisfy all hypotheses in Chapters 2 and 3 with *inflow* boundary condition.

Example 1.2.3. For (1.3), or the general (1.2), a large class of boundary-layer solutions, sufficient for the present purposes, may be generated as truncations $\bar{u}^{x_0}(x_1) := \bar{u}(x_1 - x_0)$ of *standing shock solutions*

$$u = \bar{u}(x_1), \quad \lim_{x_1 \to \pm\infty} \bar{u}(x_1) = u_\pm \tag{1.4}$$

[1]See also NASA site http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html

on the whole line $x_1 \in \mathbb{R}$, with boundary conditions $\beta_h(t) \equiv \bar{u}(0)$ (inflow) or $\beta_h(t) \equiv \bar{w}^I(0)$ (outflow) chosen to match. However, there are also many other boundary-layer solutions not connected with any shock. For more general catalogs of boundary-layer solutions of (1.3), see, e.g., [38, 54, 9, 19].

1.3 Main results

Our first main result can be described as follows in a formal fashion.

Result 1. *(One–dimensional stability; Theorems 2.1.3–2.1.4) Under general structural assumptions, necessary and sufficient condition for linearized and nonlinear one–dimensional stability of arbitrary–amplitude boundary layers of* (1.2) *is the Evans stability condition (D1) (defined in Chapter 2). In case of stability, we obtain rates of decay:*

$$\|\tilde{U}(x,t) - \bar{U}(x)\|_{L^p} \sim (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \le p \le \infty, \tag{1.5}$$
$$\|\tilde{U}(x,t) - \bar{U}(x)\|_{H^4} \sim (1+t)^{-\frac{1}{4}}. \tag{1.6}$$

The general structural assumptions required for Result 1 (precisely defined in Chapter 2) are satisfied for gas dynamics (1.3) and MHD with van der Waals equation of state under inflow or outflow conditions. Thus, in the general spirit of [60, 36, 37, 59, 23, 56], Result 1 is to reduce the questions of linearized and nonlinear stability to verification of the Evans function condition (D1), which can then be checked either numerically or by the variety of methods available for study of eigenvalue ODE; see, for example, [6, 7, 8, 5, 28, 49, 11, 3, 24, 26, 25, 9] or further discussion below.

The stability of noncharacteristic boundary layers in gas dynamics has been treated using energy estimates in, e.g., [38, 33, 52], for both "compressive" boundary layers including the truncated shock-solutions (1.4), and for "expansive" solutions analogous to rarefaction waves. However, in the case of compressive waves, these and most subsequent analyses were restricted to the *small-amplitude case*

$$\|\bar{u} - u_+\|_{L^1(\mathbb{R}^+)} \text{ sufficiently small.} \tag{1.7}$$

Examining this condition even for the special class (1.4) of truncated shock solutions, we find that it is extremely restrictive.

For, consider the one-parameter family $\bar{u}^{x_0}(x) = \bar{u}(x - x_0)$ of boundary-layers associated with a standing shock $\bar{u}$ of amplitude $\delta := |u_+ - u_-| << 1$. By center manifold analysis [47], $\bar{u} - u_+ \sim \delta e^{-c\delta x}$, hence

$$\|\bar{u} - u_+\|_{L^1(\mathbb{R}^+)} \sim e^{-c\delta x} \sim \frac{|u_+ - u(0)|}{|u_+ - u_-|}$$

in fact measures *relative amplitude* with respect to the amplitude $|u_+ - u_-|$ of the background shock solution $\bar{u}$. Thus, smallness condition (1.7) requires that the boundary layer consist of a small, nearly-constant piece of the original shock.

Result 1, extending results of [56] in the strictly parabolic case, remove this restriction, allowing applications in principle to shocks of any amplitude. In particular, in combination with the spectral stability results obtained in [9] (see Chapter 5) by asymptotic Evans function analysis, they yield stability of noncharacteristic isentropic gas-dynamical layers of sufficiently *large* amplitude. Together with further, numerical, investigations of [9] give strong evidence that in fact *all* noncharacteristic isentropic gas layers are spectrally stable, independent of amplitude, which would together with our results yield nonlinear stability. For further discussion, see Section 2.1.3

Our second and third main results concern the multi–dimensional stability.

Result 2. *(Multi–dimensional stability; Theorems 3.1.3–3.1.4) Sufficient condition for linearized and nonlinear multi–dimensional stability of boundary layers is the uniform Evans function condition (D2) (defined in Chapter 3). In case of stability, we obtain rates of decay:*

$$\begin{aligned} |\tilde{U}(t) - \bar{U}|_{L^p} &\sim (1+t)^{-\frac{d}{2}(1-1/p)+1/2p} \\ |\tilde{U}(t) - \bar{U}|_{H^s} &\sim (1+t)^{-\frac{d-1}{4}} \end{aligned} \tag{1.8}$$

for any $p \geq 2$ and dimensions $d \geq 2$.

Result 2 is established under general structural assumptions which hold for gas dynamics equations. Motivated by MHD applications for which the constant multiplicity assumption used in the course of obtaining Result 2 always fails, we establish the following extension of Result 2 by employing a rather different technique: the method of Kreiss' symmetrizers.

Result 3. *((Extended) multi–dimensional stability; Theorems 4.1.4,4.1.5, and 4.1.7) There holds an extension of Result 2 to the case where the characteristic hy-*

perbolic roots are totally nonglancing. In addition, a structural assumption (condition (H4) in Chapter 3) can be removed with price of losing a factor in the decay rates, and in dimension $d = 3$, requiring an additional (generic) assumption.

Asymptotic stability, without rates of decay, has been shown for small amplitude noncharacteristic "normal" boundary layers of the isentropic compressible Navier–Stokes equations with outflow boundary conditions and vanishing transverse velocity in [30], using energy estimates. Results 2-3, together with the spectral verification of O. Guès, G. Métivier, M. Williams, and K. Zumbrun for small-amplitude layers (see Proposition 3.1.2), recover this existing result and extend it to the general arbitrary transverse velocity, outflow or inflow, and isentropic or nonisentropic (full compressible Navier–Stokes) case and certain cases of MHD layers, in addition to giving asymptotic rates of decay. Moreover, we treat perturbations of boundary as well as initial data, as previous time-asymptotic investigations (with the exception of direct predecessors [56, 45, 46]) do not. As discussed in Appendix B.1, the type of boundary layer relevant to the drag-reduction strategy discussed in Examples 1.2.1–1.2.2 is a noncharacteristic "transverse" type with constant normal velocity, complementary to the normal type considered in [30].

Our final result concerns the one–dimensional spectral stability of compressive, or "shock-like", boundary layers of the isentropic compressible Navier–Stokes equations with γ-law pressure. The work was building on that of Barker, Humpherys, Lafitte, Rudd, and Zumbrun [3, 24] in the shock wave case, a combination of asymptotic ODE estimates and numerical Evans function computations.

Result 4a. *(Analytical results; Theorems 5.3.2–5.3.3 and Corollary 5.3.9) The convergence of the Evans function in the shock and large-amplitude limits is shown. In addition, compressive inflow/outflow boundary layers with sufficiently large amplitudes are stable.*

Result 4b. *(Numerical verification; Section 5.3.4) Our numerical computations indicate unconditional spectral stability of uniformly noncharacteristic compressive boundary-layers for isentropic Navier–Stokes equations.*

1.4 Plan of the thesis and remarks

In Chapter $i + 1$, we shall establish the Result i above, correspondingly. Except Chapter 4 whose materials depend on those of the preceding chapter, each chapter can be read alone without referring to the other.

Materials included in Chapters 2 and 3 are taken from papers with my advisor K. Zumbrun [45, 46], respectively; Chapter 4 is from [44]; and those in Chapter 5 including four figures are from a joint paper with N. Costanzino, J. Humpherys, and K. Zumbrun, [9]. The reproduction has full permission of joint authors.

Chapter 2

ONE–DIMENSIONAL STABILITY

2.1 Introduction

In this chapter, we study the one–dimensional stability of a noncharacteristic boundary layer, or stationary solution,

$$\tilde{U} = \bar{U}(x), \quad \lim_{x\to+\infty} \bar{U}(x) = U_+, \quad \bar{U}(0) = \bar{U}_0 \tag{2.1}$$

of a system of conservation laws on the quarter-plane

$$\tilde{U}_t + F(\tilde{U})_x = (B(\tilde{U})\tilde{U}_x)_x, \quad x, t > 0, \tag{2.2}$$

$\tilde{U}, F \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$, with initial data $\tilde{U}(x,0) = \tilde{U}_0(x)$ and Dirichlet type boundary conditions specified in (2.5), (2.6) below.

2.1.1 Equations and assumptions.

We consider the general hyperbolic-parabolic system of conservation laws (2.2) in conserved variable $\tilde{U}$, with

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}, \quad \sigma(b_2) \geq \theta > 0,$$

$\tilde{u} \in \mathbb{R}$, and $\tilde{v} \in \mathbb{R}^{n-1}$, where, here and elsewhere, σ denotes spectrum of a linearized operator or matrix. Here for simplicity, we have restricted to the case (as in standard gas dynamics and MHD) that the hyperbolic part (equation for $\tilde{u}$) consists of a single scalar equation. As in [36], the results extend in straightforward fashion to the case $\tilde{u} \in \mathbb{R}^k$, $k > 1$, with $\sigma(A^{11})$ strictly positive or strictly negative.

Following [37, 59], we assume that equations (2.2) can be written, alternatively, after a triangular change of coordinates

$$\tilde{W} := \tilde{W}(\tilde{U}) = \begin{pmatrix} \tilde{w}^I(\tilde{u}) \\ \tilde{w}^{II}(\tilde{u}, \tilde{v}) \end{pmatrix}, \tag{2.3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde{A}^0 \tilde{W}_t + \tilde{A} \tilde{W}_x = (\tilde{B} \tilde{W}_x)_x + \tilde{G}, \tag{2.4}$$

where

$$\tilde{A}^0 = \begin{pmatrix} \tilde{A}^0_{11} & 0 \\ 0 & \tilde{A}^0_{22} \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} \tilde{A}^{11} & \tilde{A}^{12} \\ \tilde{A}^{21} & \tilde{A}^{22} \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}, \qquad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix}$$

and, defining $\tilde{W}_+ := \tilde{W}(U_+)$,

(A1) $\tilde{A}(\tilde{W}_+), \tilde{A}^0, \tilde{A}^{11}$ are symmetric, $\tilde{A}^0 \geq \theta_0 > 0$,

(A2) no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(\tilde{W}_+)$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(\tilde{W}_+)$,

(A3) $\tilde{b} \geq \theta > 0$ and $\tilde{g}(\tilde{W}_x, \tilde{W}_x) = \mathcal{O}(|\tilde{W}_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F, B, \tilde{A}^0, \tilde{A}, \tilde{B}, \tilde{W}(\cdot), \tilde{g}(\cdot,\cdot) \in C^5$.

(H1) $\tilde{A}^{11}$ (scalar) is either strictly positive or strictly negative, that is, either $\tilde{A}^{11} \geq \theta_1 > 0$, or $\tilde{A}^{11} \leq -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF(U_+)$ are distinct and nonzero.

Condition (H1) corresponds to hyperbolic–parabolic noncharacteristicity, while (H2) is the condition for the hyperbolicity at U_+ of the associated first-order hyperbolic system obtained by dropping second-order terms. The assumptions (A1)-(A3) and

(H0)-(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [37, 9, 19, 18].

We also assume:

(B) Dirichlet boundary conditions in $\tilde{W}$-coordinates:

$$(\tilde{w}^I, \tilde{w}^{II})(0,t) = \tilde{h}(t) := (\tilde{h}_1, \tilde{h}_2)(t) \tag{2.5}$$

for the inflow case, and

$$\tilde{w}^{II}(0,t) = \tilde{h}(t) \tag{2.6}$$

for the outflow case.

This is sufficient for the main physical applications; the situation of more general, Neumann- and mixed-type boundary conditions on the parabolic variable v can be treated as discussed in [19, 18].

2.1.2 One-dimensional results.

Linearizing the equations (2.2), (B) about the boundary layer $\bar{U}$, we obtain the linearized equation

$$U_t = LU := -(\bar{A}U)_x + (\bar{B}U_x)_x, \tag{2.7}$$

where

$$\bar{B} := B(\bar{U}), \quad \bar{A}U := dF(\bar{U})U - (dB(\bar{U})U)\bar{U}_x,$$

with boundary conditions (now expressed in U-coordinates)

$$(\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)U(0,t) = h(t) := \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}(t) \tag{2.8}$$

for the inflow case, and

$$(\partial\tilde{w}^{II}/\partial\tilde{U})(\bar{U}_0)U(0,t) = h(t) \tag{2.9}$$

for the outflow case, where $(\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)$ is constant and invertible,

$$(\partial\tilde{w}^{II}/\partial\tilde{U})(\bar{U}_0) = m\begin{pmatrix} \bar{b}_1 & \bar{b}_2 \end{pmatrix}(\bar{U}_0), \tag{2.10}$$

(by (A1) and triangular structure (2.3)) is constant with $m \in \mathbb{R}^{(n-1)\times(n-1)}$ invertible, and $h := \tilde{h} - \bar{h}$.

Definition 2.1.1. *The boundary layer $\bar{U}$ is said to be linearly $X \to Y$ stable if, for some $C > 0$, the problem* (2.7) *with initial data U_0 in X and homogeneous boundary data $h \equiv 0$ has a unique global solution $U(\cdot, t)$ such that $|U(\cdot, t)|_Y \le C|U_0|_X$ for all t; it is said to be linearly asymptotically $X \to Y$ stable if also $|U(\cdot, t)|_Y \to 0$ as $t \to \infty$.*

We define the following *stability criterion*, where $D(\lambda)$ described below, denotes the Evans function associated with the linearized operator L about the layer, an analytic function analogous to the characteristic polynomial of a finite-dimensional operator, whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of L:

$$\text{There exist no zeroes of } D(\cdot) \text{ in the nonstable half-plane } Re\lambda \ge 0 \qquad \text{(D1)}$$

As discussed, e.g., in [51, 42, 19, 18], under assumptions (H0)-(H2), this is equivalent to *strong spectral stability*, $\sigma(L) \subset \{Re\lambda < 0\}$, (ii) *transversality* of $\bar{U}$ as a solution of the connection problem in the associated standing-wave ODE, and *hyperbolic stability* of an associated boundary value problem obtained by formal matched asymptotics. See [19, 18] for further discussions.

Definition 2.1.2. *The boundary layer $\bar{U}$ is said to be nonlinearly $X \to Y$ stable if, for each $\varepsilon > 0$, the problem* (2.2) *with initial data $\tilde{U}_0$ sufficiently close to the profile $\bar{U}$ in $|\cdot|_X$ has a unique global solution $\tilde{U}(\cdot, t)$ such that $|\tilde{U}(\cdot, t) - \bar{U}(\cdot)|_Y < \varepsilon$ for all t; it is said to be nonlinearly asymptotically $X \to Y$ stable if also $|\tilde{U}(\cdot, t) - \bar{U}(\cdot)|_Y \to 0$ as $t \to \infty$. We shall sometimes not explicitly define the norm X, speaking instead of stability or asymptotic stability in Y under perturbations satisfying specified smallness conditions.*

Our first main result is as follows.

Theorem 2.1.3 (Linearized stability)**.** *Assume (A1)-(A3), (H0)-(H2), and (B) with $|h(t)| \le E_0(1+t)^{-1-\epsilon}$, $|h'(t)| \le E_0(1+t)^{-1}$, for arbitrary fixed $\epsilon > 0$. Let $\bar{U}$ be a boundary layer. Then linearized $L^1 \cap L^p \to L^1 \cap L^p$ stability, $1 \le p \le \infty$, is equivalent to (D1). In the case of stability, there holds also linearized asymptotic $L^1 \cap L^p \to L^p$*

stability, $p > 1$, with rate

$$|U(\cdot,t)|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}|U_0|_{L^1\cap L^p} + CE_0(1+t)^{-\frac{1}{2}(1-1/p)}. \tag{2.11}$$

To state the pointwise nonlinear stability result, we need some notations. Denoting by

$$a_1^+ < a_2^+ < \cdots < a_n^+ \tag{2.12}$$

the eigenvalues of of the limiting convection matrix $A_+ := dF(U_+)$, define

$$\theta(x,t) := \sum_{a_j^+>0} (1+t)^{-1/2} e^{-|x-a_j^+ t|^2/Mt}, \tag{2.13}$$

$$\psi_1(x,t) := \chi(x,t) \sum_{a_j^+>0} (1+|x|+t)^{-1/2}(1+|x-a_j^+ t|)^{-1/2}, \tag{2.14}$$

and

$$\psi_2(x,t) := (1-\chi(x,t))(1+|x-a_n^+ t| + t^{1/2})^{-3/2}, \tag{2.15}$$

where $\chi(x,t) = 1$ for $x \in [0, a_n^+ t]$ and $\chi(x,t) = 0$ otherwise and $M > 0$ is a sufficiently large constant.

For simplicity, we measure the boundary data by function

$$\mathcal{B}_h(t) := \sum_{r=0}^{2} |(d/dt)^r h| \tag{2.16}$$

for the outflow case, and

$$\mathcal{B}_h(t) := \sum_{r=0}^{4} |(d/dt)^r h_1| + \sum_{r=0}^{2} |(d/dt)^r h_2| \tag{2.17}$$

for the inflow case.

Then, our next result is as follows.

Theorem 2.1.4 (Nonlinear stability)**.** *Assuming (A1)-(A3), (H0)-(H2), (B), and the linear stability condition (D1), the profile $\bar U$ is nonlinearly asymptotically stable in $L^p \cap H^4$, $p > 1$, with respect to perturbations $U_0 \in H^4$, $h \in C^4$ in initial and boundary data satisfying: $|h(t)| \le E_0(1+t)^{-1-\epsilon}$, $|h'(t)| \le E_0(1+t)^{-1}$, for arbitrary* *fixed* $\epsilon > 0$, *and*

$$\|(1+|x|^2)^{3/4} U_0\|_{H^4} \leq E_0 \quad \text{and} \quad |\mathcal{B}_h(t)| \leq E_0(1+t)^{-1/4}$$

for E_0 *sufficiently small. More precisely,*

$$\begin{aligned} |\tilde{U}(x,t) - \bar{U}(x)| &\leq CE_0(\theta + \psi_1 + \psi_2)(x,t), \\ |\tilde{U}_x(x,t) - \bar{U}_x(x)| &\leq CE_0(\theta + \psi_1 + \psi_2)(x,t), \end{aligned} \tag{2.18}$$

where $\tilde{U}(x,t)$ *denotes the solution of* (2.2) *with initial and boundary data* $\tilde{U}(x,0) = \bar{U}(x) + U_0(x)$ *and* $\tilde{U}(0,t) = \bar{U}_0 + h(t)$, *yielding the sharp rates*

$$\|\tilde{U}(x,t) - \bar{U}(x)\|_{L^p} \leq CE_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad 1 \leq p \leq \infty, \tag{2.19}$$

$$\|\tilde{U}(x,t) - \bar{U}(x)\|_{H^4} \leq CE_0(1+t)^{-\frac{1}{4}}. \tag{2.20}$$

Remark 2.1.5. *By the one dimensional Sobolev embedding, from the hypothesis on* U_0, *we automatically assume that*

$$\|U_0\|_{H^4} \leq E_0, \quad |U_0(x)| + |U_0'(x)| \leq E_0(1+|x|)^{-3/2}.$$

A crucial step in establishing Theorems 2.1.3 and 2.1.4 is to obtain pointwise bounds on the Green function $G(x,t;y)$ of the linearized evolution equations (2.7) (more properly speaking, a distribution), which we now describe. Let $a_j^+, j = 1, ..., n$ denote the eigenvalues of $A(+\infty)$, and l_j^+, r_j^+ associated left and right eigenvectors, respectively, normalized so that $l_j^+ r_k^+ = \delta_j^k$. Eigenvalues $a_j(x)$, and eigenvectors $l_j(x), r_j(x)$ correspond to large-time convection rates and modes of propagation of the linearized model (2.7).

Define time-asymptotic, scalar diffusion rates

$$\beta_j^+ := (l_j B r_j)_+, \quad j = 1, ..., n, \tag{2.21}$$

and local dissipation coefficient

$$\eta_* := -D_*(x) \tag{2.22}$$

where

$$D_*(x) := A_{12}b_2^{-1}\Big[A_{21} - A_{22}b_2^{-1}b_1 + b_2^{-1}b_1A_* + b_2\partial_x(b_2^{-1}b_1)\Big](x)$$

is an effective dissipation analogous to the effective diffusion predicted by formal, Chapman-Enskog expansion in the (dual) relaxation case,

$$A_* := A_{11} - A_{12}b_2^{-1}b_1.$$

Note that as a consequence of dissipativity, (A2), we obtain

$$\eta_*^+ > 0, \quad \beta_j^+ > 0, \quad \text{for all } j. \tag{2.23}$$

We also define modes of propagation for the reduced, hyperbolic part of system (2.7) as

$$L_* = \begin{pmatrix} 1 \\ 0_{n-1} \end{pmatrix}, \quad R_* = \begin{pmatrix} 1 \\ -b_2^{-1}b_1 \end{pmatrix} \tag{2.24}$$

We define the Green function $G(x,t;y)$ of the linearized evolution equations (2.7) with homogeneous boundary conditions (more properly speaking, a distribution), by

(i) $(\partial_t - L_x)G = 0$ in the distributional sense, for all $x, y, t > 0$;

(ii) $G(x,t;y) \to \delta(x-y)$ as $t \to 0$;

(iii) for all $y, t > 0$, $\begin{pmatrix} \bar{A}_* & 0 \\ \bar{b}_1 & \bar{b}_2 \end{pmatrix} G(0,t;y) = \begin{pmatrix} * \\ 0 \end{pmatrix}$ where $* = 0$ for the inflow case $\bar{A}_* > 0$ and $*$ is arbitrary for the outflow case $\bar{A}_* < 0$, noting that no boundary condition is needed to be prescribed on the hyperbolic part.

By standard arguments as in [36], we have the spectral resolution, or inverse Laplace transform formulae

$$e^{Lt}f = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t}(\lambda - L)^{-1} f d\lambda \tag{2.25}$$

and

$$G(x,t;y) = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(x,y)\, d\lambda \tag{2.26}$$

for any large positive η.

We prove the following pointwise bounds on the Green function $G(x,t;y)$.

Proposition 2.1.6. *Under assumptions (A1)-(A3), (H0)-(H2), (B), and (D1), we* *obtain*

$$G(x,t;y) = H(x,t;y) + \tilde{G}(x,t;y), \tag{2.27}$$

where

$$\begin{aligned} H(x,t;y) &= \frac{1}{2\pi} A_*(x)^{-1} A_*(y) \delta_{x-\bar{a}_* t}(y) e^{-\int_y^x (\eta_*/A_*)(z)dz} R_* L_*^{tr} \\ &= \mathcal{O}(e^{-\eta_0 t}) \delta_{x-\bar{a}_* t}(y) R_* L_*^{tr}, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} |\partial_x^\gamma \partial_y^\alpha \tilde{G}(x,t;y)| &\le C e^{-\eta(|x-y|+t)} \\ &+ \quad C(t^{-(|\alpha|+|\gamma|)/2} + |\alpha| e^{-\eta|y|} + |\gamma| e^{-\eta|x|}) \Big(\sum_{k=1}^n t^{-1/2} e^{-(x-y-a_k^+ t)^2/Mt} \\ &+ \sum_{a_k^+<0,\, a_j^+>0} \chi_{\{|a_k^+ t| \ge |y|\}} t^{-1/2} e^{-(x-a_j^+(t-|y/a_k^+|))^2/Mt} \Big), \end{aligned} \tag{2.29}$$

$0 \le |\alpha|, |\gamma| \le 1$, *for some* η, C, $M > 0$, *where indicator function* $\chi_{\{|a_k^+ t| \ge |y|\}}$ *is* 1 *for* $|a_k^+ t| \ge |y|$ *and* 0 *otherwise.*

Here, the averaged convection rate $\bar{a}_*(x,t)$ *in* (2.28) *denotes the time-averages over* $[0,t]$ *of* $A_*(z)$ *along backward characteristic paths* $z_* = z_*(x,t)$ *defined by*

$$\frac{dz_*}{dt} = A_*(z_*(x,t)), \quad z_*(t) = x. \tag{2.30}$$

In all equations, a_j^+, A_*, L_*, R_* *are as defined just above.*

2.1.3 Discussion

As mentioned in the Introduction, the present result extends results of [56] in the strictly parabolic case. Let us now comment briefly on the difference between our analysis and the earlier analysis [56] carried out by similar techniques based on the Evans function and stationary phase estimates on the inverse Laplace transform formula. Our analysis is in the same spirit as, and borrows heavily from this earlier work. The main new issues are technical ones connected with the more singular high-frequency/short-time behavior of hyperbolic-parabolic equations as compared to the strictly parabolic equations considered in [56]. In particular, linearized behavior in the u coordinate, $U = (u, v)$, is essentially hyperbolic, governed for short times approximately by the principle part

$$v_t + A_*(x)v_x = 0, \quad A_* := (A_0^{11})^{-1}A^{11} \tag{2.31}$$

Thus, we may expect as in the whole-line analysis of hyperbolic-parabolic equations in [36] that the associated Green function contain a delta-function component transported along the hyperbolic characteristic

$$dx/dt = A_*(x),$$

with the difference that now we must consider also a possibly-complicated interaction with the boundary.

A key point is that in fact this potential complication *does not occur*. For, in the special case occurring in continuum-mechanical systems [59] that all hyperbolic signals either enter or leave the boundary, there is no such boundary interaction and no reflected signal. For example, in the simple scalar example (2.31), the Green function on the half-line with either homogeneous inflow ($A^{11} > 0$) boundary condition $v(0) = 0$ or outflow ($A^{11} < 0$) condition $v(0)$ arbitrary, is by inspection exactly the whole-line Green function

$$g(x, t; y) = \delta_{x-\bar{a}t}(y)/A_*(x)$$

restricted to the half-line $x, y > 0$, where $\bar{a}$ is the average over $[0, t]$ of $A_*(z_*(t))$ along the backward characteristic path

$$\frac{dz_*}{dt} = A_*(z_*(x, t)), \quad z_*(t) = x.$$

Indeed, comparing the description of the homogeneous boundary-value Green function in Proposition 2.1.6 with that of the whole-line Green function in [36], we see that they are identical. However, to prove this simple observation costs us considerable care in the high-frequency analysis.

A further issue at the nonlinear level is to obtain nonlinear damping estimates using energy estimates as in [37], which are somewhat complicated by the presence of a boundary. This is necessary to prevent a loss of derivatives in the nonlinear iteration.

As in [56], we get stability also with respect to perturbations in boundary data, something that was not accounted for in earlier works on long-time stability. We mention, finally, the works [15, 42, 19, 18] in one- and multi-dimensions of a similar spirit but somewhat different technical flavor on the related small viscosity problem– for example, $\varepsilon \to 0$ in (1.3)– which establish that the Evans condition (or its multi-dimensional analog) is also sufficient for existence and stability of matched asymptotic solution as viscosity goes to zero.

2.2 Pointwise bounds on resolvent kernel G_λ

In this section, we shall establish estimates on resolvent kernel $G_\lambda(x, y)$.

2.2.1 Evans function framework

Before starting the analysis, we review the basic Evans function methods and gap/conjugation lemma.

The gap/conjugation lemma

Consider a family of first order ODE systems on the half-line:

$$\begin{aligned} W' = \mathbb{A}(x,\lambda)W, &\quad \lambda \in \Omega \quad \text{and} \quad x > 0, \\ \mathbb{B}(\lambda)W = 0, &\quad \lambda \in \Omega \quad \text{and} \quad x = 0. \end{aligned} \tag{2.32}$$

These systems of ODEs should be considered as a generalized eigenvalue equation, with λ representing frequency. We assume that the boundary matrix $\mathbb{B}$ is analytic in λ and that the coefficient matrix $\mathbb{A}$ is analytic in λ as a function from Ω into $L^\infty(x)$, C^K in x, and approaches exponentially to a limit $\mathbb{A}_+(\lambda)$ as $x \to \infty$, with uniform exponentially decay estimates

$$|(\partial/\partial x)^k(\mathbb{A} - \mathbb{A}_+)| \le C_1 e^{-\theta|x|/C_2}, \quad \text{for } x > 0,\ 0 \le k \le K, \tag{2.33}$$

C_j, $\theta > 0$, on compact subsets of Ω. Now we can state a refinement of the "Gap Lemma" of [14, 31], relating solutions of the variable-coefficient ODE to the solutions of its constant-coefficient limiting equations

$$Z' = \mathbb{A}_+(\lambda)Z \tag{2.34}$$

as $x \to +\infty$.

Lemma 2.2.1 (Conjugation Lemma [42]). *Under assumption* (2.33)*, there exists locally to any given* $\lambda_0 \in \Omega$ *a linear transformation* $P_+(x, \lambda) = I + \Theta_+(x, \lambda)$ *on* $x \geq 0$*,* Φ_+ *analytic in* λ *as functions from* Ω *to* $L^\infty[0, +\infty)$*, such that:*

(i) $|P_+|$ *and their inverses are uniformly bounded, with*

$$|(\partial/\partial\lambda)^j(\partial/\partial x)^k\Theta_+| \leq C(j)C_1C_2e^{-\theta|x|/C_2} \quad \text{for } x > 0,\ 0 \leq k \leq K+1, \tag{2.35}$$

$j \geq 0$*, where* $0 < \theta < 1$ *is an arbitrary fixed parameter, and* $C > 0$ *and the size of the neighborhood of definition depend only on* θ*,* j*, the modulus of the entries of* $\mathbb{A}$ *at* λ_0*, and the modulus of continuity of* $\mathbb{A}$ *on some neighborhood of* $\lambda_0 \in \Omega$*.*

(ii) The change of coordinates $W := P_+Z$ *reduces* (2.32) *on* $x \geq 0$ *to the asymptotic constant-coefficient equations* (2.34)*. Equivalently, solutions of* (2.32) *may be conveniently factorized as*

$$W = (I + \Theta_+)Z_+, \tag{2.36}$$

where Z_+ *are solutions of the constant-coefficient equations, and* Θ_+ *satisfy bounds.*

Proof. As described in [36], for $j = k = 0$ this is a straightforward corollary of the gap lemma as stated in [59], applied to the "lifted" matrix-valued ODE

$$P' = \mathbb{A}_+P - P\mathbb{A} + (\mathbb{A} - \mathbb{A}_+)P$$

for the conjugating matrices P_+. The x-derivative bounds $0 < k \leq K+1$ then follow from the ODE and its first K derivatives. Finally, the λ-derivative bounds follow from standard interior estimates for analytic functions. □

Definition 2.2.2. *Following [1], we define the domain of consistent splitting for the ODE system* $W' = \mathbb{A}(x, \lambda)W$ *as the (open) set of* λ *such that the limiting matrix* $\mathbb{A}_+$ *is hyperbolic (has no center subspace) and the boundary matrix* $\mathbb{B}$ *is full rank, with* $\dim S_+ = \operatorname{rank}\mathbb{B}$*.*

Lemma 2.2.3. *On any simply connected subset of the domain of consistent splitting, there exist analytic bases* $\{v_1, \ldots, v_k\}^+$ *and* $\{v_{k+1}, \ldots, v_N\}^+$ *for the subspaces* S_+ *and* U_+ *defined in Definition 2.2.2.*

Proof. By spectral separation of U_+, S_+, the associated (group) eigenprojections are analytic. The existence of analytic bases then follows by a standard result of Kato; see [32], pp. 99–102. $\square$

Corollary 2.2.4. *By the Conjugation Lemma , on the domain of consistent splitting, the stable manifold of solutions decaying as $x \to +\infty$ of (2.32) is*

$$\mathcal{S}^+ := \text{span}\,\{P_+v_1^+, \ldots, P_+v_k^+\}, \tag{2.37}$$

where $W_+^j := P_+v_j^+$ are analytic in λ and C^{K+1} in x for $\mathbb{A} \in C^K$.

Definition of the Evans Function

On any simply connected subset of the domain of consistent splitting, let $W_1^+, \ldots, W_k^+ = P_+v_1^+, \ldots, P_+v_k^+$ be the analytic basis described in Corollary 2.2.4 of the subspace $\mathcal{S}^+$ of solutions W of (2.32) satisfying the boundary condition $W \to 0$ at $+\infty$. Then, the *Evans function* for the ODE systems $W' = \mathbb{A}(x,\lambda)W$ associated with this choice of limiting bases is defined as the $k \times k$ Gramian determinant

$$\begin{aligned} D(\lambda) &:= \det\left(\mathbb{B}W_1^+, \ldots, \mathbb{B}W_k^+\right)_{|x=0,\lambda} \\ &= \det\left(\mathbb{B}P_+v_1^+, \ldots, \mathbb{B}P_+v_k^+\right)_{|x=0,\lambda}. \end{aligned} \tag{2.38}$$

Remark 2.2.5. *Note that D is independent of the choice of P_+ as, by uniqueness of stable manifolds, the exterior products (minors) $P_+v_1^+ \wedge \cdots \wedge P_+v_k^+$ are uniquely determined by their behavior as $x \to +\infty$.*

Proposition 2.2.6. *Both the Evans function and the subspace $\mathcal{S}^+$ are analytic on the entire simply connected subset of the domain of consistent splitting on which they are defined. Moreover, for λ within this region, equation (2.32) admits a nontrivial solution $W \in L^2(x > 0)$ if and only if $D(\lambda) = 0$.*

Proof. Analyticity follows by uniqueness, and local analyticity of P_+, v_k^+. Noting that the first $P_+v_j^+$ are a basis for the stable manifold of (2.32) at $x \to +\infty$, we find that the determinant of $\mathbb{B}P_+v_j^+$ vanishes if and only if $\mathbb{B}(\lambda)$ has nontrivial kernel on $\mathcal{S}_+(\lambda, 0)$, whence the second assertion follows. $\square$

Remark 2.2.7. *In the case (as here) that the ODE system describes an eigenvalue equation associated with an ordinary differential operator L, Proposition 2.2.6 implies that eigenvalues of L agree in location with zeroes of D. (Indeed, they agree also in multiplicity; see [12, 13]; Lemma 6.1, [60]; or Proposition 6.15 of [36].)*

When $\ker \mathbb{B}$ has an analytic basis $v^0_{k+1}, \dots, v^0_N$, for example, in the commonly occurring case, as here, that $\mathbb{B} \equiv$ constant, we have the following useful alternative formulation. This is the version that we will use in our analysis of the Green function and Resolvent kernel.

Proposition 2.2.8. *Let $v^0_{k+1}, \dots, v^0_N$ be an analytic basis of* $\ker \mathbb{B}$, *normalized so that* $\det\left(\mathbb{B}^*, v^0_{k+1}, \dots v^0_N\right) \equiv 1$. *Then, the solutions W^0_j of* (2.32) *determined by initial data $W^0_j(\lambda, 0) = v^0_j$ are analytic in λ and C^{K+1} in x, and*

$$D(\lambda) := \det\left(W^+_1, \dots, W^+_k, W^0_{k+1}, \dots, W^0_N\right)_{|x=0,\lambda}. \tag{2.39}$$

Proof. Analyticity/smoothness follow by analytic/smooth dependence on initial data/parameters. By the chosen normalization, and standard properties of Grammian determinants,

$$D(\lambda) = \det\left(W^+_1, \dots, W^+_k, v^0_{k+1}, \dots, v^0_N\right)_{|x=0,\lambda},$$

yielding (2.39). □

The tracking/reduction lemma

Next, consider a family of systems

$$\begin{aligned} W' = \mathbb{A}(x, p, \varepsilon)W, \quad p \in \mathcal{P},\ \varepsilon \in \mathbb{R}^+ \quad \text{and} \quad x > 0, \\ \mathbb{B}(p, \varepsilon)W = 0, \quad \lambda \in \Omega \quad \text{and} \quad x = 0 \end{aligned} \tag{2.40}$$

parametrized by p, ε, with $\varepsilon \to 0$. The main example we have in mind is (2.32) with $p = \lambda/|\lambda|$ and $\varepsilon := |\lambda|^{-1}$, in the high-frequency regime $|\lambda| \to \infty$. We assume further that by some coordinate change we can arrange that

$$\mathbb{A} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} + \Theta, \tag{2.41}$$

with

$$|\Theta| \le \delta(\varepsilon), \quad \Re(M_+ - M_-) \ge 2\eta(\varepsilon) + \alpha^\varepsilon(x), \tag{2.42}$$

$\|\alpha\|_{L^1(\mathbb{R}^+)}$ uniformly bounded for all ε sufficiently small, and

$$(\delta/\eta)(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{2.43}$$

where $\Re(Q) := (1/2)(Q + Q^*)$ denotes the symmetric part of a matrix Q.

Then, we have the following analog of Lemma 5.4.3, asserting that the approximately block-diagonalized equations (2.40) may be converted by a smooth coordinate transformation

$$\begin{pmatrix} I & \Theta^1 \\ \Theta^2 & I \end{pmatrix} \to I \quad \text{as} \quad \varepsilon \to 0$$

to exactly diagonalized form with the same leading part $\mathbb{M}$.

Lemma 2.2.9 ([36]). *Consider a system* (2.41)*, with $\tilde{F} \equiv 0$ and $\delta/\eta \to 0$ as $\varepsilon \to 0$. Then, (i) for all $0 < \epsilon \le \epsilon_0$, there exist (unique) linear transformations $\Phi_1^\epsilon(z, p)$ and $\Phi_2^\epsilon(z, p)$, possessing the same regularity with respect to the various parameters z, p, ϵ as do coefficients $M_\pm$ and Θ, for which the graphs $\{(Z_1, \Phi_2^\epsilon Z_1)\}$ and $\{(\Phi_1^\epsilon Z_2, Z_2)\}$ are invariant under the flow of* (2.41)*, and satisfying*

$$|\Phi_1^\epsilon|, \, |\Phi_2^\epsilon| \le C\delta(\epsilon)/\eta(\epsilon) \, \text{for all } z.$$

In particular, (ii) the subspace E_- of data at $z = 0$ for which the solution decays as $z \to +\infty$, given by $\operatorname{span}\{(\Phi_1^\varepsilon(0, p)v, v)\}$*, converges as $\varepsilon \to 0$ to* $\tilde{E}_- := \operatorname{span}\{(0, v)\}$.

Proof. Standard contraction mapping argument carried out on the "lifted" equations governing the flow of the conjugating matrices Φ_j^ε; see Appendix C, [36].

Remark 2.2.10. In practice, we usually have $\alpha^\varepsilon \equiv 0$, as can be obtained in general by a change of coordinates multiplying the first coordinate by exponential weight $e^{\int \alpha^\varepsilon dx}$.

2.2.2 Construction of the resolvent kernel

In this section we construct the explicit form of the resolvent kernel, which is nothing more than the Green function $G_\lambda(x, y)$ associated with the elliptic operator $(L - \lambda I)$,

where

$$(L-\lambda I)G_\lambda(\cdot,y)=\delta_y I,\qquad \begin{pmatrix}\bar{A}_* & 0\\ \bar{b}_1 & \bar{b}_2\end{pmatrix} G_\lambda(0,y)\equiv\begin{pmatrix}*\\0\end{pmatrix}\tag{2.44}$$

where $*=0$ for the inflow case and is arbitrary for the outflow case.

Let Λ be the region of consistent splitting for L. It is a standard fact (see, e.g., [He]) that the resolvent $(L-\lambda I)^{-1}$ and the Green function $G_\lambda(x,y)$ are meromorphic in λ on Λ, with isolated poles of finite order.

Writing the associated eigenvalue equation $LU-\lambda U=0$ in the form of a first-order system (2.32) as follows: $W:=(u,v,z)\in\mathbb{C}^{2n-1}$ with $z:=b_1u'+b_2v'$, and

$$\begin{aligned}
u'&=A_*^{-1}(-A_{12}b_2^{-1}z-(A_{11}'+\lambda)u-A_{12}'v),\\
v'&=b_2^{-1}z-b_2^{-1}b_1u',\\
z'&=(A_{21}-A_{22}b_2^{-1}b_1)u'+A_{22}b_2^{-1}z+A_{21}'u+(A_{22}'+\lambda)v.
\end{aligned}\tag{2.45}$$

Domain of consistent splitting

Define

$$\Lambda:=\cap\Lambda_j^+,\quad j=1,2,...,n\tag{2.46}$$

where Λ_j^+ denote the open sets bounded on the left by the algebraic curves $\lambda_j^+(\xi)$ determined by the eigenvalues of the symbols $-\xi^2B_+-i\xi A_+$ of the limiting constant-coefficient operators

$$L_+w:=B_+w''-A_+w'\tag{2.47}$$

as ξ is varied along the real axis. The curves λ_j^+ comprise the essential spectrum of operators L_+.

Lemma 2.2.11 ([36]). *The set Λ is equal to the component containing real $+\infty$ of the domain of consistent splitting for* (2.45). *Moreover, under (A1)–(A3), (H0)–(H2),*

$$\Lambda\subset\{\lambda\ :\ \Re e\lambda>-\eta|\Im m\lambda|/(1+|\Im m\lambda|),\quad \eta>0.\tag{2.48}$$

Basic construction

We first recall the following duality relation derived for the degenerate viscosity case in [36].

Lemma 2.2.12 ([60, 36])**.** *The function $W = (U, Z)$ is a solution of* (2.45) *if and only if $\tilde{W}^*\tilde{\mathcal{S}}W \equiv$ constant for any solution $\tilde{W} = (\tilde{U}, \tilde{Z})$ of the adjoint eigenvalue equation, where*

$$\tilde{\mathcal{S}} = \begin{pmatrix} -A_{11} & -A_{12} & 0 \\ -A_{21} & -A_{22} & I_r \\ -b_2^{-1}b_1 & -I_r & 0 \end{pmatrix} \tag{2.49}$$

and

$$Z = (b_1, b_2)U', \quad \tilde{Z} = (0, b_2^*)\tilde{U}'. \tag{2.50}$$

For future reference, we note the representation

$$\tilde{\mathcal{S}}^{-1} = \begin{pmatrix} -A_*^{-1} & 0 & A_*^{-1}A_{12} \\ b_2^{-1}b_1A_*^{-1} & 0 & -b_2^{-1}b_1A_*^{-1}A_{12} - I_r \\ -\tilde{A}A_*^{-1} & I_r & -A_{22} + \tilde{A}A_*^{-1}A_{12} \end{pmatrix} \tag{2.51}$$

where $\tilde{A} := A_{21} - A_{22}b_{-1}b_1$, $A_* := A_{11} - A_{12}b_2^{-1}b_1$, obtained by direct computation in [36].

Denote by

$$\Phi^0 = (\phi_{k+1}^0(x; \lambda), \cdots, \phi_{n+r}^0(x; \lambda)), \tag{2.52}$$

$$\Phi^+ = (\phi_1^+(x; \lambda), \cdots, \phi_k^+(x; \lambda) = (P_+v_1^+, \cdots, P_+v_k^+), \tag{2.53}$$

and

$$\Phi = (\Phi^+, \Phi^0), \tag{2.54}$$

the matrices whose columns span the subspaces of solutions of (2.32) that, respectively, decay at $x = +\infty$, and satisfy the prescribed boundary conditions at $x = 0$, denoting (analytically chosen) complementary subspaces by

$$\Psi^0 = (\psi_1^0(x; \lambda), \cdots, \psi_k^0(x; \lambda)), \tag{2.55}$$

$$\Psi^+ = (\psi_{k+1}^+(x; \lambda), \cdots, \psi_{n+r}^+(x; \lambda)) \tag{2.56}$$

and

$$\Psi = (\Psi^0, \Psi^+). \tag{2.57}$$

As described in the previous subsection, eigenfunctions decaying at $+\infty$ and satisfying the prescribed boundary conditions at 0 occur precisely when the subspaces span Φ^0 and span Φ^+ intersect, i.e., at zeros of the Evans function defined in (2.39):

$$D_L(\lambda) := \det(\Phi^0, \Phi^+)_{|x=0}. \tag{2.58}$$

Define the solution operator from y to x of $(L-\lambda)U = 0$, denoted by $\mathcal{F}^{y\to x}$, as

$$\mathcal{F}^{y\to x} = \Phi(x,\lambda)\Phi^{-1}(y,\lambda)$$

and the projections Π^0_y, Π^+_y on the stable manifolds at $0, +\infty$ as

$$\Pi^+_y = \begin{pmatrix}\Phi^+(y) & 0\end{pmatrix}\Phi^{-1}(y), \quad \Pi^0_y = \begin{pmatrix}0 & \Phi^0(y)\end{pmatrix}\Phi^{-1}(y).$$

With these preparations, the construction of the Resolvent kernel goes exactly as in the construction performed in [60, 36] on the whole line.

Lemma 2.2.13. *We have the the representation*

$$G_\lambda(x,y) = \begin{cases}(I_n,0)\mathcal{F}^{y\to x}\Pi^+_y\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x > y,\\ -(I_n,0)\mathcal{F}^{y\to x}\Pi^0_y\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x < y.\end{cases} \tag{2.59}$$

Moreover, on any compact subset K of $\rho(L)\cap\Lambda$,

$$|G_\lambda(x,y)| \le Ce^{\eta|x-y|}, \tag{2.60}$$

where $C > 0$ and $\eta > 0$ depend only on K, L.

We define also the dual subspaces of solutions of $(L^* - \lambda^*)\tilde{W} = 0$. We denote growing solutions

$$\tilde{\Phi}^0 = \begin{pmatrix}\tilde{\phi}^0_1(x;\lambda) & \cdots & \tilde{\phi}^0_k(x;\lambda)\end{pmatrix}, \tag{2.61}$$

$$\tilde{\Phi}^+ = \begin{pmatrix}\tilde{\phi}^+_{k+1}(x;\lambda) & \cdots & \tilde{\phi}^+_{n+r}(x;\lambda)\end{pmatrix}, \tag{2.62}$$

$\tilde{\Phi} := (\tilde{\Phi}^0, \tilde{\Phi}^+)$ and decaying solutions

$$\tilde{\Psi}^0 = \begin{pmatrix}\tilde{\psi}^0_1(x;\lambda) & \cdots & \tilde{\psi}^+_k(x;\lambda)\end{pmatrix}, \tag{2.63}$$

$$\tilde{\Psi}^+ = \begin{pmatrix}\tilde{\psi}^+_{k+1}(x;\lambda) & \cdots & \tilde{\psi}^+_{n+r}(x;\lambda),\end{pmatrix} \tag{2.64}$$

and $\tilde\Psi := (\tilde\Psi^0, \tilde\Psi^+)$, satisfying the relations

$$\begin{pmatrix}\tilde\Psi & \tilde\Phi\end{pmatrix}^*_{0,+} \tilde S \begin{pmatrix}\Psi & \Phi\end{pmatrix}_{0,+} \equiv I.$$

Then, we have

Proposition 2.2.14. *The resolvent kernel may alternatively be expressed as*

$$G_\lambda(x,y) = \begin{cases} (I_n, 0)\Phi^+(x;\lambda)M^+(\lambda)\tilde\Psi^{0*}(y;\lambda)(I_n,0)^{tr} & x > y, \\ -(I_n, 0)\Phi^0(x;\lambda)M^0(\lambda)\tilde\Psi^{+*}(y;\lambda)(I_n,0)^{tr} & x < y, \end{cases} \tag{2.65}$$

where

$$M(\lambda) := \text{diag}(M^+(\lambda), M^0(\lambda)) = \Phi^{-1}(z;\lambda)\bar{\mathcal{S}}^{-1}(z)\tilde\Psi^{-1*}(z;\lambda). \tag{2.66}$$

From Proposition 2.2.14, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant-coefficient case

Corollary 2.2.15. *On $\Lambda \cap \rho(L)$,*

$$G_\lambda(x,y) = \sum_{j,k} d^+_{jk}\phi^+_j(x;\lambda)\tilde\psi^+_k(y;\lambda)^* + \sum_k \phi^+_k(x;\lambda)\tilde\phi^+_k(y;\lambda)^* \tag{2.67}$$

for $0 \le y \le x$, and

$$G_\lambda(x,y) = \sum_{j,k} d^0_{jk}(\lambda)\phi^+_j(x;\lambda)\tilde\psi^+_k(y;\lambda)^* + \sum_k \psi^+_k(x;\lambda)\tilde\psi^+_k(y;\lambda)^* \tag{2.68}$$

for $0 \le x \le y$, where $d^{0,+}_{jk}(\lambda) = \mathcal{O}(\lambda^{-K})$ are scalar meromorphic functions with pole of order K less than or equal to the order to which the Evans function $D(\lambda)$ vanishes at $\lambda = 0$ (note that $K = 0$ under assumption (D1)).

Proof. Matrix manipulation of expression (2.66), Kramer's rule, and the definition of the Evans function; see [36]. □

Remark 2.2.16. *In the constant-coefficient case, with a choice of common bases $\Psi^{0,+} = \Phi^{+,0}$ at $0, +\infty$, the above representation* (2.2.15) *reduces to the simple formula*

$$G_\lambda(x,y) = \begin{cases} \sum_{j=k+1}^N \phi^+_j(x;\lambda)\tilde\phi^{+*}_j(y;\lambda) & x > y, \\ -\sum_{j=1}^k \psi^+_j(x;\lambda)\tilde\psi^{+*}_j(y;\lambda) & x < y. \end{cases} \tag{2.69}$$

2.2.3 High frequency estimates

We now turn to the crucial estimation of the resolvent kernel in the high-frequency regime $|\lambda| \to +\infty$, following the general approach of [36]. Define sectors

$$\Omega_P := \{\lambda \ : \ \Re e\lambda \geq -\theta_1 |\Im m\lambda| + \theta_2\}, \quad \theta_j > 0. \tag{2.70}$$

and

$$\Omega := \{\lambda \ : \ -\eta_1 \leq \Re e\lambda\} \tag{2.71}$$

with η_1 sufficiently small such that $\Omega \setminus B(0,r)$ is compactly contained in the set of consistent splitting Λ, for some small r to be chosen later. Then, we have the following crucial result analogous to the estimates on the whole line performed in [36].

Proposition 2.2.17. *Assume that (A1)-(A3), (H0)-(H2), and (B) hold. Then for any $r > 0$ and $\eta_1 = \eta_1(r) > 0$ chosen sufficiently small such that $\Omega \backslash B(0,r) \subset \Lambda \cap \rho(L)$. Moreover for $R > 0$ sufficiently large, the following decomposition holds on $\Omega \backslash B(0,R)$:*

$$G_\lambda(x,y) = H_\lambda(x,y) + P_\lambda(x,y) + \Theta_\lambda^H(x,y) + \Theta_\lambda^P(x,y), \tag{2.72}$$

where

$$H_\lambda(x,y) = \begin{cases} \chi_{\{A_*>0\}} A_*(x)^{-1} e^{\int_y^x (-\lambda/A_* - \eta_*/A_*)(z)dz} R_* L_*^{tr} & x > y, \\ \chi_{\{A_*<0\}} A_*(x)^{-1} e^{\int_y^x (-\lambda/A_* - \eta_*/A_*)(z)dz} R_* L_*^{tr} & x < y, \end{cases} \tag{2.73}$$

and

$$\begin{aligned} \Theta_\lambda^H(x,y) &= \lambda^{-1} B_\lambda(x,y;\lambda) + \lambda^{-1}(x-y) C_\lambda(x,y;\lambda), \\ \Theta_\lambda^P(x,y) &= \lambda^{-2} D_\lambda(x,y;\lambda) \end{aligned} \tag{2.74}$$

where

$$B_\lambda(x,y) = C_\lambda(x,y) = \begin{cases} \chi_{\{A_*>0\}} e^{-\int_y^x \lambda/A_*(z)dz} b_*(x,y) & x > y, \\ \chi_{\{A_*<0\}} e^{-\int_y^x \lambda/A_*(z)dz} b_*(x,y) & x < y, \end{cases} \tag{2.75}$$

with

$$b_* := e^{\int_y^x (-\eta_*/A_*)(z)dz} = \mathcal{O}(e^{-\theta|x-y|}), \tag{2.76}$$

due to (2.23), *and*

$$D_\lambda(x,y;\lambda) = \mathcal{O}(e^{-\theta(1+Re\lambda)|x-y|} + e^{-\theta|\lambda|^{1/2}|x-y|}), \tag{2.77}$$

for some uniform $\theta > 0$ independent of x, y, z, each described term separately analytic in λ, and P_λ is analytic in λ on a (larger) sector Ω_P as in (2.70)*, with θ_1 sufficiently small, and θ_2 sufficiently large, satisfying uniform bounds*

$$(\partial/\partial x)^\alpha (\partial/\partial y)^\beta P_\lambda(x,y) = \mathcal{O}(|\lambda|^{(|\alpha|+|\beta|-1)/2}) e^{-\theta|\lambda|^{1/2}|x-y|}, \quad \theta > 0, \tag{2.78}$$

for $|\alpha| + |\beta| \leq 2$ and $0 \leq |\alpha|, |\beta| \leq 1$.

Likewise, the following derivative bounds also hold:

$$\begin{aligned}(\partial/\partial x)\Theta_\lambda(x,y) =& \Big(B_x^0(x,y;\lambda) + (x-y)C_x^0(x,y;\lambda)\Big) + \lambda^{-1}\Big(B_x^1(x,y;\lambda) \\ &+ (x-y)C_x^1(x,y;\lambda) + (x-y)^2 D_x^1(x,y;\lambda)\Big) \\ &+ \lambda^{-3/2}E_x(x,y;\lambda)\end{aligned}$$

and

$$\begin{aligned}(\partial/\partial y)\Theta_\lambda(x,y) =& \Big(B_y^0(x,y;\lambda) + (x-y)C_y^0(x,y;\lambda)\Big) + \lambda^{-1}\Big(B_y^1(x,y;\lambda) \\ &+ (x-y)C_y^1(x,y;\lambda) + (x-y)^2 D_y^1(x,y;\lambda)\Big) \\ &+ \lambda^{-3/2}E_y(x,y;\lambda)\end{aligned}$$

where B_β^α, C_β^α, and D_β^1 satisfy bounds of the form (2.75)*, and E_β satisfies a bound of the form* (2.77)*.*

Proof. We shall follow closely the argument in [36], with the new feature of boundary treatments, or estimates of Φ^0, Ψ^0. Writing the associated eigenvalue equation $LU - \lambda U = 0$ in the form of a first-order system as follows: $W := (u, v, z) \in \mathbb{C}^{2n-1}$ with $z := b_1 u' + b_2 v'$, and

$$\begin{aligned} u' &= A_*^{-1}(-A_{12}b_2^{-1}z - (A_{11}' + \lambda)u - A_{12}'v), \\ v' &= b_2^{-1}z - b_2^{-1}b_1 u', \\ z' &= (A_{21} - A_{22}b_2^{-1}b_1)u' + A_{22}b_2^{-1}z + A_{21}'u + (A_{22}' + \lambda)v \end{aligned} \tag{2.79}$$

or

$$W' = AW. \tag{2.80}$$

Recall from Lemma 2.2.13 that we have the the representation

$$G_\lambda(x,y) = \begin{cases} (I_n, 0)\mathcal{F}_W^{y\to x}\Pi_W^+(y)\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x > y, \\ -(I_n, 0)\mathcal{F}_W^{y\to x}\Pi_W^0(y)\tilde{S}^{-1}(y)(I_n,0)^{tr}, & for \quad x < y. \end{cases} \tag{2.81}$$

We shall find it more convenient to use the "local" coordinates $\tilde{u} := A_* u, \tilde{v} := b_1 u + b_2 v$. yielding from (2.45):

$$\begin{aligned} \tilde{u}_x &= -\lambda A_*^{-1}\tilde{u} - (A_{12}b_2^{-1}\tilde{v})_x \\ (\tilde{v}_x)_x &= \Big[((A_{21} - A_{22}b_2^{-1}b_1 + b_2\partial_x(b_2^{-1}b_1))A_*^{-1}\tilde{u})_x \\ &\qquad + ((A_{22} + \partial_x(b_2)b_2^{-1})\tilde{v})_x + \lambda b_2^{-1}b_1 A_*^{-1}\tilde{u} + \lambda b_2^{-1}\tilde{v}\Big]. \end{aligned} \tag{2.82}$$

Following standard procedure (e.g., [1, 14, 60, 36]), performing the rescaling

$$\tilde{x} := |\lambda| x, \quad \tilde{\lambda} := \lambda/|\lambda|, \tag{2.83}$$

and changing coordinates $W \mapsto Y = \mathcal{Q}W$, where

$$Y = (\tilde{u}, \tilde{v}, \tilde{v}_x)^{tr} = (A_* u, b_1 u + b_2 v, (b_1 u + b_2 v)_x)^{tr}, \tag{2.84}$$

$$\mathcal{Q} = \begin{pmatrix} A_* & 0 & 0 \\ b_1 & b_2 & 0 \\ |\lambda|^{-1}\partial_x b_1 & |\lambda|^{-1}\partial_x b_2 & |\lambda|^{-1} I_r, \end{pmatrix} \tag{2.85}$$

and

$$\mathcal{Q}^{-1} = \begin{pmatrix} A_*^{-1} & 0 & 0 \\ -b_2^{-1}b_1 A_*^{-1} & b_2^{-1} & 0 \\ -|\lambda| b_2 \partial_x(b_2^{-1}b_1)A_*^{-1} & -|\lambda|\partial_x(b_2)b_2^{-1} & |\lambda| I_r, \end{pmatrix} \tag{2.86}$$

we obtain the first order equations

$$Y' = A(\tilde{x}, |\lambda|^{-1})Y, \quad Y := (\tilde{u}, \tilde{v}, \tilde{v}')^{tr}, \quad {}' := \partial_{\tilde{x}} \tag{2.87}$$

where

$$A(\tilde{x}, |\lambda|^{-1}) = A_0(\tilde{x}) + |\lambda|^{-1}A_1(\tilde{x}) + \mathcal{O}(|\lambda|^{-2}), \tag{2.88}$$

with

$$
\begin{aligned}
A_0(\tilde{x}) &= \begin{pmatrix} -\tilde{\lambda} A_*^{-1} & 0 & -A_{12} b_2^{-1} \\ 0 & 0 & I_r \\ 0 & 0 & 0 \end{pmatrix} \\
A_1(\tilde{x}) &= \begin{pmatrix} 0 & -\partial_x (A_{12} b_2^{-1}) & 0 \\ 0 & 0 & 0 \\ -\tilde{\lambda} d_* A_*^{-2} & \tilde{\lambda} b_2^{-1} & e_* b_2^{-1} \end{pmatrix}
\end{aligned}
\tag{2.89}
$$

$$
\begin{aligned}
d_* &:= A_{21} - A_{22} b_2^{-1} b_1 - b_2^{-1} b_1 A_* + b_2 \partial_x (b_2^{-1} b_1), \\
e_* &:= A_{22} + d_* A_*^{-1} A_{12} + \partial_x (b_2).
\end{aligned}
\tag{2.90}
$$

We will carry out the details of the lower-order estimates in Proposition 2.2.17, leaving high-order estimates and derivative bounds as brief remarks at the end. First, observe that the representation (2.81) becomes

$$
G_\lambda(x,y) = \begin{cases} (I_n, 0) \mathcal{Q}^{-1} \mathcal{F}_Y^{y \to x} \Pi_Y^+(y) \mathcal{Q} \tilde{S}^{-1}(y) (I_n, 0)^{tr}, & for \quad x > y, \\ -(I_n, 0) \mathcal{Q}^{-1} \mathcal{F}_Y^{y \to x} \Pi_Y^0(y) \mathcal{Q} \tilde{S}^{-1}(y) (I_n, 0)^{tr}, & for \quad x < y \end{cases}
\tag{2.91}
$$

where $\Pi_Y^{0,+}$ and $\mathcal{F}_Y^{y \to x}$ denote projections and flows in $Y-$coordinates.

Initial diagonalization.

Applying the formal iterative diagonalization procedure described in [36, Proposition 3.12], one obtains the approximately block-diagonalized system

$$
Z' = D(\tilde{x}, |\lambda|^{-1}) Z, \quad TZ := Y, \quad D := T^{-1} A T, \tag{2.92}
$$

$$
T(\tilde{x}, |\lambda|^{-1}) = T_0(\tilde{x}) + |\lambda|^{-1} T_1(\tilde{x}) + \cdots + |\lambda|^{-3} T_3(\tilde{x}) \tag{2.93}
$$

$$
D(\tilde{x}, |\lambda|^{-1}) = D_0(\tilde{x}) + |\lambda|^{-1} D_1(\tilde{x}) + \cdots + D_3(\tilde{x}) |\lambda|^{-3} + \mathcal{O}(|\lambda|^{-4}), \tag{2.94}
$$

where without loss of generality (since T_0 is uniquely determined up to a constant linear coordinate change)

$$
T_0 := \begin{pmatrix} 1 & 0 & -\tilde{\lambda}^{-1} A_* A_{12} b_2^{-1} \\ 0 & I_r & 0 \\ 0 & 0 & I_r \end{pmatrix}, \quad T_0^{-1} = \begin{pmatrix} 1 & 0 & \tilde{\lambda}^{-1} A_* A_{12} b_2^{-1} \\ 0 & I_r & 0 \\ 0 & 0 & I_r \end{pmatrix} \tag{2.95}
$$

and

$$D_0 := \begin{pmatrix} -\tilde{\lambda} A_*^{-1} & 0 & 0 \\ 0 & 0 & I_r \\ 0 & 0 & 0 \end{pmatrix}, \quad D_1 := \begin{pmatrix} -\eta_* A_*^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{\lambda} b_2^{-1} & * \end{pmatrix} \tag{2.96}$$

with η_* as defined in (2.22); see Proposition 3.12 [36]. (Here, the simple block upper-triangular form of A_0 has been used to deduce the above simple form of D_0, D_1.)

The parabolic block.

At this point, we have approximately diagonalized our system into a 1×1 hyperbolic block with eigenvalue $\tilde{\mu} = -\tilde{\lambda}/A_*$ of A_0, and a $2r \times 2r$ parabolic block

$$Z_p' = N Z_p \tag{2.97}$$

with

$$N := \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix} + |\lambda|^{-1} \begin{pmatrix} 0 & 0 \\ \tilde{\lambda} b_2^{-1} & * \end{pmatrix} + \mathcal{O}(|\lambda|^{-2}). \tag{2.98}$$

Balancing this matrix N by transformations $\mathcal{B} := diag\{I_r, |\lambda|^{-1/2} I_r\}$ we get

$$\tilde{M} := \mathcal{B}^{-1} N \mathcal{B} = |\lambda|^{-1/2} \tilde{M}_1 + \mathcal{O}(|\lambda|^{-1}), \quad \tilde{M}_1 := \begin{pmatrix} 0 & I_r \\ \tilde{\lambda} b_2^{-1} & 0 \end{pmatrix} \tag{2.99}$$

Observe that $\sigma(\tilde{M}_1) = \pm\sqrt{\sigma(\tilde{\lambda} b_2^{-1})}$ has a uniform spectral gap of order one. Thus, there is a well-conditioned transformation $S = S(\tilde{M}_1)$ depending continuously on $\tilde{M}_1$ such that

$$\hat{M}_1 := S^{-1} \tilde{M}_1 S = \text{diag}\{\hat{M}^-, \hat{M}^+\} \tag{2.100}$$

with $\hat{M}_1^{\pm}$ uniformly positive/negative definite, respectively. Applying this coordinate change, and noting that the "dynamic error" $S^{-1}\partial_{\tilde{x}} S$ is of order $\partial_{\tilde{x}} \tilde{M}_1 = \mathcal{O}(|\lambda|^{-1})$, we obtain the formal expansion

$$\hat{M}(\tilde{x}, |\lambda|^{-1}) = |\lambda|^{-1/2} \text{diag}\{\hat{M}_1^-, \hat{M}_1^+\} + \mathcal{O}(|\lambda|^{-1}). \tag{2.101}$$

Finally, on sector Ω_P, blocks $|\lambda|^{-1/2} \hat{M}_1^{\pm}$ are exponentially separated to order $|\lambda|^{-1/2}$. Thus, by the *reduction lemma*, Lemma 2.2.9, there is a further transformation $\hat{S} := I_{2r} + \mathcal{O}(|\lambda|^{-1/2})$ converting $\hat{M}$ to the fully diagonalized form

$$\begin{aligned} M(\tilde{x}, |\lambda|^{-1}) &:= |\lambda|^{-1/2} \hat{S}^{-1} \Big(\hat{M}_1 + \mathcal{O}(|\lambda|^{-1/2}) \Big) \hat{S} \\ &= \mathcal{O}(|\lambda|^{-1/2}) \mathrm{diag}\{M_1^-, M_1^+\} \end{aligned}$$

where $M_1^{\pm} = \hat{M}_1^{\pm} + \mathcal{O}(|\lambda|^{-1/2})$ are still uniformly positive/negative definite.

In summary, changing coordinates

$$\mathcal{B} S \hat{S} \hat{Z}_p = Z_p, \tag{2.102}$$

(2.97) yields

$$\hat{Z}_p' = \mathcal{O}(|\lambda|^{-1/2}) \begin{pmatrix} M_1^- & 0 \\ 0 & M_1^+ \end{pmatrix} \hat{Z}_p + \mathcal{O}(|\lambda|^{-3/2}) \tag{2.103}$$

Therefore the transformation

$$\mathcal{T} := (T_0 + |\lambda|^{-1} T_1) \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{B} S \hat{S} \end{pmatrix} \tag{2.104}$$

converts equations (2.87) to the following:

$$\begin{aligned} \zeta' &= -(\tilde{\lambda} A_*^{-1} + |\lambda|^{-1} \eta_* A_*^{-1}) \zeta + \mathcal{O}(|\lambda|^{-2}) \\ \rho_{\pm}' &= |\lambda|^{-1/2} M_1^{\pm} \rho_{\pm} + \mathcal{O}(|\lambda|^{-3/2}) \end{aligned} \tag{2.105}$$

by relation

$$\mathcal{T} \mathcal{Z} = Y, \quad \mathcal{Z} = (\zeta, \rho_-, \rho_+)^{tr}. \tag{2.106}$$

Then, we have the the representation

$$G_\lambda(x,y) = \begin{cases} (I_n, 0) \mathcal{Q}^{-1} \mathcal{T} \mathcal{F}_{\mathcal{Z}}^{y \to x} \Pi_{\mathcal{Z}}^+(y) \mathcal{T}^{-1} \mathcal{Q} \tilde{S}^{-1}(y) (I_n, 0)^{tr}, & for \quad x > y, \\ -(I_n, 0) \mathcal{Q}^{-1} \mathcal{T} \mathcal{F}_{\mathcal{Z}}^{y \to x} \Pi_{\mathcal{Z}}^0(y) \mathcal{T}^{-1} \mathcal{Q} \tilde{S}^{-1}(y) (I_n, 0)^{tr}, & for \quad x < y, \end{cases} \tag{2.107}$$

thanks to the fact that

$$\mathcal{F}_Y^{y \to x} = \mathcal{T} \mathcal{F}_{\mathcal{Z}}^{y \to x} \mathcal{T}^{-1}, \quad \Pi_Y^+ = \mathcal{T} \Pi_{\mathcal{Z}}^+ \mathcal{T}^{-1}. \tag{2.108}$$

Computing, we have

$$\mathcal{T} = \begin{pmatrix} 1 & |\lambda|^{-1/2} & |\lambda|^{-1/2} \\ 0 & \mathcal{O}(1) & \mathcal{O}(1) \\ 0 & |\lambda|^{-1/2} & |\lambda|^{-1/2} \end{pmatrix} \quad \mathcal{T}^{-1} = \begin{pmatrix} 1 & 0 & \tilde{\lambda}^{-1} A_* A_{12} b_2^{-1} \\ 0 & \mathcal{O}(1) & |\lambda|^{1/2} \\ 0 & \mathcal{O}(1) & |\lambda|^{1/2} \end{pmatrix}$$

and

$$(I_n, 0)\mathcal{Q}^{-1} = \begin{pmatrix} A_*^{-1} & 0 & 0 \\ -b_2^{-1} b_1 A_*^{-1} & b_2^{-1} & 0 \end{pmatrix} \tag{2.109}$$

$$(I_n, 0)\mathcal{Q}^{-1}\mathcal{T} = \begin{pmatrix} A_*^{-1} & \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2}) \\ -b_2^{-1} b_1 A_*^{-1} & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} \tag{2.110}$$

and

$$\mathcal{Q}\tilde{S}^{-1}(I_n, 0)^{tr} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ |\lambda|^{-1} & |\lambda|^{-1} I_r \end{pmatrix} \tag{2.111}$$

$$\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(I_n, 0)^{tr} = \begin{pmatrix} -1 + |\lambda|^{-1} & \mathcal{O}(|\lambda|^{-1}) \\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2}) \\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2}) \end{pmatrix} \tag{2.112}$$

Therefore now we are ready to estimate $\mathcal{F}_{\mathcal{Z}}^{y \to x} \Pi_{\mathcal{Z}}^{+}$ and $\mathcal{F}_{\mathcal{Z}}^{y \to x} \Pi_{\mathcal{Z}}^{+}$.

Estimates on projections and solution operators

We shall give estimates on the projections:

$$\Pi_{\mathcal{Z}}^{+} = (\Phi^+, 0)(\Phi^+, \Phi^0)^{-1}, \quad \Pi_{\mathcal{Z}}^{0} = (0, \Phi^0)(\Phi^+, \Phi^0)^{-1} \tag{2.113}$$

and the solution operators:

$$\mathcal{F}_{\mathcal{Z}}^{y \to x} = (\Phi^+(x), \Phi^0(x))(\Phi^+(y), \Phi^0(y))^{-1}. \tag{2.114}$$

First, let Φ^{p+}/Ψ^{p+} be the decaying/growing basis solutions of

$$\rho_-' = |\lambda|^{-1/2} M_1^- \rho_- \quad \text{and} \quad \rho_+' = |\lambda|^{-1/2} M_1^+ \rho_+ \tag{2.115}$$

and ϕ^{h+}/ψ^{h+} be the decaying/growing basis solutions of

$$\zeta' = -(\tilde{\lambda} A_*^{-1} + |\lambda|^{-1} \eta_* A_*^{-1})\zeta. \tag{2.116}$$

Lemma 2.2.18. [Inflow case] *For the inflow case $A_* > 0$, we obtain*

$$\Pi_{\mathcal{Z}}^{+} = \begin{pmatrix} 1 & 0 & -|\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+-1} \\ 0 & I_r & -\Phi^{p+}E(\lambda)\Psi^{p+-1} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.117}$$

$$\Pi_{\mathcal{Z}}^{0} = \begin{pmatrix} 0 & 0 & |\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+-1} \\ 0 & 0 & \Phi^{p+}E(\lambda)\Psi^{p+-1} \\ 0 & 0 & I_r \end{pmatrix} \tag{2.118}$$

with bounded functions $e(\lambda), E(\lambda)$, and

$$\mathcal{F}_{\mathcal{Z}}^{y\to x} = \begin{pmatrix} \phi^{h+}(x)\phi^{h+}(y)^{-1} & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & 0 \\ 0 & 0 & \Psi^{p+}(x)\Psi^{p+}(y)^{-1} \end{pmatrix} \tag{2.119}$$

Proof. We have the decaying basis solution in $\mathcal{Z}$-coordinates of the first order equations (2.105)

$$\Phi^{+} = \begin{pmatrix} \phi^{h+} & 0 \\ 0 & \Phi^{p+} \\ 0 & 0 \end{pmatrix} + \mathcal{O}(|\lambda|^{-1}). \tag{2.120}$$

Since Φ^{+} and Ψ^{+} (exactly Ψ^{p+}) form a basis solution, we can write

$$\Phi^{0}(x) = e(\lambda)\begin{pmatrix} \phi^{h+} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \Phi^{p+}(x)E(\lambda) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Psi^{p+}(x)F(\lambda) \end{pmatrix} \tag{2.121}$$

Now since $\{\psi_j^{p+}\}_j$ forms a basis, we can take $\{\psi_j^{p+}(0)\}$ to be the analytic basis for Y at $x = 0$. Also as we recall that $\mathcal{Z} = \mathcal{T}^{-1}Y$, we compute

$$\phi^{0}_{j|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 0 \\ 0 \\ \psi_j^{p+}(0) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ |\lambda|^{1/2}\psi_j^{p+}(0) \\ |\lambda|^{1/2}\psi_j^{p+}(0) \end{pmatrix} \tag{2.122}$$

This and (2.121) yield

$$\Phi^0(x) = \begin{pmatrix} e(\lambda)\phi^{h+}(x) \\ |\lambda|^{1/2}\Phi^{p+}(x)E(\lambda) \\ |\lambda|^{1/2}\Psi^{p+}(x) \end{pmatrix} + \mathcal{O}(|\lambda|^{-1/2}) \tag{2.123}$$

where

$$E(\lambda) = (E_1(\lambda), \dots, E_r(\lambda))^{tr}, \quad E_j(\lambda) = \psi_j^{p+}(0,\lambda)\Phi^{p+}(0,\lambda)^{-1} \tag{2.124}$$

and $e(\lambda), E_j(\lambda) \in \mathbb{R}^r$ are bounded functions in λ. Therefore computing, we get

$$(\Phi^+, \Phi^0)^{-1} = \begin{pmatrix} \phi^{h+^{-1}} & 0 & -|\lambda|^{-1/2}e(\lambda)\Psi^{p+^{-1}} \\ 0 & \Phi^{p+^{-1}} & -E(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & |\lambda|^{-1/2}\Psi^{p+^{-1}} \end{pmatrix} \tag{2.125}$$

and hence straightforward computations give the lemma. □

Lemma 2.2.19. [Outflow case] *For the outflow case $A_* < 0$, we obtain*

$$\Pi^+_{\mathcal{Z}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & -\Phi^{p+}E\Psi^{p+^{-1}} \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi^0_{\mathcal{Z}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \Phi^{p+}E\Psi^{p+^{-1}} \\ 0 & 0 & I_r \end{pmatrix}, \tag{2.126}$$

where $E(\lambda)$ is a bounded function in λ determined below. Moreover,

$$\mathcal{F}^{y\to x}_{\mathcal{Z}} = \begin{pmatrix} \psi^{h+}(x)\psi^{h+}(y)^{-1} & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & 0 \\ 0 & 0 & \Psi^{p+}(x)\Psi^{p+}(y)^{-1} \end{pmatrix} \tag{2.127}$$

Proof. Similarly, we have $\Phi^+ = \Phi^{p+}$ and $\Phi^0 = (\phi^{h0}, \Phi^{p0})$ where we can write

$$\Phi^0(x) = \begin{pmatrix} 0 \\ \Phi^{p+}(x)E(\lambda) \\ 0 \end{pmatrix} + e(\lambda)\begin{pmatrix} \psi^{h+} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \Psi^{p+}(x)F(\lambda) \end{pmatrix}. \tag{2.128}$$

As before, using the form of the linearized boundary conditions (2.9), we can take

$$\phi^{p0}_{j|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 0 \\ 0 \\ \psi^{p+}_j(0) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) \\ |\lambda|^{1/2}\psi^{p+}_j(0) \\ |\lambda|^{1/2}\psi^{p+}_j(0) \end{pmatrix} \tag{2.129}$$

and thus

$$\Phi^{p0}(x) = \begin{pmatrix} e(\lambda)\psi^{h+}(x) \\ |\lambda|^{1/2}\Phi^{p+}(x)E(\lambda) \\ |\lambda|^{1/2}\Psi^{p+}(x) \end{pmatrix} \tag{2.130}$$

with bounded functions $e(\lambda)$ and $E_j(\lambda) = \psi^{p+}_j(0,\lambda)\Phi^{p+}(0,\lambda)^{-1}$.

Similarly, we take

$$\phi^{h0}_{|_{x=0}} = \mathcal{T}^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and thus

$$\phi^{h0}(x) = \begin{pmatrix} \psi^{h+}(x) \\ 0 \\ 0 \end{pmatrix} \tag{2.131}$$

Putting together and computing, we obtain

$$(\Phi^+, \Phi^0) = \begin{pmatrix} 0 & \psi^{h+} & e(\lambda)\psi^{h+} \\ \Phi^{p+} & 0 & |\lambda|^{1/2}\Phi^{p+}E(\lambda) \\ 0 & 0 & |\lambda|^{1/2}\Psi^{p+} \end{pmatrix} \tag{2.132}$$

and

$$(\Phi^+, \Phi^0)^{-1} = \begin{pmatrix} 0 & \Phi^{p+^{-1}} & -E(\lambda)\Psi^{p+^{-1}} \\ \psi^{h+^{-1}} & 0 & -|\lambda|^{-1/2}e(\lambda)\Psi^{p+^{-1}} \\ 0 & 0 & |\lambda|^{-1/2}\Psi^{p+^{-1}} \end{pmatrix} \tag{2.133}$$

Direct computations yield the lemma. $\square$

Estimates on G_λ: Inflow case $A_*>0$.

Now we are ready to combine all above estimates to give the bounds on resolvent kernel G_λ. We shall work in detail for the case $x>y$. Similar estimates can be easily obtained for $x<y$. First decompose the projection as $\Pi^+_{\mathcal{Z}} = \Pi^{h+}_{\mathcal{Z}} + \Pi^{p+}_{\mathcal{Z}}$ where

$$\begin{aligned} \Pi^{h+}_{\mathcal{Z}} &= \begin{pmatrix} 1 & 0 & -|\lambda|^{-1/2}\phi^{h+}e(\lambda)\Psi^{p+-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \Pi^{p+}_{\mathcal{Z}} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_r & -\Phi^{p+}E(\lambda)\Psi^{p+-1} \\ 0 & 0 & I_r \end{pmatrix} \end{aligned} \tag{2.134}$$

Hence

$$\begin{aligned} H_\lambda(x,y) &= (I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}^{y\to x}_{\mathcal{Z}}\Pi^{h+}_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr} \\ &= \phi^{h+}(x)\phi^{h+}(y)^{-1}\begin{pmatrix} (-1+\mathcal{O}(|\lambda|^{-1}))A_*^{-1} & \mathcal{O}(|\lambda|^{-1})A_*^{-1} \\ (1+\mathcal{O}(|\lambda|^{-1}))b_2^{-1}b_1A_*^{-1} & \mathcal{O}(|\lambda|^{-1})b_2^{-1}b_1A_*^{-1} \end{pmatrix} \\ &= \phi^{h+}(x)\phi^{h+}(y)^{-1}\begin{pmatrix} -A_*^{-1}(x) & 0 \\ b_2^{-1}b_1A_*^{-1}(x) & 0 \end{pmatrix} + \mathcal{O}(|\lambda|^{-1})\phi^{h+}(x)\phi^{h+}(y)^{-1}, \\ &= \phi^{h+}(x)\phi^{h+}(y)^{-1}R_*L_*^{tr} + \mathcal{O}(|\lambda|^{-1})\phi^{h+}(x)\phi^{h+}(y)^{-1}, \end{aligned}$$

recalling that $\phi^{h+}(x)\phi^{h+}(y)^{-1}$ is the solution operator of hyperbolic equation in (2.116) and thus satisfies

$$\phi^{h+}(x)\phi^{h+}(y)^{-1} = e^{\int_{\tilde{y}}^{\tilde{x}}(-1/A_*-|\lambda|^{-1}\eta_*/A_*)(z)dz} = e^{\int_y^x(-\lambda/A_*-\eta_*/A_*)(z)dz}. \tag{2.135}$$

At the same time, computing $P_\lambda(x,y)$, we obtain

$$\begin{aligned} P_\lambda(x,y) &= (I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}^{y\to x}_{\mathcal{Z}}\Pi^{p+}_{\mathcal{Z}}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr} \\ &= \mathcal{O}(|\lambda|^{-1/2})\Phi^{p+}(x)\Phi^{p+}(y)^{-1} \end{aligned}$$

recalling that $\Phi^{p+}(x)\Phi^{p+}(y)^{-1}$ is the (stable) solution operator of parabolic equation

(2.115), with M_1^- uniformly negative definite, and thus we have an obvious estimate

$$|\Phi^{p+}(x)\Phi^{p+}(y)^{-1}| \le Ce^{-\theta|\lambda|^{-1/2}(\tilde{x}-\tilde{y})} \le Ce^{-\theta|\lambda|^{1/2}(x-y)}. \tag{2.136}$$

We therefore obtain

$$P_\lambda(x,y) = \mathcal{O}(|\lambda|^{-1/2})e^{-\theta|\lambda|^{1/2}(x-y)}. \tag{2.137}$$

Estimates on G_λ: Outflow case $A_* < 0$.

Again as above, we shall work in detail for the case $x > y$. Similar estimates can be easily obtained for $x < y$. Estimates in Lemma 2.2.19 yield

$$\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^{+}(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Phi^{p+}(x)\Phi^{p+}(y)^{-1} & -\Phi^{p+}(x)E(\lambda)\Psi^{p+}(y)^{-1} \\ 0 & 0 & 0 \end{pmatrix} \tag{2.138}$$

where $\Phi^{p+}(x)E(\lambda)\Psi^{p+}(y)^{-1} \le C\Phi^{p+}(x)\Phi^{p+}(y)^{-1}$. Observe that $\Pi_{\mathcal{Z}}^{h+} \equiv 0$. Therefore, $H_\lambda(x,y) = 0$ and

$$\begin{aligned} P_\lambda(x,y) &= (I_n,0)\mathcal{Q}^{-1}\mathcal{T}\mathcal{F}_{\mathcal{Z}}^{y\to x}\Pi_{\mathcal{Z}}^{p+}(y)\mathcal{T}^{-1}\mathcal{Q}\tilde{S}^{-1}(y)(I_n,0)^{tr} \\ &= \Phi^{p+}(x)\Phi^{p+}(y)^{-1}\begin{pmatrix} \mathcal{O}(|\lambda|^{-1}) & \mathcal{O}(|\lambda|^{-1}) \\ \mathcal{O}(|\lambda|^{-1/2}) & \mathcal{O}(|\lambda|^{-1/2}) \end{pmatrix} \\ &\le C|\lambda|^{-1/2}e^{-\theta|\lambda|^{1/2}(x-y)} \end{aligned}$$

We thus complete the proof of estimates H_λ and of P_λ appearing in Proposition 2.2.17.

Derivative estimates.

Derivative estimates now follow in a straightforward fashion, by differentiation of (2.107), noting from the approximately decoupled equations that differentiation of the flow brings down a factor (to absorbable error) of λ in hyperbolic modes, $\lambda^{1/2}$ in parabolic modes. This completes the proof of Proposition 2.2.17. □

2.2.4 Low frequency estimates

Our goal in this section is the estimation of the resolvent kernel in the critical regime $|\lambda| \to 0$, i.e., the large time behavior of the Green function G, or global behavior in space and time. We are basically following the same treatment as that carried out for viscous shock waves of strictly parabolic conservation laws in [60, 36]; we refer to those references for details. In the low frequency case the behavior is essentially governed by the limiting far-field equation

$$U_t = L_+ U := -A_+ U_x + B_+ U_{xx} \tag{2.139}$$

Lemma 2.2.20 ([36]). *Assuming (A1)–(A3), (H0)-(H2), for $|\lambda|$ sufficiently small, the eigenvalue equation $(L_+ - \lambda)W = 0$ associated with the limiting, constant-coefficient operator L_+, considered as a first-order system $W' = \mathbb{A}_+ W$, $W = (u, v, v')$, has a basis of $2n-1$ solutions $\bar{W}_j^+ = e^{\mathbb{A}_+(\lambda)x} V_j(\lambda)$, consisting of $n-1$ "fast" modes (not necessarily eigenmodes)*

$$|e^{\mathbb{A}_+(\lambda)x} V_j| \le C e^{-\theta|x|}, \quad \theta > 0, \tag{2.140}$$

and n analytic "slow" (eigen-)modes

$$\begin{aligned} e^{\mathbb{A}_+(\lambda)x} V_j &= e^{\mu_j(\lambda)x} V_j, \\ \mu_{n-1+j}^+(\lambda) &:= -\lambda / a_j^+ + \lambda^2 \beta_j^+ / {a_j^+}^3 + \mathcal{O}(\lambda^3), \\ V_{n-1+j}^+(\lambda) &:= r_j^+ + \mathcal{O}(\lambda), \end{aligned} \tag{2.141}$$

where a_j^+, l_j^+, r_j^+, β_j^+ are defined as in Proposition 2.1.6. The same is true for the adjoint eigenvalue equation

$$(L_+ - \lambda)^* Z = 0,$$

i.e, it has a basis of solutions $\tilde{\bar{W}}_j^+ = e^{-\mathbb{A}_+^*(\lambda)x} \tilde{V}_j(\lambda)$ with $n-1$ analytic "fast" modes

$$|e^{-\mathbb{A}_+^*(\lambda)x} \tilde{V}_j| \le C e^{-\theta|x|}, \quad \theta > 0, \tag{2.142}$$

and n analytic "slow" (eigen-)modes

$$\tilde{V}_{n-1+j}^+(\lambda) = l_j^+ + \mathcal{O}(\lambda). \tag{2.143}$$

Proof. Standard matrix perturbation theory; see [36], Appendix B. □

Also we recall from the representation of G_λ in Corollary 2.2.15:

Proposition 2.2.21. *Assuming (A1)–(A3), (H0)-(H2), let K be the order of the pole of G_λ at $\lambda = 0$ and r be sufficiently small that there are no other poles in $B(0, r)$. Then for $\lambda \in \Omega_\theta$ such that $|\lambda| \le r$ and we have*

$$G_\lambda(x, y) = \sum_{j,k} d^+_{jk}(\lambda)\phi^+_j(x)\tilde{\psi}^+_k(y) + \sum_k \phi^+_k(x)\tilde{\phi}^+_k(y), \tag{2.144}$$

for $x > y > 0$, and

$$G_\lambda(x, y) = \sum_{j,k} d^0_{jk}(\lambda)\phi^+_j(x)\tilde{\psi}^+_k(y) + \sum_k \psi^+_k(x)\tilde{\psi}^+_k(y), \tag{2.145}$$

for $0 < x < y$, where $d^{0,+}_{jk}(\lambda) = \mathcal{O}(\lambda^{-K})$ are scalar meromorphic functions, moreover $K \le$ order of vanishing of the Evans function $D(\lambda)$ at $\lambda = 0$.

Proof. See [60, Proposition 7.1] for the first statement and theorem 6.3 for the second statement linking order K of the pole to multiplicity of the zero of the Evans Function. □

Our main result of this section is then:

Proposition 2.2.22. *Assume (A1)–(A3), (H0)-(H2), and (D1). Then, for $r > 0$ sufficiently small, the resolvent kernel G_λ associated with the linearized evolution equation* (2.139) *satisfies, for $0 \le y \le x$:*

$$\begin{aligned} &|\partial_x^\gamma \partial_y^\alpha G_\lambda(x, y)| \\ &\qquad \le C(|\lambda|^\gamma + e^{-\theta|x|})(|\lambda|^\alpha + e^{-\theta|y|})\Big(\sum_{a_k^+>0} \big|e^{(-\lambda/a_k^+ + \lambda^2\beta_k^+/a_k^{+3})(x-y)}\big| \\ &\quad + \sum_{a_k^+<0,\, a_j^+>0} \big|e^{(-\lambda/a_j^+ + \lambda^2\beta_j^+/a_j^{+3})x + (\lambda/a_k^+ - \lambda^2\beta_k^+/a_k^{+3})y}\big|\Big), \end{aligned} \tag{2.146}$$

$0 \le |\alpha|, |\gamma| \le 1$, $\theta > 0$, with similar bounds for $0 \le x \le y$. Moreover, each term in the summation on the righthand side of (2.146) *bounds a separately analytic function.*

Proof. By condition (D1), $D(\lambda)$ does not vanish on $Re(\lambda) \geq 0$, hence, by continuity, on $|\lambda| \leq r$. Thus, according to Proposition 2.2.21, all $|d_{jk}(\lambda)|$ are uniformly bounded on $|\lambda| \leq r$, and thus it is enough to find estimates for fast and slow modes ϕ_j^+, $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. By applying Lemma 2.2.20 and using (2.53) we find:

$$\begin{pmatrix} \phi_j^+ \\ \partial_x \phi_j^+ \end{pmatrix} = e^{\mathbb{A}_+(\lambda)x} P^+ \begin{pmatrix} v_j \\ \mu_j v_j \end{pmatrix} = e^{\mathbb{A}_+(\lambda)x}(I + \Theta) \begin{pmatrix} v_j \\ \mu_j v_j \end{pmatrix} \tag{2.147}$$

and similarly for $\tilde{\phi}_j^+$, ψ_j^+ and $\tilde{\psi}_j^+$. Now using (5.73) and the fact, by Lemma 2.2.20, that $e^{\mu_j(\lambda)x}$ is of order $e^{-(\lambda/a_j^+ + \lambda^2 \beta_j^+/a_j^{+3} + \mathcal{O}(\lambda^3))x}$ for slow modes and order $e^{-\theta|x|}$ for fast modes, so by substituting this and corresponding dual estimates in (2.147) and grouping terms, we obtain the result.

□

2.3 Pointwise bounds on Green function $G(x, t; y)$

In this section, we prove the pointwise bounds on the Green function G following the general approach of [36] in the whole-line, shock, case. Our starting point is the representation

$$G(x, t; y) = \frac{1}{2\pi i} P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(x, y)\, d\lambda \tag{2.148}$$

where η is any sufficiently large positive real number.

Case I. $|x - y|/t$ **large.** We first treat the simple case that $|x - y|/t \geq S$, S sufficiently large. Fixing x, y, t, set $\lambda = \eta + i\xi$, for $\eta > 0$ sufficiently large. Applying Proposition 2.2.17, we obtain the decomposition

$$\begin{aligned} G(x, t; y) &= \frac{1}{2\pi i} P.V \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} \Big[H_\lambda + \Theta_\lambda^H + P_\lambda + \Theta_\lambda^P \Big](x, y) d\lambda \\ &=: I + II + III + IV. \end{aligned}$$

For definiteness considering the inflow case $A_* > 0$ and taking $x > y$, we estimate each term in turn.

Term I. Computing,

$$
\begin{aligned}
I &= \frac{1}{2\pi i} P.V \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x,y) d\lambda \\
&= \frac{1}{2\pi} A_*(x)^{-1} e^{\eta(t - \int_y^x 1/A_*(z)dz)} e^{-\int_y^x (\eta_*/A_*)(z)dz} P.V \int_{-\infty}^{+\infty} e^{i\xi(t - \int_y^x 1/A_*(z)dz)} d\xi \\
&= \frac{1}{2\pi} A_*(x)^{-1} \delta(t - \int_y^x 1/A_*(z)dz) e^{-\int_y^x (\eta_*/A_*)(z)dz} \\
&= \frac{1}{2\pi} A_*(x)^{-1} A_*(y) \delta_{x - \bar{a}_* t}(y) e^{-\int_y^x (\eta_*/A_*)(z)dz}
\end{aligned}
$$

where $\bar{a}_*$ is defined as in Proposition 2.1.6. Noting that $\bar{a}_* \geq \inf_x A_*(x) > 0$ and $\eta_*^+ > 0$, we get $e^{-\int_y^x (\eta_*/A_*)(z)dz} = \mathcal{O}(e^{-\theta(x-y)})$ and thus

$$
I = \mathcal{O}(e^{-\theta t}) \delta_{x - \bar{a}_* t}(y), \tag{2.149}
$$

vanishing for $|x - y|/t$ large.

Term II. Similar calculations show that the "hyperbolic error term" II also vanishes. For example, the term $e^{\lambda t} \lambda^{-1} B(x,y;\lambda)$ contributes

$$
\frac{1}{2\pi} e^{\eta(t - \int_y^x 1/A_*(z)dz)} e^{-\int_y^x (\eta_*/A_*)(z)dz} P.V \int_{-\infty}^{+\infty} (\eta + i\xi)^{-1} e^{i\xi(t - \int_y^x 1/A_*(z)dz)} d\xi.
$$

The integral though not absolutely convergent, is integrable and uniformly bounded as a principal value integral, for all real η bounded away from zero, by explicit computation. On the other hand

$$
e^{\eta(t - \int_y^x 1/A_*(z)dz)} \leq e^{\eta(t - |x-y|/\min_z A_*(z))} \leq e^{\eta t(1 - S/\min_z A_*(z))} \to 0,
$$

as $\eta \to +\infty$, for S sufficiently large. Thus, we find that the above integral term goes to zero. Likewise, the result applies for the term of $e^{\lambda t} C(x,y;\lambda)$, since $(x - y) e^{-\int_y^x (\eta_*/A_*)(z)dz} \leq C(x-y) e^{-\theta(x-y)}$ is also bounded. Thus, each term of II vanishes as $\eta \to +\infty$.

Term III. The parabolic term III may be treated exactly as in the strictly parabolic case [60]. Precisely, we may first deform the contour in the principle value integral to

$$\int_{\Gamma_1\cup\Gamma_2} e^{\lambda t} P_\lambda(x,y)\, d\lambda, \tag{2.150}$$

where $\Gamma_1 := \partial B(0,R) \cap \bar{\Omega}_P$ and $\Gamma_2 := \partial\Omega_P \setminus B(0,R)$, recalling the parabolic sector Ω_P defined in (2.70). Choose

$$\bar{\alpha} := \frac{|x-y|}{2\theta t}, \quad R := \theta\bar{\alpha}^2, \tag{2.151}$$

where θ is as in (2.78). Note that the intersection of Γ with the real axis is $\lambda_{min} = R = \theta\bar{\alpha}^2$. By the large $|\lambda|$ estimates of Proposition 2.2.17, we have for all $\lambda \in \Gamma_1 \cup \Gamma_2$ that

$$|P_\lambda(x,y)| \le C|\lambda|^{-1/2} e^{-\theta|\lambda|^{1/2}|x-y|}.$$

Further, we have

$$\begin{aligned} Re\lambda &\le R(1-\eta\omega^2), \quad \lambda \in \Gamma_1, \\ Re\lambda &\le Re\lambda_0 - \eta(|Im\lambda| - |Im\lambda_0|), \quad \lambda \in \Gamma_2 \end{aligned} \tag{2.152}$$

for R sufficiently large, where ω is the argument of λ and λ_0 and λ_0^* are the two points of intersection of Γ_1 and Γ_2, for some $\eta > 0$ independent of $\bar{\alpha}$. Combining these estimates, we obtain

$$\begin{aligned} \Big|\int_{\Gamma_1} e^{\lambda t} P_\lambda d\lambda\Big| &\le C\int_{\Gamma_1} |\lambda|^{-1/2}\, e^{Re\lambda t - \theta|\lambda|^{1/2}|x-y|}\, d\lambda \\ &\le Ce^{-\theta\bar{\alpha}^2 t}\int_{-arg(\lambda_0)}^{+arg(\lambda_0)} R^{-1/2} e^{-\theta R\eta\omega^2 t}\, Rd\omega \\ &\le Ct^{-1/2}e^{-\theta\bar{\alpha}^2 t}. \end{aligned} \tag{2.153}$$

Likewise,

$$
\begin{aligned}
|\int_{\Gamma_2} e^{\lambda t} P_\lambda d\lambda| &\le \int_{\Gamma_2} C|\lambda|^{-1/2}\, Ce^{Re\lambda t - \theta|\lambda|^{1/2}|x-y|} d\lambda \\
&\le Ce^{Re(\lambda_0)t - \theta|\lambda_0|^{1/2}|x-y|} \int_{\Gamma_2} |\lambda|^{-1/2} e^{(Re\lambda - Re\lambda_0)t}\ |d\lambda| \\
&\le Ce^{-\theta\bar{\alpha}^2 t} \int_{\Gamma_2} |Im\,\lambda|^{-1/2} e^{-\eta|Im\,\lambda - Im\,\lambda_0|t}\ |d\,Im\lambda| \\
&\le Ct^{-1/2} e^{-\theta\bar{\alpha}^2 t}.
\end{aligned}
\tag{2.154}
$$

Combining these last two estimates, we have

$$
III \le Ct^{-1/2} e^{-\theta\bar{\alpha}^2 t/2} e^{-(x-y)^2/8\theta t} \le Ct^{-1/2} e^{-\eta t} e^{-(x-y)^2/8\theta t}, \tag{2.155}
$$

for $\eta > 0$ independent of $\bar{\alpha}$. Observing that $|x-at|/2t \le |x-y|/t \le 2|x-at|/t$ for any bounded a, for $|x-y|/t$ sufficiently large, we find that III may be absorbed in any summand $t^{-1/2} e^{-(x-y-a_k^+ t)^2/Mt}$.

Term IV. Similarly, as in the treatment of the term III, the principle value integral for the "parabolic error term IV may be shifted to $\eta = R = \theta\bar{\alpha}^2$, $\bar{\alpha}$ as above. This yields an estimate

$$
|IV| \le Ce^{-\theta\bar{\alpha}^2 t} \int_{-\infty}^{+\infty} |\eta_0 + i\xi|^{-2} d\xi \le Ce^{-\theta\bar{\alpha}^2 t},
$$

absorbed in $\mathcal{O}(e^{-\eta t} e^{-|x-y|^2/Mt})$ for all t.

Case II. $|x-y|/t$ bounded. We now turn to the critical case where $|x-y|/t \le S$, for some fixed S.

Decomposition of the contour: We begin by converting the contour integral (2.148) into a more convenient form decomposing high, intermediate, and low frequency contributions.

We first observe that L has no spectrum on the portion of Ω lying outside the rectangle

$$
\mathcal{R} := \{\lambda\ :\ -\eta_1 \le \Re\lambda \le \eta, -R \le \Im\lambda \le R\} \tag{2.156}
$$

for $\eta > 0$, $R > 0$ sufficiently large, hence G_λ is analytic on this region. Since, also, H_λ is analytic on the whole complex plane, contours involving either one of these contributions may be arbitrarily deformed within $\Omega \setminus \mathcal{R}$ without affecting the result, by Cauchy's theorem. Likewise, P_λ is analytic on $\Omega_P \setminus \mathcal{R}$, and so contours involving this contribution may be arbitrarily deformed within this region. Thus, we obtain

Observation 2.3.1 ([36])**.** *Assume (A1)–(A3), (H0)-(H2), and (D1). Then, the principle value integral* (2.148) *may be replaced by*

$$G(x,t;y) = I_a + I_b + I_c + II_a + II_b + III \tag{2.157}$$

where

$$\begin{aligned}
I_a &:= P.V. \int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} H_\lambda(x,y) d\lambda \\
I_b &:= P.V \Big(\int_{-\eta_1 - i\infty}^{-\eta_1 - iR} + \int_{-\eta_1 + iR}^{-\eta_1 + i\infty} \Big) e^{\lambda t} (G_\lambda - H_\lambda - P_\lambda)(x,y) d\lambda \\
I_c &: = \int_{\Gamma_2} e^{\lambda t} P_\lambda(x,y) d\lambda \\
II_a &:= \Big(\int_{-\eta_1 - iR}^{-\eta_1 - ir/2} + \int_{-\eta_1 + ir/2}^{-\eta_1 + iR} \Big) e^{\lambda t} G_\lambda(x,y) d\lambda \\
II_b &:= - \int_{-\eta_1 - iR}^{-\eta_1 + iR} e^{\lambda t} H_\lambda(x,y) d\lambda \\
III &:= \int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y) d\lambda
\end{aligned}$$

with

$$\begin{aligned}
\Gamma_1 :=&[-\eta_1 - ir/2, \eta - ir/2] \cup [\eta - ir/2, \eta + ir/2] \cup [\eta + ir/2, -\eta_1 + ir/2] \\
\Gamma_2 :=&\partial\Omega_P \setminus \Omega,
\end{aligned}$$

for any $\eta, r > 0$, R sufficiently large, and η_1 sufficiently small with respect to r.

Using the above decomposition (2.157), we shall estimate in turn the high-frequency contributions I_a, I_b, and I_c, the intermediate-frequency contributions II_a and II_b, and the low-frequency contributions III.

High-frequency contribution. We first carry out the straightforward estimation of the high-frequency terms I_a, I_b, and I_c. The principal term I_a has already been computed in (2.149) to be $H(x,t;y)$. Likewise, calculations similar to those of *Term II* show that the term I_b is time-exponentially small. For example, the term $e^{\lambda t}\lambda^{-1}B(x,y;\lambda)$ contributes

$$P.V.\Big(\int_{-\infty}^{-R}+\int_{R}^{+\infty}\Big)(-\eta_1+i\xi)^{-1}e^{i\xi(t-\int_y^x 1/A_*(z)dz)}d\xi \\ \times e^{-\eta_1(t-\int_y^x 1/A_*(z)dz)}e^{-\int_y^x(\eta_*/A_*)(z)dz} \tag{2.158}$$

where

$$P.V.\Big(\int_{-\infty}^{-R}+\int_{R}^{+\infty}\Big)(-\eta_1+i\xi)^{-1}e^{i\xi(t-\int_y^x 1/A_*(z)dz)}d\xi<\infty \tag{2.159}$$

and

$$e^{\eta_1\int_y^x 1/A_*(z)dz}e^{-\int_y^x(\eta_*/A_*)(z)dz}\le Ce^{\frac{\eta_1|x-y|}{\min_z A_*(z)}}e^{-\theta|x-y|}\le Ce^{-\theta|x-y|/2}, \tag{2.160}$$

for η_1 sufficiently small. This contributes in the term $\mathcal{O}(e^{-\eta_1(t+|x-y|)})$ of R. Likewise, the contributions of terms $e^{\lambda t}\lambda^{-1}(x-y)C(x,y;\lambda)$ and $e^{\lambda t}\lambda^{-2}D(x,y;\lambda)$ split into the product of a convergent, uniformly bounded integral in ξ, a bounded factor analogous to (2.160), and the factor $e^{-\eta_1 t}$, giving the result.

The term I_c may be estimated exactly as was term III in the large $|x-y|/t$ case, to obtain contribution $\mathcal{O}(t^{-1/2}e^{-\eta_1 t})$ absorbable again in the residual term $\mathcal{O}(e^{-\eta t}e^{-|x-y|^2/Mt})$ for $t\ge\epsilon$, any $\epsilon>0$, and by any summand $\mathcal{O}(t^{-1/2}(1+t)^{-1/2}e^{-(x-y-a_k^+)^2/Mt})e^{-\eta(x+y}$ for t small.

Intermediate-frequency contribution. Error term II_b is time-exponentially small for η_1 sufficiently small, by the same calculation as in (2.158)-(2.160), hence negligible. Likewise, term II_a by the basic estimate (2.60) is seen to be time-exponentially small of order $\mathcal{O}(e^{-\eta_1 t})$ for any $\eta_1>0$ sufficiently small that the associated contour lies in the resolvent set of L.

Low-frequency contribution. It remains to estimate the low-frequency term III, which is of essentially the same form as the low-frequency contribution analyzed in [60, 56] in the strictly parabolic case, in that the contour is the same and the resolvent kernel G_λ satisfies same bounds (with no E_λ term) in this regime. Thus, we may conclude from these previous analyses that III gives contribution as claimed, exactly as in the strictly parabolic case. For completeness, we indicate the main features of the argument here.

Bounded time. For t bounded, we can use the medium-λ bounds $|G_\lambda|$, $|G_{\lambda_x}|$, $|G_{\lambda_y}| \le C$ to obtain $|\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda| \le C_2 |\Gamma_1|$. This contribution is order $Ce^{-\eta t}$ for bounded time, hence can be absorbed.

Large time. For t large, we must instead estimate $\int_{\Gamma_1} e^{\lambda t} G_\lambda d\lambda$ using the small-$|\lambda|$ expansions. First, observe that, all coefficient functions $d_{jk}(\lambda)$ are uniformly bounded (since $|\lambda|$ is bounded in this case).

Case II(i). $(0 < y < x)$. By our low-frequency estimates in Proposition 2.2.21, we have

$$\begin{aligned}\int_{\Gamma_1} e^{\lambda t} G_\lambda(x,y)\, d\lambda = &\int_{\Gamma_1} \sum_{j,k} e^{\lambda t} d_{jk} \phi_j^+(x) \tilde{\psi}_k^+(y) d\lambda \\ &+ \int_{\Gamma_1} \sum_{k} e^{\lambda t} \phi_k^+(x) \tilde{\phi}_k^+(y) d\lambda,\end{aligned} \tag{2.161}$$

where each d_{jk} is analytic, hence bounded. We estimate separately each of the terms

$$\int_{\Gamma_1} e^{\lambda t} d_{jk} \phi_j^+(x) \tilde{\psi}_k^+(y) d\lambda$$

on the righthand side of (2.161). Estimates for terms

$$\int_{\Gamma_1} e^{\lambda t} \phi_k^+(x) \tilde{\phi}_k^+(y) d\lambda$$

go similarly.

Case II(ia). First, consider the critical case $a_j^+ > 0$, $a_k^+ < 0$. For this case,

$$|d_{jk} \phi_j^+(x) \tilde{\psi}_k^+(y)| \le C e^{Re(\rho_j^+ x - \nu_k^+ y)},$$

where

$$\begin{cases} \nu_k^+(\lambda) = -\lambda / a_k^+ + \lambda^2 \beta_k^+ / (a_k^+)^3 + \mathcal{O}(\lambda^3) \\ \rho_j^+(\lambda) = -\lambda / a_j^+ + \lambda^2 \beta_j^+ / (a_j^+)^3 + \mathcal{O}(\lambda^3). \end{cases}$$

Set

$$\bar{\alpha} = \frac{a_k^+ x/a_j^+ - y - a_k^+ t}{2t}, \quad p := \frac{\beta_j^+ a_k^+ x/(a_j^+)^3 - \beta_k^+ y/(a_k^+)^2}{t} < 0.$$

Define Γ'_{1a} to be the portion contained in Ω_θ of the hyperbola

$$\begin{aligned} &Re(\rho_j^+ x - \nu_k^+ y) + \mathcal{O}(\lambda^3)(|x| + |y|) \\ &\quad = (1/a_k^+) Re[\lambda(-a_k^+ x/a_j^+ + y) + \lambda^2(x\beta_j^+ a_k^+/(a_j^+)^3 - y\beta_k^+/(a_k^+)^2)] \\ &\quad \equiv \text{constant} \\ &\quad = (1/a_k^+)[(\lambda_{min}(-a_k^+ x/a_j^+ + y) + \lambda_{min}^2(x\beta_j^+ a_k^+/(a_j^+)^3 - y\beta_k^+/(a_k^+)^2)], \end{aligned} \tag{2.162}$$

where

$$\lambda_{min} := \begin{cases} \frac{\bar{\alpha}}{p} & if \quad |\frac{\bar{\alpha}}{p}| \le \epsilon \\ \pm\epsilon & if \quad \frac{\bar{\alpha}}{p} \gtrless \epsilon \end{cases} \tag{2.163}$$

Denoting by λ_1, λ_1^*, the intersections of this hyperbola with $\partial\Omega_\theta$, define Γ'_{1_b} to be the union of $\lambda_1\lambda_0$ and $\lambda_0^*\lambda_1^*$, and define $\Gamma'_1 = \Gamma'_{1_a} \cup \Gamma'_{1_b}$. Note that $\lambda = \bar{\alpha}/p$ minimizes the left hand side of (2.162) for λ real. Note also that that p is bounded for $\bar{\alpha}$ sufficiently small, since $\bar{\alpha} \le \epsilon$ implies that

$$(|a_k^+ x/a_j^+| + |y|)/t \le 2|a_k^+| + 2\epsilon$$

i.e. $(|x| + |y|)/t$ is controlled by $\bar{\alpha}$.

With these definitions, we readily obtain that

$$\begin{aligned} Re(\lambda t + \rho_j^+ x - \nu_k^+ y) &\le -(t/a_k^-)(\bar{\alpha}^2/4p) - \eta Im(\lambda)^2 t \\ &\le -\bar{\alpha}^2 t/M - \eta Im(\lambda)^2 t, \end{aligned} \tag{2.164}$$

for $\lambda \in \Gamma'_{1_a}$ (note: here, we have used the crucial fact that $\bar{\alpha}$ controls $(|x| + |y|)/t$, in bounding the error term $\mathcal{O}(\lambda^3)(|x| + |y|)/t$ arising from expansion Likewise, we obtain for any q that

$$\int_{\Gamma'_{1_a}} |\lambda|^q e^{Re(\lambda t + \rho_j^+ x - \nu_k^- y)} d\lambda \le C t^{-\frac{1}{2} - \frac{q}{2}} e^{-\bar{\alpha}^2 t/M}, \tag{2.165}$$

for suitably large C, $M > 0$ (depending on q). Observing that

$$\bar{\alpha} = (a_k^+/a_j^+)(x - a_j^+(t - |y/a_k^+|))/2t,$$

we find that the contribution of (2.165) can be absorbed in the described bounds for $t \geq |y/a_k^-|$. At the same time, we find that $\bar{\alpha} \geq x > 0$ for $t \leq |y/a_k^+|$, whence

$$\bar{\alpha} \geq (x - y - a_j^+ t)/Mt + |x|/M,$$

for some $\epsilon > 0$ sufficiently small and $M > 0$ sufficiently large.

This gives

$$e^{-\bar{\alpha}^2/|p|} \leq e^{-(x-y-a_k^+ t)^2/Mt} e^{-\eta|x|}$$

provided $|x|/t > a_j^+$, a contribution which can again be absorbed. On the other hand, if $t \leq |x/a_j^+|$, we can use the dual estimate

$$\begin{aligned} \bar{\alpha} &= (-y - a_k^+(t - |x/a_j^+|))/2t \\ &\geq (x - y - a_k^+ t)/Mt + |y|/M, \end{aligned} \tag{2.166}$$

together with $|y| \geq |a_k^- t|$, to obtain

$$e^{-\bar{\alpha}^2/|p|} \leq e^{-(x-y-a_j^+ t)^2/Mt} e^{-\eta|y|},$$

a contribution that can likewise be absorbed.

Case II(ib). In case $a_j^+ < 0$ or $a_k^+ > 0$, terms $|\varphi_j^+| \leq Ce^{-\eta|x|}$ and $|\tilde{\psi}_j^+| \leq Ce^{-\eta|y|}$ are strictly smaller than those already treated in Case II(ia), so may be absorbed in previous terms.

Case II(ii) $(0 < x < y)$**.** The case $0 < x < y$ can be treated very similarly to the previous one; see [60] for details. This completes the proof of Case II, and the theorem.

2.4 Energy estimates

2.4.1 Energy estimate I

We shall require the following energy estimate adapted from [37, 58]. Define the nonlinear perturbation variables $U = (u, v)$ by

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x). \tag{2.167}$$

Proposition 2.4.1. *Under the hypotheses of Theorem 2.1.4, let $U_0 \in H^4$ and $U = (u,v)^T$ be a solution of (2.2) and (2.167). Suppose that, for $0 \le t \le T$, the $W_x^{2,\infty}$ norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then*

$$\|U(t)\|_{H^4}^2 \le Ce^{-\theta t}\|U_0\|_{H^4}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big(\|U(\tau)\|_{L^2}^2 + \mathcal{B}_h(\tau)^2\Big)d\tau \tag{2.168}$$

for all $0 \le t \le T$, where the boundary operator $\mathcal{B}_h$ is defined in Theorem 2.1.4.

Proof. Observe that a straightforward calculation shows that $|U|_{H^r} \sim |W|_{H^r}$,

$$W = \tilde{W} - \bar{W} := W(\tilde{U}) - W(\bar{U}), \tag{2.169}$$

for $0 \le r \le 4$, provided $|U|_{W^{2,\infty}}$ remains bounded, hence it is sufficient to prove a corresponding bound in the special variable W. We first carry out a complete proof in the more straightforward case with conditions (A1)-(A3) replaced by the following global versions, indicating afterward by a few brief remarks the changes needed to carry out the proof in the general case.

(A1') $\tilde{A}(\tilde{W}), \tilde{A}^0, \tilde{A}^{11}$ are symmetric, $\tilde{A}^0 \ge \theta_0 > 0$,

(A2') no eigenvector of $\tilde{A}(\tilde{A}^0)^{-1}(\tilde{W})$ lies in the kernel of $\tilde{B}(\tilde{A}^0)^{-1}(\tilde{W})$,

(A3') $\tilde{W} = \begin{pmatrix} \tilde{w}^I \\ \tilde{w}^{II} \end{pmatrix}, \tilde{B} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b} \end{pmatrix}, \tilde{b} \ge \theta > 0$, and $\tilde{G} \equiv 0$.

Substituting (2.169) into (2.4), we obtain the quasilinear perturbation equation

$$A^0 W_t + AW_x = (BW_x)_x + M_1\bar{W}_x + (M_2\bar{W}_x)_x \tag{2.170}$$

where $A^0 := A^0(W+\bar{W})$ is positive definite symmetric, $A := A(W+\bar{W})$ is symmetric,

$$M_1 = A(W+\bar{W}) - A(\bar{W}) = \Big(\int_0^1 dA(\bar{W}+\theta W)d\theta\Big)W,$$
$$M_2 = B(W+\bar{W}) - B(\bar{W}) = \begin{pmatrix} 0 & 0 \\ 0 & (\int_0^1 db(\bar{W}+\theta W)d\theta)W \end{pmatrix}.$$

As shown in [37], we have bounds

$$|A^0| \le C, \quad |A^0_t| \le C|W_t| \le C(|W_x| + |w^{II}_{xx}|) \le C\zeta, \tag{2.171}$$

$$|\partial_x A^0| + |\partial_x^2 A^0| \le C(\sum_{k=1}^{2} |\partial_x^k W| + |\bar{W}_x|) \le C(\zeta + |\bar{W}_x|). \tag{2.172}$$

We have the same bounds for A, B, K, and also due to the form of M_1, M_2,

$$|M_1|, |M_2| \le C(\zeta + |\bar{W}_x|)|W|. \tag{2.173}$$

Note that thanks to Lemma A.1.1 we have the bound on the profile: $|\bar{W}_x| \le Ce^{-\theta|x|}$, as $x \to +\infty$.

The following results assert that hyperbolic effects can compensate for degenerate viscosity B, as revealed by the existence of a compensating matrix K.

Lemma 2.4.2 ([34])**.** *Assuming (A1'), condition (A2') is equivalent to the following:*

(K1) There exists a smooth skew-symmetric matrix $K(W)$ such that

$$\Re e(K(A^0)^{-1}A + B)(W) \ge \theta_2 > 0. \tag{2.174}$$

Define α by the ODE

$$\alpha_x = -\text{sign}(A^{11})c_*|\bar{W}_x|\alpha, \quad \alpha(0) = 1 \tag{2.175}$$

where $c_* > 0$ is a large constant to be chosen later. Note that we have

$$(\alpha_x/\alpha)A^{11} \le -c_*\theta_1|\bar{W}_x| =: -\omega(x) \tag{2.176}$$

and

$$|\alpha_x/\alpha| \le c_*|\bar{W}_x| = \theta_1^{-1}\omega(x). \tag{2.177}$$

In what follows, we shall use $\langle\cdot,\cdot\rangle$ as the α-weighted L^2 inner product defined as

$$\langle f, g\rangle = \langle \alpha f, g\rangle_{L^2}$$

and $\|f\|_s = \sum_{i=0}^{s}\langle \frac{d^{(i)}}{dx^i}f, \frac{d^{(i)}}{dx^i}f\rangle^{1/2}$ as the norm in weighted H^s space. Note that for any symmetric operator S,

$$\langle Sf_x, f\rangle = -\frac{1}{2}\langle (S_x + (\alpha_x/\alpha)S)f, f\rangle - \frac{1}{2}S_0 f_0 . f_0.$$

Note that in what follows, we shall pay attention to keeping track of c_*. For constants independent of c_*, we simply write them as C.

Zeroth order "Friedrichs-type" estimate

First employing integration by parts yields, and using estimates (2.171), (2.172), and then (2.176), we obtain

$$\begin{aligned}
&-\langle AW_x, W\rangle \\
&\quad = \frac{1}{2}\langle (A_x + (\alpha_x/\alpha)A)W, W\rangle + \frac{1}{2}A_0 W(0)\cdot W(0) \\
&\quad \le \frac{1}{2}\langle (\alpha_x/\alpha)A^{11}w^I, w^I\rangle + C\langle (\zeta + |\bar{W}_x|)|W| + \omega(x)|w^{II}|, |W|\rangle + J_b^0 \\
&\quad \le -\frac{1}{2}\langle \omega(x)w^I, w^I\rangle + C(\zeta\|w^I\|_0^2 + \langle |\bar{W}_x|w^I, w^I\rangle) + C(c_*)\|w^{II}\|_0^2 + J_b^0
\end{aligned}$$

where J_b^0 denotes the boundary term $\frac{1}{2}A_0 W(0)\cdot W(0)$. The term $\langle |\bar{W}_x|w^I, w^I\rangle$ may be easily absorbed into the first term of the right-hand side, since for c_* sufficiently large,

$$\langle |\bar{W}_x|w^I, w^I\rangle \le (c_*\theta_1)^{-1}\langle \omega(x)w^I, w^I\rangle \le \frac{1}{4C}\langle \omega(x)w^I, w^I\rangle. \tag{2.178}$$

Also, integration by parts yields

$$\begin{aligned}\langle (BW_x)_x, W\rangle &= -\langle BW_x, W_x\rangle - \langle (\alpha_x/\alpha)BW_x, W\rangle - B_0 W_x(0)\cdot W(0)\\ &\le -\theta\|w_x^{II}\|_0^2 + C\langle \omega(x) w_x^{II}, w^{II}\rangle - b_0 w_x^{II}(0)\cdot w^{II}(0)\\ &\le -\theta\|w_x^{II}\|_0^2 + C(c_*)\|w^{II}\|_0^2 - b_0 w_x^{II}(0)\cdot w^{II}(0).\end{aligned}$$

where we used the fact that $BW_x \cdot W = bw_x^{II}\cdot w^{II}$, noting that B has block-diagonal form with the first block identical to zero. Similarly, recalling that $M_2 = B(W + \bar W) - B(\bar W)$, we have

$$\begin{aligned}&\langle (M_2\bar W_x)_x, W\rangle\\ &\quad = -\langle M_2\bar W_x, W_x\rangle - \langle (\alpha_x/\alpha)M_2\bar W_x, W\rangle - M_2(0)\bar W_x(0)\cdot W(0)\\ &\quad \le C\langle |\bar W_x||W|, |w_x^{II}|\rangle + C\langle \omega(x)|W|, w^{II}\rangle - m_2(0)\bar W_x(0)\cdot w^{II}(0)\\ &\quad \le \xi\|w_x^{II}\|_0^2 + C\Big(\epsilon\langle \omega(x)w^I, w^I\rangle + C(c_*)\|w^{II}\|_0^2\Big) - m_2(0)\bar W_x(0)\cdot w^{II}(0)\end{aligned}$$

for any small ξ, ϵ. Note that C is independent of c_*. Therefore, for $\xi = \theta/2$ and c_* sufficiently large, combining all above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle A^0 W, W\rangle\\ &\quad = \langle A^0 W_t, W\rangle + \frac{1}{2}\langle A_t^0 W, W\rangle\\ &\quad = \langle -AW_x + (BW_x)_x + M_1\bar W_x + (M_2\bar W_x)_x, W\rangle + \frac{1}{2}\langle A_t^0 W, W\rangle\\ &\quad \le -\frac{1}{4}[\langle \omega(x)w^I, w^I\rangle + \theta\|w_x^{II}\|_0^2] + C\zeta\|w^I\|_0^2 + C(c_*)\|w^{II}\|_0^2 + I_b^0\end{aligned} \tag{2.179}$$

where the boundary term

$$I_b^0 := \frac{1}{2}A_0 W(0)\cdot W(0) - b_0 w_x^{II}(0) w^{II}(0) - M_2(0)\bar W_x(0)\cdot W(0) \tag{2.180}$$

which, in the outflow case (thanks to the negative definiteness of A_{11}), is estimated as

$$I_b^0 \le -\frac{\theta_1}{2}|w^I(0)|^2 + C(|w^{II}(0)|^2 + |w_x^{II}(0)||w^{II}(0)|), \tag{2.181}$$

and similarly in the inflow case, estimated as

$$I_b^0 \le C(|W(0)|^2 + |w_x^{II}(0)||w^{II}(0)|). \tag{2.182}$$

Therefore together with these boundary treatments, (2.179) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle A^0 W, W\rangle \\
&\quad \le -\frac{1}{4}[\langle \omega(x) w^I, w^I\rangle + \theta \|w_x^{II}\|_0^2] + C\zeta\|w^I\|_0^2 + C(c_*)\|w^{II}\|_0^2 + I_b^0.
\end{aligned} \tag{2.183}$$

First order "Friedrichs-type" estimate

Similarly as above, we need the following key estimate, computing by the use of integration by parts, (2.178), and c_* being sufficiently large,

$$\begin{aligned}
-\langle W_x, AW_{xx}\rangle &= \frac{1}{2}\langle W_x, (A_x + (\alpha_x/\alpha)A)W_x\rangle + \frac{1}{2}A_0 W_x(0)\cdot W_x(0) \\
&\le -\frac{1}{4}\langle \omega(x) w_x^I, w_x^I\rangle + C\zeta\|w_x^I\|_0^2 + Cc_*^2\|w_x^{II}\|_0^2 \\
&\quad + \frac{1}{2}A_0 W_x(0)\cdot W_x(0).
\end{aligned} \tag{2.184}$$

We deal with the boundary term later. Now let us compute

$$\frac{1}{2}\frac{d}{dt}\langle A^0 W_x, W_x\rangle = \langle W_x, (A^0 W_t)_x\rangle - \langle W_x, A_x^0 W_t\rangle + \frac{1}{2}\langle A_t^0 W_x, W_x\rangle. \tag{2.185}$$

We control each term in turn. By (2.171) and (2.172), we first have

$$\langle A_t^0 W_x, W_x\rangle \le C\zeta\|W_x\|_0^2$$

and by multiplying $(A^0)^{-1}$ into (2.170),

$$\begin{aligned}
|\langle W_x, A_x^0 W_t\rangle| \le& C\langle(\zeta + |\bar{W}_x|)|W_x|, (|W_x| + |w_{xx}^{II}| + |W|)\rangle \\
\le& \xi\|w_{xx}^{II}\|_0^2 + C\langle(\zeta + |\bar{W}_x|)w_x^I, w_x^I\rangle \\
&+ C\langle(\zeta + |\bar{W}_x|)w^I, w^I\rangle + C\|w^{II}\|_1^2,
\end{aligned}$$

where the term $\langle|\bar{W}_x|w_x^I, w_x^I\rangle$ may be treated in the same way as was $\langle|\bar{W}_x|w^I, w^I\rangle$ in (2.178). Using (2.170), we write the first term in the right-hand side of (2.185) as

$$\begin{aligned}
\langle W_x, (A^0W_t)_x\rangle =&\langle W_x, [-AW_x + (BW_x)_x + M_1\bar{W}_x + (M_2\bar{W}_x)_x]_x\rangle \\
=&-\langle W_x, AW_{xx}\rangle + \langle W_x, -A_xW_x + (M_1\bar{W}_x)_x\rangle \\
&-\langle W_{xx} + (\alpha_x/\alpha)W_x, [(BW_x)_x + (M_2\bar{W}_x)_x]\rangle \\
&-W_x(0).[(BW_x)_x + (M_2\bar{W}_x)_x](0) \\
\le&-\frac{1}{4}\Big[\langle\omega(x)w_x^I, w_x^I\rangle + \theta\|w_{xx}^{II}\|_0^2\Big] \\
&+C\Big[\zeta\|w^I\|_1^2 + C(c_*)\|w_x^{II}\|_0^2 + \langle|\bar{W}_x|w^I, w^I\rangle\Big] + I_b^1
\end{aligned}$$

where I_b^1 denotes the boundary terms

$$I_b^1 := \frac{1}{2}A_0W_x(0)\cdot W_x(0) - W_x(0)\cdot[(BW_x)_x + (M_2\bar{W}_x)_x](0), \tag{2.186}$$

and we have used estimates (2.184),(2.178) for w^I, w_x^I, and Young's inequality to obtain:

$$\begin{aligned}
\langle W_x, -A_xW_x + (M_1\bar{W}_x)_x\rangle &\le C\langle(\zeta + |\bar{W}_x|)|W_x|, |W_x| + |W|\rangle. \\
-\langle W_{xx} + (\alpha_x/\alpha)W_x, (BW_x)_x\rangle &\le \\
&-\theta\|w_{xx}^{II}\|_0^2 + C\langle|w_{xx}^{II}| + \omega(x)|w_x^{II}|, (\zeta + |\bar{W}_x|)|w_x^{II}|\rangle \\
-\langle W_{xx} + (\alpha_x/\alpha)W_x, (M_2\bar{W}_x)_x\rangle &\le \\
&C\langle|w_{xx}^{II}| + \omega(x)|w_x^{II}|, (\zeta + |\bar{W}_x|)(|W_x| + |W|)\rangle.
\end{aligned}$$

Putting these estimates together into (2.185), we have obtained

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle A^0W_x, W_x\rangle + \frac{1}{4}\theta\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle\omega(x)w_x^I, w_x^I\rangle& \\
\le C\Big[\zeta\|w^I\|_1^2 + \langle|\bar{W}_x|w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big] + I_b^1.&
\end{aligned} \tag{2.187}$$

Let us now treat the boundary term. First observe that using the parabolic equations, noting that A^0 is the diagonal-block form, we can estimate

$$\begin{aligned}
&\left|W_x(0)\cdot[(BW_x)_x+(M_2\bar{W}_x)_x](0)\right| \\
&\qquad\qquad\qquad = \left|w_x^{II}(0)\cdot[(bw_x^{II})_x+(M_2^{22}\bar{W}_x)_x](0)\right| \\
&\qquad\qquad\qquad = \left|w_x^{II}(0)\cdot[A_2^0 w_t^{II}+A_{21}w_x^I+A_{22}w_x^{II}-M_1\bar{W}_x](0)\right| \\
&\qquad\qquad\qquad \le \epsilon|w_x^{II}(0)|^2+C(|W(0)|^2+|w_x^{II}(0)|^2+|w_t^{II}(0)|^2).
\end{aligned}$$

For the first term in I_b, we consider each inflow/outflow case separately. For the outflow case, since $A^{11}\le -\theta_1<0$, we get

$$A_0W_x(0)\cdot W_x(0)\le -\frac{\theta_1}{2}|w_x^I(0)|^2+C|w_x^{II}(0)|^2.$$

Therefore

$$I_b^1\le -\frac{\theta_1}{2}|w_x^I(0)|^2+C(|W(0)|^2+|w_x^{II}(0)|^2+|w_t^{II}(0)|^2). \tag{2.188}$$

Meanwhile, for the inflow case, since $A^{11}\ge\theta_1>0$, we have

$$|A_0W_x(0)\cdot W_x(0)|\le C|W_x(0)|^2.$$

In this case, the invertibility of A^{11} allows us to use the hyperbolic equation to derive

$$|w_x^I(0)|\le C(|w_t^I(0)|+|w_x^{II}(0)|).$$

Therefore we get

$$I_b^1\le C(|W(0)|^2+|W_t(0)|^2+|w_x^{II}(0)|^2). \tag{2.189}$$

Now applying the standard Sobolev inequality (applies for $\alpha-$weighted norms as long as $|\alpha_x/\alpha|$ is uniformly bounded):

$$|w(0)|^2\le C\|w\|_{L^2}(\|w_x\|_{L^2}+\|w\|_{L^2}) \tag{2.190}$$

to control the term $|w_x^{II}(0)|^2$ in I_b^1 in both cases. We get

$$|w_x^{II}(0)|^2 \leq \epsilon' \|w_{xx}^{II}\|_0^2 + C\|w_x^{II}\|_0^2. \tag{2.191}$$

Using this with $\epsilon' = \theta/8$, (2.186), and (2.188), the estimate (2.187) reads

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A^0 W_x, W_x\rangle + \frac{\theta}{8}\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle \omega(x) w_x^I, w_x^I\rangle \\ \leq C\Big(\zeta\|w^I\|_1^2 + \langle |\bar{W}_x| w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big) + I_b^1\end{aligned} \tag{2.192}$$

where the boundary term I_b^1 is estimated as

$$I_b^1 \leq -\frac{\theta_1}{2}|w_x^I(0)|^2 + C(|W(0)|^2 + |w_t^{II}(0)|^2) \tag{2.193}$$

for the outflow case, and similarly

$$I_b^1 \leq C(|W(0)|^2 + |W_t(0)|^2) \tag{2.194}$$

for the inflow case.

Higher order "Friedrichs-type" estimate

Similarly as above, we shall now derive an estimate for $\langle A^0 \partial_x^k W, \partial_x^k W\rangle$, $k = 2, 3, 4$. We need the following key estimate. Integration by parts and (2.176) give

$$\begin{aligned}-\langle \partial_x^k W, A\partial_x^{k+1} W\rangle =& \frac{1}{2}\langle \partial_x^k W, (A_x + (\alpha_x/\alpha)A)\partial_x^k W\rangle + \frac{1}{2}A_0 \partial_x^k W(0)\cdot \partial_x^k W(0) \\ \leq& -\frac{1}{4}\langle \omega(x)\partial_x^k w^I, \partial_x^k w^I\rangle + C\zeta\|\partial_x^k w^I\|_0^2 \\ &+ Cc_*^2\|\partial_x^k w^{II}\|_0^2 + \frac{1}{2}A_0\partial_x^k W(0)\cdot \partial_x^k W(0).\end{aligned}$$

We compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A^0\partial_x^k W, \partial_x^k W\rangle =& \frac{1}{2}\langle A_t^0 \partial_x^k W, \partial_x^k W\rangle + \langle A^0\partial_x^k W, \partial_x^k W_t\rangle \\ =& \frac{1}{2}\langle A_t^0 \partial_x^k W, \partial_x^k W\rangle + \langle A^0 \partial_x^k W, \partial_x^k[(A^0)^{-1} \\ & (-AW_x + (BW_x)_x) + M_1\bar{W}_x + (M_2\bar{W}_x)_x]\rangle.\end{aligned} \tag{2.195}$$

We shall estimate each term in turn. First, $|\langle A^0_t \partial_x^k W, \partial_x^k W\rangle| \le C\zeta \|\partial_x^k W\|_0^2$, and

$$
\begin{aligned}
&\langle A^0 \partial_x^k W, \partial_x^k [-(A^0)^{-1} A W_x]\rangle \\
&= \langle A^0 \partial_x^k W, \sum_{i=0}^{k} \partial_x^i [-(A^0)^{-1} A] \partial_x^{k-i+1} W\rangle \\
&= -\langle \partial_x^k W, A \partial_x^{k+1} W\rangle + \sum_{i=1}^{k} \langle A^0 \partial_x^k W, \partial_x^i [-(A^0)^{-1} A] \partial_x^{k-i+1} W\rangle
\end{aligned}
$$

where we have

$$
\left| \partial_x^i [-(A^0)^{-1} A] \right| \le C \sum_{\sum \alpha_j = i} \prod_{1 \le j \le i} |\partial_x^{\alpha_j} W|. \tag{2.196}
$$

Using the hypothesis on the boundedness of solutions in $W^{2,\infty}$ and weak Moser inequality [57, Lemma 1.5], we get

$$
\begin{aligned}
&|\langle A^0 \partial_x^k W, \partial_x^i [-(A^0)^{-1} A] \partial_x^{k-i+1} W\rangle| \le \\
&\qquad C\Big(\|w^{II}\|_k^2 + \zeta \|w^I\|_k^2 + \sum_{i=1}^{k} \langle |\bar{W}_x| \partial_x^i w^I, \partial_x^i w^I\rangle \Big).
\end{aligned}
$$

This, (2.195), similar treatment (2.178) for $\langle |\bar{W}_x| \partial_x^k w^I, \partial_x^k w^I\rangle$ with c_* being sufficiently large give

$$
\begin{aligned}
\langle A^0 \partial_x^k W, \partial_x^k [-(A^0)^{-1} A W_x]\rangle &\le -\frac{1}{4} \langle \omega \partial_x^k w^I, \partial_x^k w^I\rangle + \frac{1}{2} A_0 \partial_x^k W(0) \cdot \partial_x^k W(0) \\
&\quad + C\Big(\|w^{II}\|_k^2 + \zeta \|w^I\|_k^2 + \sum_{i=1}^{k-1} \langle |\bar{W}_x| \partial_x^i w^I, \partial_x^i w^I\rangle \Big)
\end{aligned} \tag{2.197}
$$

Next, similarly, we obtain

$$
|\langle A^0 \partial_x^k W, \partial_x^k [(A^0)^{-1} M_1 \bar{W}_x]\rangle| \le C\Big(\|w^{II}\|_k^2 + \zeta \|w^I\|_k^2 + \sum_{i=1}^{k} \langle |\bar{W}_x| \partial_x^i w^I, \partial_x^i w^I\rangle \Big).
$$

Finally, we compute and

$$
\begin{aligned}
\langle A^0\partial_x^k W, &\partial_x^k[(A^0)^{-1}(BW_x+M_2\bar{W}_x)_x]\rangle\\
&=\sum_{i=0}^{k}\langle A^0\partial_x^k W, \partial_x^i[(A^0)^{-1}]\partial_x^{k-i+1}(BW_x+M_2\bar{W}_x)\rangle\\
&=\langle \partial_x^k W, \partial_x^{k+1}(BW_x+M_2\bar{W}_x)\rangle\\
&\quad+\sum_{i=1}^{k}\langle A^0\partial_x^k W, \partial_x^i[(A^0)^{-1}]\partial_x^{k-i+1}(BW_x+M_2\bar{W}_x)\rangle\\
&\le -\langle \partial_x^{k+1}W+(\alpha_x/\alpha)\partial_x^k W, \partial_x^k(BW_x+M_2\bar{W}_x)\rangle\\
&\quad-\partial_x^k[b\partial_x w^{II}+M_2^{22}\bar{W}_x](0)\partial_x^k w^{II}(0)\\
&\quad+\xi\|\partial_x^{k+1}w^{II}\|_0^2+C\Big(c_*^2\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^{k}\langle|\bar{W}_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big)\\
&\le -\frac{\theta}{2}\|\partial_x^{k+1}w^{II}\|_0^2-\partial_x^k[b\partial_x w^{II}+M_2^{22}\bar{W}_x](0)\partial_x^k w^{II}(0)\\
&\qquad+C\Big(c_*^2\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^{k}\langle|\bar{W}_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big)
\end{aligned}
$$

where in the last inequality we used the special form of B and M_2 to get

$$
\begin{aligned}
&\langle \partial_x^{k+1}W+(\alpha_x/\alpha)\partial_x^k W, \partial_x^k(BW_x+M_2\bar{W}_x)\rangle\\
&\le \langle|\partial_x^{k+1}w^{II}|+\omega(x)|\partial_x^k w^{II}|, |\partial_x^k(bw_x^{II}+\Pi_2M_2\bar{W}_x)|\rangle\\
&\le -\theta\|\partial_x^{k+1}w^{II}\|_0^2+C\Big(C(c_*)\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^{k}\langle|\bar{W}_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big).
\end{aligned}
$$

Note that in the last inequality, there is no term of $\langle\omega(x)\partial_x^i w^I, \partial_x^i w^I\rangle$ because of the presence of $|\bar{W}_x|$ in term of $\Pi_2 M_2$.

Put all these estimates into (2.195) together, we have obtained

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\langle A^0\partial_x^k W, \partial_x^k W\rangle+\frac{1}{4}\theta\|\partial_x^{k+1}w^{II}\|_0^2+\frac{1}{4}\langle\omega(x)\partial_x^k w^I, \partial_x^k w^I\rangle\\
&\le C\Big(C(c_*)\|w^{II}\|_k^2+\zeta\|w^I\|_k^2+\sum_{i=1}^{k-1}\langle|\bar{W}_x|\partial_x^i w^I, \partial_x^i w^I\rangle\Big)+I_b
\end{aligned}
\tag{2.198}
$$

where the boundary term

$$I_b := \frac{1}{2}A_0\partial_x^k W(0)\cdot\partial_x^k W(0) - \partial_x^k[b\partial_x w^{II} + M_2^{22}\bar{W}_x](0)\partial_x^k w^{II}(0). \tag{2.199}$$

For this boundary term, we shall treat the same as we did before. First using the parabolic equations with noting that A^0 is the diagonal-block matrix $\mathrm{diag}(A_1^0, A_2^0)$, we can write

$$\begin{aligned}&\partial_x^k[b\partial_x w^{II} + M_2^{22}\bar{W}_x](0)\\&\quad = \partial_x^{k-1}[A_2^0(0)w_t^{II}(0,t) + A_{21}w_x^I(0) + A_{22}w_x^{II}(0) - \Pi_2 M_1(0)\bar{W}_x(0)].\end{aligned} \tag{2.200}$$

Therefore we get

$$\begin{aligned}|\partial_x^k&[b\partial_x w^{II} + M_2^{22}\bar{W}_x](0)\partial_x^k w^{II}(0)|\\&\le C|\partial_x^k w^{II}(0)|\Big[|\partial_x^{k-1}w_t^{II}(0)| + \sum_{i=0}^{k}(|\partial_x^i w^{II}(0)| + |\partial_x^i w^I(0)|)\Big]\\&\le \epsilon\sum_{i=0}^{k}|\partial_x^i w^I(0)|^2 + C\sum_{i=1}^{k}|\partial_x^i w^{II}(0)|^2 &(2.201)\\&\quad + C|\partial_x^k w^{II}(0)||\partial_x^{k-1}w_t^{II}(0)| &(2.202)\end{aligned}$$

for any ϵ small. To deal with the term of w_t^{II}, for simplicity, assume $k = 3$. By solving the parabolic-part equations and using the invertibility of b, we obtain

$$\begin{aligned}|\partial_x^2 w_t^{II}| = |\partial_t w_{xx}^{II}| &\le C(|w_{tt}^{II}| + |W_t| + |W_x| + |W_{xt}|)\\|W_{xt}| &\le C(|W| + |W_x| + |W_{xx}| + |w_{xxx}^{II}|).\end{aligned} \tag{2.203}$$

Since for case $k = 3$ we have a good term $\|\partial_x^4 w^{II}\|_0$ (see (2.198)), the term $|w_{xxx}^{II}(0)|$ resulting from the boundary treatment is easily treated via Sobolev embedding inequality. Hence all terms in a form $\partial_x^r w^{II}(0)$ are easily estimated. Meanwhile, using the hyperbolic-part equations, we have

$$|w_t^I| \le C(|W| + |W_x|). \tag{2.204}$$

Employing Young's inequality to the last term in (2.201), we obtain

$$
\begin{aligned}
&|\partial_x^k[b\partial_x w^{II} + M_2^{22}\bar{W}_x](0)\partial_x^k w^{II}(0)| \\
&\qquad \le \epsilon \sum_{i=0}^{k} |\partial_x^i w^I(0)|^2 + C(\sum_{i=0}^{k} |\partial_x^i w^{II}(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2) \qquad (2.205)
\end{aligned}
$$

To deal with the term of w^I, we need to consider two cases separately. When $A^{11} \le -\theta_1 < 0$, we get

$$
A_0\partial_x^k W(0) \cdot \partial_x^k W(0) \le -\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2 + C|\partial_x^k w^{II}(0)|^2.
$$

Therefore

$$
\begin{aligned}
I_b^k \le &-\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2 + C(\sum_{i=0}^{k-1} |\partial_x^i w^I(0)|^2 \\
&+ \sum_{i=0}^{k} |\partial_x^i w^{II}(0)|^2 + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2). \qquad (2.206)
\end{aligned}
$$

Meanwhile, for the case $A^{11} \ge \theta_1 > 0$, we have

$$
|A_0\partial_x^k W(0) \cdot \partial_x^k W(0)| \le C(|\partial_x^k w^I(0)|^2 + |\partial_x^k w^{II}(0)|^2).
$$

The invertibility of A^{11} allows us to use the hyperbolic equation to derive

$$
|\partial_x^k w^I(0)| \le C(\sum_{i=0}^{k}(|\partial_x^i w^{II}(0)|^2 + |\partial_t^i w^I(0)|^2) + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2).
$$

Therefore in the case of $A^{11} \ge \theta_1 > 0$, we get

$$
I_b^k \le C(\sum_{i=0}^{k}(|\partial_x^i w^{II}(0)|^2 + |\partial_t^i w^I(0)|^2) + |w_t^{II}(0)|^2 + |w_{tt}^{II}(0)|^2). \qquad (2.207)
$$

Employing the boundary estimates into (2.198), we have obtained

$$\begin{aligned}&\frac{d}{dt}\langle A^0\partial_x^k W,\partial_x^k W\rangle+\theta\|\partial_x^{k+1}w^{II}\|_0^2+c_*\theta_1\langle|\bar{W}_x|\partial_x^k w^I,\partial_x^k w^I\rangle\\&\quad\le C\Big(\zeta\|w^I\|_k^2+c_*^2\|w^{II}\|_k^2+\sum_{j=0}^{k-1}\langle|\bar{W}_x|\partial_x^j w^I,\partial_x^j w^I\rangle\Big)+I_b^k\end{aligned}\tag{2.208}$$

where, after absorbing the terms of $|\partial_x^r w^{II}(0)|$ via Sobolev embedding, the boundary term I_b^k satisfies

$$I_b^k\le-\frac{\theta_1}{2}|\partial_x^k w^I(0)|^2+C(\sum_{i=0}^{k-1}|\partial_x^i w^I(0)|^2+|w_t^{II}(0)|^2+|w_{tt}^{II}(0)|^2)\tag{2.209}$$

for outflow case, and

$$I_b^k\le C(\sum_{i=0}^{k}|\partial_t^i w^I(0)|^2+|w_t^{II}(0)|^2+|w_{tt}^{II}(0)|^2)\tag{2.210}$$

for the inflow case.

We shall establish an Kawashima-type estimate to bound the term $\|w^I\|_k^2$ appearing on the left hand side of the above.

"Kawashima-type" estimate

Let K be the skew-symmetry in (2.174). Integration by parts and skew-symmetry property of K yield

$$\begin{aligned}\langle KW_{xt},W\rangle&=-\langle KW_t,W_x\rangle-\langle(K_x+(\alpha_x/\alpha)K)W_t,W\rangle-K_0W_0\cdot(W_0)_t\\&=\langle KW_x,W_t\rangle+\langle(K_x+(\alpha_x/\alpha)K)W,W_t\rangle-K_0W_0\cdot(W_0)_t.\end{aligned}$$

Using this, we compute

$$
\begin{aligned}
\frac{d}{dt}&\langle KW_x, W\rangle = \\
&\langle K_t W_x + KW_{xt}, W\rangle + \langle KW_x, W_t\rangle \\
=&\langle K_t W_x, W\rangle + \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, W_t\rangle \\
&- K_0 W_0 \cdot (W_0)_t \\
=&\langle K_t W_x, W\rangle + \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, -(A^0)^{-1}AW_x\rangle \\
&+ \langle 2KW_x + (K_x + (\alpha_x/\alpha)K)W, (A^0)^{-1}(BW_x)_x \\
&+ M_1\bar{W}_x + (M_2\bar{W}_x)_x\rangle - K_0 W_0 \cdot (W_0)_t \\
\leq& -2\langle K(A^0)^{-1}AW_x, W_x\rangle + \xi\|w_x^I\|_0^2 - K_0 W_0 \cdot (W_0)_t \\
&+ C\Big(C(c_*)\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 + \langle\omega(x)w^I, w^I\rangle + \langle\omega(x)w_x^I, w_x^I\rangle\Big).
\end{aligned}
$$

Using (2.174), we get

$$
\langle K(A^0)^{-1}AW_x, W_x\rangle \geq \theta_2\|w_x^I\|_0^2 - C(c_0)\|w_x^{II}\|_0^2,
$$

and thus obtain from the above estimate with $\xi = \theta_2/2$

$$
\begin{aligned}
\frac{d}{dt}\langle KW_x, W\rangle \leq& -\frac{\theta_2}{2}\|w_x^I\|_0^2 + C\Big(C(c_*)\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 \\
&+ \langle\omega(x)w^I, w^I\rangle + \langle\omega(x)w_x^I, w_x^I\rangle\Big) - K_0 W_0 \cdot (W_0)_t. \qquad (2.211)
\end{aligned}
$$

Higher order "Kawashima-type" estimate

With similar calculations, we shall obtain an estimate for $\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1}W\rangle, k \geq 1$. We compute

$$
\begin{aligned}
\langle K\partial_x^k W_t, \partial_x^{k-1}W\rangle =& \langle K\partial_x^k W, \partial_x^{k-1}W_t\rangle \\
&+ \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1}W, \partial_x^{k-1}W_t\rangle - K\partial_x^{k-1}W_t \cdot \partial_x^{k-1}W(0).
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1} W\rangle &= \langle K_t\partial_x^k W, \partial_x^{k-1} W\rangle + \langle 2K\partial_x^k W, \partial_x^{k-1} W_t\rangle \\
&\quad + \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1} W, \partial_x^{k-1} W_t\rangle - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0) \\
&= \langle 2K\partial_x^k W, \partial_x^{k-1}[(-A^0)^{-1}(AW_x + (BW_x)_x + M_1\bar{W}_x + (M_2\bar{W}_x)_x)]\rangle \\
&\quad + \langle (K_x + (\alpha_x/\alpha)K)\partial_x^{k-1} W, \\
&\qquad \partial_x^{k-1}[(-A^0)^{-1}(AW_x + (BW_x)_x + M_1\bar{W}_x + (M_2\bar{W}_x)_x)]\rangle \\
&\quad - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0) \\
&\le -2\langle K(A^0)^{-1} A\partial_x^k W, \partial_x^k W\rangle + \epsilon\|w^I\|_k^2 + Cc_*^2\|w^{II}\|_{k+1}^2 \\
&\quad + C\zeta\|w^I\|_0^2 + C\sum_{l=1}^{k}\langle \omega(x)\partial_x^l w^I, \partial_x^l w^I\rangle - K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0)
\end{aligned}
$$

for ϵ small.

Using (2.174), we obtain from the above

$$
\begin{aligned}
\frac{d}{dt}\langle K\partial_x^k W, \partial_x^{k-1} W\rangle \le& -\frac{\theta_2}{3}\|\partial_x^k w^I\|_0^2 + Cc_*^2\|w^{II}\|_{k+1}^2 + \epsilon\|w^I\|_{k-1} &(2.212)\\
&+ C\zeta\|w^I\|_0^2 + C\sum_{l=1}^{k}\langle \omega(x)\partial_x^l w^I, \partial_x^l w^I\rangle \\
&- K\partial_x^{k-1} W_t \cdot \partial_x^{k-1} W(0). &(2.213)
\end{aligned}
$$

Final estimates

We are ready to conclude our result. First combining the estimate (2.192) with (2.183), we easily obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle\Big) \\
&\quad \le -\Big(\frac{\theta}{8}\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle \omega(x) w_x^I, w_x^I\rangle\Big) \\
&\qquad + C\Big(\zeta\|w^I\|_1^2 + \langle |\bar{W}_x| w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big) + I_b^1 \\
&\qquad - \frac{M}{4}\Big(\langle \omega(x) w^I, w^I\rangle + \theta\|w_x^{II}\|_0^2\Big) + CM\zeta\|w^I\|_0^2 \\
&\qquad\quad + MC(c_*)\|w^{II}\|_0^2 + MI_b^0
\end{aligned}
$$

By choosing M sufficiently large such that $M\theta \gg C(c_*)$, and noting that $c_*\theta_1|\bar{W}_x| \le \omega(x)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0W_x, W_x\rangle + M\langle A^0 W, W\rangle\Big)\\
&\quad \le -\Big(\theta\|w^{II}\|_2^2 + \langle \omega(x)w^I, w^I\rangle + \langle \omega(x)w_x^I, w_x^I\rangle\Big) \\
&\qquad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) + I_b^1 + MI_b^0.
\end{aligned} \tag{2.214}$$

We shall treat the boundary terms later. Now we employ the estimate (2.211) to absorb the term $\|w^I\|_1$ into the left hand side. Indeed, fixing c_* large as above, adding (2.214) with (2.211) times ϵ, and choosing ϵ, ζ sufficiently small such that $\epsilon C(c_*) \ll \theta, \epsilon \ll 1$ and $\zeta \ll \epsilon\theta_2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle KW_x, W\rangle\Big)\\
&\le -\Big(\theta\|w^{II}\|_2^2 + \langle \omega(x)w^I, w^I\rangle + \langle \omega(x)w_x^I, w_x^I\rangle\Big)\\
&\quad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) - \frac{\theta_2\epsilon}{2}\|w_x^I\|_0^2\\
&\quad + C\epsilon\Big(C(c_*)\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 + \langle \omega(x)w^I, w^I\rangle + \langle \omega(x)w_x^I, w_x^I\rangle\Big)\\
&\quad + I_b^1 + MI_b^0 - \epsilon K_0W_0\cdot(W_0)_t\\
&\le -\frac{1}{2}\Big(\theta\|w^{II}\|_2^2 + \theta_2\epsilon\|w_x^I\|_0^2\Big) + C(c_*)\Big(\zeta\|w^I\|_0^2 + \|w^{II}\|_0^2\Big) + I_b
\end{aligned}$$

where $I_b := I_b^1 + MI_b^0 - \epsilon K_0W_0\cdot(W_0)_t$.

By a view of boundary terms I_b^0, I_b^1, we treat the term I_b in each inflow/outflow case separately. Recalling the inequality (2.191), $|w_x^{II}(0)| \le C\|w^{II}\|_2$. Thus, using this, for the inflow case we have

$$\begin{aligned}
I_b &\le M|W(0)|^2 + C|W_t(0)|^2 + M|w_x^{II}(0)||w^{II}(0)|\\
&\le \frac{\theta}{2}\|w^{II}\|_2^2 + M^2|W(0)|^2 + C|W_t(0)|^2.
\end{aligned}$$

Meanwhile, for the outflow case, with $M\theta_1 \gg 1$ and $K_0W_0\cdot(W_0)_t \sim w_0^{II}w_{0t}^I + w_0^Iw_{0t}^{II}$, we have I_b is bounded by

$$-\frac{\theta_1}{2}(|w_x^I(0)|^2+|w^I(0)|^2)$$
$$+C(|w_t^{II}(0)|^2+|w^{II}(0)|^2)+\epsilon(|w_x^{II}(0)|^2+|w_t^I(0)|^2)$$

which, together with ϵ being sufficiently small and the facts that

$$|w_t^I(0)|\le C(|w_x^I(0)|+|w_x^{II}(0)|+|W(0)|)$$

obtained from solving the hyperbolic equation and the embedding inequality

$$|w_x^{II}(0)|\le C\|w^{II}\|_2,$$

yields

$$I_b\le -\frac{\theta_1}{2}(|w_x^I(0)|^2+|w^I(0)|^2)+\frac{\theta}{2}\|w^{II}\|_2^2+C(|w^{II}(0)|^2+|w_t^{II}(0)|^2)$$

for the outflow case. Now by Cauchy-Schwarz's inequality and by positivity definite of A^0, it is easy to see that

$$\mathcal{E}:=\langle A^0W_x,W_x\rangle+M\langle A^0W,W\rangle+\epsilon\langle KW_x,W\rangle\sim\|W\|_{H^1_\alpha}^2\sim\|W\|_{H^1}^2. \qquad (2.215)$$

The last equivalence is due to the fact that α is bounded above and below away from zero. Thus the above gives

$$\frac{d}{dt}\mathcal{E}(W)(t)\le-\theta_3\mathcal{E}(W)(t)+C(c_*)\Big(\|W(t)\|_{L^2}^2+\mathcal{B}_1(t)^2\Big),$$

for some positive constant θ_3, which by the Gronwall inequality yields

$$\|W(t)\|_{H^1}^2\le Ce^{-\theta t}\|W_0\|_{H^1}^2+C(c_*)\int_0^t e^{-\theta(t-\tau)}\Big(\|W(\tau)\|_{L^2}^2+\mathcal{B}_1(\tau)^2\Big)d\tau, \qquad (2.216)$$

where $W(x,0)=W_0(x)$ and

$$\mathcal{B}_1(\tau)^2:=\mathcal{O}(|W(0,\tau)|^2+|W_t(0,\tau)|^2)=\mathcal{O}(|(h_1,h_2)|^2+|(h_1,h_2)_t|^2) \qquad (2.217)$$

for the inflow case, and

$$\mathcal{B}_1(\tau)^2 := \mathcal{O}(|w^{II}(0,\tau)|^2 + |w_t^{II}(0,\tau)|^2) = \mathcal{O}(|h|^2 + |h_t|^2) \tag{2.218}$$

for the outflow case.

Similarly, by induction, we shall derive the same estimates for W in H^s. To do that, let us define

$$\begin{aligned} \mathcal{E}_1(W) &:= \langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle K W_x, W\rangle \\ \mathcal{E}_k(W) &:= \langle A^0 \partial_x^k W, \partial_x^k W\rangle + M\mathcal{E}_{k-1}(W) + \epsilon\langle K\partial_x^k W, \partial_x^{k-1} W\rangle. \end{aligned}$$

Then by Cauchy-Schwarz inequality, it is easy to see that $\mathcal{E}_k(W) \sim \|W\|_{H^k}^2$, and by induction, we obtain

$$\frac{d}{dt}\mathcal{E}_s(W)(t) \le -\theta_3\mathcal{E}_s(W)(t) + C(c_*)(\|W(t)\|_{L^2}^2 + \mathcal{B}_h(t)^2),$$

for some positive constant θ_3, which by the Gronwall inequality yields

$$\|W(t)\|_{H^s}^2 \le Ce^{-\theta t}\|W_0\|_{H^s}^2 + C(c_*)\int_0^t e^{-\theta(t-\tau)}(\|W(\tau)\|_{L^2}^2 + \mathcal{B}_h(\tau)^2)d\tau, \tag{2.219}$$

where $W(x,0) = W_0(x)$ and $\mathcal{B}_h$ is defined as in (2.16) and (2.17).

The general case

Following [37], the general case that hypotheses (A1)-(A3) hold can easily be covered via following simple observations. First, we may express matrix A in (2.170) as

$$A(W+\bar W) = \hat A + (\zeta + |\bar W_x|)\begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix} \tag{2.220}$$

where $\hat A$ is a symmetric matrix obeying the same derivative bounds as described for A, identical to A in the 11 block and obtained in other blocks jk by

$$\begin{aligned} A^{jk}(W+\bar W) &= A^{jk}(\bar W) + A^{jk}(W+\bar W) - A^{jk}(\bar W) \\ &= A^{jk}(W_+) + \mathcal{O}(|W_x| + |\bar W_x|) = A^{jk}(W_+) + \mathcal{O}(\zeta + |\bar W_x|). \end{aligned} \tag{2.221}$$

Replacing A by $\hat{A}$ in the k^{th} order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as

$$\langle \partial_x^k \mathcal{O}(\zeta + |\bar{W}_x|)|W|, |\partial_x^{k+1} w^{II}|\rangle,$$

plus lower order terms, easily absorbed using Young's inequality, and boundary terms

$$\mathcal{O}(\sum_{i=0}^{k} |\partial_x^i w^{II}(0)||\partial_x^k w^I(0)|)$$

resulting from the use of integration by parts as we deal with the $12-$block. However these boundary terms were already treated somewhere as before (see (2.201)). Hence we can recover the same Friedrichs-type estimates obtained above. Thus we may relax $(A1')$ to $(A1)$.

The second observation is that, because of the favorable terms

$$c_* \theta_1 \langle |\bar{W}_x| \partial_x^k w^I, \partial_x^k w^I \rangle$$

occurring in the lefthand sides of the Friedrichs-type estimates (2.208), we need the Kawashima-type bound only to control the contribution to $|\partial_x^k w^I|^2$ coming from x near $+\infty$; more precisely, we require from this estimate only a favorable term

$$-\theta_2 \langle (1 - \mathcal{O}(\zeta + |\bar{W}_x|)) \partial_x^k w^I, \partial_x^k w^I \rangle$$

rather than $\theta_2 \|\partial_x^k w^I\|_0^2$ as in (2.212). But, this may easily be obtained by substituting for K a skew-symmetric matrix-valued function $\hat{K} := K(W_+)$, and using the fact that

$$\Re e(K(A^0)^{-1}A + B)(W_+) \geq \theta_2 > 0,$$

and same as (2.221), $K = \hat{K} + \mathcal{O}(\zeta + |\bar{W}_x|)$, we have

$$\Re e(K(A^0)^{-1}A + B)(W) \geq \theta_2(1 - \mathcal{O}(\zeta + |\bar{W}_x|)) > 0.$$

Thus we may relax $(A2')$ to $(A2)$.

Finally, notice that the term $g(\tilde{W}_x) - g(\bar{W}_x)$ in the perturbation equation may be

Taylor expanded as

$$\begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_x) + g_1(\bar{W}_x, \tilde{W}_x) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix}$$

The first term, since it vanishes in the first component and since $|\bar{W}_x|$ decays at plus spatial infinity, yields by Young's inequality the estimate

$$\Big\langle \begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_x) + g_1(\bar{W}_x, \tilde{W}_x) \end{pmatrix}, \begin{pmatrix} w_x^I \\ w_x^{II} \end{pmatrix} \Big\rangle \leq C\Big(\langle(\zeta + |\bar{W}_x|)w_x^I, w_x^I\rangle + \|w_x^{II}\|_0^2\Big)$$

which can be treated in the Friedrichs-type estimates. The $(0, O(|W_x|^2)^T$ nonlinear term may be treated as other source terms in the energy estimates. Specifically, the worst-case term

$$\langle \partial_x^k W, \partial_x^k \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix} \rangle = -\langle \partial_x^{k+1} w^{II}, \partial_x^{k-1}\mathcal{O}(|W_x|^2)\rangle - \partial_x^k w^{II}(0)\partial_x^{k-1}\mathcal{O}(|W_x|^2)(0)$$

may be bounded by

$$\|\partial_x^{k+1} w^{II}\|_{L^2}\|W\|_{W^{2,\infty}}\|W\|_{H^k} - \partial_x^k w^{II}(0)\partial_x^{k-1}\mathcal{O}(|W_x|^2)(0).$$

The boundary term will contribute to energy estimates in the form (2.199) of I_b, and thus we may use the parabolic equations to get rid of this term as we did in (2.200). Thus, we may relax $(A3')$ to $(A3)$, completing the proof of the general case $(A1) - (A3)$ and the proposition. □

2.4.2 Energy estimate II

We require also the following estimate:

Lemma 2.4.3 ([22]). *Under the hypotheses of Theorem 2.1.4, let* $E_0 := \|(1 + |x|^2)^{3/4} U_0\|_{H^4}$*, and suppose that, for* $0 \leq t \leq T$*, the* $W^{2,\infty}$ *norm of the solution* U *of* (2.225) *remains bounded by some constant* $C > 0$*. Then, for all* $0 \leq t \leq T$*,*

$$\|(1 + |x|^2)^{3/4} U(x,t)\|_{H^4}^2 \leq M E_0 e^{Mt}. \tag{2.222}$$

Proof. This follows by standard Friedrichs symmetrizer estimates carried out in the weighted H^4 norm. □

Remark 2.4.4. *An immediate consequence of Lemma 2.4.3, by Sobolev embedding: $W^{3,\infty} \subset H^4$ and equation (2.225), is that, if E_0 and $\|U\|_{H^4}$ are uniformly bounded on $[0,T]$, then*

$$(1+|x|)^{3/2}\Big[|U|+|U_t|+|U_x|+|U_{xt}|\Big](x,t) \tag{2.223}$$

is uniformly bounded on $[0,T]$ as well.

2.5 Stability analysis

In this section, we shall prove Theorems 2.1.3 and 2.1.4. Following [23, 36], define the nonlinear perturbation $U=(u,v)$ by

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x), \tag{2.224}$$

we obtain

$$U_t - LU = Q(U,U_x)_x, \tag{2.225}$$

where linearized operator

$$LU := -(AU)_x + (BU_x)_x \tag{2.226}$$

where

$$AU := dF(\bar{U})U - (dB(\bar{U})U)\bar{U}_x, \quad B = B(\bar{U})$$

and the second-order Taylor remainder:

$$Q(U,U_x) = F(\bar{U}+U) - F(\bar{U}) + A(\bar{U})U + (B(\bar{U}+U) - B(\bar{U}))U_x$$

satisfying

$$\begin{aligned} |Q(U,U_x)| &\le C(|U||U_x| + |U|^2) \\ |\Pi_1 Q(U,U_x)_x| &\le C(|U||U_x| + |U|^2) \\ |Q(U,U_x)_x| &\le C(|U||U_{xx}| + |U_x|^2 + |U||U_x|) \\ |Q(U,U_x)_{xx}| &\le C(|U||U_{xx}| + |U||U_{xxx}| + |U_x||U_{xx}| + |U_x|^2) \end{aligned} \tag{2.227}$$

so long as $|U|$ remains bounded.

For boundary conditions written in $U-$coordinates, (B) gives

$$\begin{aligned} h(t) = \tilde{h}(t) - \bar{h} &= (\tilde{W}(U + \bar{U}) - \tilde{W}(\bar{U}))(0, t) \\ &= (\partial \tilde{W}/\partial \tilde{U})(\bar{U}_0)U(0, t) + \mathcal{O}(|U(0, t)|^2). \end{aligned} \tag{2.228}$$

in inflow case and

$$\begin{aligned} h(t) = \tilde{h}(t) - \bar{h} &= (\tilde{w}^{II}(U + \bar{U}) - \tilde{w}^{II}(\bar{U}))(0, t) \\ &= (\partial \tilde{w}^{II}/\partial \tilde{U})(\bar{U}_0)U(0, t) + \mathcal{O}(|U(0, t)|^2) \\ &= m \begin{pmatrix} \bar{b}_1 & \bar{b}_2 \end{pmatrix} (\bar{U}_0)U(0, t) + \mathcal{O}(|U(0, t)|^2) \\ &= mB(\bar{U}_0)U(0, t) + \mathcal{O}(|U(0, t)|^2). \end{aligned} \tag{2.229}$$

2.5.1 Integral formulation

We obtain the following:

Lemma 2.5.1 (Integral formulation)**.** *We have*

$$\begin{aligned} U(x, t) = &\int_0^\infty G(x, t; y)U_0(y)\, dy \\ &+ \int_0^t \Big(\tilde{G}_y(x, t - s; 0)BU(0, s) + G(x, t - s; 0)AU(0, s) \Big)\, ds \\ &+ \int_0^t \int_0^\infty H(x, t - s; y)\Pi_1 Q(U, U_y)_y(y, s)\, dy\, ds \\ &- \int_0^t \int_0^\infty \tilde{G}_y(x, t - s; y)\Pi_2 Q(U, U_y)(y, s)\, dy\, ds \end{aligned} \tag{2.230}$$

where $U(y, 0) = U_0(y)$*.*

Proof. From the duality (see [60, Lemma 4.3]), we find that $G(x, t - s; y)$ considered as a function of y, s satisfies the adjoint equation

$$(\partial_s - L_y)^* G^*(x, t - s; y) = 0, \tag{2.231}$$

or

$$-G_s - (GA)_y + GA_y = (G_y B)_y. \tag{2.232}$$

in the distributional sense, for all $x, y, t > s > 0$, where the adjoint operator of L_y is defined by

$$L_y^* V := V_y^* A + (V_y^* B)_y, \tag{2.233}$$

with $V^* = V^{tr}$.

Likewise, for boundary conditions, we have, by duality

(iii') for all $x, t > 0$, $G(x,t;0) \equiv 0$ in the outflow case $\bar{A}_* < 0$; and $G(x,t;0)B = 0$ in the inflow case $\bar{A}_* > 0$, noting that no boundary condition need be applied on the hyperbolic part for the adjoint equations in the inflow case.

Thus, integrating G against (2.225), we obtain for any classical solution that

$$\begin{aligned} \int_0^t \int_0^\infty & G(x,t-s;y) Q(U,U_y)_y(y,s)\, dy\, ds = \\ & \int_0^t \int_0^\infty G(x,t-s;y)(\partial_s - L_y)U(y,s)\, dy\, ds \\ & =: I_1 + I_2. \end{aligned} \tag{2.234}$$

Integrating by parts and using the boundary conditions (iii') on the boundary $y = 0$, we get

$$\begin{aligned} I_1 &= \int_0^t \int_0^\infty G(x,t-s;y)\partial_s U(y,s)\, dy\, ds \\ &= \int_0^t \int_0^\infty \partial_s G(x,t-s;y) U(y,s)\, dy\, ds \\ &\quad + \int_0^\infty G(x,0;y)U(y,t)\, dy - \int_0^\infty G(x,t;y)U(y,0)\, dy \end{aligned}$$

where note that

$$U(x,t) = \int_0^\infty G(x,0;y)U(y,t)\, dy$$

and also

$$\begin{aligned}
I_2 &= \int_0^t \int_0^\infty G(x,t-s;y)(-L_y)U(y,s)\,dy\,ds \\
&= \int_0^t \int_0^\infty G(x,t-s;y)((AU)_y - (BU_y)_y)(y,s)\,dy\,ds \\
&= \int_0^t \int_0^\infty (-G_yA - (G_yB)_y)U(y,s)\,dy\,ds \\
&\quad - \int_0^t G_y(x,t-s;0)BU(0,s)ds - \int_0^t G(x,t-s;0)AU(0,s)ds
\end{aligned}$$

Combining these estimates, and noting that $G_yB = \tilde{G}_yB$ since $HB \equiv 0$, we obtain (2.230) by rearranging and integrating by parts the last term of

$$\begin{aligned}
&\int_0^t \int_0^\infty G(x,t-s;y)Q(U,U_y)_y(y,s)\,dy\,ds \\
&\qquad = \int_0^t \int_0^\infty (H+\tilde{G})(x,t-s;y)Q(U,U_y)_y(y,s)\,dy\,ds
\end{aligned} \tag{2.235}$$

□

As an expression for U_x, we obtain the following.

Lemma 2.5.2 (Integral formulation for U_x). *We have*

$$\begin{aligned}
U_x(x,t) &= \int_0^\infty G_x(x,t;y)U_0(y)\,dy - \int_0^t H(x,t-s;0)\Pi_1 Q(U,U_y)_y(0,s)\,ds \\
&+ \int_0^t \Big[\tilde{G}_{xy}(x,t-s;0)BU(0,s) + G_x(x,t-s;0)AU(0,s)\Big]\,ds \\
&+ \int_0^t \int_0^\infty (H_x - H_y)(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds \\
&- \int_0^t \int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_{yy}(y,s)\,dy\,ds \\
&- \int_0^{t-1} \int_0^\infty \tilde{G}_{xy}(x,t-s;y)\Pi_2 Q(U,U_y)(y,s)\,dy\,ds \\
&+ \int_{t-1}^t \int_0^\infty \tilde{G}_x(x,t-s;y)\Pi_2 Q(U,U_y)_y(y,s)\,dy\,ds
\end{aligned} \tag{2.236}$$

where $U(y,0) = U_0(y)$.

Proof. Differentiating the formulation (2.230) for $U(x,t)$ with respect to x and noting that

$$\int_0^t \int_0^\infty H_x \phi \, dy \, ds = \int_0^t \int_0^\infty (H_x - H_y)\phi \, dy \, ds - \int_0^t \int_0^\infty H(x, t-s; y)\phi_y(y,s) \, dy \, ds - \int_0^t H(x, t-s; 0)\phi(0,s) ds$$

and

$$\int_0^t \int_0^\infty \tilde{G}_{xy}\psi \, dy \, ds = \int_0^{t-1} \int_0^\infty \tilde{G}_{xy}\psi \, dy \, ds - \int_{t-1}^t \int_0^\infty \tilde{G}_x \psi_y \, dy \, ds - \int_{t-1}^t \tilde{G}_x(x, t-s; 0)\psi(0,s) ds$$

are valid for any smooth functions ϕ, ψ, we obtain the lemma. □

2.5.2 Convolution estimates

To establish stability, we use the following lemmas whose proof was given in [23, 22, 50] and will be recalled in Appendix A.2, for sake of completeness.

Lemma 2.5.3 (Linear estimates I). *Under the assumptions of Theorem 2.1.4,*

$$\begin{aligned} \int_0^{+\infty} |\tilde{G}(x,t;y)|(1+|y|)^{-3/2} \, dy \le C(\theta + \psi_1 + \psi_2)(x,t), \\ \int_0^{+\infty} |\tilde{G}_x(x,t;y)|(1+|y|)^{-3/2} \, dy \le C(\theta + \psi_1 + \psi_2)(x,t), \end{aligned} \tag{2.237}$$

for $0 \le t \le +\infty$*, some* $C > 0$*.*

Lemma 2.5.4 (Linear estimates II). *Under the assumptions of Theorem 2.1.4, if* $|U_0(x)| + |\partial_x U_0(x)| \le E_0(1+|x|)^{-3/2}, E_0 > 0$*, then, for some* $\theta > 0$*,*

$$\begin{aligned} \int_0^{+\infty} H(x,t;y)U_0(y) \, dy \le CE_0 e^{-\theta t}(1+|x|)^{-3/2}, \\ \int_0^{+\infty} H_x(x,t;y)U_0(y) \, dy \le CE_0 e^{-\theta t}(1+|x|)^{-3/2}, \end{aligned} \tag{2.238}$$

and so both are dominated by $CE_0(\psi_1 + \psi_2)$*, for* $0 \le t \le +\infty$*, some* $C > 0$*.*

Lemma 2.5.5 (Nonlinear estimates I). *Under the assumptions of Theorem 2.1.4,*

$$\begin{aligned} \int_0^t \int_0^{+\infty} |\tilde{G}_y(x, t-s; y)| \Psi(y, s)\, dy ds &\le C(\theta + \psi_1 + \psi_2)(x, t), \\ \int_0^{t-1} \int_0^{+\infty} |\tilde{G}_{xy}(x, t-s; y)| \Psi(y, s)\, dy ds &\le C(\theta + \psi_1 + \psi_2)(x, t), \end{aligned} \tag{2.239}$$

for $0 \le t \le +\infty$*, some* $C > 0$*, where*

$$\Psi(y, s) := (\theta + \psi_1 + \psi_2)^2 (y, s). \tag{2.240}$$

Lemma 2.5.6 (Nonlinear estimates II). *Under the assumptions of Theorem 2.1.4,*

$$\begin{aligned} \int_0^t \int_0^{+\infty} H(x, t-s; y) \Upsilon(y, s)\, dy ds &\le C(\psi_1 + \psi_2)(x, t) \\ \int_0^t \int_0^{+\infty} (H_x - H_y)(x, t-s; y) \Upsilon(y, s)\, dy ds &\le C(\psi_1 + \psi_2)(x, t) \\ \int_{t-1}^t \int_0^{+\infty} |\tilde{G}_x(x, t-s; y)| \Upsilon(y, s)\, dy ds &\le C(\psi_1 + \psi_2)(x, t) \end{aligned} \tag{2.241}$$

for all $0 < t < +\infty$*, some* $C > 0$*, where*

$$\Upsilon(y, s) := s^{-1/4} (\theta + \psi_1 + \psi_2)(y, s) \tag{2.242}$$

We require also the following estimate accounting boundary effects.

Lemma 2.5.7 (Boundary estimates I). *Under the assumptions of Theorem 2.1.4, if* $|h(t)| + |h'(t)| \le E_0 (1 + t)^{-1}$*,*

$$\begin{aligned} \int_0^t H(x, t-s; 0) h(s)\, ds &\le C E_0 (\psi_1 + \psi_2)(x, t) \\ \int_0^t H_x(x, t-s; 0) h(s)\, ds &\le C E_0 (\psi_1 + \psi_2)(x, t), \end{aligned} \tag{2.243}$$

for $0 \le t \le +\infty$*, some* $C > 0$*.*

Proof. Note that $H(x, t; 0) \equiv 0$ for the outflow case $A_* < 0$. Consider the inflow case

$A_* > 0$ (and thus $\bar{a}_* > 0$). We have

$$\begin{aligned}
&\Big|\int_0^t H(x,t-s;0)h(s)\,ds\Big| \\
&\qquad\qquad = e^{-\eta_0 x/\bar{a}_*}|h(-\frac{1}{\bar{a}_*}(x-\bar{a}_* t))| \\
&\qquad\qquad \le e^{-\eta_0|x|}(1+|x-\bar{a}_* t|)^{-1} \le CE_0(\psi_1+\psi_2)(x,t), \\
&\Big|\int_0^t H_x(x,t-s;0)h(s)\,ds\Big| \\
&\qquad\qquad \le e^{-\eta_0 x/\bar{a}_*}\Big(|h|+|h'|\Big)(-\frac{1}{\bar{a}_*}(x-\bar{a}_* t))| \\
&\qquad\qquad \le e^{-\eta_0|x|}(1+|x-\bar{a}_* t|)^{-1} \le CE_0(\psi_1+\psi_2)(x,t),
\end{aligned}$$

which completes the proof of the lemma. □

Lemma 2.5.8 (Boundary estimates II). *Under the assumptions of Theorem 2.1.4, if* $|h(t)| \le E_0(1+t)^{-1-\epsilon}$ *and* $|h'(t)| \le E_0(1+t)^{-1}$,

$$\begin{aligned}
&\Big|\int_0^t \Big(\tilde{G}_y(x,t-s;0)Bh(s)+G(x,t-s;0)Ah(s)\Big)\,ds\Big| \\
&\qquad\qquad \le CE_0(\theta+\psi_1+\psi_2)(x,t) \\
&\Big|\int_0^t \Big(\tilde{G}_{xy}(x,t-s;0)Bh(s)+G_x(x,t-s;0)Ah(s)\Big)\,ds\Big| \\
&\qquad\qquad \le CE_0(\theta+\psi_1+\psi_2)(x,t)
\end{aligned} \tag{2.244}$$

for $0 \le t \le +\infty$, *some* $C > 0$.

Proof. We first give the estimate on $\int_0^{t-1}$, where $G_y(x,t-s;0)$ and $\tilde{G}_{xy}(x,t-s;0)$ are nonsingular. We have

$$\Big|\int_0^{t-1} \tilde{G}_y(x,t-s;0)Bh(s)\,ds\Big| \le C\int_1^t |\tilde{G}_y(x,\tau;0)|(1+t-\tau)^{-1-\epsilon}\,d\tau. \tag{2.245}$$

We shall estimate the integral for each term $(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}$, appearing in $\tilde{G}_y(x,\tau;0)$, and omit the $\mathcal{O}(e^{-\eta(x+t)})$ term, which is negligible. First, for $a_k < 0$, using $e^{-|x-a_k\tau|^2/M\tau} \le e^{-x^2/Mt}e^{-\eta\tau}$ for some $\eta > 0$, we have

$$\begin{aligned}\int_1^t &(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le e^{-x^2/Mt}\Big(\int_1^{t/2}+\int_{t/2}^t\Big)(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-\eta\tau}\,d\tau \\ &\le e^{-x^2/Mt}\Big((1+t)^{-1}+(1+t)^{-1/2}e^{-\eta t}\Big),\end{aligned} \tag{2.246}$$

which is clearly bounded by $C(\theta+\psi_1)(x,t)$. For $a_k > 0$, we consider three distinct regions depending on x and t. First for $x \ge a_k t$, we further divide the estimates into two cases: $(1, t/2)$ and $(t/2, t)$. For $\tau \in (1, t/2)$, we have $e^{-|x-a_k\tau|^2/M\tau} \le e^{-x^2/Mt}e^{-\eta\tau}$ for some $\eta > 0$ and thus as above the integral is bounded by $C(\theta+\psi_1)(x,t)$. For $\tau \in (t/2, t)$, we write $x - a_k\tau = x - a_k t + a_k(t-\tau)$ and thus

$$\begin{aligned}\int_{t/2}^t &(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le e^{-(x-a_kt)^2/Mt}\int_{t/2}^t(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon}e^{-a_k(t-\tau)^2/M\tau}\,d\tau \\ &\le C(1+t)^{-1/2}e^{-(x-a_kt)^2/Mt}\int_{t/2}^t(1+t-\tau)^{-1-\epsilon}\,d\tau \le C\theta(x,t).\end{aligned}$$

Next, consider the case: $x \le a_k t/2$. Divide the analysis into cases: $(1, 3t/4)$ and $(3t/4, t)$. For $\tau \in (1, 3t/4)$, use the change of variable $s := (x - a_k\tau)/\sqrt{\tau}$ to get

$$\begin{aligned}\int_1^{3t/4} &(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le (1+t)^{-1}\int_1^{3t/4}(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le (1+t)^{-1}\int_{-\infty}^{+\infty}e^{-s^2/M}\,ds \le (1+t)^{-1},\end{aligned} \tag{2.247}$$

which is bounded by $C\psi_1(x,t)$. For $\tau \in (3t/4, t)$, we have $e^{-|x-a_k\tau|^2/M\tau} \le e^{-\eta\tau}$ for some $\eta > 0$ and thus

$$\begin{aligned}\int_{3t/4}^{t} &(1+\tau)^{-1/2}(1+t-\tau)^{-1}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le (1+t)^{-1/2}\int_{3t/4}^{t} e^{-\eta\tau}\,d\tau \le C(1+t)^{-1/2}e^{-\eta t} \le C\theta(x,t).\end{aligned} \tag{2.248}$$

Finally, consider the case $x \in (a_k t/2, a_k t)$. We write $x = aa_k t$ with $a := \frac{x}{a_k t}$. We again divide the estimate into three regions: $(1, at), (at, \frac{1+a}{2}t)$, and $(\frac{1+a}{2}t, t)$. For $\tau \in (1, at)$, we have $(1+t-\tau)^{-1} \le C(1+t)^{-1} \le C\psi_1(x,t)$ and

$$\int_{1}^{at} (1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \le \int_0^{+\infty} e^{-s^2/M}\,ds \le C. \tag{2.249}$$

For $\tau \in (at, \frac{1+a}{2}t)$, we have $(1+t-\tau)^{-1} \le C(1+|x-a_k t|)^{-1}$ and by change of variable $s := (x-a_k\tau)/\tau$,

$$\begin{aligned}\int_{at}^{\frac{1+a}{2}t} &(1+\tau)^{-1/2}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le \int_0^{\frac{1-a}{1+a}} e^{-\tau^2/M}\,d\tau \le C(1-a) \le Ct^{-1}|x-a_k t|.\end{aligned} \tag{2.250}$$

Thus the integral is bounded by $Ct^{-1} \le C\psi_1(x,t)$. For $\tau \in (\frac{1+a}{2}t, t)$, we have $|x - a_k\tau| \ge |x - a_k\frac{1+a}{2}t| = \frac{a_k}{2}|1-a|t = \frac{|x-a_k t|}{2}$, and thus

$$\begin{aligned}\int_{\frac{1+a}{2}t}^{t} &(1+\tau)^{-1/2}(1+t-\tau)^{-1-\epsilon}e^{-|x-a_k\tau|^2/M\tau}\,d\tau \\ &\le (1+t)^{-1/2}e^{-|x-a_k t|^2/2Mt}\int_{\frac{1+a}{2}t}^{t}(1+t-\tau)^{-1-\epsilon}\,d\tau \\ &\le C(1+t)^{-1/2}e^{-|x-a_k t|^2/2Mt} \le C\theta(x,t).\end{aligned} \tag{2.251}$$

Therefore, combining all these estimates, we obtain

$$\Big|\int_0^{t-1} \tilde{G}_y(x, t-s; 0)Bh(s)\,ds\Big| \le C(\theta + \psi_1)(x,t). \tag{2.252}$$

We also have similar estimates for G_{xy} on the nonsingular part $\int_0^{t-1}$.

Next, to bound the singular part $\int_{t-1}^{t}$, we integrate (2.232) in y from 0 to $+\infty$ to obtain

$$\tilde{G}_y B + GA = -\int_0^{+\infty} G(x, t-s; y) A_y \, dy + \int_0^{+\infty} G_s(x, t-s; y)\, dy. \tag{2.253}$$

Substituting in the lefthand side of (2.244), and integrating by parts in s, we obtain

$$\begin{aligned} \int_{t-1}^{t} (\tilde{G}_y B + GA) h(s)\, ds = & \int_0^1 \Big(\int_0^{+\infty} A_y(y) G(x, \tau; y)\, dy \Big) h(t-\tau)\, d\tau \\ & - \int_0^1 \Big(\int_0^{+\infty} G(x, \tau; y)\, dy \Big) h'(t-\tau)\, d\tau \\ & + \Big(\int_0^{+\infty} G(x, 1; y)\, dy \Big) h(t-1), \end{aligned} \tag{2.254}$$

which by $\int |G| dy \le C$ is bounded by $\max_{0\le\tau\le 1}(|h| + |h'|)(t-\tau)$.

Combining this with the following more straightforward estimate (for large x, $|x| > a_n^+ t$)

$$\begin{aligned} \Big| \int_{t-1}^{t} \tilde{G}_y(x, t-s; 0) B h(s)\, ds \Big| & \le \int_0^1 |\tilde{G}_y(x, \tau; 0)| B h(t-\tau)\, d\tau \\ & \le C \max_{0\le\tau\le1} |h(t-\tau)| \int_0^1 \tau^{-1/2} e^{-|x|^2/C\tau}\, d\tau \\ & \le C \max_{0\le\tau\le1} |h(t-\tau)| \int_0^1 \tau^{-1} e^{-|x|^2/C\tau}\, d\tau \\ & = C|x|^{-2} \max_{0\le\tau\le1} |h(t-\tau)| \\ & \quad \times \int_0^1 (|x|^2/\tau) e^{-|x|^2/C\tau}\, d\tau \\ & \le C \max_{0\le\tau\le1} |h(t-\tau)| |x|^{-2}, \end{aligned} \tag{2.255}$$

$$\begin{aligned} \Big| \int_{t-1}^{t} \tilde{G}(x, t-s; 0) A h(s)\, ds \Big| & \le \int_0^1 |\tilde{G}(x, \tau; 0)| A h(t-\tau)\, d\tau \\ & \le C \max_{0\le\tau\le1} |h(t-\tau)| \int_0^1 \tau^{-1/2} e^{-|x|^2/C\tau}\, d\tau \\ & \le C \max_{0\le\tau\le1} |h(t-\tau)| |x|^{-2}, \end{aligned} \tag{2.256}$$

and the estimate (2.243) for H term (thus together with (2.256) for $G = \tilde{G} + H$), we find that the contribution from $\int_{t-1}^{t}$ has norm bounded by

$$\max_{0\le\tau\le 1}(|h|+|h'|)(t-\tau)(1+|x|)^{-2}\le CE_0(\psi_1+\psi_2)(x,t).$$

Combining this estimate with the one for $\int_0^{t-1}$, we obtain the first inequality in (2.244). For second inequality, we first differentiate (2.254) with respect to x to get

$$\begin{aligned}\int_{t-1}^{t}(\tilde G_{xy}B+G_xA)h(s)\,ds=&\int_0^1\Big(\int_0^{+\infty}A_y(y)G_x(x,\tau;y)\,dy\Big)h(t-\tau)\,d\tau\\&-\int_0^1\Big(\int_0^{+\infty}G_x(x,\tau;y)\,dy\Big)h'(t-\tau)\,d\tau\\&+\Big(\int_0^{+\infty}G_x(x,1;y)\,dy\Big)h(t-1),\end{aligned}\tag{2.257}$$

which, by $\int_0^1\int|G_x|dyd\tau\le C\int_0^1\tau^{-1/2}d\tau\le C$, is bounded by $\max_{0\le\tau\le 1}(|h|+|h'|)(t-\tau)$, similarly as above.

For the large x, clearly we still have similar estimates as (2.255) and (2.256) for $\tilde G_{xy}$ and $\tilde G_x$. These, estimate (2.243) for H_x, and (2.257) yield the contribution from $\int_{t-1}^{t}$ as above, which together with the estimate for $\int_0^{t-1}$ completes the proof of (2.244). □

2.5.3 Linearized stability

In this subsection, we shall give the proof of Theorem 2.1.3. We first need the following estimates:

Lemma 2.5.9 ([37]). *Under the assumptions of Theorem 2.1.3,*

$$\begin{aligned}\Big|\int_0^{+\infty}\tilde G(\cdot,t;y)f(y)\,dy\Big|_{L^p}&\le C(1+t)^{-\frac12(1-1/r)}|f|_{L^q},\\\Big|\int_0^{+\infty}H(\cdot,t;y)f(y)\,dy\Big|_{L^p}&\le Ce^{-\eta t}|f|_{L^p},\end{aligned}\tag{2.258}$$

for all $t\ge 0$, some C, $\eta>0$, for any $1\le q\le p$ and $f\in L^q\cap L^p$, where $1/r+1/q=1+1/p$.

Lemma 2.5.10. *Under the assumptions of Theorem 2.1.3, if* $|h(t)| \le E_0(1+t)^{-1-\epsilon}$*,*

$$\Big| \int_0^t \Big(\tilde{G}_y(x,t-s;0)Bh(s) + G(x,t-s;0)Ah(s) \Big)\, ds \Big|_{L^p} \le CE_0(1+t)^{-\frac{1}{2}(1-1/p)} \tag{2.259}$$

for $0 \le t \le +\infty$*, some* $C > 0$*.*

Proof. This follows at once by the boundary estimate (2.244) and the fact that $|(\theta + \psi_1 + \psi_2)(\cdot, t)|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}$. □

Proof of Theorem 2.1.3. Sufficiency of (D1) for linearized stability (the main point here) follows easily by applying the above lemmas to the following representation for solution $U(x,t)$ of the linearized equations (2.7)

$$\begin{aligned} U(x,t) = & \int_0^\infty G(x,t;y)U_0(y)\, dy \\ & + \int_0^t \Big(\tilde{G}_y(x,t-s;0)BU(0,s) + G(x,t-s;0)AU(0,s) \Big)\, ds \end{aligned}$$

where $U(y,0) = U_0(y)$ and $|U(0,s)| \le C|h(s)| \le C(1+s)^{-1-\epsilon}$ by (2.8) in the inflow case, and $|BU(0,s)| \le C|h(s)| \le C(1+s)^{-1-\epsilon}$ by (2.9) in the outflow case, noting that $G(x,t;0) \equiv 0$ in this case. Necessity follows by a much simpler argument, restricting x, y to a bounded set and letting $t \to \infty$, noting that G is given by the ODE evolution of the spectral projection onto the finite set of zeros of D in $\Re\lambda \ge 0$, necessarily nondecaying, plus an $O(e^{-\eta t})$ error, $\eta > 0$, from which we find that asymptotic decay implies nonexistence of any such zeros; see Proposition 7.7 and Corollary 7.8, [36] for details. □

2.5.4 Nonlinear argument

In this subsection, we shall give the proof of Theorem 2.1.4. In fact, with the above preparations, the proof of nonlinear stability is also straightforward.

Lemma 2.5.11 (H^4 local theory)**.** *Under the hypotheses of Theorem 2.1.4, then, for* T *sufficiently small depending on the* H^4*−norm of* U_0*, there exists a unique solution* $U(x,t) \in L^\infty(0,T;H^4(x))$ *of* (2.225) *satisfying*

$$|U(t)|_{H^4} \le C|U_0|_{H^4} \tag{2.260}$$

for all $0 \le t \le T$*.*

Proof. Short-time existence, uniqueness, and stability are described in [58, 57], using a standard (bounded high norm, contractive low norm) contraction mapping argument. We omit the details. □

Lemma 2.5.12. *Under the hypotheses of Theorem 2.1.4, let* $U \in L^\infty(0,T;H^4(x))$ *satisfy* (2.225) *on* $[0,T]$*, and define*

$$\zeta(t) := \sup_{x, 0\le s\le t} \Big[(|U| + |U_x|)(\theta + \psi_1 + \psi_2)^{-1}(x,t)\Big]. \tag{2.261}$$

If $\zeta(T)$ *and* $|U_0|_{H^4}$ *are bounded by* ζ_0 *sufficiently small, then, for some* $\epsilon > 0$*, (i) the solution* U*, and thus* ζ *extends to* $[0, T+\epsilon]$*, and (ii)* ζ *is bounded and continuous on* $[0, T+\epsilon]$*.*

Proof. Boundedness and smallness of $|U(t)|_{H^4}$ on $[0,T]$ follow by Proposition 2.4.1, provided smallness of $\zeta(T)$ and $|U_0|_{H^4}$. By Lemma 2.5.11, this implies the existence, boundedness of $|U(t)|_{H^4}$ on $[0, T+\epsilon]$, for some $\epsilon > 0$, and thus, by Lemma 2.4.3, boundedness and continuity of ζ on $[0, T+\epsilon]$. □

Proof of Theorem 2.1.4. We shall establish:

Claim. For all $t \ge 0$ for which a solution exists with ζ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \le C_2(E_0 + \zeta(t)^2). \tag{2.262}$$

From this result, provided $E_0 < 1/4C_2^2$, we have that by continuous induction

$$\zeta(t) < 2C_2E_0 \tag{2.263}$$

for all $t \ge 0$. From (2.263) and the definition of ζ in (2.261) we then obtain the bounds of (2.18). Thus, it remains only to establish the claim above.

Proof of Claim. We must show that $(|U| + |U_x|)(\theta + \psi_1 + \psi_2)^{-1}$ is bounded by $C(E_0 + \zeta(t)^2)$, for some $C > 0$, all $0 \le s \le t$, so long as ζ remains sufficiently small. First we need an estimate for $U(0,s)$ and $U_s(0,s)$. For the inflow case, by boundary condition estimate (2.228) and by the hypotheses on $h(s)$, we have

$$|U(0,s)| \le C(h(s) + |U(0,s)|^2) \le C(E_0(1+s)^{-1-\epsilon} + |U(0,s)|^2) \tag{2.264}$$

from which by continuity of $|U(0,t)|$ (Remark 2.4.4) and smallness of E_0, we obtain a similar estimate to (2.263):

$$|U(0,s)| \le CE_0(1+s)^{-1-\epsilon}. \tag{2.265}$$

Similarly for an estimate of $U_t(0,t)$, by taking the derivative of (2.228), we get

$$\begin{aligned} |U_s(0,s)| &\le C(h'(s) + |U||U_s|(0,s)) \\ &\le C(E_0(1+s)^{-1} + |U(0,s)||U_s(0,s)|) \\ &\le C(E_0(1+s)^{-1} + |U_s(0,s)|^2) \end{aligned} \tag{2.266}$$

which by the same argument as above yields

$$|U_s(0,s)| \le CE_0(1+s)^{-1}. \tag{2.267}$$

Next, for the outflow case with boundary condition (2.229), we have

$$\begin{aligned} |BU(0,s)| &\le CE_0(1+s)^{-1-\epsilon} + \mathcal{O}(|U(0,s)|^2) \\ |(BU)_s(0,s)| &\le CE_0(1+s)^{-1} + \mathcal{O}(|U||U_s|(0,s)). \end{aligned} \tag{2.268}$$

Now by (2.261), we have for all $t \ge 0$ and some $C > 0$ that

$$|U(x,t)| + |U_x(x,t)| \le \zeta(t)(\theta + \psi_1 + \psi_2)(x,t), \tag{2.269}$$

and therefore

$$\begin{aligned} |Q(U,U_y)(y,s)| &\le C\zeta(t)^2\Psi(y,s) \\ |\Pi_1 Q(U,U_y)_y(y,s)| &\le C\zeta(t)^2\Psi(y,s) \end{aligned} \tag{2.270}$$

with $\Psi = (\theta + \psi_1 + \psi_2)^2$ as defined in (2.240), for $0 \le s \le t$.

As an estimate for $U(x,t)$, we use the representation (2.230) of $U(x,t)$:

$$
\begin{aligned}
|U(x,t)| = &\Big|\int_0^\infty G(x,t;y)U_0(y)\,dy\Big| \\
&+\Big|\int_0^t (\tilde{G}_y(x,t-s;0)BU(0,s)+G(x,t-s;0)AU(0,s))\,ds\Big| \\
&+\Big|\int_0^t\int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds\Big| \\
&+\Big|\int_0^t\int_0^\infty \tilde{G}_y(x,t-s;y)\Pi_2 Q(U,U_y)(y,s)\,dy\,ds\Big|,
\end{aligned}
$$

where by applying Lemmas 2.5.3-2.5.6 together with (2.270), we have

$$
\begin{aligned}
&\Big|\int_0^\infty G(x,t;y)g(y)\,dy\Big| \\
&\qquad\le E_0\int_0^\infty (|\tilde{G}(x,t;y)|+|H(x,t;y)|)(1+|y|)^{-3/2}\,dy \\
&\qquad\le CE_0(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{2.271}
$$

$$
\begin{aligned}
&\Big|\int_0^t\int_0^\infty \tilde{G}_y(x,t-s;y)Q(U,U_y)(y,s)\,dy\,ds\Big| \\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty |\tilde{G}_y(x,t-s;y)|\Psi(y,s)\,dy\,ds \\
&\qquad\le C\zeta(t)^2(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{2.272}
$$

$$
\begin{aligned}
&\Big|\int_0^t\int_0^\infty H(x,t-s;y)\Pi_1 Q(U,U_y)_y(y,s)\,dy\,ds\Big| \\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty H(x,t-s;y)(\theta+\psi_1+\psi_2)^2\,dy\,ds \\
&\qquad\le C\zeta(t)^2\int_0^t\int_0^\infty H(x,t-s;y)\Upsilon(y,s)\,dy\,ds \\
&\qquad\le C\zeta(t)^2(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{2.273}
$$

and, for the boundary term, we apply the estimate (2.265) and Lemma 2.5.8, yielding

$$
\begin{aligned}
&\Big|\int_0^t (\tilde{G}_y(x,t-s;0)BU(0,s)+G(x,t-s;0)AU(0,s))\,ds\Big| \\
&\qquad\le C(E_0+\zeta(t)^2)(\theta+\psi_1+\psi_2)(x,t)
\end{aligned}
\tag{2.274}
$$

for the inflow. Whereas, for the outflow case, noting that $G(x, t-s; 0) \equiv 0$ in the outflow case, we apply the estimate (2.268), (2.269) and Lemma 2.5.8 to give the same estimate as above, yielding

$$\Big| \int_0^t \tilde{G}_y(x, t-s; 0) BU(0, s)\, ds \Big| \le C(E_0 + \zeta(t)^2)(\theta + \psi_1 + \psi_2)(x, t)$$

where we used (2.269) for $|U(0, s)| \le \zeta(t)(1+s)^{-1}$ and thus by (2.268), $|BU(0, s)| \le C(E_0 + \zeta(t)^2)(1+s)^{-1-\epsilon}$.

Therefore, combining the above estimates, we obtain

$$|U(x, t)|(\theta + \psi_1 + \psi_2)^{-1}(x, t) \le C(E_0 + \zeta(t)^2). \tag{2.275}$$

To derive the same estimate for $|U_x(x, t)|$, we first obtain by using Proposition 2.4.1,

$$\begin{aligned} |U(t)|_{H^4}^2 &\le Ce^{-\theta t}|U_0|_{H^4}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big[|U(\tau)|_{L^2}^2 + \mathcal{B}_h(\tau)^2\Big] d\tau \\ &\le C(E_0 + \zeta(t)^2)t^{-1/2}, \end{aligned}$$

where $\mathcal{B}_h$ is the boundary function defined in Proposition 2.4.1, and thus by the one dimensional Sobolev embedding: $|U(t)|_{W^{3,\infty}} \le C|U(t)|_{H^4}$,

$$\begin{aligned} |Q(U, U_x)_x| &\le C(\zeta^2(t) + 4C^2E_0^2)\Upsilon \\ |Q(U, U_x)_{xx}| &\le C(\zeta^2(t) + 4C^2E_0^2)\Upsilon \end{aligned} \tag{2.276}$$

where $\Upsilon = t^{-1/4}(\theta + \psi_1 + \psi_2)$.

Now again applying Lemmas 2.5.3-2.5.8 together with (2.276), (2.267), and (2.268), we have obtained the desired estimate, that is, bounded by $(\zeta^2(t) + CE_0)(\theta + \psi_1 + \psi_2)(x, t)$, for most terms in the formulation (2.236) of $U_x(x, t)$, except one boundary term:

$$\int_0^t H(x, t-s; 0)|\Pi_1 Q(U, U_y)_y(0, s)|\, ds,$$

which is bounded by $CE_0(\psi_1 + \psi_2)(x, t)$ by using (2.227), (2.269), and Lemma 2.5.7, and noting that

$$|\Pi_1 Q(U, U_y)_y(0,s)| \le \zeta(t)|h(s)|(\theta + \psi_1 + \psi_2)(0,s) \le C\zeta(t)|h(s)|.$$

Therefore, together with (2.275), we have obtained

$$(|U(x,t)| + |U_x(x,t)|)(\theta + \psi_1 + \psi_2)^{-1}(x,t) \le C(E_0 + \zeta(t)^2) \tag{2.277}$$

as claimed, which completes the proof of Theorem 2.1.4. □

Chapter 3

MULTI–DIMENSIONAL STABILITY

3.1 Introduction

In this chapter, we study the multi–dimensional stability of a noncharacteristic boundary layer, or stationary solution,

$$\tilde{U} = \bar{U}(x_1), \quad \lim_{z\to+\infty} \bar{U}(z) = U_+, \quad \bar{U}(0) = \bar{U}_0 \tag{3.1}$$

of a system of conservation laws on the quarter-space

$$\tilde{U}_t + \sum_j F^j(\tilde{U})_{x_j} = \sum_{jk} (B^{jk}(\tilde{U})\tilde{U}_{x_k})_{x_j}, \quad x \in \mathbb{R}^d_+ = \{x_1 > 0\}, \quad t > 0, \tag{3.2}$$

$\tilde{U}, F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n\times n}$, with initial data $\tilde{U}(x,0) = \tilde{U}_0(x)$ and Dirichlet type boundary conditions specified in (3.5), (3.6) below.

3.1.1 Equations and assumptions

We consider the general hyperbolic-parabolic system of conservation laws (3.2) in conserved variable $\tilde{U}$, with

$$\tilde{U} = \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},$$

$\tilde u \in \mathbb{R}^{n-r}$, and $\tilde v \in \mathbb{R}^r$, where

$$\Re\sigma \sum_{jk} b_2^{jk}\xi_j\xi_k \ge \theta|\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^n\backslash\{0\}.$$

Following [37, 59, 57], we assume that equations (3.2) can be written, alternatively, after a triangular change of coordinates

$$\tilde W := \tilde W(\tilde U) = \begin{pmatrix} \tilde w^I(\tilde u) \\ \tilde w^{II}(\tilde u, \tilde v) \end{pmatrix}, \tag{3.3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde A^0 \tilde W_t + \sum_j \tilde A^j \tilde W_{x_j} = \sum_{jk} (\tilde B^{jk} \tilde W_{x_k})_{x_j} + \tilde G, \tag{3.4}$$

where

$$\tilde A^0 = \begin{pmatrix} \tilde A^0_{11} & 0 \\ 0 & \tilde A^0_{22} \end{pmatrix}, \qquad \tilde A^j = \begin{pmatrix} \tilde A^j_{11} & \tilde A^j_{12} \\ \tilde A^j_{21} & \tilde A^j_{22} \end{pmatrix}, \qquad \tilde B^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde b^{jk} \end{pmatrix}, \quad \tilde G = \begin{pmatrix} 0 \\ \tilde g \end{pmatrix}$$

and, defining $\tilde W_\pm := \tilde W(U_\pm)$,

(A1) $\tilde A^j(\tilde W_+), \tilde A^0, \tilde A^1_{11}$ are symmetric, $\tilde A^0 \ge \theta_0 > 0$,

(A2) no eigenvector of $\sum_j \xi_j \tilde A^j (\tilde A^0)^{-1}(\tilde W_+)$ lies in the kernel of $\sum_{jk} \xi_j\xi_k \tilde B^{jk}(\tilde A^0)^{-1}(\tilde W_+)$, for each $\xi \in \mathbb{R}^d \setminus \{0\}$,

(A3) $\sum \tilde b^{jk}\xi_j\xi_k \ge \theta|\xi|^2$, $\theta > 0$, and $\tilde g(\tilde W_x, \tilde W_x) = \mathcal{O}(|\tilde W_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F^j, B^{jk}, \tilde A^0, \tilde A^j, \tilde B^{jk}, \tilde W(\cdot), \tilde g(\cdot,\cdot) \in C^{s+1}$, with $s \ge [(d-1)/2] + 4$ in our analysis of linearized stability, and $s \ge s(d) := [(d-1)/2] + 7$ in our analysis of nonlinear stability.

(H1) $\tilde A^{11}_1$ is either strictly positive or strictly negative, that is, either $\tilde A^{11}_1 \ge \theta_1 > 0$, or $\tilde A^{11}_1 \le -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF^1(U_+)$ are distinct and nonzero.

(H3) The eigenvalues of $\sum_j dF^j_+ \xi_j$ have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

(H4) The set of branch points of the eigenvalues of $(\tilde{A}^1)^{-1}(i\tau\tilde{A}^0 + \sum_{j\neq 1} i\xi_j \tilde{A}^j)_+$, $\tau \in \mathbb{R}$, $\tilde{\xi} \in \mathbb{R}^{d-1}$ is the (possibly intersecting) union of finitely many smooth curves $\tau = \eta^+_q(\tilde{\xi})$, on which the branching eigenvalue has constant multiplicity s_q (by definition ≥ 2).

Condition (H1) corresponds to hyperbolic–parabolic noncharacteristicity, while (H2) is the condition for the hyperbolicity at U_+ of the associated first-order hyperbolic system obtained by dropping second-order terms. The assumptions (A1)-(A3) and (H0)-(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [37, 9, 19, 18]. Condition (H3) holds always for gas dynamics, but fails always for MHD in dimension $d \geq 2$. Condition (H4) is a technical requirement of the analysis introduced in [58]. It is satisfied always in dimension $d = 2$ or for rotationally invariant systems in dimensions $d \geq 2$, for which it serves only to define notation; in particular, it holds always for gas dynamics.

We also assume:

(B) Dirichlet boundary conditions in $\tilde{W}$-coordinates:

$$(\tilde{w}^I, \tilde{w}^{II})(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) := (\tilde{h}_1, \tilde{h}_2)(\tilde{x}, t) \tag{3.5}$$

for the inflow case, and

$$\tilde{w}^{II}(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) \tag{3.6}$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

This is sufficient for the main physical applications; the situation of more general, Neumann and mixed-type boundary conditions on the parabolic variable v can be treated as discussed in [19, 18].

3.1.2 The Evans condition and strong spectral stability

The linearized equations of (3.2), (B) about $\bar{U}$ are

$$U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j} \tag{3.7}$$

with initial data $U(0) = U_0$ and boundary conditions in (linearized) $\tilde{W}$-coordinates of

$$W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$, where $W := (\partial\tilde{W}/\partial U)(\bar{U})U$.

A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re\lambda > 0$ of the linearized operator L about the wave. As described in Section 3.2.1, this is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re\lambda > 0$ of the *Evans function*

$$D_L(\tilde{\xi}, \lambda)$$

(defined in (3.21)), a Wronskian associated with the Fourier-transformed eigenvalue ODE.

Definition 3.1.1. We define *strong spectral stability* as *uniform Evans stability*:

$$|D_L(\tilde{\xi}, \lambda)| \geq \theta(C) > 0 \tag{D2}$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$.

For the class of equations we consider, this is equivalent to the uniform Evans condition of [19, 18], which includes an additional high-frequency condition that for these equations is always satisfied (see Proposition 3.8, [19]). A fundamental result proved in [19] is that small-amplitude noncharacteristic boundary-layers are always strongly spectrally stable.[1]

Proposition 3.1.2 ([19])**.** *Assuming (A1)-(A3), (H0)-(H3), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small satisfy the strong spectral stability condition (D2).

As demonstrated in [54], stability of large-amplitude boundary layers may fail for the class of equations considered here, even in a single space dimension, so there is

[1] The result of [19] applies also to more general types of boundary conditions and in some situations to systems with variable multiplicity characteristics, including, in some parameter ranges, MHD.

no such general theorem in the large-amplitude case. Stability of large-amplitude boundary-layers may be checked efficiently by numerical Evans computations as in [5, 6, 7, 8, 28, 3, 24, 9, 26, 25].

3.1.3 Multi-dimensional results I

Our main results are as follows.

Theorem 3.1.3 (Linearized stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D2), we obtain asymptotic* $L^1\cap H^{[(d-1)/2]+5}\to L^p$ *stability of* (3.7) *in dimension* $d\geq 2$*, and any* $2\leq p\leq\infty$*, with rate of decay*

$$\begin{aligned} |U(t)|_{L^2} &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1\cap H^3}+E_0),\\ |U(t)|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}(|U_0|_{L^1\cap H^{[(d-1)/2]+5}}+E_0), \end{aligned} \tag{3.8}$$

provided that the initial perturbations U_0 *are in* $L^1\cap H^3$ *for* $p=2$*, or in* $L^1\cap H^{[(d-1)/2]+5}$ *for* $p>2$*, and boundary perturbations* h *satisfy*

$$\begin{aligned} |h(t)|_{L^2_{\tilde x}} &\leq E_0(1+t)^{-(d+1)/4},\\ |h(t)|_{L^\infty_{\tilde x}} &\leq E_0(1+t)^{-d/2}\\ |\mathcal{D}_h(t)|_{L^1_{\tilde x}\cap H^{[(d-1)/2]+5}_{\tilde x}} &\leq E_0(1+t)^{-d/2-\epsilon}, \end{aligned} \tag{3.9}$$

where $\mathcal{D}_h(t):=|h_t|+|h_{\tilde x}|+|h_{\tilde x\tilde x}|$*,* E_0 *is some positive constant, and* $\epsilon>0$ *is arbitrary small for the case* $d=2$ *and* $\epsilon=0$ *for* $d\geq 3$*.*

Theorem 3.1.4 (Nonlinear stability). *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D2), we obtain asymptotic* $L^1\cap H^s\to L^p\cap H^s$ *stability of* $\bar U$ *as a solution of* (3.2) *in dimension* $d\geq 2$*, for* $s\geq s(d)$ *as defined in (H0), and any* $2\leq p\leq\infty$*, with rate of decay*

$$\begin{aligned} |\tilde U(t)-\bar U|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}(|U_0|_{L^1\cap H^s}+E_0)\\ |\tilde U(t)-\bar U|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1\cap H^s}+E_0), \end{aligned} \tag{3.10}$$

provided that the initial perturbations $U_0:=\tilde U_0-\bar U$ *are sufficiently small in* $L^1\cap H^s$ *and boundary perturbations* $h(t):=\tilde h(t)-W(\bar U_0)$ *satisfy* (3.9) *and*

$$\mathcal{B}_h(t)\leq E_0(1+t)^{-\frac{d-1}{4}}, \tag{3.11}$$

with sufficiently small E_0, where the boundary measure $\mathcal{B}_h$ is defined as

$$\mathcal{B}_h(t) := |h|_{H^s(\tilde{x})} + \sum_{i=0}^{[(s+1)/2]} |\partial_t^i h|_{L^2(\tilde{x})} \tag{3.12}$$

for the outflow case, and similarly

$$\mathcal{B}_h(t) := |h|_{H^s(\tilde{x})} + \sum_{i=0}^{[(s+1)/2]} |\partial_t^i h_2|_{L^2(\tilde{x})} + \sum_{i=0}^{s} |\partial_t^i h_1|_{L^2(\tilde{x})} \tag{3.13}$$

for the inflow case.

Combining Theorem 3.1.4 and Proposition 3.1.2, we obtain the following small-amplitude stability result, applying in particular to the motivating situation of Example 1.2.1.

Corollary 3.1.5. *Assuming (A1)-(A3), (H0)-(H4), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 3.1.3 and 3.1.4.

Remark 3.1.6. The obtained rate of decay in L^2 may be recognized as that of a $(d-1)$-dimensional heat kernel, and the obtained rate of decay in L^∞ as that of a d-dimensional heat kernel. We believe that the sharp rate of decay in L^2 is rather that of a d-dimensional heat kernel and the sharp rate of decay in L^∞ dependent on the characteristic structure of the associated inviscid equations, as in the constant-coefficient case [20, 21].

Remark 3.1.7. In one dimension, strong spectral stability is necessary for linearized asymptotic stability; see Theorem 2.1.3. However, in multi-dimensions, it appears likely that, as in the shock case [59], there are intermediate possibilities between strong and weak spectral stability for which linearized stability might hold with degraded rates of decay. In any case, the gap between the necessary weak spectral and the sufficient strong spectral stability conditions concerns only pure imaginary spectra

$\Re\lambda = 0$ on the boundary between strictly stable and unstable half-planes, so this should not interfere with investigation of physical stability regions.

3.1.4 Discussion

The large-amplitude asymptotic stability result of Theorem 3.1.4 extends to multi dimensions corresponding one-dimensional results of [56, 45] or Theorem 2.1.3, reducing the problem of stability to verification of a numerically checkable Evans condition. See also the related, but technically rather different, work on the small viscosity limit in [42, 19, 18]. By a combination of numerical Evans function computations and asymptotic ODE estimates, spectral stability has been checked for *arbitrary amplitude* noncharacteristic boundary layers of the one-dimensional isentropic compressible Navier–Stokes equations in [9]. Extensions to the nonisentropic and multi-dimensional case should be possible by the methods used in [26] and [25] respectively to treat the related shock stability problem.

This (investigation of large-amplitude spectral stability) would be a very interesting direction for further investigation. In particular, note that it is large-amplitude stability that is relevant to drag-reduction at flight speeds, since the transverse relative velocity (i.e., velocity parallel to the airfoil) is zero at the wing surface and flight speed outside a thin boundary layer, so that variation across the boundary layer is substantial. We discuss this problem further in Appendix B.1 for the model isentropic case.

Our method of analysis follows the basic approach introduced in [58, 59, 57] for the study of multi-dimensional shock stability and we are able to make use of much of that analysis without modification. However, there are some new difficulties to be overcome in the boundary-layer case.

The main new difficulty is that the boundary-layer case is analogous to the *undercompressive shock* case rather than the more favorable *Lax shock* case emphasized in [59], in that $G_{y_1} \not\sim t^{-1/2}G$ as in the Lax shock case but rather $G_{y_1} \sim (e^{-\theta|y_1|} + t^{-1/2})G$, $\theta > 0$, as in the undercompressive case. This is a significant difficulty; indeed, for this reason, the undercompressive shock analysis was carried out in [59] only in nonphysical dimensions $d \geq 4$. On the other hand, there is no translational invariance in the boundary layer problem, so no zero-eigenvalue and no pole of the resolvent kernel at the origin for the one-dimensional operator, and in this sense G is somewhat better in the boundary layer than in the shock case.

Thus, the difficulty of the present problem is roughly intermediate to that of the Lax and undercompressive shock cases. Though the undercompressive shock case is still open in multi-dimensions for $d \leq 3$, the slight advantage afforded by lack of pole terms allows us to close the argument in the boundary-layer case. Specifically, thanks to the absence of pole terms, we are able to get a slightly improved rate of decay in $L^\infty(x_1)$ norms, though our $L^2(x_1)$ estimates remain the same as in the shock case. By keeping track of these improved sup norm bounds throughout the proof, we are able to close the argument without using detailed pointwise bounds as in the one-dimensional analyses of [23, 50].

Other difficulties include the appearance of boundary terms in integrations by parts, which makes the auxiliary energy estimates by which we control high-frequency effects considerably more difficult in the boundary-layer than in the shock-layer case, and the treatment of boundary perturbations. In terms of the homogeneous Green function G, boundary perturbations lead by a standard duality argument to contributions consisting of integrals on the boundary of perturbations against various derivatives of G, and these are a bit too singular as time goes to zero to be absolutely integrable. Following the strategy introduced in [56, 45], we instead use duality to convert these to less singular integrals over the whole space, that *are* absolutely integrable in time. However, we make a key improvement here over the treatment in [56, 45], integrating against an exponentially decaying test function to obtain terms of exactly the same form already treated for the homogeneous problem. This is necessary for us in the multi-dimensional case, for which we have insufficient information about individual parts of the solution operator to estimate them separately as in [56, 45], but makes things much more transparent also in the one-dimensional case.

Among physical systems, our hypotheses here appear to apply to and essentially only to the case of compressible Navier–Stokes equations with inflow or outflow boundary conditions. In Chapter 4, we will establish an extension to more general situations such as MHD equations by a rather different technique: the method of Kreiss' symmetrizers. In addition, the above technical assumption (H4) which appears to be crucial in current analyses can be dropped in our analysis of Chapter 4. See also [19, 18] for the recent results on the related small-viscosity problem.

Finally, as pointed out in Remark 3.1.7, the strong spectral stability condition does not appear to be necessary for asymptotic stability. It would be interesting to develop a refined stability condition similarly as was done in [54, 58, 59, 57] for the shock case.

3.2 Resolvent kernel: construction and low-frequency bounds

In this section, we briefly recal the construction of resolvent kernel and then establish the pointwise low-frequency bounds on $G_{\tilde{\xi},\lambda}$, by appropriately modifying the proof in [59] in the boundary layer context [56, 45] or Chapter 2.

3.2.1 Construction

We construct a representation for the family of elliptic Green distributions $G_{\tilde{\xi},\lambda}(x_1, y_1)$,

$$G_{\tilde{\xi},\lambda}(\cdot, y_1) := (L_{\tilde{\xi}} - \lambda)^{-1}\delta_{y_1}(\cdot), \tag{3.14}$$

associated with the ordinary differential operators $(L_{\tilde{\xi}} - \lambda)$, i.e. the resolvent kernel of the Fourier transform $L_{\tilde{\xi}}$ of the linearized operator L of (3.7). To do so, we study the homogeneous eigenvalue equation $(L_{\tilde{\xi}} - \lambda)U = 0$, or

$$\begin{aligned}\overbrace{(B^{11}U')' - (A^1U)'}^{L_0 U} - i\sum_{j\neq 1} A^j\xi_j U + i\sum_{j\neq 1} B^{j1}\xi_j U' \\ + i\sum_{k\neq 1}(B^{1k}\xi_k U)' - \sum_{j,k\neq 1} B^{jk}\xi_j\xi_k U - \lambda U = 0,\end{aligned} \tag{3.15}$$

with boundary conditions (translated from those in W-coordinates)

$$\begin{pmatrix} A^1_{11} - A^1_{12}(b^{11}_2)^{-1}b^{11}_1 & 0 \\ b^{11}_1 & b^{11}_2 \end{pmatrix} U(0) \equiv \begin{pmatrix} * \\ 0 \end{pmatrix} \tag{3.16}$$

where $* = 0$ for the inflow case and is arbitrary for the outflow case.

Define

$$\Lambda^{\tilde{\xi}} := \bigcap_{j=1}^{n} \Lambda^+_j(\tilde{\xi})$$

where $\Lambda^+_j(\tilde{\xi})$ denote the open sets bounded on the left by the algebraic curves $\lambda^+_j(\xi_1, \tilde{\xi})$

determined by the eigenvalues of the symbols $-\xi^2 B_+ - i\xi A_+$ of the limiting constant-coefficient operators

$$L_{\tilde{\xi}+} w := B_+ w'' - A_+ w'$$

as ξ_1 is varied along the real axis, with $\tilde{\xi}$ held fixed. The curves $\lambda_j^+(\cdot, \tilde{\xi})$ comprise the essential spectrum of operators $L_{\tilde{\xi}+}$. Let Λ denote the set of $(\tilde{\xi}, \lambda)$ such that $\lambda \in \Lambda^{\tilde{\xi}}$.

For $(\tilde{\xi}, \lambda) \in \Lambda^{\tilde{\xi}}$, introduce locally analytically chosen (in $\tilde{\xi}$, λ) matrices

$$\Phi^+ = (\phi_1^+, \cdots, \phi_k^+), \quad \Phi^0 = (\phi_{k+1}^0, \cdots, \phi_{n+r}^0), \tag{3.17}$$

and

$$\Phi = (\Phi^+, \Phi^0), \tag{3.18}$$

whose columns span the subspaces of solutions of (4.67) that, respectively, decay at $x = +\infty$ and satisfy the prescribed boundary conditions at $x = 0$, and locally analytically chosen matrices

$$\Psi^0 = (\psi_1^0, \cdots, \psi_k^0), \quad \Psi^+ = (\psi_{k+1}^+, \cdots, \psi_{n+r}^+) \tag{3.19}$$

and

$$\Psi = (\Psi^0, \Psi^+). \tag{3.20}$$

whose columns span complementary subspaces. The existence of such matrices is guaranteed by the general Evans function framework of [1, 14, 36]; see in particular [59, 45]. That dimensions sum to $n + r$ follows by a general result of [19]; see also [54].

The Evans function

Following [1, 14, 54], we define on Λ the *Evans function*

$$D_L(\tilde{\xi}, \lambda) := \det(\Phi^0, \Phi^+)_{|x=0}. \tag{3.21}$$

Evidently, eigenfunctions decaying at $+\infty$ and satisfying the prescribed boundary conditions at $x_1 = 0$ occur precisely when the subspaces $\operatorname{span}\Phi^0$ and $\operatorname{span}\Phi^+$ intersect, i.e., at zeros of the Evans function

$$D_L(\tilde{\xi}, \lambda) = 0.$$

The Evans function as constructed here is locally analytic in $(\tilde{\xi}, \lambda)$, which is all that we need for our analysis; we prescribe different versions of the Evans function as needed on different neighborhoods of Λ. Note that Λ includes all of $\{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$, so that Definition 3.1.1 is well-defined and equivalent to simple nonvanishing, away from the origin $(\tilde{\xi}, \lambda) = (0, 0)$. To make sense of this definition near the origin, we must insist that the matrices Φ^j in (3.21) remain *uniformly bounded*, a condition that can always be achieved by limiting the neighborhood of definition.

For the class of equations we consider, the Evans function may in fact be extended continuously along rays through the origin [51, 42, 19, 18].

Basic representation formulae

Define the solution operator from y_1 to x_1 of ODE $(L_{\tilde{\xi}} - \lambda)U = 0$, denoted by $\mathcal{F}^{y_1 \to x_1}$, as

$$\mathcal{F}^{y_1 \to x_1} = \Phi(x_1, \lambda)\Phi^{-1}(y_1, \lambda)$$

and the projections $\Pi^0_{y_1}, \Pi^+_{y_1}$ on the stable manifolds at $0, +\infty$ as

$$\Pi^+_{y_1} = \begin{pmatrix} \Phi^+(y_1) & 0 \end{pmatrix} \Phi^{-1}(y_1), \quad \Pi^0_{y_1} = \begin{pmatrix} 0 & \Phi^0(y_1) \end{pmatrix} \Phi^{-1}(y_1).$$

We define also the dual subspaces of solutions of $(L^*_{\tilde{\xi}} - \lambda^*)\tilde{W} = 0$. We denote growing solutions

$$\tilde{\Phi}^0 = (\tilde{\phi}^0_1, \cdots, \tilde{\phi}^0_k), \quad \tilde{\Phi}^+ = (\tilde{\phi}^+_{k+1}, \cdots, \tilde{\phi}^+_{n+r}), \tag{3.22}$$

$\tilde{\Phi} := (\tilde{\Phi}^0, \tilde{\Phi}^+)$ and decaying solutions

$$\tilde{\Psi}^0 = (\tilde{\psi}^0_1, \cdots, \tilde{\psi}^+_k), \quad \tilde{\Psi}^+ = (\tilde{\psi}^+_{k+1}, \cdots, \tilde{\psi}^+_{n+r}), \tag{3.23}$$

and $\tilde{\Psi} := (\tilde{\Psi}^0, \tilde{\Psi}^+)$, satisfying the relations

$$\begin{pmatrix} \tilde{\Psi} & \tilde{\Phi} \end{pmatrix}^*_{0,+} \bar{\mathcal{S}}^{\tilde{\xi}} \begin{pmatrix} \Psi & \Phi \end{pmatrix}_{0,+} \equiv I, \tag{3.24}$$

where

$$\bar{\mathcal{S}}^{\tilde\xi} = \begin{pmatrix} -A^1 + iB^{1\tilde\xi} + iB^{\tilde\xi 1} & \begin{pmatrix} 0 \\ I_r \end{pmatrix} \\ \begin{pmatrix} -(b_2^{11})^{-1} b_I^{11} & -I_r \end{pmatrix} & 0 \end{pmatrix}. \tag{3.25}$$

With these preparations, the construction of the Resolvent kernel goes exactly as in the construction performed in [60, 36, 59] on the whole line and [56, 45] on the half line, yielding the following basic representation formulae; for a proof, see [36, 45] or Lemma 2.2.13 and Proposition 2.2.14.

Proposition 3.2.1. *We have the following representation*

$$G_{\tilde\xi,\lambda}(x_1, y_1) = \begin{cases} (I_n, 0)\mathcal{F}^{y_1 \to x_1} \Pi^+_{y_1} (\bar{S}^{\tilde\xi})^{-1}(y_1)(I_n, 0)^{tr}, & for \quad x_1 > y_1, \\ -(I_n, 0)\mathcal{F}^{y_1 \to x_1} \Pi^0_{y_1} (\bar{S}^{\tilde\xi})^{-1}(y_1)(I_n, 0)^{tr}, & for \quad x_1 < y_1. \end{cases} \tag{3.26}$$

Proposition 3.2.2. *The resolvent kernel may alternatively be expressed as*

$$G_{\tilde\xi,\lambda}(x_1, y_1) = \begin{cases} (I_n, 0)\Phi^+(x_1;\lambda)M^+(\lambda)\tilde\Psi^{0*}(y_1;\lambda)(I_n, 0)^{tr} & x_1 > y_1, \\ -(I_n, 0)\Phi^0(x_1;\lambda)M^0(\lambda)\tilde\Psi^{+*}(y_1;\lambda)(I_n, 0)^{tr} & x_1 < y_1, \end{cases}$$

where

$$M(\lambda) := \operatorname{diag}(M^+(\lambda), M^0(\lambda)) = \Phi^{-1}(z;\lambda)(\bar{\mathcal{S}}^{\tilde\xi})^{-1}(z)\tilde\Psi^{-1*}(z;\lambda). \tag{3.27}$$

Scattering decomposition

From Propositions 3.2.1 and 3.2.2, we obtain the following scattering decomposition, generalizing the Fourier transform representation in the constant-coefficient case, from which we will obtain pointwise bounds in the low-frequency regime.

Corollary 3.2.3. *On $\Lambda^{\tilde\xi} \cap \rho(L_{\tilde\xi})$,*

$$G_{\tilde\xi,\lambda}(x_1, y_1) = \sum_{j,k} d^+_{jk}\phi^+_j(x_1;\lambda)\tilde\psi^+_k(y_1;\lambda)^* + \sum_k \phi^+_k(x_1;\lambda)\tilde\phi^+_k(y_1;\lambda)^* \tag{3.28}$$

for $0 \le y_1 \le x_1$, and

$$G_{\tilde\xi,\lambda}(x_1, y_1) = \sum_{j,k} d^0_{jk}\phi^+_j(x_1;\lambda)\tilde\psi^+_k(y_1;\lambda)^* - \sum_k \psi^+_k(x_1;\lambda)\tilde\psi^+_k(y_1;\lambda)^* \tag{3.29}$$

for $0 \le x_1 \le y_1$, *where*

$$d_{jk}^{0,+}(\lambda) = (I,0)\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\Psi^+. \tag{3.30}$$

Proof. For $0 \le x_1 \le y_1$, we obtain the preliminary representation

$$G_{\tilde\xi,\lambda}(x_1,y_1) = \sum_{j,k} d_{jk}^0(\lambda)\phi_j^+(x_1;\lambda)\tilde\psi_k^+(y_1;\lambda)^* + \sum_{jk} e_{jk}^0\psi_j^+(x_1;\lambda)\tilde\psi_k^+(y_1;\lambda)^*$$

from which, together with duality (3.24), representation (3.26), and the fact that $\Pi_0 = I - \Pi_+$, we have

$$\begin{aligned}\begin{pmatrix}d^0\\ e^0\end{pmatrix} &= -\begin{pmatrix}\tilde\Phi^+ & \tilde\Psi^+\end{pmatrix}^* A\Pi_0\Psi^+\\ &= -\begin{pmatrix}\Phi^+ & \Psi^+\end{pmatrix}^{-1}\left[I - \begin{pmatrix}\Phi^+ & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\right]\Psi^+\\ &= \begin{pmatrix}0\\ -I_k\end{pmatrix} + \begin{pmatrix}I_{n-k} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\Psi^+.\end{aligned} \tag{3.31}$$

Similarly, for $0 \le y_1 \le x_1$, we obtain the preliminary representation

$$G_{\tilde\xi,\lambda}(x_1,y_1) = \sum_{j,k} d_{jk}^+(\lambda)\phi_j^+(x_1;\lambda)\tilde\psi_k^+(y_1;\lambda)^* + \sum_{jk} e_{jk}^+\phi_j^+(x_1;\lambda)\tilde\phi_k^+(y_1;\lambda)^*$$

from which, together with duality (3.24) and representation (3.26), we have

$$\begin{aligned}\begin{pmatrix}d^+\\ e^+\end{pmatrix} &= \tilde\Phi^{+*}A\Pi_+\begin{pmatrix}\Psi^+ & \Phi^+\end{pmatrix}\\ &= (\Phi^+)^{-1}\begin{pmatrix}\Phi^+ & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\begin{pmatrix}\Psi^+ & \Phi^+\end{pmatrix}\\ &= \begin{pmatrix}I & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\begin{pmatrix}\Psi^+ & \Phi^+\end{pmatrix}\\ &= \begin{pmatrix}I_{n-k} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\Phi^+ & \Phi^0\end{pmatrix}^{-1}\Psi^+ + \begin{pmatrix}0 & 0\\ I_k & 0\end{pmatrix}\begin{pmatrix}0 & I_k\\ 0 & 0\end{pmatrix}.\end{aligned} \tag{3.32}$$

□

Remark 3.2.4. In the constant-coefficient case, with a choice of common bases

$\Psi^{0,+} = \Phi^{+,0}$ at $0, +\infty$, the above representation reduces to the simple formula

$$G_{\tilde{\xi},\lambda}(x_1, y_1) = \begin{cases} \sum_{j=k+1}^{N} \phi_j^+(x_1;\lambda)\tilde{\phi}_j^{+*}(y_1;\lambda) & x_1 > y_1, \\ -\sum_{j=1}^{k} \psi_j^+(x_1;\lambda)\tilde{\psi}_j^{+*}(y_1;\lambda) & x_1 < y_1. \end{cases} \tag{3.33}$$

3.2.2 Pointwise low-frequency bounds

We obtain pointwise low-frequency bounds on the resolvent kernel $G_{\tilde{\xi},\lambda}(x_1, y_1)$ by appealing to the detailed analysis of [58, 59, 17] in the viscous shock case. Restrict attention to the surface

$$\Gamma^{\tilde{\xi}} := \{\lambda \ : \ \Re e\lambda = -\theta_1(|\tilde{\xi}|^2 + |\Im m\lambda|^2)\}, \tag{3.34}$$

for $\theta_1 > 0$ sufficiently small.

Proposition 3.2.5 ([59])**.** *Under the hypotheses of Theorem 3.1.4, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, $\theta_1 > 0$, and $\theta > 0$ sufficiently small, there hold:*

$$|G_{\tilde{\xi},\lambda}(x_1, y_1)| \le C\gamma_2 e^{-\theta\rho^2|x_1-y_1|}. \tag{3.35}$$

and

$$|\partial_{y_1}^{\beta} G_{\tilde{\xi},\lambda}(x_1, y_1)| \le C\gamma_2(\rho^\beta + \beta e^{-\theta y_1})e^{-\theta\rho^2|x_1-y_1|} \tag{3.36}$$

where

$$\gamma_2 := 1 + \sum_j \Big[\rho^{-1}|\Im m\lambda - \eta_j^+(\tilde{\xi})| + \rho\Big]^{1/s_j - 1}, \tag{3.37}$$

and $s_j, \eta_j^+(\tilde{\xi})$ are as defined in (H4).

Proof. This follows by a simplified version of the analysis of [59], Section 5 in the viscous shock case, replacing Φ^-, Ψ^- with Φ^0, Ψ^0, omitting the refined derivative bounds of Lemmas 5.23 and 5.27 describing special properties of the Lax and overcompressive shock case (not relevant here), and setting $\ell = 0$, or $\tilde{\gamma} \equiv 1$ in definition (5.128). Here, ℓ is the multiplicity to which the Evans function vanishes at the origin, $(\tilde{\xi}, \lambda) = (0, 0)$, evidently zero under assumption (D2). The key modes Φ^+, Ψ^+ at plus spatial infinity are the same for the boundary-layer as for the shock case.

This leads to the pointwise bounds (5.37)–(5.38) given in Proposition 5.10 of [59] in case $\alpha = 1$, $\gamma_1 \equiv 1$ corresponding to the uniformly stable undercompressive shock case, but without the first $O(\rho^{-1})$, or "pole", terms appearing on the righthand side, which derive from cases $\tilde\gamma \sim \rho^{-1}$ not arising here. But, these are exactly the claimed bounds (3.35)–(3.37).

We omit the (substantial) details of this computation, referring the reader to [59]. However, the basic idea is, starting with the scattering decomposition of Corollary 3.2.1, to note, first, that the normal modes Φ^j, Ψ^j, $\tilde\Phi^j$, $\tilde\Psi^j$ can be approximated up to an exponentially trivial coordinate change by solutions of the constant-coefficient limiting system at $x \to +\infty$ (the conjugation lemma of [42]) and, second, that the coefficients M_{jk}, d_{jk} may be well-estimated through formulae (3.27) and (3.30) using Kramer's rule and the assumed lower bound on the Evans function $|D|$ appearing in the denominator. This is relatively straightforward away from the branch points $\Im\lambda = \eta_j(\tilde\xi)$ or "glancing set" of hyperbolic theory; the treatment near these points involves some delicate matrix perturbation theory applied to the limiting constant-coefficient system at $x \to +\infty$ followed by careful bookkeeping in the application of Kramer's rule. □

3.3 Linearized estimates

We next establish estimates on the linearized inhomogeneous problem

$$U_t - LU = f \tag{3.38}$$

with initial data $U(0) = U_0$ and Dirichlet boundary conditions as usual in $\tilde W$-coordinates:

$$W(0,\tilde x,t) := (w^I, w^{II})^T(0,\tilde x,t) = h \tag{3.39}$$

for the inflow case, and

$$w^{II}(0,\tilde x,t) = h \tag{3.40}$$

for the outflow case, with $x = (x_1, \tilde x) \in \mathbb{R}^d$.

3.3.1 Resolvent bounds

Our first step is to estimate solutions of the resolvent equation with homogeneous boundary data $\hat h \equiv 0$.

Proposition 3.3.1 (High-frequency bounds)**.** *Given (A1)-(A2), (H0)-(H2), and homogeneous boundary conditions (B), for some* R, C *sufficiently large and* $\theta > 0$ *sufficiently small,*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\hat{f}|_{\hat{H}^1(x_1)} \leq C|\hat{f}|_{\hat{H}^1(x_1)}, \tag{3.41}$$

and

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\hat{f}|_{L^2(x_1)} \leq \frac{C}{|\lambda|^{1/2}}|\hat{f}|_{\hat{H}^1(x_1)}, \tag{3.42}$$

for all $|(\tilde{\xi}, \lambda)| \geq R$ *and* $\Re e\lambda \geq -\theta$*, where* $\hat{f}$ *is the Fourier transform of* f *in variable* $\tilde{x}$ *and* $|\hat{f}|_{\hat{H}^1(x_1)} := |(1 + |\partial_{x_1}| + |\tilde{\xi}|)\hat{f}|_{L^2(x_1)}$.

Proof. First observe that a Laplace-Fourier transformed version with respect to variables $(\lambda, \tilde{x})$ of the nonlinear energy estimate in Section 3.4 with $s = 1$, carried out on the linearized equations written in W-coordinates, yields

$$(\Re e\lambda + \theta_1)|(1 + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C\Big(|W|^2 + (1 + |\tilde{\xi}|^2)|W||\hat{f}| + |\partial_{x_1}W||\partial_{x_1}\hat{f}|\Big) \tag{3.43}$$

for some C big and $\theta_1 > 0$ sufficiently small, where $|.|$ denotes $|.|_{L^2(x_1)}$. Applying Young's inequality, we obtain

$$(\Re e\lambda + \theta_1)|(1 + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C|W|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)\hat{f}|^2. \tag{3.44}$$

On the other hand, taking the imaginary part of the L^2 inner product of U against $\lambda U = f + LU$, we have also the standard estimate

$$|\Im m\lambda||U|_{L^2}^2 \leq C|U|_{H^1}^2 + C|f|_{L^2}^2, \tag{3.45}$$

and thus, taking the Fourier transform in $\tilde{x}$, we obtain

$$|\Im m\lambda||W|^2 \leq C|\hat{f}|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)W|^2. \tag{3.46}$$

Therefore, taking $\theta = \theta_1/2$, we obtain from (3.44) and (3.46)

$$|(1 + |\lambda|^{1/2} + |\tilde{\xi}| + |\partial_{x_1}|)W|^2 \leq C|W|^2 + C|(1 + |\tilde{\xi}| + |\partial_{x_1}|)\hat{f}|^2, \tag{3.47}$$

for any $\Re e\lambda \geq -\theta$. Now take R sufficiently large such that $|W|^2$ on the right hand side of the above can be absorbed into the left hand side, and thus, for all $|(\tilde{\xi}, \lambda)| \geq R$

and $\Re e\lambda \geq -\theta$,

$$|(1+|\lambda|^{1/2}+|\tilde{\xi}|+|\partial_{x_1}|)W|^2 \leq C|(1+|\tilde{\xi}|+|\partial_{x_1}|)\hat{f}|^2, \tag{3.48}$$

for some large $C>0$, which gives the result. □

We next have the following:

Proposition 3.3.2 (Mid-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and strong spectral stability (D2),*

$$|(L_{\tilde{\xi}}-\lambda)^{-1}|_{\hat{H}^1(x_1)} \leq C, \quad \text{for } R^{-1} \leq |(\tilde{\xi},\lambda)| \leq R \text{ and } \Re e\lambda \geq -\theta, \tag{3.49}$$

for any R and $C = C(R)$ sufficiently large and $\theta = \theta(R) > 0$ sufficiently small, where $|\hat{f}|_{\hat{H}^1(x_1)}$ is defined as in Proposition 3.3.1.

Proof. Immediate, by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$ is analytic with respect to H^1 in $(\tilde{\xi},\lambda)$; see Proposition 4.8, [57]. □

We next obtain the following resolvent bound for low-frequency regions as a direct consequence of pointwise bounds on the resolvent kernel, obtained in Proposition 3.2.5.

Proposition 3.3.3 (Low-frequency bounds). *Under the hypotheses of Theorem 3.1.4, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi},\lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde{\xi}}-\lambda)^{-1}\partial_{x_1}^{\beta}\hat{f}|_{L^p(x_1)} \leq C\gamma_2\rho^{-2/p}\Big[\rho^{\beta}|\hat{f}|_{L^1(x_1)}+\beta|\hat{f}|_{L^\infty(x_1)}\Big] \tag{3.50}$$

for all $2 \leq p \leq \infty$, $\beta = 0,1$, where γ_2 is as defined in (3.37)*.*

Proof. Using the convolution inequality $|g*h|_{L^p} \leq |g|_{L^p}|h|_{L^1}$ and noticing that

$$|\partial_{y_1}^{\beta}G_{\tilde{\xi},\lambda}(x_1,y_1)| \leq C\gamma_2(\rho^{\beta}+\beta e^{-\theta y_1})e^{-\theta\rho^2|x_1-y_1|},$$

we obtain

$$
\begin{aligned}
&|(L_{\tilde\xi}-\lambda)^{-1}\partial_{x_1}^{\beta}\hat f|_{L^p(x_1)}\\
&\quad=\Big|\int \partial_{y_1}^{\beta}G_{\tilde\xi,\lambda}(x_1,y_1)\hat f(y_1,\tilde\xi)\,dy_1\Big|_{L^p(x_1)}+\beta|G_{\tilde\xi,\lambda}(x_1,0)\hat f(0,\tilde\xi)|_{L^p(x_1)}\\
&\quad\le\Big|\int C\gamma_2(\rho^{\beta}+\beta e^{-\theta y_1})e^{-\theta\rho^2|x_1-y_1|}|\hat f(y_1,\tilde\xi)|\,dy_1\Big|_{L^p}+C\gamma_2\beta|\hat f(0,\tilde\xi)||e^{-\theta\rho^2 x_1}|_{L^p(x_1)}\\
&\quad\le C\gamma_2\rho^{-2/p}\Big[\rho^{\beta}|\hat f|_{L^1(x_1)}+\beta|\hat f|_{L^\infty(x_1)}\Big]
\end{aligned}
$$

as claimed. □

Remark 3.3.4. The above L^p bounds may alternatively be obtained directly by the argument of Section 12, [17], using quite different Kreiss symmetrizer techniques, again omitting pole terms arising from vanishing of the Evans function at the origin, and also the auxiliary problem construction of Section 12.6 used to obtain sharpened bounds in the Lax or overcompressive shock case (not relevant here).

3.3.2 Estimates on homogeneous solution operators

Define low- and high-frequency parts of the linearized solution operator $\mathcal{S}(t)$ of the linearized problem with homogeneous boundary and forcing data, f, $h\equiv 0$, as

$$
\mathcal{S}_1(t):=\frac{1}{(2\pi i)^d}\int_{|\tilde\xi|\le r}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}}e^{\lambda t+i\tilde\xi\cdot\tilde x}(L_{\tilde\xi}-\lambda)^{-1}d\lambda d\tilde\xi \tag{3.51}
$$

and

$$
\mathcal{S}_2(t):=e^{Lt}-\mathcal{S}_1(t). \tag{3.52}
$$

Then we obtain the following:

Proposition 3.3.5 (Low-frequency estimate)**.** *Under the hypotheses of Theorem 3.1.4, for $\beta=(\beta_1,\beta')$ with $\beta_1=0,1$,*

$$
\begin{aligned}
|\mathcal{S}_1(t)\partial_x^{\beta}f|_{L^2_{\tilde x}}&\le C(1+t)^{-(d-1)/4-|\beta|/2}|f|_{L^1_x}+C\beta_1(1+t)^{-(d-1)/4}|f|_{L^{1,\infty}_{\tilde x,x_1}},\\
|\mathcal{S}_1(t)\partial_x^{\beta}f|_{L^{2,\infty}_{\tilde x,x_1}}&\le C(1+t)^{-(d+1)/4-|\beta|/2}|f|_{L^1_x}+C\beta_1(1+t)^{-(d+1)/4}|f|_{L^{1,\infty}_{\tilde x,x_1}},\\
|\mathcal{S}_1(t)\partial_x^{\beta}f|_{L^{\infty}_{\tilde x,x_1}}&\le C(1+t)^{-d/2-|\beta|/2}|f|_{L^1_x}+C\beta_1(1+t)^{-d/2}|f|_{L^{1,\infty}_{\tilde x,x_1}},
\end{aligned} \tag{3.53}
$$

where $|\cdot|_{L^{p,q}_{\tilde x,x_1}}$ denotes the norm in $L^p(\tilde x;L^q(x_1))$.

Proof. The proof will follow closely the treatment of the shock case in [59]. Let $\hat{u}(x_1, \tilde{\xi}, \lambda)$ denote the solution of $(L_{\tilde{\xi}} - \lambda)\hat{u} = \hat{f}$, where $\hat{f}(x_1, \tilde{\xi})$ denotes Fourier transform of f, and

$$u(x,t) := \mathcal{S}_1(t)f = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}|\le r} \oint_{\Gamma^{\tilde{\xi}}\cap\{|\lambda|\le r\}} e^{\lambda t + i\tilde{\xi}\cdot\tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda d\tilde{\xi}.$$

Recalling the resolvent estimates in Proposition 3.3.3, we have

$$|\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^p(x_1)} \le C\gamma_2 \rho^{-2/p} |\hat{f}|_{L^1(x_1)} \le C\gamma_2 \rho^{-2/p} |f|_{L^1(x)}$$

where γ_2 is as defined in (3.37).

Therefore, using Parseval's identity, Fubini's theorem, and the triangle inequality, we may estimate

$$\begin{aligned}
|u|^2_{L^2(x_1,\tilde{x})}(t) &= \frac{1}{(2\pi)^{2d}} \int_{x_1} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}\cap\{|\lambda|\le r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \Big|^2 d\tilde{\xi} dx_1 \\
&= \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}\cap\{|\lambda|\le r\}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \Big|^2_{L^2(x_1)} d\tilde{\xi} \\
&\le \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}\cap\{|\lambda|\le r\}} e^{\Re e \lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^2(x_1)} d\lambda \Big|^2 d\tilde{\xi} \\
&\le C|f|^2_{L^1(x)} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}\cap\{|\lambda|\le r\}} e^{\Re e \lambda t} \gamma_2 \rho^{-1} d\lambda \Big|^2 d\tilde{\xi}.
\end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde{\xi}}$ by

$$\lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R},$$

and observing that by (3.37),

$$\begin{aligned}
\gamma_2 \rho^{-1} &\le (|k| + |\tilde{\xi}|)^{-1} \Big[1 + \sum_j \Big(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \Big)^{1/s_j - 1} \Big] \\
&\le (|k| + |\tilde{\xi}|)^{-1} \Big[1 + \sum_j \Big(\frac{|k - \tau_j(\tilde{\xi})|}{\rho} \Big)^{\epsilon - 1} \Big],
\end{aligned} \tag{3.54}$$

where $\epsilon := \frac{1}{\max_j s_j}$ ($0 < \epsilon < 1$ chosen arbitrarily if there are no singularities), we estimate

$$\begin{aligned}
\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2\rho^{-1}d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi}\Big|\int_{\mathbb{R}} e^{-\theta_1(k^2+|\tilde\xi|^2)t}\gamma_2\rho^{-1}dk\Big|^2 d\tilde\xi\\
&\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\
&\quad + \sum_j \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k-\tau_j(\tilde\xi)|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\
&\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\
&\le Ct^{-(d-1)/2}.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
|u|^2_{L^{2,\infty}_{\tilde x,x_1}}(t) &= \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t}\hat u(x_1,\tilde\xi,\lambda)d\lambda\Big|^2_{L^\infty(x_1)} d\tilde\xi\\
&\le \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^\infty(x_1)}d\lambda\Big|^2 d\tilde\xi\\
&\le C|f|^2_{L^1(x)}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2 d\lambda\Big|^2 d\tilde\xi
\end{aligned}$$

where

$$\begin{aligned}
\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2 d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}dk\Big|^2 d\tilde\xi\\
&\quad + \sum_j \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{2-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k-\tau_j(\tilde\xi)|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\
&\le Ct^{-(d+1)/2} + C\int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{2-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\
&\le Ct^{-(d+1)/2}.
\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}
|u|_{L^\infty_{\tilde x,x_1}}(t) &\le \frac{1}{(2\pi)^d}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t}\hat u(x_1,\tilde\xi,\lambda)d\lambda\Big|_{L^\infty(x_1)} d\tilde\xi\\
&\le \frac{1}{(2\pi)^d}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^\infty(x_1)}d\lambda d\tilde\xi\\
&\le C|f|_{L^1(x)}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2 d\lambda d\tilde\xi
\end{aligned}$$

where as above we have

$$\begin{aligned}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2 d\lambda d\tilde\xi &\le \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}\int_{\mathbb{R}} e^{-\theta_1 k^2 t} dk d\tilde\xi \\ &\quad + \sum_j \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{1-\epsilon}\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k-\tau_j(\tilde\xi)|^{\epsilon-1} dk d\tilde\xi \\ &\le Ct^{-d/2} + C\int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{1-\epsilon}\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1} dk d\tilde\xi \\ &\le Ct^{-d/2}.\end{aligned}$$

The x_1-derivative bounds follow similarly by using the resolvent bounds in Proposition 3.3.3 with $\beta_1 = 1$. The $\tilde{x}$-derivative bounds are straightforward by the fact that $\widehat{\partial_{\tilde x}^{\tilde\beta} f} = (i\tilde\xi)^{\tilde\beta}\hat f$.

Finally, each of the above integrals is bounded by $C|f|_{L^1(x)}$ as the product of $|f|_{L^1(x)}$ times the integral quantities $\gamma_2\rho^{-1}$, γ_2 over a bounded domain, hence we may replace t by $(1+t)$ in the above estimates. □

Next, we obtain estimates on the high-frequency part $\mathcal{S}_2(t)$ of the linearized solution operator. Recall that $\mathcal{S}_2(t) = \mathcal{S}(t) - \mathcal{S}_1(t)$, where

$$\mathcal{S}(t) = \frac{1}{(2\pi i)^d}\int_{\mathbb{R}^{d-1}} e^{i\tilde\xi\cdot\tilde x} e^{L_{\tilde\xi} t} d\tilde\xi$$

and

$$\mathcal{S}_1(t) = \frac{1}{(2\pi i)^d}\int_{|\tilde\xi|\le r}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t + i\tilde\xi\cdot\tilde x}(L_{\tilde\xi}-\lambda)^{-1} d\lambda d\tilde\xi.$$

Then according to [57, Corollary 4.11], we can write

$$\begin{aligned}\mathcal{S}_2(t)f = \frac{1}{(2\pi i)^d}\text{P.V.}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\int_{\mathbb{R}^{d-1}} &\chi_{|\tilde\xi|^2+|\Im m\lambda|^2\ge\theta_1+\theta_2} \\ &\times e^{i\tilde\xi\cdot\tilde x+\lambda t}(\lambda - L_{\tilde\xi})^{-1}\hat f(x_1,\tilde\xi) d\tilde\xi d\lambda.\end{aligned} \tag{3.55}$$

Proposition 3.3.6 (High-frequency estimate)**.** *Given (A1)-(A2), (H0)-(H2), (D2), and homogeneous boundary conditions (B), for* $0 \le |\alpha| \le s-3$, *s as in (H0),*

$$\begin{aligned}|\mathcal{S}_2(t)f|_{L^2_x} &\le Ce^{-\theta_1 t}|f|_{H^3_x}, \\ |\partial_x^\alpha \mathcal{S}_2(t)f|_{L^2_x} &\le Ce^{-\theta_1 t}|f|_{H^{|\alpha|+3}_x}.\end{aligned} \tag{3.56}$$

Proof. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(L_{\tilde{\xi}})$ of the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$, for all $f \in \mathcal{D}(L_{\tilde{\xi}})$,

$$(\lambda - L_{\tilde{\xi}})^{-1} f = \lambda^{-1}(\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} f + \lambda^{-1} f. \tag{3.57}$$

Using this identity and (3.55), we estimate

$$\begin{aligned}
\mathcal{S}_2(t) f = & \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2 + |\Im m \lambda|^2 \geq \theta_1 + \theta_2} \\
& \times e^{i\tilde{\xi}\cdot\tilde{x} + \lambda t} \lambda^{-1} (\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\
& + \frac{1}{(2\pi i)^d} \text{P.V.} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2 + |\Im m \lambda|^2 \geq \theta_1 + \theta_2} \\
& \times e^{i\tilde{\xi}\cdot\tilde{x} + \lambda t} \lambda^{-1} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda \\
=: & \, S_1 + S_2,
\end{aligned} \tag{3.58}$$

where, by Plancherel's identity and Propositions 3.3.6 and 3.3.2, we have

$$\begin{aligned}
|S_1|_{L^2(\tilde{x}, x_1)} &\leq C \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} |\lambda|^{-1} |e^{\lambda t}| |(\lambda - L_{\tilde{\xi}})^{-1} L_{\tilde{\xi}} \hat{f}|_{L^2(\tilde{\xi}, x_1)} |d\lambda| \\
&\leq C e^{-\theta_1 t} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} |\lambda|^{-3/2} \Big| (1 + |\tilde{\xi}|) |L_{\tilde{\xi}} \hat{f}|_{H^1(x_1)} \Big|_{L^2(\tilde{\xi})} |d\lambda| \\
&\leq C e^{-\theta_1 t} |f|_{H^3_x}
\end{aligned}$$

and

$$\begin{aligned}
|S_2|_{L^2_x} \leq & \frac{1}{(2\pi)^d} \Big| \text{P.V.} \int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x}\cdot\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \Big|_{L^2} \\
& + \frac{1}{(2\pi)^d} \Big| \text{P.V.} \int_{-\theta_1 - ir}^{-\theta_1 + ir} \lambda^{-1} e^{\lambda t} d\lambda \int_{\mathbb{R}^{d-1}} e^{i\tilde{x}\cdot\tilde{\xi}} \hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} \Big|_{L^2} \\
\leq & \, C e^{-\theta_1 t} |f|_{L^2_x},
\end{aligned} \tag{3.59}$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the first inequality stated in the proposition. Derivative bounds follow similarly. □

Remark 3.3.7. Here, we have used the $\lambda^{1/2}$ improvement in (3.42) over (3.41) together with modifications introduced in [35] to greatly simplify the original high-frequency argument given in [59] for the shock case.

3.3.3 Boundary estimates

For the purpose of studying the nonzero boundary perturbation, we need the following proposition. For $h := h(\tilde{x}, t)$, define

$$\mathcal{D}_h(t) := (|h_t| + |h_{\tilde{x}}| + |h_{\tilde{x}\tilde{x}}|)(t), \tag{3.60}$$

and

$$\Gamma h(t) := \int_0^t \int_{\mathbb{R}^{d-1}} \Big(\sum_k G_{y_k} B^{k1} + GA^1\Big)(x, t-s; 0, \tilde{y}) h(\tilde{y}, s)\, d\tilde{y} ds, \tag{3.61}$$

where $G(x, t; y)$ is the Green function of $\partial_t - L$. This boundary term will appear when we write down the Duhamel formulas for the linearized and nonlinear equations (see (3.73) and (3.130)). Noting that for the outflow case, the fact that $G(x, t; 0, \tilde{y}) \equiv 0$ simplifies Γh to

$$\Gamma h(t) = \int_0^t \int_{\mathbb{R}^{d-1}} G_{y_1}(x, t-s; 0, \tilde{y}) B^{11} h\, d\tilde{y} ds. \tag{3.62}$$

Therefore when dealing with the outflow case, instead of putting assumptions on h itself as in the inflow case, we make assumptions on $B^{11}h$, matching with the hypotheses on W-coordinates.

Proposition 3.3.8. *Assume that $h = h(\tilde{x}, t)$ satisfies*

$$\begin{aligned} |h(t)|_{L^2_{\tilde{x}}} &\le E_0(1+t)^{-(d+1)/4}, \\ |h(t)|_{L^\infty_{\tilde{x}}} &\le E_0(1+t)^{-d/2} \\ |\mathcal{D}_h(t)|_{L^1_{\tilde{x}} \cap H^{|\gamma|+3}_{\tilde{x}}} &\le E_0(1+t)^{-d/2-\epsilon}, \end{aligned} \tag{3.63}$$

for some positive constant E_0; here $|\gamma| = [(d-1)/2] + 2$, and $\epsilon > 0$ is arbitrary small for $d = 2$ and $\epsilon = 0$ for $d \ge 3$. For the outflow case, we replace these assumptions on *h by those on $B^{11}h$. Then we obtain*

$$
\begin{aligned}
|\Gamma h(t)|_{L^2} &\le CE_0(1+t)^{-(d-1)/4},\\
|\Gamma h(t)|_{L^{2,\infty}_{\tilde{x},x_1}} &\le CE_0(1+t)^{-(d+1)/4},\\
|\Gamma h(t)|_{L^\infty} &\le CE_0(1+t)^{-d/2},
\end{aligned}
\tag{3.64}
$$

and derivative bounds

$$
\begin{aligned}
|\partial_x \Gamma h(t)|_{L^{2,\infty}_{\tilde{x},x_1}} &\le CE_0(1+t)^{-(d+1)/4},\\
|\partial^2_{\tilde{x}} \Gamma h(t)|_{L^{2,\infty}_{\tilde{x},x_1}} &\le CE_0(1+t)^{-(d+1)/4},
\end{aligned}
\tag{3.65}
$$

for all $t \ge 0$.

Proof. We first recall that $G(x,t-s;y)$ is a solution of $(\partial_s - L_y)^* G^* = 0$, that is,

$$
-G_s - \sum_j (GA^j)_{y_j} + \sum_j GA^j_{y_j} = \sum_{jk} (G_{y_k} B^{kj})_{y_j}. \tag{3.66}
$$

Integrating this on $\mathbb{R}^d_+ \times [0,t]$ against

$$
g(y_1,\tilde{y},s) := e^{-y_1} h(\tilde{y},s), \tag{3.67}
$$

and integrating by parts twice, we obtain

$$
\begin{aligned}
\Gamma h = &-\int_0^t \int_{\mathbb{R}^d_+} \Big(\sum_{jk} G_{y_k} B^{kj} + \sum_j GA^j\Big) g_{y_j}\, dyds\\
&- \int_0^t \int_{\mathbb{R}^d_+} \Big(-G_s + \sum_j GA^j_{y_j}\Big) g(y,s)\, dyds
\end{aligned}
$$

where, recalling that

$$
\mathcal{S}(t)f(x) = \int_{\mathbb{R}^d_+} G(x,t;y) f(y)\, dy,
$$

we get

$$
\begin{aligned}
-\int_0^t \int_{\mathbb{R}^d_+} &\sum_{jk} \Big(G_{y_k} B^{kj} + \sum_j GA^j\Big) g_{y_j}\, dyds\\
&= -\int_0^t \mathcal{S}(t-s)\Big(-\sum_{jk} (B^{kj} g_{x_j})_{x_k} + \sum_j A^j g_{x_j}\Big) ds
\end{aligned}
$$

and

$$-\int_0^t\int_{\mathbb{R}^d_+}\Big(-G_s+\sum_j GA^j_{y_j}\Big)g(y,s)dyds$$
$$=-\int_0^t \mathcal{S}(t-s)\Big(g_s+\sum_j A^j_{x_j}g\Big)ds+g(x,t)-\mathcal{S}(t)g(x,0).$$

Therefore combining all these estimates yields

$$\Gamma h=g(x,t)-\mathcal{S}(t)g_0-\int_0^t\mathcal{S}(t-s)(g_s-L_xg(x,s))ds \tag{3.68}$$

with $g_0(x):=g(x,0)$ and $L_xg=-\sum_j(A^jg)_{x_j}+\sum_{jk}(B^{jk}g_{x_k})_{x_j}$.

Now we are ready to employ estimates obtained in the previous section on the solution operator $\mathcal{S}(t)=\mathcal{S}_1(t)+\mathcal{S}_2(t)$. Noting that

$$|g|_{L^p_x}\le C|h|_{L^p_{\tilde{x}}},$$

we estimate

$$\begin{aligned}|\Gamma h|_{L^2}&\le|g|_{L^2}+|\mathcal{S}_1(t)g_0|_{L^2}+|\mathcal{S}_2(t)g_0|_{L^2}\\&\quad+\int_0^t|\mathcal{S}_1(t-s)(g_s-Lg)|_{L^2}+|\mathcal{S}_2(t-s)(g_s-Lg)|_{L^2}ds\\&\le|h(t)|_{L^2_{\tilde{x}}}+C(1+t)^{-\frac{d-1}{4}}|g_0|_{L^1}+Ce^{-\eta t}|g_0|_{H^3}\\&\quad+\int_0^t(1+t-s)^{-(d-1)/4}(|g_s|+|Lg|)_{L^1}+e^{-\theta(t-s)}(|g_s|+|Lg|)_{H^3}ds\\&\le|h(t)|_{L^2_{\tilde{x}}}+C(1+t)^{-\frac{d-1}{4}}|h_0|_{L^1_{\tilde{x}}\cap H^3_{\tilde{x}}}\\&\quad+\int_0^t(1+t-s)^{-(d-1)/4}|\mathcal{D}_h(s)|_{L^1_{\tilde{x}}}+e^{-\theta(t-s)}|\mathcal{D}_h(s)|_{H^3_{\tilde{x}}}ds\\&\le CE_0(1+t)^{-\frac{d-1}{4}}\end{aligned}$$

and similarly we also obtain

$$\begin{aligned}|\Gamma h|_{L^{2,\infty}_{\tilde{x},x_1}}&\le|h(t)|_{L^2_{\tilde{x}}}+C(1+t)^{-\frac{d+1}{4}}|h_0|_{L^1_{\tilde{x}}\cap H^4_{\tilde{x}}}\\&\quad+C\int_0^t(1+t-s)^{-(d+1)/4}|\mathcal{D}_h(s)|_{L^1_{\tilde{x}}}+e^{-\theta(t-s)}|\mathcal{D}_h(s)|_{H^4_{\tilde{x}}}ds\\&\le CE_0(1+t)^{-\frac{d+1}{4}}\end{aligned}\tag{3.69}$$

and

$$
\begin{aligned}
|\Gamma h|_{L^\infty} \le |h(t)|_{L^\infty_{\tilde x}} &+ C(1+t)^{-\frac{d}{2}}|h_0|_{L^1_{\tilde x}\cap H^{|\gamma|+3}_{\tilde x}} \\
&+ C\int_0^t (1+t-s)^{-d/2}|\mathcal{D}_h(s)|_{L^1_{\tilde x}} + e^{-\theta(t-s)}|\mathcal{D}_h(s)|_{H^{|\gamma|+3}_{\tilde x}}ds \\
&\le CE_0(1+t)^{-\frac{d}{2}}.
\end{aligned}
\tag{3.70}
$$

Similar bounds hold for derivatives.

This completes the proof of the proposition. □

3.3.4 Duhamel formula

The following integral representation formula expresses the solution of the inhomogeneous equation (3.38) in terms of the homogeneous solution operator $\mathcal{S}$ for f, $h \equiv 0$.

Lemma 3.3.9 (Integral formulation)**.** *Solutions U of (3.38) may be expressed as*

$$
U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)f(\cdot,s) + \Gamma U(0,\tilde x,t) \tag{3.71}
$$

where $U(x,0)=U_0(x)$,

$$
\Gamma U(0,\tilde x,t) := \int_0^t \int_{\mathbb{R}^{d-1}} (\sum_j G_{y_j}B^{j1} + GA^1)(x,t-s;0,\tilde y)U(0,\tilde y,s)\,d\tilde y ds, \tag{3.72}
$$

and $G(\cdot,t;y)=\mathcal{S}(t)\delta_y(\cdot)$ is the Green function of $\partial_t - L$.

Proof. Integrating on $\mathbb{R}^d_+$ the linearized equations

$$
(\partial_s - L_y)U = f
$$

against $G(x,t-s;y)$ and using the fact that by duality

$$
(\partial_s - L_y)^* G^*(x,t-s;y) \equiv 0,
$$

we easily obtain the lemma as in the one-dimensional case (see [56, 45]), recalling that

$$
\mathcal{S}(t)f = \int_{\mathbb{R}^d_+} G(x,t;y)f(y)dy.
$$

□

3.3.5 Proof of linearized stability

Proof of Theorem 3.1.3. Writing the Duhamel formula for the linearized equations

$$U(x,t) = \mathcal{S}(t)U_0 + \Gamma h(\tilde{x},t), \tag{3.73}$$

with Γh defined in (3.61), where $U(x,0) = U_0(x)$ and $U(0,\tilde{x},t) = h(\tilde{x},t)$, and applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, we obtain

$$\begin{aligned} |U(t)|_{L^2} &\le |\mathcal{S}_1(t)U_0|_{L^2} + |\mathcal{S}_2(t)U_0|_{L^2} + |\Gamma h(t)|_{L^2} \\ &\le C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1} + Ce^{-\eta t}|U_0|_{H^3} + CE_0(1+t)^{-(d-1)/4} \\ &\le C(1+t)^{-\frac{d-1}{4}}(|U_0|_{L^1\cap H^3} + E_0) \end{aligned} \tag{3.74}$$

and

$$\begin{aligned} |U(t)|_{L^\infty} &\le |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty} + |\Gamma h(t)|_{L^\infty} \\ &\le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + C|\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}} + CE_0(1+t)^{-d/2} \\ &\le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + Ce^{-\eta t}|U_0|_{H^{[(d-1)/2]+5}} + CE_0(1+t)^{-d/2} \\ &\le C(1+t)^{-\frac{d}{2}}(|U_0|_{L^1\cap H^{[(d-1)/2]+5}} + E_0). \end{aligned} \tag{3.75}$$

These prove the bounds as stated in the theorem for $p = 2$ and $p = \infty$. For $2 < p < \infty$, we use the interpolation inequality between L^2 and L^∞. □

3.4 Energy estimates

For the analysis of nonlinear stability, we need the following energy estimate adapted from [37, 45, 57]. Define the nonlinear perturbation variables $U = (u,v)$ by

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x_1). \tag{3.76}$$

Proposition 3.4.1. *Under the hypotheses of Theorem 3.1.4, let* $U_0 \in H^s$ *and* $U = (u,v)^T$ *be a solution of* (3.2) *and* (3.76). *Suppose that, for* $0 \le t \le T$, *the* $W_x^{2,\infty}$

norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then

$$|U(t)|_{H^s}^2 \le Ce^{-\theta t}|U_0|_{H^s}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big(|U(\tau)|_{L^2}^2 + |\mathcal{B}_h(\tau)|^2\Big)d\tau \tag{3.77}$$

for all $0 \le t \le T$, where the boundary term $\mathcal{B}_h$ is defined as in Theorem 3.1.4.

Proof. Observe that a straightforward calculation shows that $|U|_{H^r} \sim |W|_{H^r}$,

$$W = \tilde{W} - \bar{W} := W(\tilde{U}) - W(\bar{U}), \tag{3.78}$$

for $0 \le r \le s$, provided $|U|_{W^{2,\infty}}$ remains bounded, hence it is sufficient to prove a corresponding bound in the special variable W. We first carry out a complete proof in the more straightforward case with conditions (A1)-(A3) replaced by the following global versions, indicating afterward by a few brief remarks the changes needed to carry out the proof in the general case.

(A1') $\tilde{A}^j(\tilde{W}), \tilde{A}^0, \tilde{A}^1_{11}$ are symmetric, $\tilde{A}^0 \ge \theta_0 > 0$,

(A2') Same as (A2),

(A3') $\tilde{W} = \begin{pmatrix} \tilde{w}^I \\ \tilde{w}^{II} \end{pmatrix}, \quad \tilde{B}^{jk} = \tilde{B}^{kj} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{b}^{jk} \end{pmatrix}, \quad \sum \xi_j\xi_k\tilde{b}^{jk} \ge \theta|\xi|^2$, and $\tilde{G} \equiv 0$.

Substituting (3.78) into (3.4), we obtain the quasilinear perturbation equation

$$A^0W_t + \sum_j A^jW_{x_j} = \sum_{jk}(B^{jk}W_{x_k})_{x_j} + M_1\bar{W}_{x_1} + \sum_j(M_2^j\bar{W}_{x_1})_{x_j} \tag{3.79}$$

where $A^0 := A^0(W + \bar{W})$ is symmetric positive definite, $A^j := A^j(W + \bar{W})$ are symmetric,

$$M_1 = A^1(W + \bar{W}) - A^1(\bar{W}) = \Big(\int_0^1 dA^1(\bar{W} + \theta W)d\theta\Big)W,$$
$$M_2^j = B^{j1}(W + \bar{W}) - B^{j1}(\bar{W}) = \begin{pmatrix} 0 & 0 \\ 0 & (\int_0^1 db^{j1}(\bar{W} + \theta W)d\theta)W \end{pmatrix}.$$

As shown in [37], we have bounds

$$|A^0| \le C, \quad |A^0_t| \le C|W_t| \le C(|W_x| + |w^{II}_{xx}|) \le C\zeta, \tag{3.80}$$

$$|\partial_x A^0| + |\partial_x^2 A^0| \le C(\sum_{k=1}^{2} |\partial_x^k W| + |\bar{W}_{x_1}|) \le C(\zeta + |\bar{W}_{x_1}|). \tag{3.81}$$

We have the same bounds for A^j, B^{jk}, and also due to the form of M_1, M_2,

$$|M_1|, |M_2| \le C(\zeta + |\bar{W}_{x_1}|)|W|. \tag{3.82}$$

Note that thanks to Lemma A.1.1 we have the bound on the profile: $|\bar{W}_{x_1}| \le Ce^{-\theta|x_1|}$, as $x_1 \to +\infty$.

The following results assert that hyperbolic effects can compensate for degenerate viscosity B, as revealed by the existence of a compensating matrix K.

Lemma 3.4.2 ([34]). *Assuming (A1'), condition (A2') is equivalent to the following:*

(K1) There exist smooth skew-symmetric "compensating matrices" $K(\xi)$, homogeneous degree one in ξ, such that

$$\Re e\Big(\sum_{j,k} \xi_j \xi_k B^{jk} - K(\xi)(A^0)^{-1} \sum_k \xi_k A^k\Big)(W_+) \ge \theta_2 |\xi|^2 > 0 \tag{3.83}$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Define α by the ODE

$$\alpha_{x_1} = -\mathrm{sign}(A^1_{11}) c_* |\bar{W}_{x_1}| \alpha, \quad \alpha(0) = 1 \tag{3.84}$$

where $c_* > 0$ is a large constant to be chosen later. Note that we have

$$(\alpha_{x_1}/\alpha) A^1_{11} \le -c_* \theta_1 |\bar{W}_{x_1}| =: -\omega(x_1) \tag{3.85}$$

and

$$|\alpha_{x_1}/\alpha| \le c_* |\bar{W}_{x_1}| = \theta_1^{-1} \omega(x_1). \tag{3.86}$$

In what follows, we shall use $\langle\cdot,\cdot\rangle$ as the α-weighted L^2 inner product defined as

$$\langle f,g\rangle = \langle \alpha f, g\rangle_{L^2(\mathbb{R}^d_+)}$$

and

$$\|f\|_s = \sum_{i=0}^{s}\sum_{|\alpha|=i}\left\langle \partial_x^\alpha f, \partial_x^\alpha f\right\rangle^{1/2}$$

as the norm in weighted H^s space. Note that for any symmetric operator S,

$$\begin{aligned}
\langle S f_{x_j}, f\rangle &= -\frac{1}{2}\langle S_{x_j} f, f\rangle, \quad j\neq 1\\
\langle S f_{x_1}, f\rangle &= -\frac{1}{2}\langle (S_{x_1} + (\alpha_{x_1}/\alpha)S)f, f\rangle - \frac{1}{2}\langle Sf, f\rangle_0,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_0$ denotes the integration on $\mathbb{R}^d_0 := \{x_1 = 0\}\times\mathbb{R}^{d-1}$. Also we define

$$\|f\|_{0,s} = \|f\|_{H^s(\mathbb{R}^d_0)} = \sum_{i=0}^{s}\sum_{|\alpha|=i}\left\langle \partial_{\tilde{x}}^\alpha f, \partial_{\tilde{x}}^\alpha f\right\rangle_0^{1/2}.$$

Note that in what follows, we shall pay attention to keeping track of c_*. For constants independent of c_*, we simply write them as C. Also, for simplicity, the sum symbol will sometimes be dropped where it is no confusion. We write $\|f_x\| = \sum_j \|f_{x_j}\|$ and $\|\partial_x^k f\| = \sum_{|\alpha|=k}\|\partial_x^\alpha f\|$.

Zeroth order "Friedrichs-type" estimate

First, by integration by parts and estimates (3.80), (3.81), and then (3.85), we obtain for $j\neq 1$,

$$-\langle A^j W_{x_j}, W\rangle = \frac{1}{2}\langle A^j_{x_j} W, W\rangle \le C\langle(\zeta + |\bar{W}_{x_1}|)w^I, w^I\rangle + C\|w^{II}\|_0^2$$

and for $j = 1$,

$$\begin{aligned}
-\langle A^1 W_{x_1}, W\rangle &= \frac{1}{2}\langle (A^1_{x_1} + (\alpha_{x_1}/\alpha)A^1)W, W\rangle + \frac{1}{2}\langle A^1 W, W\rangle_0\\
&\le \frac{1}{2}\langle(\alpha_{x_1}/\alpha)A^1_{11}w^I, w^I\rangle + C\langle(\zeta + |\bar{W}_{x_1}|)|W| + \omega(x_1)|w^{II}|, |W|\rangle + J_b^0\\
&\le -\frac{1}{2}\langle\omega(x)w^I, w^I\rangle + C\langle(\zeta + |\bar{W}_{x_1}|)w^I, w^I\rangle + C(c_*)\|w^{II}\|_0^2 + J_b^0,
\end{aligned}$$

where J_b^0 denotes the boundary term $\frac{1}{2}\langle A^1 W, W\rangle_0$. The term $\langle |\bar{W}_{x_1}| w^I, w^I\rangle$ may be easily absorbed into the first term of the right-hand side, since for c_* sufficiently large,

$$\langle |\bar{W}_{x_1}| w^I, w^I\rangle \le (c_*\theta_1)^{-1}\langle \omega(x_1) w^I, w^I\rangle \le \frac{1}{4C}\langle \omega(x_1) w^I, w^I\rangle. \tag{3.87}$$

Also, integration by parts yields

$$\begin{aligned}\langle (B^{jk}W_{x_k})_{x_j}, W\rangle &= -\langle B^{jk}W_{x_k}, W_{x_j}\rangle - \langle (\alpha_{x_1}/\alpha) B^{1k} W_{x_k}, W\rangle - \langle B^{1k}W_{x_k}, W\rangle_0 \\ &\le -\theta \|w_x^{II}\|_0^2 + C\langle \omega(x_1) w_x^{II}, w^{II}\rangle - \langle b^{1k} w_{x_k}^{II}, w^{II}\rangle_0 \\ &\le -\theta \|w_x^{II}\|_0^2 + C(c_*)\|w^{II}\|_0^2 - \langle b^{1k} w_{x_k}^{II}, w^{II}\rangle_0.\end{aligned}$$

where we used the fact that $B^{jk}W_x \cdot W = b^{jk} w_x^{II} \cdot w^{II}$, noting that B has block-diagonal form with the first block identical to zero. Similarly, recalling that $M_2^j = B^{j1}(W+\bar{W}) - B^{j1}(\bar{W})$, we have

$$\begin{aligned}\langle (M_2^j \bar{W}_{x_1})_{x_j}, W\rangle &= -\langle M_2^j \bar{W}_{x_1}, W_{x_j}\rangle - \langle (\alpha_{x_1}/\alpha) M_2^1 \bar{W}_{x_1}, W\rangle - \langle M_2^1 \bar{W}_{x_1}, W\rangle_0 \\ &\le C\langle |\bar{W}_{x_1}||W|, |w_x^{II}|\rangle + C\langle \omega(x_1)|W|, w^{II}\rangle - \langle m_2^1 \bar{W}_{x_1}, w^{II}\rangle_0 \\ &\le \xi \|w_x^{II}\|_0^2 + C\Big(\epsilon\langle \omega(x_1) w^I, w^I\rangle + C(c_*)\|w^{II}\|_0^2\Big) - \langle m_2^1 \bar{W}_{x_1}, w^{II}\rangle_0\end{aligned}$$

for any small ξ, ϵ. Note that C is independent of c_*. Therefore, for $\xi = \theta/2$ and c_* sufficiently large, combining all above estimates, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A^0 W, W\rangle &= \langle A^0 W_t, W\rangle + \frac{1}{2}\langle A_t^0 W, W\rangle \\ &= \langle -A^j W_{x_j} + (B^{jk}W_{x_k})_{x_j} + M_1\bar{W}_{x_1} + (M_2^j \bar{W}_{x_1})_{x_j}, W\rangle + \frac{1}{2}\langle A_t^0 W, W\rangle \\ &\le -\frac{1}{4}[\langle \omega(x_1) w^I, w^I\rangle + \theta\|w_x^{II}\|_0^2] + C\zeta\|w^I\|_0^2 + C(c_*)\|w^{II}\|_0^2 + I_b^0\end{aligned} \tag{3.88}$$

where the boundary term

$$I_b^0 := \frac{1}{2}\langle A^1 W, W\rangle_0 - \langle b^{1k} w_{x_k}^{II}, w^{II}\rangle_0 - \langle m_2^1 \bar{W}_{x_1}, w^{II}\rangle_0 \tag{3.89}$$

which, in the outflow case (thanks to the negative definiteness of A_{11}^1), is estimated as

$$I_b^0 \le -\frac{\theta_1}{2}\|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w_x^{II}\|_{0,0}\|w^{II}\|_{0,0}), \tag{3.90}$$

and similarly in the inflow case, estimated as

$$I_b^0 \leq C(\|W\|_{0,0}^2 + \|w_x^{II}\|_{0,0}\|w^{II}\|_{0,0}). \tag{3.91}$$

Here we recall that $\|\cdot\|_{0,s} := \|\cdot\|_{H^s(\mathbb{R}_0^d)}$.

First order "Friedrichs-type" estimate

Similarly as above, we need the following key estimate, computing by the use of integration by parts, (3.87), and c_* being sufficiently large,

$$\begin{aligned}
-\sum_j &\langle W_{x_i}, A^j W_{x_i x_j}\rangle \\
&= \frac{1}{2}\sum_j \langle W_{x_i}, A^j_{x_j} W_{x_i}\rangle + \frac{1}{2}\langle W_{x_i}, (\alpha_{x_1}/\alpha) A^1 W_{x_i}\rangle + \frac{1}{2}\langle W_{x_i}, A^1 W_{x_i}\rangle_0 \\
&\leq -\frac{1}{4}\langle \omega(x_1) w_x^I, w_x^I\rangle + C\zeta\|w_x^I\|_0^2 + Cc_*^2\|w_x^{II}\|_0^2 + \frac{1}{2}\langle W_{x_i}, A^1 W_{x_i}\rangle_0.
\end{aligned} \tag{3.92}$$

We deal with the boundary term later. Now let us compute

$$\frac{1}{2}\frac{d}{dt}\langle A^0 W_{x_i}, W_{x_i}\rangle = \langle W_{x_i}, (A^0 W_t)_{x_i}\rangle - \langle W_{x_i}, A^0_{x_i} W_t\rangle + \frac{1}{2}\langle A^0_t W_{x_i}, W_{x_i}\rangle. \tag{3.93}$$

We control each term in turn. By (3.80) and (3.81), we first have

$$\langle A_t^0 W_{x_i}, W_{x_i}\rangle \leq C\zeta\|W_x\|_0^2$$

and by multiplying $(A^0)^{-1}$ into (3.79),

$$\begin{aligned}
|\langle W_{x_i}, A^0_{x_i} W_t\rangle| \leq& C\langle(\zeta + |\bar{W}_{x_1}|)|W_x|, (|W_x| + |w_{xx}^{II}| + |W|)\rangle \\
\leq& \xi\|w_{xx}^{II}\|_0^2 + C\langle(\zeta + |\bar{W}_{x_1}|)w_x^I, w_x^I\rangle + C\langle(\zeta + |\bar{W}_{x_1}|)w^I, w^I\rangle + C\|w^{II}\|_1^2,
\end{aligned}$$

where the term $\langle|\bar{W}_{x_1}|w^I_x, w^I_x\rangle$ may be treated in the same way as was $\langle|\bar{W}_{x_1}|w^I, w^I\rangle$ in (3.87). Using (3.79), we write the first term in the right-hand side of (3.93) as

$$\begin{aligned}
\langle W_{x_i}, (A^0 W_t)_{x_i}\rangle =& \langle W_{x_i}, [-A^j W_{x_j} + (B^{jk} W_{x_k})_{x_j} + M_1 \bar{W}_{x_1} + (M_2^j \bar{W}_{x_1})_{x_j}]_{x_i}\rangle \\
=& -\langle W_{x_i}, A^j W_{x_i x_j}\rangle + \langle W_{x_i}, -A^j_{x_i} W_{x_j} + (M_1 \bar{W}_{x_1})_{x_i}\rangle \\
& - \langle W_{x_i x_j}, [(B^{jk} W_{x_k})_{x_i} + (M_2^j \bar{W}_{x_1})_{x_i}]\rangle \\
& - \langle (\alpha_{x_1}/\alpha) W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}]\rangle \\
& - \langle W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}]\rangle_0 \\
\leq& -\frac{1}{4}\Big[\langle \omega(x_1) w^I_x, w^I_x\rangle + \theta \|w^{II}_{xx}\|_0^2\Big] \\
& + C\Big[\zeta \|w^I\|_1^2 + C(c_*)\|w^{II}_x\|_0^2 + \langle |\bar{W}_{x_1}| w^I, w^I\rangle\Big] + I_b^1
\end{aligned}$$

where I_b^1 denotes the boundary terms

$$\begin{aligned}
I_b^1 :=& \frac{1}{2}\langle W_{x_i}, A^1 W_{x_i}\rangle_0 - \langle W_{x_i}, [(B^{1k} W_{x_k})_{x_i} + (M_2^1 \bar{W}_{x_1})_{x_i}]\rangle_0 \\
=& \frac{1}{2}\langle W_{x_i}, A^1 W_{x_i}\rangle_0 - \langle w^{II}_{x_i}, [(b^{1k} w^{II}_{x_k})_{x_i} + (m_2^1 \bar{W}_{x_1})_{x_i}]\rangle_0,
\end{aligned} \tag{3.94}$$

and we have used (A3) for each fixed i and $\xi_j = (W_{x_i})_{x_j}$ to get

$$\sum_{jk} \langle W_{x_i x_j}, B^{jk} W_{x_k x_i}\rangle \geq \theta \sum_j \|W_{x_i x_j}\|_0^2, \tag{3.95}$$

and estimates (3.92),(3.87) for w^I, w^I_x, and Young's inequality to obtain:

$$\begin{aligned}
&\langle W_x, -A^j_x W_x + (M_1 \bar{W}_{x_1})_x\rangle \leq C\langle (\zeta + |\bar{W}_{x_1}|)|W_x|, |W_x| + |W|\rangle. \\
&-\langle W_{xx} + (\alpha_{x_1}/\alpha) W_x, (B^{jk} W_x)_x\rangle \leq \\
&\qquad -\theta \|w^{II}_{xx}\|_0^2 + C\langle |w^{II}_{xx}| + \omega(x_1)|w^{II}_x|, (\zeta + |\bar{W}_{x_1}|)|w^{II}_x|\rangle \\
&-\langle W_{xx} + (\alpha_{x_1}/\alpha) W_x, (M_2^j \bar{W}_{x_1})_x\rangle \leq \\
&\qquad C\langle |w^{II}_{xx}| + \omega(x_1)|w^{II}_x|, (\zeta + |\bar{W}_{x_1}|)(|W_x| + |W|)\rangle.
\end{aligned}$$

Putting these estimates together into (3.93), we have obtained

$$\frac{1}{2}\frac{d}{dt}\langle A^0 W_x, W_x\rangle + \frac{1}{4}\theta\|w^{II}_{xx}\|_0^2 + \frac{1}{4}\langle \omega(x_1) w^I_x, w^I_x\rangle \\ \leq C\Big[\zeta\|w^I\|_1^2 + \langle|\bar{W}_{x_1}|w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big] + I_b^1. \tag{3.96}$$

Let us now treat the boundary term. First observe that using the parabolic equations, noting that A^0 is the diagonal-block form, we can estimate

$$(b^{jk} w^{II}_{x_k})_{x_j}(0,\tilde{x},t) \leq C\Big(|w^{II}_t| + |W_{x_j}| + |W|\Big)(0,\tilde{x},t) \tag{3.97}$$

and thus for $i \neq 1$

$$\begin{aligned} &\langle w^{II}_{x_i}, [(b^{1k} w^{II}_{x_k})_{x_i} + (m^1_2 \bar{W}_{x_1})_{x_i}]\rangle_0 \\ &\qquad \leq \int_{\mathbb{R}^d_0} |w^{II}_{x_i x_i}|\big(|W| + |w^{II}_{x_k}|\big) \\ &\qquad \leq C\int_{\mathbb{R}^d_0}\Big(|W|^2 + |w^{II}_x|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \end{aligned} \tag{3.98}$$

and for $i = 1$, using $b^{1k} = b^{k1}$, (3.97), and recalling here that we always use the sum convention,

$$\begin{aligned} \sum_k (b^{1k} w^{II}_{x_k})_{x_1} &= \frac{1}{2}\Big((b^{1k} w^{II}_{x_k})_{x_1} + (b^{j1} w^{II}_{x_1})_{x_j} + b^{1k}_{x_1} w^{II}_{x_k} - b^{j1}_{x_j} w^{II}_{x_1}\Big) \\ &= \frac{1}{2}\Big((b^{jk} w^{II}_{x_k})_{x_j} + b^{1k}_{x_1} w^{II}_{x_k} - b^{j1}_{x_j} w^{II}_{x_1} - \sum_{j\neq 1;\, k\neq 1}(b^{jk} w^{II}_{x_k})_{x_j}\Big) \\ &\leq C\big(|w^{II}_t| + |W| + |W_{x_j}| + |w^{II}_{\tilde{x}\tilde{x}}|\big). \end{aligned} \tag{3.99}$$

Therefore

$$\begin{aligned} &\langle w^{II}_{x_1},[(b^{1k} w^{II}_{x_k})_{x_1} + (m^1_2 \bar{W}_{x_1})_{x_1}]\rangle_0 \\ &\qquad \leq \epsilon\int_{\mathbb{R}^d_0}|w^I_x|^2 + C\int_{\mathbb{R}^d_0}\Big(|w^{II}_t|^2 + |W|^2 + |w^{II}_x|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \end{aligned}$$

For the first term in I_b^1, we consider each inflow/outflow case separately. For the outflow case, since $A^1_{11} \le -\theta_1 < 0$, we get

$$A^1 W_x \cdot W_x \le -\frac{\theta_1}{2}|w^I_x|^2 + C|w^{II}_x|^2.$$

Therefore

$$I^1_b \le -\frac{\theta_1}{2}\int_{\mathbb{R}^d_0} |w^I_x|^2 + \int_{\mathbb{R}^d_0} \Big(|W|^2 + |w^{II}_x|^2 + |w^{II}_t|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big). \tag{3.100}$$

Meanwhile, for the inflow case, since $A^1_{11} \ge \theta_1 > 0$, we have

$$|A^1 W_x \cdot W_x| \le C|W_x|^2.$$

In this case, the invertibility of A^1_{11} allows us to use the hyperbolic equation to derive

$$|w^I_{x_1}| \le C(|w^I_t| + |w^{II}_x| + |w^I_{\tilde{x}}|).$$

Therefore we get

$$I^1_b \le \int_{\mathbb{R}^d_0} \Big(|W|^2 + |W_t|^2 + |w^I_{\tilde{x}}|^2 + |w^{II}_x|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big). \tag{3.101}$$

Now apply the standard Sobolev inequality

$$|w(0)|^2 \le C\|w\|_{L^2(\mathbb{R})}(\|w_x\|_{L^2(\mathbb{R})} + \|w\|_{L^2(\mathbb{R})}) \tag{3.102}$$

to control the term $|w^{II}_{x_1}(0)|^2$ in I^1_b in both cases. We get

$$\int_{\mathbb{R}^d_0} |w^{II}_{x_1}|^2 \le \epsilon'\|w^{II}_{xx}\|^2_0 + C\|w^{II}_x\|^2_0. \tag{3.103}$$

Using this with $\epsilon' = \theta/8$, (3.94), and (3.100), the estimate (3.96) reads

$$\begin{aligned} \frac{d}{dt}\langle A^0 W_x, W_x\rangle + \|w^{II}_{xx}\|^2_0 + \langle \omega(x_1) w^I_x, w^I_x\rangle \\ \le C\Big(\zeta\|w^I\|^2_1 + \langle|\bar{W}_{x_1}|w^I, w^I\rangle + C(c_*)\|w^{II}\|^2_1\Big) + I^1_b \end{aligned} \tag{3.104}$$

where the (new) boundary term I_b^1 satisfies

$$I_b^1 \le -\frac{\theta_1}{2}\int_{\mathbb{R}_0^d} |w_x^I|^2 + C\int_{\mathbb{R}_0^d}\Big(|W|^2 + |w_{\tilde{x}}^{II}|^2 + |w_t^{II}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2\Big) \tag{3.105}$$

for the outflow case, and

$$I_b^1 \le \int_{\mathbb{R}_0^d}\Big(|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w_{\tilde{x}\tilde{x}}^{II}|^2\Big) \tag{3.106}$$

for the inflow case.

Higher order "Friedrichs-type" estimate

For any fixed multi-index $\alpha = (\alpha_{x_1}, \cdots, \alpha_{x_d})$, $\alpha_1 = 0, 1$, $|\alpha| = k = 2, ..., s$, by computing $\frac{d}{dt}\langle A^0\partial_x^\alpha W, \partial_x^\alpha W\rangle$ and following the same spirit as the above subsection, we easily obtain

$$\begin{aligned}&\frac{d}{dt}\langle A^0\partial_x^\alpha W, \partial_x^\alpha W\rangle + \theta\|\partial_x^{\alpha+1}w^{II}\|_0^2 + \langle \omega(x_1)\partial_x^\alpha w^I, \partial_x^\alpha w^I\rangle \\ &\qquad \le C\Big(C(c_*)\|w^{II}\|_k^2 + \zeta\|w^I\|_k^2 + \sum_{i=1}^{k-1}\langle|\bar{W}_{x_1}|\partial_x^i w^I, \partial_x^i w^I\rangle\Big) + I_b^\alpha\end{aligned} \tag{3.107}$$

where

$$\partial_x^\alpha := \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}, \quad \partial_x^{\alpha+1} := \sum_j \partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}\partial_{x_j}, \quad \partial_x^i = \sum_{|\beta|=i}\partial_{x_1}^{\beta_1}\cdots\partial_{x_d}^{\beta_d}$$

and the boundary term I_b^α satisfies

$$I_b^\alpha \le -\frac{\theta_1}{2}\int_{\mathbb{R}_0^d}|\partial_x^\alpha w^I|^2 + C\int_{\mathbb{R}_0^d}\Big(\sum_{i=1}^{[(k+1)/2]}|\partial_t^i w^{II}|^2 + \sum_{i=0}^{k-1}|\partial_x^i w^I|^2 + \sum_{i=0}^{k}|\partial_{\tilde{x}}^i w^{II}|^2\Big) \tag{3.108}$$

for the outflow case, and

$$I_b^\alpha \le \int_{\mathbb{R}_0^d}\Big(\sum_{i=0}^{k}|\partial_t^i w^I|^2 + \sum_{i=1}^{[(k+1)/2]}|\partial_t^i w^{II}|^2 + \sum_{i=0}^{k}|\partial_{\tilde{x}}^i W|^2\Big) \tag{3.109}$$

for the inflow case.

Now for α with $\alpha_1 = 2, ..., s$ we observe that the estimate (3.107) still holds. Indeed, using integration by parts and computing $\frac{d}{dt}\langle A^0\partial_x^\alpha W, \partial_x^\alpha W\rangle$ as above leaves the boundary terms as

$$I_b^\alpha := \frac{1}{2}\langle \partial_x^\alpha W, A^1\partial_x^\alpha W\rangle_0 - \langle \partial_x^\alpha w^{II}, \partial_x^\alpha[(b^{1k}w_{x_k}^{II}) + (m_2^1\bar{W}_{x_1})]\rangle_0. \tag{3.110}$$

Then we can use the parabolic equations to solve

$$w_{x_1x_1}^{II} = (b^{11})^{-1}\Big(A_2^0 w_t^{II} + A_2^j W_{x_j} - (b^{jk}w_{x_k}^{II})_{x_j} - b_{x_1}^{11}w_{x_1}^{II} - M_1\bar{W}_{x_1} - (m_2^j\bar{W}_{x_1})_{x_j}\Big).$$

Using this we can reduce the order of derivative with respect to x_1 in ∂_x^α to one, with the same spirit as (3.98) and (3.99). Finally we use the Sobolev embedding similar to (3.103) to obtain the estimate for the normal derivative ∂_{x_1}, and get the estimate for I_b^α as claimed in (3.108) and (3.109).

We recall next the following Kawashima-type estimate, presented in [59], to bound the term $\|w^I\|_k^2$ appearing on the left hand side of (3.107).

"Kawashima-type" estimate

Let $K(\xi)$ be the skew-symmetry in (3.83). Using Plancherel's identity and the equations (3.79), we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle K(\partial_x)\partial_x^r W, \partial_x^r\rangle &= \frac{1}{2}\frac{d}{dt}\langle iK(\xi)(i\xi)^r\hat{W}, (i\xi)^r\hat{W}\rangle \\ &= \langle iK(\xi)(i\xi)^r\hat{W}, (i\xi)^r\hat{W}_t\rangle \\ &= \langle (i\xi)^r\hat{W}, -K(\xi)(A_+^0)^{-1}\sum_j \xi_j A_+^j(i\xi)^r\hat{W}\rangle \\ &\quad + \langle iK(\xi)(i\xi)^r\hat{W}, (i\xi)^r\hat{H}\rangle,\end{aligned} \tag{3.111}$$

where

$$\begin{aligned}H :=& \sum_j \Big((A_+^0)^{-1}A_+^j - (A^0)^{-1}A^j\Big)W_{x_j} \\ &+ (A^0)^{-1}\Big(\sum_{jk}(B^{jk}W_{x_k})_{x_j} + M_1\bar{W}_{x_1} + \sum_j(M_2^j\bar{W}_{x_1})_{x_j}\Big).\end{aligned} \tag{3.112}$$

By using the fact that $|(A_+^0)^{-1}A_+^j - (A^0)^{-1}A^j| = \mathcal{O}(\zeta + |\bar{W}_{x_1}|)$, we can easily obtain

$$\|\partial_x^r H\|_0^2 \le C\|w^{II}\|_{r+2}^2 + C\sum_{k=0}^{r+1}\langle(\zeta+|\bar{W}_{x_1}|)\partial_x^k w^I, \partial_x^k w^I\rangle.$$

Meanwhile, applying (3.83) into the first term of the last line in (3.111), we get

$$\begin{aligned}\langle(i\xi)^r\hat{W}, -K(\xi)(A_+^0)^{-1}\sum_j \xi_j A_+^j (i\xi)^r\hat{W}\rangle \\ \ge \theta\||\xi|^{r+1}\hat{W}\|_0^2 - C\||\xi|^{r+1}\hat{w}^{II}\|_0^2 \\ = \theta\|\partial_x^{r+1}w^I\|_0^2 - C\|\partial_x^{r+1}w^{II}\|_0^2.\end{aligned}$$

Putting these estimates together into (3.111), we have obtained the high order "Kawashima-type" estimate:

$$\begin{aligned}\frac{d}{dt}\langle K(\partial_x)\partial_x^r W, \partial_x^r W\rangle \le& -\theta\|\partial_x^{r+1}w^I\|_0^2 + C\|w^{II}\|_{r+2}^2 \\ &+ C\sum_{i=0}^{r+1}\langle(\zeta+|\bar{W}_{x_1}|)\partial_x^i w^I, \partial_x^i w^I\rangle\end{aligned} \tag{3.113}$$

Final estimates

We are ready to conclude our result. First combining the estimate (3.104) with (3.88), we easily obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle\Big) \\ &\le -\Big(\frac{\theta}{8}\|w_{xx}^{II}\|_0^2 + \frac{1}{4}\langle\omega(x_1)w_x^I, w_x^I\rangle\Big) \\ &\quad + C\Big(\zeta\|w^I\|_1^2 + \langle|\bar{W}_{x_1}|w^I, w^I\rangle + C(c_*)\|w^{II}\|_1^2\Big) + I_b^1 \\ &\quad - \frac{M}{4}\Big(\langle\omega(x_1)w^I, w^I\rangle + \theta\|w_x^{II}\|_0^2\Big) + CM\zeta\|w^I\|_0^2 + MC(c_*)\|w^{II}\|_0^2 + MI_b^0\end{aligned}$$

By choosing M sufficiently large such that $M\theta \gg C(c_*)$, and noting that $c_*\theta_1|\bar{W}_{x_1}| \le$

$\omega(x_1)$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0W_x, W_x\rangle + M\langle A^0W, W\rangle\Big)\\
&\quad\le -\Big(\theta\|w^{II}\|_2^2 + \langle\omega(x_1)w^I, w^I\rangle + \langle\omega(x_1)w^I_x, w^I_x\rangle\Big)\\
&\qquad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) + I_b^1 + MI_b^0.
\end{aligned}
\tag{3.114}
$$

We shall treat the boundary terms later. Now we use the estimate (3.113) (for $r=0$) to absorb the term $\|\partial_x w^I\|_0$ into the left hand side. Indeed, fixing c_* large as above, adding (3.114) with (3.113) times ϵ, and choosing ϵ, ζ sufficiently small such that $\epsilon C(c_*) \ll \theta, \epsilon \ll 1$ and $\zeta \ll \epsilon\theta_2$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle A^0W_x, W_x\rangle + M\langle A^0W, W\rangle + \epsilon\langle KW_x, W\rangle\Big)\\
&\quad\le -\Big(\theta\|w^{II}\|_2^2 + \langle\omega(x_1)w^I, w^I\rangle + \langle\omega(x_1)w^I_x, w^I_x\rangle\Big)\\
&\qquad + C\Big(\zeta\|w^I\|_1^2 + C(c_*)\|w^{II}\|_0^2\Big) - \frac{\theta_2\epsilon}{2}\|w^I_x\|_0^2\\
&\qquad + C\epsilon\Big(\|w^{II}\|_2^2 + \zeta\|w^I\|_0^2 + \langle\omega(x_1)w^I, w^I\rangle + \langle\omega(x_1)w^I_x, w^I_x\rangle\Big) + I_b^1 + MI_b^0\\
&\quad\le -\frac{1}{2}\Big(\theta\|w^{II}\|_2^2 + \theta_2\epsilon\|w^I_x\|_0^2\Big) + C(c_*)\|W\|_0^2 + I_b
\end{aligned}
$$

where $I_b := I_b^1 + MI_b^0$.

In view of boundary terms I_b^0 and I_b^1, we treat the term I_b in each inflow/outflow case separately. Recall the inequality (3.103), $\|w^{II}_{x_1}\|_{0,0} \le C\|w^{II}\|_2$. Thus, using this, for the inflow case we have

$$
I_b^0 \le C(\|W\|_{0,0}^2 + \|w^{II}_x\|_{0,0}\|w^{II}\|_{0,0}) \le C(\|W\|_{0,0}^2 + \|w^{II}_{\tilde{x}}\|_{0,0}^2 + \epsilon\|w^{II}\|_2^2) \tag{3.115}
$$

and for the outflow case,

$$
\begin{aligned}
I_b^0 &\le -\frac{\theta_1}{2}\|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w^{II}_x\|_{0,0}\|w^{II}\|_{0,0})\\
&\le -\frac{\theta_1}{2}\|w^I\|_{0,0}^2 + C(\|w^{II}\|_{0,0}^2 + \|w^{II}_{\tilde{x}}\|_{0,0}^2 + \epsilon\|w^{II}\|_2^2).
\end{aligned}
\tag{3.116}
$$

Therefore these together with (3.105) and (3.106), using the good estimate of

$\|w^{II}_{xx}\|^2_0$, yield

$$I_b \le -\frac{\theta_1}{2}\int_{\mathbb{R}^d_0}(|w^I|^2 + |w^I_x|^2) + C\int_{\mathbb{R}^d_0}\Big(|w^{II}|^2 + |w^{II}_{\tilde{x}}|^2 + |w^{II}_t|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \tag{3.117}$$

for the outflow case, and

$$I^1_b \le \int_{\mathbb{R}^d_0}\Big(|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \tag{3.118}$$

for the inflow case.

Now by Cauchy-Schwarz's inequality, $|K(\xi)| \le C|\xi|$, and positive definiteness of A^0, it is easy to see that

$$\mathcal{E} := \langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle K(\partial_x)W, W\rangle \sim \|W\|^2_{H^1_\alpha} \sim \|W\|^2_{H^1}. \tag{3.119}$$

The last equivalence is due to the fact that α is bounded above and below away from zero. Thus the above yields

$$\frac{d}{dt}\mathcal{E}(W)(t) \le -\theta_3\mathcal{E}(W)(t) + C(c_*)\Big(\|W(t)\|^2_{L^2} + |\mathcal{B}_1(t)|^2\Big),$$

for some positive constant θ_3, which by the Gronwall inequality implies

$$\|W(t)\|^2_{H^1} \le Ce^{-\theta t}\|W_0\|^2_{H^1} + C(c_*)\int_0^t e^{-\theta(t-\tau)}\Big(\|W(\tau)\|^2_{L^2} + |\mathcal{B}_1(\tau)|^2\Big)d\tau, \tag{3.120}$$

where $W(x,0) = W_0(x)$ and

$$|\mathcal{B}_1(\tau)|^2 := \int_{\mathbb{R}^d_0}\Big(|W|^2 + |W_t|^2 + |W_{\tilde{x}}|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \tag{3.121}$$

for the inflow case, and

$$|\mathcal{B}_1(\tau)|^2 := \int_{\mathbb{R}^d_0}\Big(|w^{II}|^2 + |w^{II}_{\tilde{x}}|^2 + |w^{II}_t|^2 + |w^{II}_{\tilde{x}\tilde{x}}|^2\Big) \tag{3.122}$$

for the outflow case.

Similarly, by induction, we can derive the same estimates for W in H^s. To do that, let us define

$$\mathcal{E}_1(W) := \langle A^0 W_x, W_x\rangle + M\langle A^0 W, W\rangle + \epsilon\langle K W_x, W\rangle$$
$$\mathcal{E}_k(W) := \langle A^0 \partial_x^k W, \partial_x^k W\rangle + M\mathcal{E}_{k-1}(W) + \epsilon\langle K\partial_x^k W, \partial_x^{k-1} W\rangle, \qquad k \le s.$$

Then similarly by the Cauchy-Schwarz inequality, $\mathcal{E}_s(W) \sim \|W\|^2_{H^s}$, and by induction, we obtain

$$\frac{d}{dt}\mathcal{E}_s(W)(t) \le -\theta_3 \mathcal{E}_s(W)(t) + C(c_*)(\|W(t)\|^2_{L^2} + |\mathcal{B}_h(t)|^2),$$

for some positive constant θ_3, which by the Gronwall inequality yields

$$\|W(t)\|^2_{H^s} \le Ce^{-\theta t}\|W_0\|^2_{H^s} + C(c_*)\int_0^t e^{-\theta(t-\tau)}(\|W(\tau)\|^2_{L^2} + |\mathcal{B}_h(\tau)|^2)d\tau, \tag{3.123}$$

where $W(x,0) = W_0(x)$, and $\mathcal{B}_h$ are defined as in (3.12) and (3.13).

The general case

Following [37, 59], the general case that hypotheses (A1)-(A3) hold can easily be covered via following simple observations. First, we may express matrix A in (3.79) as

$$A^j(W + \bar{W}) = \hat{A}^j + (\zeta + |\bar{W}_{x_1}|)\begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1)\end{pmatrix}, \tag{3.124}$$

where $\hat{A}^j$ is a symmetric matrix obeying the same derivative bounds as described for A^j, $\hat{A}^1$ identical to A^1 in the 11 block and obtained in other blocks kl by

$$\begin{aligned} A^1_{kl}(W + \bar{W}) &= A^1_{kl}(\bar{W}) + A^1_{kl}(W + \bar{W}) - A^1_{kl}(\bar{W}) \\ &= A^1_{kl}(W_+) + \mathcal{O}(|W_x| + |\bar{W}_{x_1}|) \\ &= A^1_{kl}(W_+) + \mathcal{O}(\zeta + |\bar{W}_{x_1}|) \end{aligned} \tag{3.125}$$

and meanwhile, $\hat{A}^j$, $j \ne 1$, obtained by $A^j = A^j(W_+) + \mathcal{O}(\zeta + |\bar{W}_{x_1}|)$, similarly as in (3.125).

Replacing A^j by $\hat{A}^j$ in the k^{th} order Friedrichs-type bounds above, we find that the resulting error terms may be expressed as

$$\langle \partial_x^k \mathcal{O}(\zeta + |\bar{W}_{x_1}|)|W|, |\partial_x^{k+1} w^{II}|\rangle,$$

plus lower order terms, easily absorbed using Young's inequality, and boundary terms

$$\mathcal{O}(\sum_{i=0}^{k} |\partial_x^i w^{II}(0)||\partial_x^k w^I(0)|)$$

resulting from the use of integration by parts as we deal with the 12-block. However these boundary terms were already treated somewhere as before. Hence we can recover the same Friedrichs-type estimates obtained above. Thus we may relax $(A1')$ to $(A1)$.

Next, to relax $(A3')$ to $(A3)$, first we show that the symmetry condition $B^{jk} = B^{kj}$ is not necessary. Indeed, by writing

$$\sum_{jk} (B^{jk} W_{x_k})_{x_j} = \sum_{jk} \Big(\frac{1}{2}(B^{jk} + B^{kj}) W_{x_k}\Big)_{x_j} + \frac{1}{2}\sum_{jk} (B^{jk} - B^{kj})_{x_j} W_{x_k},$$

we can just replace B^{jk} by $\tilde{B}^{jk} := \frac{1}{2}(B^{jk} + B^{kj})$, satisfying the same $(A3')$, and thus still obtain the energy estimates as before, with a harmless error term (last term in the above identity). Next notice that the term $g(\tilde{W}_x) - g(\bar{W}_{x_1})$ in the perturbation equation may be Taylor expanded as

$$\begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_{x_1}) + g_1(\bar{W}_{x_1}, \tilde{W}_x) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix}$$

The first term, since it vanishes in the first component and since $|\bar{W}_x|$ decays at plus spatial infinity, yields by Young's inequality the estimate

$$\Big\langle \begin{pmatrix} 0 \\ g_1(\tilde{W}_x, \bar{W}_{x_1}) + g_1(\bar{W}_{x_1}, \tilde{W}_x) \end{pmatrix}, \begin{pmatrix} w_x^I \\ w_x^{II} \end{pmatrix} \Big\rangle \le C\Big(\langle (\zeta + |\bar{W}_{x_1}|) w_x^I, w_x^I\rangle + \|w_x^{II}\|_0^2\Big)$$

which can be treated in the Friedrichs-type estimates. The $(0, O(|W_x|^2)^T$ nonlinear term may be treated as other source terms in the energy estimates. Specifically, the worst-case term

$$\left\langle \partial_x^k W, \partial_x^k \begin{pmatrix} 0 \\ \mathcal{O}(|W_x|^2) \end{pmatrix} \right\rangle$$
$$= -\langle \partial_x^{k+1} w^{II}, \partial_x^{k-1}\mathcal{O}(|W_x|^2)\rangle - \partial_x^k w^{II}(0)\partial_x^{k-1}\mathcal{O}(|W_x|^2)(0)$$

may be bounded by

$$\|\partial_x^{k+1} w^{II}\|_{L^2}\|W\|_{W^{2,\infty}}\|W\|_{H^k} - \partial_x^k w^{II}(0)\partial_x^{k-1}\mathcal{O}(|W_x|^2)(0).$$

The boundary term will contribute to energy estimates in the form (3.110) of I_b^α, and thus we may use the parabolic equations to get rid of this term as we did in (3.98), (3.99). Thus, we may relax $(A3')$ to $(A3)$, completing the proof of the general case $(A1)-(A3)$ and the proposition. □

3.5 Nonlinear stability

Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$U_t - LU = \sum_j Q^j(U, U_x)_{x_j}, \tag{3.126}$$

where

$$\begin{aligned} Q^j(U,U_x) &= \mathcal{O}(|U||U_x| + |U|^2) \\ Q^j(U,U_x)_{x_j} &= \mathcal{O}(|U||U_x| + |U||U_{xx}| + |U_x|^2) \\ Q^j(U,U_x)_{x_j x_k} &= \mathcal{O}(|U||U_{xx}| + |U_x||U_{xx}| + |U_x|^2 + |U||U_{xxx}|) \end{aligned} \tag{3.127}$$

so long as $|U|$ remains bounded.

For boundary conditions written in U-coordinates, (B) gives

$$\begin{aligned} h = \tilde{h} - \bar{h} &= (\tilde{W}(U+\bar{U}) - \tilde{W}(\bar{U}))(0,\tilde{x},t) \\ &= (\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)U(0,\tilde{x},t) + \mathcal{O}(|U(0,\tilde{x},t)|^2). \end{aligned} \tag{3.128}$$

in inflow case, where $(\partial\tilde{W}/\partial\tilde{U})(\bar{U}_0)$ is constant and invertible, and

$$\begin{aligned} h = \tilde{h} - \bar{h} &= (\tilde{w}^{II}(U+\bar{U}) - \tilde{w}^{II}(\bar{U}))(0,\tilde{x},t) \\ &= (\partial\tilde{w}^{II}/\partial\tilde{U})(\bar{U}_0)U(0,\tilde{x},t) + \mathcal{O}(|U(0,\tilde{x},t)|^2) \\ &= m\begin{pmatrix}\bar{b}_1 & \bar{b}_2\end{pmatrix}(\bar{U}_0)U(0,\tilde{x},t) + \mathcal{O}(|U(0,\tilde{x},t)|^2) \\ &= mB(\bar{U}_0)U(0,\tilde{x},t) + \mathcal{O}(|U(0,\tilde{x},t)|^2) \end{aligned} \tag{3.129}$$

for some invertible constant matrix m.

Applying Lemma 3.3.9 to (3.126), we obtain

$$U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)\sum_j \partial_{x_j}Q^j(U,U_x)ds + \Gamma U(0,\tilde{x},t) \tag{3.130}$$

where $U(x,0) = U_0(x)$,

$$\Gamma U(0,\tilde{x},t) := \int_0^t \int_{\mathbb{R}^{d-1}} (\sum_j G_{y_j}B^{j1} + GA^1)(x,t-s;0,\tilde{y})U(0,\tilde{y},s)\,d\tilde{y}ds, \tag{3.131}$$

and G is the Green function of $\partial_t - L$.

Proof of Theorem 3.1.4. Define

$$\begin{aligned} \zeta(t) := \sup_s \Big(&|U(s)|_{L^2_x}(1+s)^{\frac{d-1}{4}} + |U(s)|_{L^\infty_x}(1+s)^{\frac{d}{2}} \\ &+ (|U(s)| + |U_x(s)| + |\partial^2_{\tilde{x}}U(s)|)_{L^{2,\infty}_{\tilde{x},x_1}}(1+s)^{\frac{d+1}{4}}\Big). \end{aligned} \tag{3.132}$$

We shall prove here that for all $t \geq 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \leq C(|U_0|_{L^1\cap H^s} + E_0 + \zeta(t)^2). \tag{3.133}$$

This bound together with continuity of $\zeta(t)$ implies that

$$\zeta(t) \leq 2C(|U_0|_{L^1\cap H^s} + E_0) \tag{3.134}$$

for $t \geq 0$, provided that $|U_0|_{L^1\cap H^s} + E_0 < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s, and the standard principle of continuation, there exists a solution $U \in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0,T)$ be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta > 0$. By Proposition 3.4.1, and the Sobolev embeding inequality $|U|_{W^{2,\infty}} \le C|U|_{H^s}$, we have

$$\begin{aligned} |U(t)|_{H^s}^2 &\le Ce^{-\theta t}|U_0|_{H^s}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big(|U(\tau)|_{L^2}^2 + |\mathcal{B}_h(\tau)|^2\Big)d\tau \\ &\le C(|U_0|_{H^s}^2 + E_0^2 + \zeta(t)^2)(1+t)^{-(d-1)/2}. \end{aligned} \tag{3.135}$$

and so the solution continues so long as ζ remains small, with bound (3.134), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (3.133). First by (3.130), we obtain

$$\begin{aligned} |U(t)|_{L^2} \le & |\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ & + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds + |\Gamma U(0,\tilde{x},t)|_{L^2} \\ \le & I_1 + I_2 + I_3 + |\Gamma U(0,\tilde{x},t)|_{L^2} \end{aligned} \tag{3.136}$$

where

$$\begin{aligned} I_1 :=& |\mathcal{S}(t)U_0|_{L^2} \le C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^3}, \\ I_2 :=& \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ \le& C\int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d-1}{4}}|Q^j(s)|_{L^{1,\infty}_{\tilde{x},x_1}}ds \\ \le& C\int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}}|U|_{H^1}^2 + (1+t-s)^{-\frac{d-1}{4}}\Big(|U|_{L^{2,\infty}_{\tilde{x},x_1}}^2 + |U_x|_{L^{2,\infty}_{\tilde{x},x_1}}^2\Big)ds \\ \le& C(|U_0|_{H^s}^2 + \zeta(t)^2)\int_0^t \Big[(1+t-s)^{-\frac{d-1}{4}-\frac{1}{2}}(1+s)^{-\frac{d-1}{2}} \\ &\quad + (1+t-s)^{-\frac{d-1}{4}}(1+s)^{-\frac{d+1}{2}}\Big]ds \\ \le& C(1+t)^{-\frac{d-1}{4}}(|U_0|_{H^s}^2 + \zeta(t)^2) \end{aligned}$$

and

$$
\begin{aligned}
I_3 :&= \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L^2} ds \\
&\le \int_0^t e^{-\theta(t-s)} |\partial_{x_j} Q^j(s)|_{H^3} ds \\
&\le C \int_0^t e^{-\theta(t-s)} (|U|_{L^\infty} + |U_x|_{L^\infty}) |U|_{H^5} ds \\
&\le C \int_0^t e^{-\theta(t-s)} |U|_{H^s}^2 ds \\
&\le C(|U_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\theta(t-s)} (1+s)^{-\frac{d-1}{2}} ds \\
&\le C(1+t)^{-\frac{d-1}{2}} (|U_0|_{H^s}^2 + \zeta(t)^2).
\end{aligned}
$$

Meanwhile, for the boundary term $|\Gamma U(0,\tilde{x},t)|_{L^2}$, we treat two cases separately. First for the inflow case, then by (3.128) we have

$$
|U(0,\tilde{x},t)| \le C|h(\tilde{x},t)| + \mathcal{O}(|U(0,\tilde{x},t)|^2),
$$

and thus $|U(0,\tilde{x},t)| \le C|h(\tilde{x},t)|$, provided that $|h|$ is sufficiently small. Therefore under the hypotheses on h in Theorem 3.1.4, Proposition 3.3.8 yields

$$
|\Gamma U(0,\cdot,\cdot)|_{L^2_x} \le C E_0 (1+t)^{-\frac{d-1}{4}}.
$$

Now for the outflow case, recall that in this case $G(x,t;0,\tilde{y}) \equiv 0$. Thus (3.131) simplifies to

$$
\Gamma U(0,\tilde{x},t) = \int_0^t \int_{\mathbb{R}^{d-1}} G_{y_1}(x,t-s;0,\tilde{y}) B^{11} U(0,\tilde{y},s)\, d\tilde{y} ds. \tag{3.137}
$$

To deal with this term, we shall use Proposition 3.3.8 as in the inflow case. In view of (3.129),

$$
|B^{11}U(0,\tilde{y},s)| \le C|h(\tilde{y},t)| + \mathcal{O}(|U(0,\tilde{y},s)|^2),
$$

and assumptions on h are imposed as in Theorem 3.1.3, so that (3.63) is satisfied. To check the last term $\mathcal{O}(|U(0)|^2)$, using the definition (3.132) of $\zeta(t)$, we have

$$
\begin{aligned}
|\mathcal{O}(|U(0,\tilde{y},s)|^2)|_{L^2} &\le C|U|_{L^\infty} |U|_{L^{2,\infty}_{\tilde{x},x_1}} \le C\zeta^2(t)(1+s)^{-\frac{d}{2}-\frac{d+1}{4}} \\
|\mathcal{O}(|U(0,\tilde{y},s)|^2)|_{L^\infty} &\le C|U|_{L^\infty}^2 \le C\zeta^2(t)(1+s)^{-d}
\end{aligned}
$$

and for the term $\mathcal{D}_h$ with h replaced by $\mathcal{O}(|U(0,\tilde{y},s)|^2)$, using the standard Hölder inequality to get

$$\begin{aligned}|\mathcal{D}_h|_{L^1_{\tilde{x}}} &\le C(|U|^2_{L^{2,\infty}} + |U_x|^2_{L^{2,\infty}} + |U_{\tilde{x}\tilde{x}}|^2_{L^{2,\infty}}) \le C\zeta^2(t)(1+s)^{-\frac{d+1}{2}}\\ |\mathcal{D}_h|_{H^{[(d-1)/2]+5}_{\tilde{x}}} &\le C|U|_{L^\infty}|U|_{H^s} \le C\zeta^2(t)(1+s)^{-d/2-(d-1)/4}.\end{aligned}$$

We remark here that Sobolev bounds (3.135) are not good enough for estimates of $\mathcal{D}_h$ in L^1, requiring a decay at rate $(1+t)^{-d/2-\epsilon}$ for the two-dimensional case (see Proposition 3.3.8). This is exactly why we have to keep track of $U_{\tilde{x}\tilde{x}}$ in $L^{2,\infty}$ norm in $\zeta(t)$ as well, to gain a bound as above for $\mathcal{D}_h$.

Therefore applying Proposition 3.3.8, we also obtain (3.137) for the outflow case. Combining these above estimates yields

$$|U(t)|_{L^2}(1+t)^{\frac{d-1}{4}} \le C(|U_0|_{L^1\cap H^s} + E_0 + \zeta(t)^2). \tag{3.138}$$

Next, we estimate

$$\begin{aligned}|U(t)|_{L^{2,\infty}_{\tilde{x},x_1}} \le& |\mathcal{S}(t)U_0|_{L^{2,\infty}_{\tilde{x},x_1}} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds\\ &+ \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds + |\Gamma U(0,\tilde{x},t)|_{L^{2,\infty}_{\tilde{x},x_1}}\\ \le& J_1 + J_2 + J_3 + |\Gamma U(0,\tilde{x},t)|_{L^{2,\infty}_{\tilde{x},x_1}}\end{aligned} \tag{3.139}$$

where

$$\begin{aligned}J_1 :=& |\mathcal{S}(t)U_0|_{L^{2,\infty}_{\tilde{x},x_1}} \le C(1+t)^{-\frac{d+1}{4}}|U_0|_{L^1\cap H^4}\\ J_2 :=& \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds\\ \le& C\int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d+1}{4}}|Q^j(s)|_{L^{1,\infty}_{\tilde{x},x_1}}\,ds\\ \le& C\int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}}|U|^2_{H^1} + (1+t-s)^{-\frac{d+1}{4}}\Big(|U|^2_{L^{2,\infty}_{\tilde{x},x_1}} + |U_x|^2_{L^{2,\infty}_{\tilde{x},x_1}}\Big)ds\\ \le& C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t (1+t-s)^{-\frac{d+1}{4}-\frac{1}{2}}(1+s)^{-\frac{d-1}{2}}\\ &\quad + (1+t-s)^{-\frac{d+1}{4}}(1+s)^{-\frac{d+1}{2}}\,ds\\ \le& C(1+t)^{-\frac{d+1}{4}}(|U_0|^2_{H^s} + \zeta(t)^2)\end{aligned}$$

and (by Moser's inequality)

$$\begin{aligned}
J_3 := &\int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds \\
\leq &\, C\int_0^t e^{-\theta(t-s)}|\partial_{x_j}Q^j(s)|_{H^4}ds \\
\leq &\, C\int_0^t e^{-\theta(t-s)}|U|_{L^\infty_x}|U|_{H^6}ds \\
\leq &\, C(|U_0|^2_{H^s}+\zeta(t)^2)\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{d}{2}}(1+s)^{-\frac{d-1}{4}}ds \\
\leq &\, C(1+t)^{-\frac{d+1}{4}}(|U_0|^2_{H^s}+\zeta(t)^2).
\end{aligned}$$

These estimates together with similar treatment for the boundary term yield

$$|U(t)|_{L^{2,\infty}_{\tilde{x},x_1}}(1+t)^{\frac{d+1}{4}} \leq C(|U_0|_{L^1\cap H^s}+E_0+\zeta(t)^2). \tag{3.140}$$

Similarly, we have the same estimate for $|U_x(t)|_{L^{2,\infty}_{\tilde{x},x_1}}$. Indeed, we have

$$\begin{aligned}
|U_x(t)|_{L^{2,\infty}_{\tilde{x},x_1}} \leq &|\partial_x\mathcal{S}(t)U_0|_{L^{2,\infty}_{\tilde{x},x_1}}+\int_0^t |\partial_x\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds \\
&+\int_0^t |\partial_x\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^{2,\infty}_{\tilde{x},x_1}}\,ds+|\partial_x\Gamma U(0,\tilde{x},t)|_{L^{2,\infty}_{\tilde{x},x_1}} \\
\leq &K_1+K_2+K_3+|\partial_x\Gamma U(0,\tilde{x},t)|_{L^{2,\infty}_{\tilde{x},x_1}}
\end{aligned} \tag{3.141}$$

where K_2 and K_3 are treated exactly in the same way as the treatment of J_2, J_3, yet in the first term of K_2 it is a bit better by a factor $t^{-1/2}$. Similar bounds hold for $|U_{\tilde{x}\tilde{x}}|$ in $L^{2,\infty}$, noting that there are no higher derivatives in x_1 involved and thus similar to those in (3.139).

Finally, we estimate the L^∞ norm of U. By Duhamel's formula (3.130), we obtain

$$\begin{aligned}
|U(t)|_{L^\infty} \leq &|\mathcal{S}(t)U_0|_{L^\infty}+\int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^\infty}ds \\
&+\int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^\infty}ds+|\Gamma U(0,\tilde{x},t)|_{L^\infty} \\
\leq &L_1+L_2+L_3+|\Gamma U(0,\tilde{x},t)|_{L^\infty}
\end{aligned} \tag{3.142}$$

where the boundary term is treated in the same way as above, and for $|\gamma| = [(d-$

1)/2] + 2,

$$\begin{aligned}
L_1 :&= |\mathcal{S}(t)U_0|_{L^\infty} \le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1\cap H^{|\gamma|+3}},\\
L_2 :&= \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^\infty}ds\\
&\le C\int_0^t (1+t-s)^{-\frac{d}{2}-\frac{1}{2}}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d}{2}}|Q^j(s)|_{L^{1,\infty}_{\tilde{x},x_1}}ds\\
&\le C\int_0^t (1+t-s)^{-\frac{d}{2}-\frac{1}{2}}|U|^2_{H^1} + (1+t-s)^{-\frac{d}{2}}\Big(|U|^2_{L^{2,\infty}_{\tilde{x},x_1}} + |U_x|^2_{L^{2,\infty}_{\tilde{x},x_1}}\Big)ds\\
&\le C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t \Big[(1+t-s)^{-\frac{d}{2}-\frac{1}{2}}(1+s)^{-\frac{d-1}{2}}\\
&\qquad + (1+t-s)^{-\frac{d}{2}}(1+s)^{-\frac{d+1}{2}}\Big]ds\\
&\le C(1+t)^{-\frac{d}{2}}(|U_0|^2_{H^s} + \zeta(t)^2)
\end{aligned}$$

and (again by Moser's inequality),

$$\begin{aligned}
L_3 :&= \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^\infty}ds\\
&\le \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{H^{|\gamma|}}ds\\
&\le \int_0^t e^{-\theta(t-s)}|\partial_x Q^j(s)|_{H^{|\gamma|+3}}ds\\
&\le C\int_0^t e^{-\theta(t-s)}|U|_{L^\infty}|U|_{H^{|\gamma|+5}}ds\\
&\le C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{d}{2}}(1+s)^{-\frac{d-1}{4}}ds\\
&\le C(1+t)^{-\frac{d}{2}}(|U_0|^2_{H^s} + \zeta(t)^2).
\end{aligned}$$

Therefore we have obtained

$$|U(t)|_{L^\infty_x}(1+t)^{\frac{d}{2}} \le C(|U_0|_{L^1\cap H^s} + E_0 + \zeta(t)^2) \tag{3.143}$$

and thus completed the proof of claim (3.133), and the main theorem. □

Chapter 4

STABILITY FOR SYSTEMS WITH VARIABLE MULTIPLICITIES

4.1 Introduction

4.1.1 Refined assumptions

Multi–dimensional stability results obtained in previous chapter do not apply to MHD layers for which the constant multiplicity assumption (H3) always fails to hold in dimensions $d \geq 2$. In this chapter, we are able to extend previous results to certain MHD layers where the following alternative hypothesis (H3′) holds.

Alternative Hypothesis H3′. We assume that

(H3′) The eigenvalues of $\sum_j \xi_j dF^j(U_+)$ are either semisimple and of constant multiplicity or totally nonglancing in the sense of [18], Definition 4.3.

Remark 4.1.1. Here we stress that we are able to drop the structural assumption (H4), which is needed for the earlier analyses of [58, 59, 57, 46] or Chapter 3.

In the treatment of the three–dimensional case, the analysis turns out to be quite delicate and we are able to establish the stability under the following additional (generic) hypothesis.

Additional Hypothesis H4′ (in 3D). We assume that

(H4$'$) In the case the eigenvalue $\lambda_k(\xi)$ of $\sum_j \xi_j dF^j(U_+)$ is semisimple and of constant multiplicity, we assume further that $\nabla_{\tilde{\xi}}\lambda_k \neq 0$ when $\partial_{\xi_1}\lambda_k = 0$, $\xi \neq 0$.

Remark 4.1.2. Genericity of our additional structural assumption (H4$'$) is clear. Indeed, violation of the condition would require d equations: $\partial_{\xi_j}\lambda_k(\xi) = 0$ for all $j = 1, \cdots, d$, whereas only $d-1$ parameters in $\xi \in \mathbb{R}^d \setminus \{0\}$ are varied as ξ may be constrained in the unit sphere S^d by homogeneity of $\lambda(\xi)$ in ξ. For further discussion, see Remark 4.3.4.

However, we have the following counterexample of K. Zumbrun in the two–dimensional case for which the hypothesis (H4$'$) fails. Counterexamples for higher–dimensional cases can be constructed similarly.

Counterexample 4.1.3. Let

$$A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

Then both A_1 and A_2 are clearly symmetric and do not commute. However, at $\xi_1 = 0$, the matrix $\xi_1 A_1 + \xi_2 A_2$ has an eigenvalue ($\lambda(\xi) \equiv 0$) such that $\nabla\lambda = 0$, violating (H4$'$).

4.1.2 Multi-dimensional results II

Our main results are as follows.

Theorem 4.1.4 (Linearized stability). *Assuming (A1)-(A3), (H0)-(H2), (H3$'$), (H4$'$), (B), and (D2), we obtain the asymptotic $L^1 \cap H^{[(d-1)/2]+2} \to L^p$ stability in dimensions $d \geq 3$, and any $2 \leq p \leq \infty$, with rates of decay*

$$\begin{aligned} |U(t)|_{L^2} &\leq C(1+t)^{-\frac{d-2}{4}-\epsilon}|U_0|_{L^1\cap L^2}, \\ |U(t)|_{L^p} &\leq C(1+t)^{-\frac{d-1}{2}(1-1/p)+\frac{1}{2p}-\epsilon}|U_0|_{L^1\cap H^{[(d-1)/2]+2}}, \end{aligned} \tag{4.2}$$

for some $\epsilon > 0$, provided that the initial perturbations U_0 are in $L^1 \cap H^{[(d-1)/2]+2}$, and zero boundary perturbations.

Theorem 4.1.5 (Nonlinear stability). *Assuming (A1)-(A3), (H0)-(H2), (H3$'$), (H4$'$), (B), and (D2), we obtain the asymptotic $L^1 \cap H^s \to L^p \cap H^s$ stability in dimensions*

$d \geq 3$, for $s \geq s(d)$ as defined in (H0), and any $2 \leq p \leq \infty$, with rates of decay

$$
\begin{aligned}
|\tilde{U}(t) - \bar{U}|_{L^p} &\leq C(1+t)^{-\frac{d-1}{2}(1-1/p)+\frac{1}{2p}-\epsilon}|U_0|_{L^1\cap H^s} \\
|\tilde{U}(t) - \bar{U}|_{H^s} &\leq C(1+t)^{-\frac{d-2}{4}-\epsilon}|U_0|_{L^1\cap H^s},
\end{aligned}
\tag{4.3}
$$

for some $\epsilon > 0$, provided that the initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ are sufficiently small in $L^1 \cap H^s$ and zero boundary perturbations.

Remark 4.1.6. As will be seen in the proof, the assumption (H4′) can be dropped in the case $d \geq 4$, though we then lose the factor $t^{-\epsilon}$ in the decay rate.

Our final main result gives the stability for the two–dimensional case that is not covered by the above theorems. We remark here that as shown in [58, 59], Hypothesis (H4) is automatically satisfied in dimensions $d = 1, 2$ and in any dimension for rotationally invariant problems. Thus, in treating the two–dimensional case, we assume this hypothesis without making any further restriction on structure of the systems. Also since the proof does not depend on dimension d, we state the theorem in a general form as follows.

Theorem 4.1.7 (Two-dimensional case or cases with (H4))**.** *Assume (A1)-(A3), (H0)-(H2), (H3′), (H4), (B), and (D2). We obtain asymptotic $L^1 \cap H^s \to L^p \cap H^s$ stability of $\bar{U}$ as a solution of (3.2) in dimension $d \geq 2$, for $s \geq s(d)$ as defined in (H0), and any $2 \leq p \leq \infty$, with rates of decay*

$$
\begin{aligned}
|\tilde{U}(t) - \bar{U}|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}|U_0|_{L^1\cap H^s} \\
|\tilde{U}(t) - \bar{U}|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^s},
\end{aligned}
\tag{4.4}
$$

provided that the initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ are sufficiently small in $L^1 \cap H^s$ and zero boundary perturbations. Similar statement holds for linearized stability.

Remark 4.1.8. The same results can be also obtained for nonzero boundary perturbations as treated in Chapter 3. In fact, though a bit of tricky, it was shown that estimates on solution operator (see Proposition 4.2.1) for homogenous boundary conditions are enough to treat nonzero boundary perturbations. Thus for sake of simplicity, we only treat zero boundary perturbations in this chapter.

Combining Theorems 4.1.4, 4.1.5, 4.1.7 and Proposition 3.1.2, we obtain the following small-amplitude stability result.

Corollary 4.1.9. *Assuming (A1)-(A3), (H0)-(H2), (H3′), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 4.1.4, 4.1.5, and 4.1.7.

4.2 Nonlinear stability

The linearized equations of (3.2) about the profile $\bar{U}$ are

$$U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j} \tag{4.5}$$

with initial data $U(0) = U_0$. Then, we obtain the following proposition, extending Proposition 3.5 of [46] under our weaker assumptions.

Proposition 4.2.1. *Under the hypotheses of Theorem 4.1.5, the solution operator $\mathcal{S}(t) := e^{Lt}$ of the linearized equations may be decomposed into low frequency and high frequency parts (see below) as $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ satisfying*

$$\begin{aligned}
|\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde{x}}^{\tilde{\beta}} f|_{L^2_x} &\le C(1+t)^{-(d-2)/4-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-2)/4-\epsilon/2}|f|_{L^{1,\infty}_{\tilde{x},x_1}} \\
|\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde{x}}^{\tilde{\beta}} f|_{L^{2,\infty}_{\tilde{x},x_1}} &\le C(1+t)^{-(d-1)/4-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-1)/4-\epsilon/2}|f|_{L^{1,\infty}_{\tilde{x},x_1}} \\
|\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde{x}}^{\tilde{\beta}} f|_{L^\infty_x} &\le C(1+t)^{-(d-1)/2-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-1)/2-\epsilon/2}|f|_{L^{1,\infty}_{\tilde{x},x_1}}
\end{aligned} \tag{4.6}$$

for some $\epsilon > 0$ and $\beta = (\beta_1, \tilde{\beta})$ with $\beta_1 = 0, 1$, and

$$|\partial_{x_1}^{\gamma_1}\partial_{\tilde{x}}^{\tilde{\gamma}} \mathcal{S}_2(t) f|_{L^2} \le Ce^{-\theta_1 t}|f|_{H^{|\gamma_1|+|\tilde{\gamma}|+3}}, \tag{4.7}$$

for $\gamma = (\gamma_1, \tilde{\gamma})$ with $\gamma_1 = 0, 1$.

We shall give a proof of Proposition 4.2.1 in Section 4.3. For the rest of this section, we give a rather straightforward proof of the first two main theorems using estimates of the solution operator stated in Proposition 4.2.1, following nonlinear arguments of [59, 46].

4.2.1 Proof of linearized stability

Applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ obtained in Proposition 4.2.1, we obtain

$$
\begin{aligned}
|U(t)|_{L^2} &\le |\mathcal{S}_1(t)U_0|_{L^2} + |\mathcal{S}_2(t)U_0|_{L^2} \\
&\le C(1+t)^{-\frac{d-2}{4}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde{x},x_1}}] + Ce^{-\eta t}|U_0|_{H^3} \\
&\le C(1+t)^{-\frac{d-2}{4}-\frac{\epsilon}{2}}|U_0|_{L^1\cap H^3}
\end{aligned}
\tag{4.8}
$$

and (together with Sobolev embedding)

$$
\begin{aligned}
|U(t)|_{L^\infty} &\le |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde{x},x_1}}] + C|\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde{x},x_1}}] + Ce^{-\eta t}|U_0|_{H^{[(d-1)/2]+2}} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}|U_0|_{L^1\cap H^{[(d-1)/2]+2}}.
\end{aligned}
\tag{4.9}
$$

These prove the bounds as stated in the theorem for $p=2$ and $p=\infty$. For $2<p<\infty$, we use the interpolation inequality between L^2 and L^∞.

4.2.2 Proof of nonlinear stability

Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$
U_t - LU = \sum_j Q^j(U,U_x)_{x_j}, \tag{4.10}
$$

where

$$
\begin{aligned}
Q^j(U,U_x) &= \mathcal{O}(|U||U_x| + |U|^2) \\
Q^j(U,U_x)_{x_j} &= \mathcal{O}(|U||U_x| + |U||U_{xx}| + |U_x|^2) \\
Q^j(U,U_x)_{x_j x_k} &= \mathcal{O}(|U||U_{xx}| + |U_x||U_{xx}| + |U_x|^2 + |U||U_{xxx}|)
\end{aligned}
\tag{4.11}
$$

so long as $|U|$ remains bounded.

Applying the Duhamel principle to (4.10), we obtain

$$
U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)\sum_j \partial_{x_j} Q^j(U,U_x)ds \tag{4.12}
$$

where $U(x,0)=U_0(x)$.

Proof of Theorem 4.1.5. Define

$$\begin{aligned}\zeta(t) := \sup_s \Big(&|U(s)|_{L^2_x}(1+s)^{\frac{d-2}{4}+\epsilon} + |U(s)|_{L^\infty_x}(1+s)^{\frac{d-1}{2}+\epsilon} \\ &+ (|U(s)|+|U_x(s)|)_{L^{2,\infty}_{\tilde{x},x_1}}(1+s)^{\frac{d-1}{4}+\epsilon}\Big).\end{aligned} \tag{4.13}$$

We shall prove here that for all $t\geq 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \leq C(|U_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{4.14}$$

This bound together with continuity of $\zeta(t)$ implies that

$$\zeta(t) \leq 2C|U_0|_{L^1\cap H^s} \tag{4.15}$$

for $t\geq 0$, provided that $|U_0|_{L^1\cap H^s} < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s, and the standard principle of continuation, there exists a solution $U\in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0,T)$ be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta>0$. Recalling the following energy estimate obtained in Proposition 3.4.1 and the Sobolev embeding inequality $|U|_{W^{2,\infty}}\leq C|U|_{H^s}$, we have

$$\begin{aligned}|U(t)|^2_{H^s} &\leq Ce^{-\theta t}|U_0|^2_{H^s} + C\int_0^t e^{-\theta(t-\tau)}|U(\tau)|^2_{L^2}d\tau \\ &\leq C(|U_0|^2_{H^s}+\zeta(t)^2)(1+t)^{-(d-2)/2-2\epsilon}.\end{aligned} \tag{4.16}$$

and so the solution continues so long as ζ remains small, with bound (4.15), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (4.14). First by (4.12), we obtain

$$\begin{aligned}|U(t)|_{L^2} \leq &|\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ &+ \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds\end{aligned} \tag{4.17}$$

where $|\mathcal{S}(t)U_0|_{L^2} \le C(1+t)^{-\frac{d-1}{4}-\epsilon}|U_0|_{L^1\cap H^3}$ and

$$\begin{aligned}
&\int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\
&\quad\le C\int_0^t (1+t-s)^{-\frac{d-2}{4}-\frac{1}{2}-\epsilon}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d-2}{4}-\epsilon}|Q^j(s)|_{L^{1,\infty}_{\tilde{x},x_1}}ds \\
&\quad\le C\int_0^t (1+t-s)^{-\frac{d-2}{4}-\frac{1}{2}-\epsilon}|U|^2_{H^1} + (1+t-s)^{-\frac{d-2}{4}-\epsilon}\Big(|U|^2_{L^{2,\infty}_{\tilde{x},x_1}} + |U_x|^2_{L^{2,\infty}_{\tilde{x},x_1}}\Big)ds \\
&\quad\le C(|U_0|^2_{H^s}+\zeta(t)^2)\int_0^t \Big[(1+t-s)^{-\frac{d-2}{4}-\frac{1}{2}-\epsilon}(1+s)^{-\frac{d-2}{2}-2\epsilon} \\
&\qquad\quad + (1+t-s)^{-\frac{d-2}{4}-\epsilon}(1+s)^{-\frac{d-1}{2}-2\epsilon}\Big]ds \\
&\quad\le C(1+t)^{-\frac{d-2}{4}-\epsilon}(|U_0|^2_{H^s}+\zeta(t)^2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\
&\quad\le \int_0^t e^{-\theta(t-s)}|\partial_{x_j}Q^j(s)|_{H^3}ds \\
&\quad\le C\int_0^t e^{-\theta(t-s)}|U|^2_{H^s}ds \\
&\quad\le C(|U_0|^2_{H^s}+\zeta(t)^2)\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{d-2}{2}-2\epsilon}ds \\
&\quad\le C(1+t)^{-\frac{d-2}{2}-2\epsilon}(|U_0|^2_{H^s}+\zeta(t)^2).
\end{aligned}$$

Therefore, combining these above estimates yields

$$|U(t)|_{L^2}(1+t)^{\frac{d-2}{4}+\epsilon} \le C(|U_0|_{L^1\cap H^s}+\zeta(t)^2). \tag{4.18}$$

Similarly, we can obtain estimates for other norms of U in definition of ζ, and finish the proof of claim (4.14) and thus the main theorem. □

Remark 4.2.2. The decaying factor $t^{-\epsilon}$ is crucial in above analysis when $d = 3$. In fact, the main difficulty here comparing with the shock cases in [43] is to obtain a refined bound of solutions in L^∞. See further discussion in Section 4.3 below.

4.3 Linearized estimates

In this section, we shall give a proof of Proposition 4.2.1 or bounds on $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, where we use the same decomposition of solution operator $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ as in [58, 59].

4.3.1 High–frequency estimate

We first observe that our relaxed Hypothesis (H3′) and the dropped Hypothesis (H4) only play a role in low–frequency regimes. Thus, in course of obtaining the high–frequency estimate (4.7), we make here the same assumptions as were made in the previous chapter, and therefore the same estimate remains valid as claimed in (4.7) under our current assumptions. We omit to repeat its proof here, and refer the reader to Proposition 3.3.6.

In the remaining of this section, we shall focus on proving the bounds on low-frequency part $\mathcal{S}_1(t)$ of linearized solution operator.

Taking the Fourier transform in $\tilde{x} := (x_2, \ldots, x_d)$ of linearized equation (4.5), we obtain a family of eigenvalue ODE

$$\begin{aligned}\lambda U = L_{\tilde{\xi}} U := \overbrace{(B_{11}U')' - (A_1 U)'}^{L_0 U} - i\sum_{j\neq 1} A_j \xi_j U + i\sum_{j\neq 1} B_{j1}\xi_j U' \\ + i\sum_{k\neq 1}(B_{1k}\xi_k U)' - \sum_{j,k\neq 1} B_{jk}\xi_j\xi_k U.\end{aligned} \tag{4.19}$$

4.3.2 The GMWZ's L^2 stability estimate

Let $U = (u^I, u^{II})^T$ be a solution of resolvent equation $(L_{\tilde{\xi}} - \lambda)U = f$. Following [59, 18], consider the variable W as usual

$$W := \begin{pmatrix} w^I \\ w^{II} \\ w^{II}_{x_1} \end{pmatrix}$$

with $w^I := A_* u^I, w^{II} := b_1^{11} u^I + b_2^{11} u^{II}$, $A_* := A_{11}^1 - A_{12}^1 (b_2^{11})^{-1} b_1^{11}$. Then we can write equations of W as a first order system

$$\begin{aligned} \partial_{x_1} W &= \mathcal{G}(x_1, \lambda, \tilde{\xi}) W + F \\ \Gamma W &= 0 \text{ on } x_1 = 0. \end{aligned} \tag{4.20}$$

For small or bounded frequencies $(\lambda, \tilde{\xi})$, we use the MZ conjugation lemma (see [42, 41]). That is, given any $(\underline{\lambda}, \underline{\tilde{\xi}}) \in \mathbb{R}^{d+1}$, there is a smooth invertible matrix $\Phi(x_1, \lambda, \tilde{\xi})$ for $x_1 \geq 0$ and $(\lambda, \tilde{\xi})$ in a small neighborhood of $(\underline{\lambda}, \underline{\tilde{\xi}})$, such that (4.20) is equivalent to

$$\partial_{x_1} Y = \mathcal{G}_+(\lambda, \tilde{\xi}) Y + \tilde{F}, \quad \tilde{\Gamma}(\lambda, \tilde{\xi}) Y = 0 \tag{4.21}$$

where $\mathcal{G}_+(\lambda, \tilde{\xi}) := \tilde{\mathcal{G}}(+\infty, \lambda, \tilde{\xi}), W = \Phi Y, \tilde{F} = \Phi^{-1} F$ and $\tilde{\Gamma} Y := \Gamma \Phi Y$.

Next, there are smooth matrices $V(\lambda, \tilde{\xi})$ such that

$$V^{-1} \mathcal{G}_+ V = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} \tag{4.22}$$

with blocks $H(\lambda, \tilde{\xi})$ and $P(\lambda, \tilde{\xi})$ satisfying the eigenvalues μ of P in $\{|\Re e \mu| \geq c > 0\}$ and

$$\begin{aligned} H(\lambda, \tilde{\xi}) &= H_0(\lambda, \tilde{\xi}) + \mathcal{O}(\rho^2) \\ H_0(\lambda, \tilde{\xi}) : &= -(A_+^1)^{-1} \Big((i\tau + \gamma) A_+^0 + \sum_{j=2}^d i \xi_j A_+^j \Big), \end{aligned}$$

with $\lambda = \gamma + i\tau$. We later often use the polar coordinate notation $\zeta = (\tau, \gamma, \tilde{\xi}), \zeta = \rho \hat{\zeta}$, where $\hat{\zeta} = (\hat{\tau}, \hat{\gamma}, \hat{\tilde{\xi}})$ and $\hat{\zeta} \in S^d$.

Define variables $Z = (u_H, u_P)^T$ as $W = \Phi Y = \Phi V Z$, $\bar{\Gamma} Z := \Gamma \Phi V Z$, and $(f_H, f_P)^T = V^{-1} \tilde{F}$. We have

$$\partial_{x_1} \begin{pmatrix} u_H \\ u_P \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} u_H \\ u_P \end{pmatrix} + \begin{pmatrix} f_H \\ f_P \end{pmatrix}, \quad \bar{\Gamma} Z = 0. \tag{4.23}$$

Then the maximal stability estimate for the low frequency regimes in [18] states that

$$(\gamma + \rho^2) |u_H|_{L^2}^2 + |u_P|_{L^2}^2 + |u_H(0)|^2 + |u_P(0)|^2 \lesssim \langle |f_H|, |u_H| \rangle + \langle |f_P|, |u_P| \rangle, \tag{4.24}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard L^2 product over $[0,\infty)$, that is,

$$\langle f,g\rangle = \int_0^\infty f(x_1)g(x_1)dx_1, \qquad \forall\ f,g\in L^2(0,\infty).$$

We note that in the final step there in [17], the standard Young's inequality has been used to absorb all terms of (u_H,u_P) into the left-hand side, leaving the L^2 norm of F alone in the right hand side. For our purpose, we shall keep it as stated in (4.24). Here, by $f\lesssim g$, we mean $f\le Cg$, for some C independent of parameter ρ.

We remark here that the Kreiss' symmetrizers in [18] were constructed in a full neighborhood of the basepoint $(\underline{\xi},\underline{\lambda})$ even for $\Re\underline{\lambda}=0$ (see, e.g., Theorem 3.7, [18]). Thus, the estimate (4.24) is in fact available in any region of

$$\gamma \ge -\theta(|\tau|^2+|\tilde{\xi}|^2) \tag{4.25}$$

for θ sufficiently small. In what follows, we shall always assume that λ remains in the general region of (4.25).

In addition, as shown in [17], all of coordinate transformation matrices are uniformly bounded. Thus a bound on $Z=(u_H,u_P)^T$ would yield a corresponding bound on the solution U.

4.3.3 L^2 and L^∞ resolvent bounds

Changing variables as above and taking the inner product of each equation in (4.23) against u_H and u_P, respectively, and integrating the results over $[0,x_1]$, for $x_1>0$, we obtain

$$\begin{aligned}\frac{1}{2}|u_H(x_1)|^2 &= \frac{1}{2}|u_H(0)|^2+\Re e\int_0^{x_1}(H(\lambda,\tilde{\xi})u_H\cdot u_H+f_H\cdot u_H)dz,\\ \frac{1}{2}|u_P(x_1)|^2 &= \frac{1}{2}|u_P(0)|^2+\Re e\int_0^{x_1}(P(\lambda,\tilde{\xi})u_P\cdot u_P+f_P\cdot u_P)dz.\end{aligned} \tag{4.26}$$

This together with the facts that $|H|\le C\rho$ and $|P|\le C$ yields

$$\begin{aligned}|u_H|^2_{L^\infty(x_1)} &\lesssim |u_H(0)|^2+\rho|u_H|^2_{L^2}+\langle|f_H|,|u_H|\rangle,\\ |u_P|^2_{L^\infty(x_1)} &\lesssim |u_P(0)|^2+|u_P|^2_{L^2}+\langle|f_P|,|u_P|\rangle,\end{aligned} \tag{4.27}$$

and thus in view of (4.24) gives

$$(\gamma+\rho^2)|u_H|_{L^2}^2+|u_P|_{L^2}^2+\rho|u_H|_{L^\infty}^2+|u_P|_{L^\infty}^2\lesssim\langle|f_H|,|u_H|\rangle+\langle|f_P|,|u_P|\rangle. \tag{4.28}$$

Now applying the Young's inequality, we get

$$\langle|f_H|,|u_H|\rangle+\langle|f_P|,|u_P|\rangle\le(\epsilon|u_P|_{L^\infty}^2+C_\epsilon|f_P|_{L^1}^2)+\Big(\epsilon(\hat\gamma+\rho)|u_H|_{L^\infty}^2+\frac{C_\epsilon}{\hat\gamma+\rho}|f_H|_{L^1}^2\Big)$$

and thus for ϵ sufficiently small, together with (4.28),

$$(\gamma+\rho^2)|u_H|_{L^2}^2+|u_P|_{L^2}^2+(\hat\gamma+\rho)|u_H|_{L^\infty}^2+|u_P|_{L^\infty}^2\lesssim\frac{1}{\hat\gamma+\rho}|f_H|_{L^1}^2+|f_P|_{L^1}^2. \tag{4.29}$$

Therefore in term of $Z=(u_H,u_P)^t$,

$$|Z|_{L^\infty(x_1)}\le C(\hat\gamma+\rho)^{-1}|f|_{L^1}\quad\text{and}\quad|Z|_{L^2(x_1)}\le C(\hat\gamma+\rho)^{-3/2}|f|_{L^1}. \tag{4.30}$$

Unfortunately, unlike the shock cases (see [43]), bounds (4.30) are not enough for our need to close the analysis in dimension $d=3$. See Remark 4.2.2. In the following subsection, we shall derive better bounds for Z in both L^∞ and L^2 norms.

4.3.4 Refined L^2 and L^∞ resolvent bounds

With the same notations as above, we prove in this subsection that there hold refined resolvent bounds:

$$\begin{aligned}|Z|_{L^\infty(x_1)}&\lesssim(\hat\gamma+\rho)^{-1+\epsilon}(|f|_{L^1}+|f|_{L^\infty})\\|Z|_{L^2(x_1)}&\lesssim(\hat\gamma+\rho)^{-3/2+\epsilon}(|f|_{L^1}+|f|_{L^\infty})\end{aligned} \tag{4.31}$$

for some small $\epsilon>0$. We stress here that a refined factor ρ^ϵ in L^∞ is crucial in our analysis for three-dimensional case. See Remark 4.2.2.

Assumption (H3′) implies the following block structure (see [41, 18]).

Proposition 4.3.1 (Block structure; [18]). *For all $\underline{\hat\zeta}$ with $\underline{\hat\gamma}\ge0$ there is a neighborhood ω of $(\underline{\hat\zeta},0)$ in $S^d\times\overline{\mathbb{R}}_+$ and there are C^∞ matrices $T(\hat\zeta,\rho)$ on ω such that $T^{-1}H_0T$ has the block diagonal structure*

$$T^{-1}H_0T=H_B(\hat\zeta,\rho)=\rho\hat H_B(\hat\zeta,\rho) \tag{4.32}$$

with

$$\hat{H}_B(\hat{\zeta},\rho) = \begin{bmatrix} Q_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & Q_p \end{bmatrix} (\hat{\zeta},\rho) \tag{4.33}$$

with diagonal blocks Q_k of size $\nu_k \times \nu_k$ such that:

(i) (Elliptic modes) $\Re Q_k$ is either positive definite or negative definite.

(ii) (Hyperbolic modes) $\nu_k = 1$, $\Re Q_k = 0$ when $\hat{\gamma} = \rho = 0$, and $\partial_{\hat{\gamma}}(\Re Q_k)\partial_\rho(\Re Q_k) > 0$.

(iii) (Glancing modes) $\nu_k > 1$, Q_k has the following form:

$$Q_k(\hat{\zeta},\rho) = i(\underline{\mu}_k \mathrm{Id} + J) + i\sigma Q_k'(\hat{\tilde{\xi}}) + \mathcal{O}(\hat{\gamma}+\rho), \tag{4.34}$$

where $\sigma := |\hat{\tilde{\xi}} - \underline{\hat{\tilde{\xi}}}|$,

$$J := \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & 0 & 0 \end{bmatrix}, \qquad Q_k'(\hat{\tilde{\xi}}) := \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ q_2 & 0 & \cdots & 0 \\ & & \cdots & \\ q_{\nu_k} & 0 & \cdots & 0 \end{bmatrix} \tag{4.35}$$

$q_{\nu_k} \neq 0$, and the lower left hand corner a of Q_k satisfies $\partial_{\hat{\gamma}}(\Re a)\partial_\rho(\Re a) > 0$.

(iv) (Totally nonglancing modes) $\nu_k > 1$, eigenvalue of Q_k, when $\hat{\gamma} = \rho = 0$, is totally nonglancing, see Definition 4.3, [18].

Proof. For a proof, see for example [40], Theorem 8.3.1. It is also straightforward to see that for the case (iii),

$$q_{\nu_k}(\underline{\hat{\tilde{\xi}}}) = |\nabla_{\tilde{\xi}} D_k(\underline{\zeta}, \underline{\xi}_1)| = c|\nabla_{\tilde{\xi}}\lambda_k(\xi)|,$$

where c is a nonzero constant, $D_k(\zeta,\xi_1)$ is defined as $\det(iQ_k(\zeta) + \xi_1 Id)$, and $\lambda_k(\xi)$ is the zero of $D_k(\zeta,\xi_1)$ (recalling $\zeta = (\lambda,\tilde{\xi})$) satisfying

$$\partial_{\xi_1}\lambda_k = ... = \partial_{\xi_1}^{\nu_k-1}\lambda_k = 0, \quad \partial_{\xi_1}^{\nu_k}\lambda_k \neq 0 \qquad \text{at } (\underline{\tilde{\xi}}, \underline{\xi}_1).$$

Thus, assumption (H4′) guarantees the nonvanishing of q_{ν_k}. We skip the proof of other facts. □

We shall treat each mode in turn. The following simple lemma may be found useful.

Lemma 4.3.2. *Let U be a solution of $\partial_z U = QU + F$ with $U(+\infty) = 0$. Assume that there is a positive [resp., negative] symmetric matrix S such that*

$$\Re SQ := \frac{1}{2}(SQ + Q^* S^*) \geq \theta Id \tag{4.36}$$

for some $\theta > 0$, and $S \geq Id$ [resp., $-S \geq Id$]. Then there holds

$$\begin{aligned} |U|^2_{L^\infty} + \theta |U|^2_{L^2} &\lesssim |F|^2_{L^1} \\ \text{[resp., } |U|^2_{L^\infty} + \theta |U|^2_{L^2} &\lesssim |U(0)|^2 + |F|^2_{L^1} \text{].} \end{aligned} \tag{4.37}$$

Proof. Taking the inner product of the equation of U against SU and integrating the result over $[x_1, \infty]$ for the first case [resp., $[0, x_1]$ for the second case], we easily obtain the lemma. □

Thanks to Proposition 4.3.1, we can decompose U as follows

$$U = u_P + u_{H_e} + u_{H_h} + u_{H_g} + u_{H_t}, \tag{4.38}$$

corresponding to parabolic, elliptic, hyperbolic, glancing, or totally nonglancing modes.

Parabolic modes

Since spectrum of P is away from the imaginary axis, we can assume that

$$P(\lambda, \tilde{\xi}) = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$$

with $\pm \Re P_\pm \geq c > 0$. Therefore applying Lemma 4.3.2 with $S = Id$ or $-Id$ yields

$$\begin{aligned} &|u_{P_+}|^2_{L^\infty} + |u_{P_+}|^2_{L^2} \lesssim |F_{P_+}|^2_{L^1}, \\ &|u_{P_-}|^2_{L^\infty} + |u_{P_-}|^2_{L^2} \lesssim |u_{P_-}(0)|^2 + |F_{P_-}|^2_{L^1}. \end{aligned} \tag{4.39}$$

Elliptic modes

This is case (i) in Proposition 4.3.1 when the spectrum of Q_k lies in

$$\{\Re e\mu > \delta\} \qquad [\text{resp., } \{\Re e\mu < -\delta\}].$$

In this case, there are positive symmetric matrices $S^k(\hat{\zeta}, \rho)$, C^∞ on a neighborhood ω of $(\underline{\hat{\zeta}}, 0)$ and such that

$$\Re S^k Q^k \geq cId \qquad [\text{resp., } -\Re S^k Q^k \geq cId]$$

for $c > 0$. Thus, Lemma 4.3.2 again yields

$$\begin{aligned} |u_{H_{e+}}|^2_{L^\infty} + \rho|u_{H_{e+}}|^2_{L^2} &\lesssim |F_{H_{e+}}|^2_{L^1}, \\ |u_{H_{e-}}|^2_{L^\infty} + \rho|u_{H_{e-}}|^2_{L^2} &\lesssim |u_{H_{e-}}(0)|^2 + |F_{H_{e-}}|^2_{L^1}. \end{aligned} \tag{4.40}$$

Hyperbolic modes

This is case (ii) in Proposition 4.3.1. In this case, as shown in [40] we can write

$$Q^k(\hat{\zeta}, \rho) = q^k(\hat{\zeta})Id + \rho\mathcal{R}^k(\hat{\zeta}, \rho) \tag{4.41}$$

where q^k is purely imaginary when $\hat{\gamma} = 0$, $\dot{q}^k := \partial_{\hat{\gamma}}\Re e q^k(\hat{\zeta})$ does not vanish, and the spectrum of $\dot{q}^k\mathcal{R}^k(\hat{\zeta}, 0)$ is contained in the half space $\{\Re e\mu > 0\}$. Therefore, when $\dot{q}^k > 0$ [resp., $\dot{q}^k < 0$] and $(\zeta, \hat{\gamma})$ is sufficiently close to $(\hat{\zeta}, 0)$we have positive symmetric matrices $S^k(\hat{\zeta}, \rho)$ satisfying

$$\Re S^k Q^k \geq c(\hat{\gamma} + \rho)Id \qquad [\text{resp., } -\Re S^k Q^k \geq c(\hat{\gamma} + \rho)Id]$$

for $c > 0$. Thus, again by Lemma 4.3.2, we obtain

$$\begin{aligned} |u_{H_{h+}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{h+}}|^2_{L^2} &\lesssim |F_{H_{h+}}|^2_{L^1}, \\ |u_{H_{h-}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{h-}}|^2_{L^2} &\lesssim |u_{H_{h-}}(0)|^2 + |F_{H_{h-}}|^2_{L^1}. \end{aligned} \tag{4.42}$$

Totally nonglancing modes

This is case (iv) in Proposition 4.3.1. As constructed in [18], there exist symmetrizers S^k that are positive [resp. negative] definite when the mode is totally incoming [resp.

outgoing]. Denote $u_{H_{t+}}$ [resp., $u_{H_{t-}}$] associated with totally incoming [resp. outgoing] modes. Then similarly as in above, we also have

$$\begin{aligned}|u_{H_{t+}}|^2_{L^\infty} + (\gamma+\rho^2)|u_{H_{t+}}|^2_{L^2} &\lesssim |F_{H_{t+}}|^2_{L^1},\\ |u_{H_{t-}}|^2_{L^\infty} + (\gamma+\rho^2)|u_{H_{t-}}|^2_{L^2} &\lesssim |u_{H_{t-}}(0)|^2 + |F_{H_{t-}}|^2_{L^1}.\end{aligned} \tag{4.43}$$

Thus, putting these estimates together with noting that the stability estimate (4.24) already gives a bound on $|u(0)|$, we easily obtain sharp bounds on u in L^∞ and L^2 for all above cases:

$$|u_k|^2_{L^\infty} + \rho^2|u_k|^2_{L^2} \lesssim |f|^2_{L^1} + |u_{H_g}|_{L^\infty}|f|_{L^1}, \tag{4.44}$$

for all $k = P, H_e, H_h, H_t$.

Glancing modes

Hence, we remain to consider the final case: case (iii) in Proposition 4.3.1. Recall (4.34)

$$Q_k(\hat\zeta,\rho) = i(\underline{\mu}_k \mathrm{Id} + J) + i\sigma Q_k'(\hat{\tilde\xi}) + \mathcal{O}(\hat\gamma+\rho) \tag{4.45}$$

on a neighborhood of $(\underline{\zeta}, 0)$, where $\sigma = |\hat{\tilde\xi} - \underline{\hat{\tilde\xi}}|$. We consider two cases.

Case a. $\sigma \lesssim (\hat\gamma+\rho)^\epsilon$ for some small $\epsilon > 0$. Recall that we consider the reduced system:

$$\partial_{x_1} u_k = \rho Q_k(\hat\zeta,\rho)u_k + f_k \tag{4.46}$$

with $Q_k(\hat\zeta,\rho)$ having a form as in (4.45). It is clear that the L^p norm of u_k remains unchanged under the transformation u_k to $u_k e^{-i\underline{\mu}_k x_1}$. Thus, we can assume that $\underline{\mu}_k = 0$. Note that we have the following bounds by (4.30)

$$|u_k|_{L^\infty(x_1)} \lesssim (\hat\gamma+\rho)^{-1}|f|_{L^1} \quad \text{and} \quad |u_k|_{L^2(x_1)} \lesssim (\hat\gamma+\rho)^{-3/2}|f|_{L^1}. \tag{4.47}$$

To prove the refined bounds (4.31), we first observe that

$$|\partial_{x_1} u_k|_{L^\infty} \lesssim \rho|u_k|_{L^\infty} + |f_k|_{L^\infty} \lesssim |f|_{L^1} + |f|_{L^\infty},$$

where the last inequality is due to (4.47). Now, write $u_k = (u_{k,1},\cdots,u_{k,\nu_k})$. Thanks to the special form of Q_k in (4.45), we have

$$\partial_{x_1} u_{k,\nu_k} = i\rho\sigma Q_k'(\hat{\hat{\xi}})u_k + \mathcal{O}(\gamma+\rho^2)u_k + f_k. \tag{4.48}$$

Taking inner product of the equation (4.48) against $\partial_{x_1} u_{k,\nu_k}$, we easily obtain by applying the standard Young's inequality:

$$|\partial_{x_1} u_{k,\nu_k}|^2_{L^2} \lesssim \rho^2(\hat\gamma+\rho)^{2\epsilon}|u_k|^2_{L^2} + |f_k|_{L^1}|\partial_{x_1} u_{k,\nu_k}|_{L^\infty} \lesssim (\hat\gamma+\rho)^{-1+2\epsilon}|f|^2_{L^1} + |f|^2_{L^\infty}. \tag{4.49}$$

Similarly, for u_{k,ν_k-1} satisfying

$$\partial_{x_1} u_{k,\nu_k-1} = i\rho\sigma Q_k'(\hat{\hat{\xi}})u_k + i\rho u_{k,\nu_k} + \mathcal{O}(\gamma+\rho^2)u_k + f_k,$$

we have

$$|\partial_{x_1} u_{k,\nu_k-1}|^2_{L^2} \lesssim \rho^2(\hat\gamma+\rho)^{2\epsilon}|u_k|^2_{L^2} + \rho| < u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} > | + |f_k|_{L^1}|\partial_{x_1} u_{k,\nu_k}|_{L^\infty}. \tag{4.50}$$

Here, integration by parts and Young's inequality yield

$$\rho| < u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} > | \lesssim \rho|\partial_{x_1} u_{k,\nu_k}|_{L^2}|u_{k,\nu_k-1}|_{L^2} + \rho|u_k(0)|^2.$$

Thus, using the refined bound (4.49) and noting that

$$|u_k(0)|^2 \lesssim | < f, u_k > | \lesssim |f|_{L^1}|u_k|_{L^\infty} \lesssim (\hat\gamma+\rho)^{-1}|f|^2_{L^1},$$

we obtain

$$\rho| < u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} > | \lesssim \rho(\hat\gamma+\rho)^{-2+\epsilon}(|f|^2_{L^1} + |f|^2_{L^\infty})$$

Therefore, applying this estimate into (4.50), we get

$$|\partial_{x_1} u_{k,\nu_k-1}|^2_{L^2} \lesssim (\hat\gamma+\rho)^{-1+\epsilon}(|f|^2_{L^1} + |f|^2_{L^\infty}). \tag{4.51}$$

Using this refined bound, we can estimate the same for u_{k,ν_k-2}, u_{k,ν_k-3}, and so on. Thus, we obtain a refined bound for u_k:

$$|\partial_{x_1} u_k|^2_{L^2} \lesssim \rho^{-1+\epsilon}(|f|^2_{L^1} + |f|^2_{L^\infty}) \tag{4.52}$$

where ϵ may be changed in each finite step and smaller than the original one. This and the standard Sobolev imbedding yield

$$|u_k|_{L^\infty}^2 \lesssim |u_k|_{L^2}|\partial_{x_1} u_k|_{L^2} \lesssim (\hat\gamma+\rho)^{-2+\epsilon}(|f|_{L^1}^2+|f|_{L^\infty}^2) \tag{4.53}$$

which proves the L^∞ refined bound in (4.31) for Z. Using (4.53) into (4.28), we also obtain the refined bound in L^2 as claimed in (4.31):

$$|u_k|_{L^2}^2 \lesssim (\hat\gamma+\rho)^{-3+\epsilon}(|f|_{L^1}^2+|f|_{L^\infty}^2), \tag{4.54}$$

for some $\epsilon > 0$.

Case b. $\sigma \gtrsim (\hat\gamma+\rho)^\epsilon$ for some small ϵ in $(0,1/2)$. We shall diagonalize this block. Recall that

$$Q_k(\hat\zeta,\rho) = i\underline{\mu}_k \mathrm{Id} + i\begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ \sigma q_{\nu_k} & & 0 & 0 \end{bmatrix} + \mathcal{O}(\sigma). \tag{4.55}$$

Following [58, 59, 17], we diagonalize this glancing block by

$$u'_{H_g} := T_{H_g}^{-1} u_{H_g},$$

where $u_{H_g} := u_{H_{g+}} + u_{H_{g-}}$. Here $u_{H_{g\pm}}$ are defined as the projections of u_{H_g} onto the growing (resp. decaying) eigenspaces of $Q_k(\hat\zeta,\rho)$ in (4.55). We recall the following whose proof can be found in [58, 59] or Lemma 12.1, [17].

Lemma 4.3.3 (Lemma 12.1, [17])**.** *The diagonalizing transformation T_{H_g} may be chosen so that*

$$|T_{H_g}| \le C, \qquad |T_{H_g}^{-1}| \le C\beta, \qquad |T^{-1}_{H_g|_{H_{g-}}}| \le C\alpha \tag{4.56}$$

where α, β are defined as

$$\beta := \sigma^{-1+1/\nu_k}, \qquad \alpha := \sigma^{(1-[(\nu_k+1)/2])/\nu_k}, \tag{4.57}$$

and $T^{-1}_{H_g|_{H_{g-}}}$ denotes the restriction of $T^{-1}_{H_g}$ to subspace H_{g-}. In particular, $\beta\alpha^{-2} \ge 1$.

Simple calculations show that eigenvalues of Q_k are

$$\alpha_{k,j} = i\underline{\mu_k} + \pi_{k,j} + o(\sigma^{1/\nu_k}), \qquad j = 0, 1, ..., s-1. \tag{4.58}$$

Here, $\pi_{k,j} = \epsilon^j i(q_{\nu_k}\sigma)^{1/\nu_k}$, with $\epsilon = 1^{1/\nu_k}$. We can further change of coordinates if necessary to assume that

$$Q'_k := T_{H_g}^{-1} Q_k T_{H_g} = \mathrm{diag}(\alpha_{k,1}, \cdots, \alpha_{k,l}, \alpha_{k,l+1}, \cdots, \alpha_{k,\nu_k}) \tag{4.59}$$

with

$$\begin{aligned} -\Re e\ \alpha_{k,j} > 0, \quad j = 1, ..., l, \\ \Re e\ \alpha_{k,j} > 0, \quad j = l+1, ..., \nu_k. \end{aligned} \tag{4.60}$$

Hence, applying Lemma 4.3.2 to equations of u'_{H_g} with $S = Id$ or $S = -Id$, we easily obtain

$$\begin{aligned} |u'_{H_{g+}}|^2_{L^\infty} + \rho \min_j |\Re e\ \alpha_{k,j}||u'_{H_{g+}}|^2_{L^2} \lesssim |F'_{H_{g+}}|^2_{L^1}, \\ |u'_{H_{g-}}|^2_{L^\infty} + \rho \min_j |\Re e\ \alpha_{k,j}||u'_{H_{g-}}|^2_{L^2} \lesssim |u'_{H_{g-}}(0)|^2 + |F'_{H_{g-}}|^2_{L^1}. \end{aligned} \tag{4.61}$$

The diagonalized boundary condition $\Gamma' := \Gamma_a T_{H_g}$. By computing, we observe that

$$|\Gamma' u'_{H_{g-}}| = |\Gamma u_{H_{g-}}| \geq C^{-1}|u_{H_{g-}}| \geq \frac{C^{-1}|u'_{H_{g-}}|}{|T^{-1}_{H_g|_{H_{g-}}}|} \geq C^{-1}\alpha^{-1}||u'_{H_{g-}}|.$$

Thus,

$$|u'_{H_{g-}}| \leq C\alpha|\Gamma' u'_{H_{g-}}| \leq C\alpha(|\Gamma' u'| + |\Gamma' u'_+|) \leq C\alpha|u'_+|. \tag{4.62}$$

Using this estimate, (4.56), and (4.44), the estimate (4.61) yields

$$\alpha^{-2}|u_{H_g}|^2_{L^\infty} + \rho\alpha^{-2} \min_j |\Re e\ \alpha_{k,j}||u_{H_g}|^2_{L^2} \lesssim \beta^2|f|^2_{L^1}. \tag{4.63}$$

Recalling that α, β are defined as in (4.57) and the fact that we are in the case of $\sigma \geq \rho^\epsilon$ for some small $\epsilon > 0$, we get

$$|u_{H_g}|_{L^\infty} \leq C\alpha\beta|f|_{L^1} \leq C(\hat{\gamma} + \rho)^{-2\epsilon}|f|_{L^1}, \tag{4.64}$$

from which we obtain the refined bounds (4.31) for this case as well.

Remark 4.3.4. In case b) above, we use the nonvanishing of q_{ν_k} to make sure that σq_{ν_k} is order of σ in the neighborhood ω of $(\underline{\hat{\zeta}}, 0)$ so that the lower left hand entry of Q_k dominates and thus we can be sure to diagonalize the block. Otherwise, the other entries of Q_k in (4.55) may dominate and the behavior is not clear. The nonvanishing of q_{ν_k} is guaranteed by our additional Hypothesis (H4′) as shown in the proof of Proposition 4.3.1. This is only place in the paper where the assumption (H4′) is used.

4.3.5 $L^1 \to L^p$ estimates

We establish the $L^1 \to L^p$ resolvent bounds for solutions of eigenvalue equations $(L_{\tilde{\xi}} - \lambda)U = f$ in the low frequency regime; specifically, we are interested in regime of parameters restricting to the surface

$$\Gamma^{\tilde{\xi}} := \{\lambda \ : \ \Re e\lambda = -\theta_1(|\tilde{\xi}|^2 + |\Im m\lambda|^2)\}, \tag{4.65}$$

for $\theta_1 > 0$ and $|(\tilde{\xi}, \lambda)|$ sufficiently small. The curve $\Gamma^{\tilde{\xi}}$ was introduced in [58, equation (4.26)]. Introducing $\Gamma^{\tilde{\xi}}$ is in fact regarded as a key to the analysis of long-time stability in multidimensions. The main point here is that even though λ enters into the stable complex half-plane ($\{\Re e\lambda < 0\}$), $\Gamma^{\tilde{\xi}}$ remains outside of the essential spectrum of limiting linearized operators $L_{\tilde{\xi},\pm}$; see [59, Lemma 2.21].

We obtain the following:

Proposition 4.3.5 (Low-frequency bounds)**.** *Under the hypotheses of Theorem 4.1.5, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\partial_{x_1}^{\beta} f|_{L^p(x_1)} \le C\rho^{-1-1/p+\epsilon}[\rho^{\beta}|f|_{L^1(x_1)} + |f|_{L^\infty(x_1)}], \tag{4.66}$$

for all $2 \le p \le \infty$, $\beta = 0, 1$, and $\epsilon > 0$.

Proof. Recalling that $W = \Phi V Z$ and all coordinate transformation matrices are uniformly bounded, the refined bounds of Z therefore imply improved bounds for W and thus U. Bounds for L^p, $2 < p < \infty$, are obtained by interpolation inequality between L^2 and L^∞. Hence, we have proved the bounds for $\beta = 0$ as claimed.

For $\beta = 1$, we expect that $\partial_{x_1} f$ plays a role as "ρf" forcing. Recall that the eigenvalue equations $(L_{\tilde{\xi}} - \lambda)U = \partial_{x_1} f$ read

$$\begin{aligned}\overbrace{(B^{11}U_{x_1})_{x_1} - (A^1U)_{x_1}}^{L_0U} &-i\sum_{j\neq 1} A^j\xi_j U + i\sum_{j\neq 1} B^{j1}\xi_j U_{x_1} \\ &+ i\sum_{k\neq 1}(B^{1k}\xi_k U)_{x_1} - \sum_{j,k\neq 1} B^{jk}\xi_j\xi_k U - \lambda U = \partial_{x_1} f.\end{aligned} \tag{4.67}$$

Now modifying the nice argument of Kreiss-Kreiss presented in [30, 17], we write $U = V + U_1$, where V satisfies

$$(L_0 - \lambda_0)V = \partial_{x_1} f, \qquad x_1 \geq 0, \tag{4.68}$$

for $\lambda_0 = \rho$. Noting that A^1 and B^{11} depend on x_1 only, we thus can apply here the one–dimensional Green kernel bounds investigated in [56, 45].

Let $G^0_{\lambda_0}$ be the Green kernel of $\lambda_0 - L_0$. Observe that our assumptions as projected on one–dimensional situations (i.e., $\tilde{\xi} = 0$) are still the same as those in [45]. Thus, we apply Proposition 2.22 in [45] for (4.68), noting that $\lambda_0 = \rho$ is sufficiently small. After a simplification, we simply obtain

$$|\partial_{y_1} G^0_{\lambda_0}(x_1, y_1)| \leq Ce^{-\rho|x_1 - y_1|}(\rho + e^{-\theta|y_1|}). \tag{4.69}$$

Hence, employing Hausdorff-Young's inequality, we obtain

$$|V|_{L^p(x_1)} + |V_{x_1}|_{L^p(x_1)} \leq C\rho^{-1/p}[\rho|f|_{L^1(x_1)} + |f|_{L^\infty(x_1)}], \tag{4.70}$$

for all $1 \leq p \leq \infty$.

Now from $U_1 = U - V$ and equations of U and V, we observe that U_1 satisfies

$$(L_{\tilde{\xi}} - \lambda)U_1 = L(V, V_{x_1}), \tag{4.71}$$

where

$$\begin{aligned}L(V, V_{x_1}) :&= i\sum_{j\neq 1} A^j\xi_j V - i\sum_{j\neq 1} B^{j1}\xi_j V_{x_1} - i\sum_{k\neq 1}(B^{1k}\xi_k V)_{x_1} + \sum_{j,k\neq 1} B^{jk}\xi_j\xi_k V + (\lambda - \lambda_0)V \\ &= \rho\mathcal{O}(|V| + |V_{x_1}|).\end{aligned}$$

Therefore applying the result which we just proved for $\beta = 0$ to the equations (4.71), we obtain

$$\begin{aligned} |U_1|_{L^p(x_1)} &\le C\rho^{-1-1/p+\epsilon}\Big[|L(V,V_{x_1})|_{L^1(x_1)} + |L(V,V_{x_1})|_{L^\infty(x_1)}\Big] \\ &\le C\rho^{-1-1/p+\epsilon}\rho\Big[|V|_{L^q} + |V_{x_1}|_{L^q}\Big] \\ &\le C\rho^{-1/p+\epsilon}[|f|_{L^1(x_1)} + \rho^{-1}|f|_{L^\infty(x_1)}]. \end{aligned} \tag{4.72}$$

Bounds on V and U_1 clearly give our claimed bounds on U by triangle inequality:

$$|U|_{L^p} \le |V|_{L^p} + |U_1|_{L^p}.$$

We obtain the proposition for the case $\beta = 1$, and thus complete the proof.

□

4.3.6 Estimates on the solution operator

In this subsection, we complete the proof of Proposition 4.2.1. As mentioned earlier, it suffices to prove the bounds for $\mathcal{S}_1(t)$, where the low frequency solution operator $\mathcal{S}_1(t)$ is defined as

$$\mathcal{S}_1(t) := \frac{1}{(2\pi i)^d}\int_{|\tilde\xi|\le r}\oint_{\Gamma^{\tilde\xi}} e^{\lambda t + i\tilde\xi\cdot\tilde x}(L_{\tilde\xi} - \lambda)^{-1} d\lambda d\tilde\xi. \tag{4.73}$$

Proof of bounds on $\mathcal{S}_1(t)$. Let $\hat u(x_1,\tilde\xi,\lambda)$ denote the solution of $(L_{\tilde\xi} - \lambda)\hat u = \hat f$, where $\hat f(x_1,\tilde\xi)$ denotes Fourier transform of f, and

$$u(x,t) := \mathcal{S}_1(t)f = \frac{1}{(2\pi i)^d}\int_{|\tilde\xi|\le r}\oint_{\Gamma^{\tilde\xi}} e^{\lambda t + i\tilde\xi\cdot\tilde x}(L_{\tilde\xi} - \lambda)^{-1}\hat f(x_1,\tilde\xi) d\lambda d\tilde\xi.$$

Using Parseval's identity, Fubini's theorem, the triangle inequality, and Proposition 4.3.5, we may estimate

$$\begin{aligned}|u|^2_{L^2(x_1,\tilde x)}(t) &= \frac{1}{(2\pi)^{2d}}\int_{x_1}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\lambda t}\hat u(x_1,\tilde\xi,\lambda)d\lambda\Big|^2 d\tilde\xi dx_1\\ &\le \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^2(x_1)}d\lambda\Big|^2 d\tilde\xi\\ &\le C[|f|_{L^1(x)}+|f|_{L^{1,\infty}_{\tilde x,x_1}}]^2\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-3/2+\epsilon}d\lambda\Big|^2 d\tilde\xi.\end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde\xi}$ by

$$\lambda(\tilde\xi,k)=ik-\theta_1(k^2+|\tilde\xi|^2),\quad k\in\mathbb{R},$$

we estimate

$$\begin{aligned}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-3/2+\epsilon}d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi}\Big|\int_{\mathbb{R}} e^{-\theta_1(k^2+|\tilde\xi|^2)t}\rho^{-3/2+\epsilon}dk\Big|^2 d\tilde\xi\\ &\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-1}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\ &\le Ct^{-(d-2)/2-\epsilon},\end{aligned}$$

noting that $\int_{\mathbb{R}^{d-1}} e^{-\theta|x|^2}|x|^{-\alpha}dx$ is finite, provided $\alpha<d-1$.

Similarly, we estimate

$$\begin{aligned}|u|^2_{L^{2,\infty}_{\tilde x,x_1}}(t) &\le \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^\infty(x_1)}d\lambda\Big|^2 d\tilde\xi\\ &\le C[|f|_{L^1(x)}+|f|_{L^{1,\infty}_{\tilde x,x_1}}]^2\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda\Big|^2 d\tilde\xi\end{aligned}$$

where, parametrizing $\Gamma^{\tilde\xi}$ as above, we have

$$\begin{aligned}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\ &\le Ct^{-(d-1)/2-\epsilon}.\end{aligned}$$

Finally, we estimate

$$|u|_{L^\infty_{\tilde x,x_1}}(t)\le \frac{1}{(2\pi)^d}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^\infty(x_1)}d\lambda d\tilde\xi$$
$$\le C[|f|_{L^1(x)}+|f|_{L^{1,\infty}_{\tilde x,x_1}}]\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda d\tilde\xi$$

where, parametrizing $\Gamma^{\tilde\xi}$ as above, we have

$$\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda d\tilde\xi \le \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}\int_{\mathbb R} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk d\tilde\xi$$
$$\le Ct^{-(d-1)/2-\epsilon/2}.$$

The x_1-derivative bounds follow similarly by using the version of the $L^1\to L^p$ estimates for $\beta_1=1$. The $\tilde x-$derivative bounds are straightforward by the fact that $\widehat{\partial^{\tilde\beta}_{\tilde x} f}=(i\tilde\xi)^{\tilde\beta}\hat f$. □

4.4 Two–dimensional case or cases with (H4)

In this section, we give an immediate proof of Theorem 4.1.7. Notice that the only assumption we make here that differs from those in Chapter 3 is the relaxed Hypothesis (H3′), treating the case of totally nonglancing characteristic roots, which is only involved in low–frequency estimates. That is to say, we only need to establish the $L^1\to L^p$ bounds in low-frequency regimes for this new case.

Proposition 4.4.1 (Low-frequency bounds; Proposition 3.3.3)**.** *Under the hypotheses of Theorem 4.1.7, for $\lambda\in\Gamma^{\tilde\xi}$ (see (3.37)) and $\rho:=|(\tilde\xi,\lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde\xi}-\lambda)^{-1}\partial^\beta_{x_1}f|_{L^p(x_1)}\le C\gamma_2\rho^{-2/p}\Big[\rho^\beta|\hat f|_{L^1(x_1)}+\beta|\hat f|_{L^\infty(x_1)}\Big], \tag{4.74}$$

for all $2\le p\le\infty$, $\beta=0,1$, and γ_2 is the diagonalization error (see [59], (5.40)) defined as

$$\gamma_2:=1+\sum_{j,\pm}\Big[\rho^{-1}|\Im m\lambda-\eta_j^\pm(\tilde\xi)|+\rho\Big]^{1/s_j-1}, \tag{4.75}$$

with $\eta_j^\pm, s_j$ as in (H4).

Proof. We only need to treat the new case: the totally nonglancing blocks Q_t^k. But this is already treated in our previous subsection, Subsection 4.3.4, yielding

$$\begin{aligned} |u_{H_{t+}}|^2_{L^\infty} + \rho^2 |u_{H_{t+}}|^2_{L^2} &\lesssim |F_{H_{t+}}|^2_{L^1}, \\ |u_{H_{t-}}|^2_{L^\infty} + \rho^2 |u_{H_{t-}}|^2_{L^2} &\lesssim |u_{H_{t-}}(0)|^2 + |F_{H_{t-}}|^2_{L^1}, \end{aligned} \tag{4.76}$$

where the boundary term $|u_{H_{t-}}(0)|^2$ can be treated by applying the L^2 stability estimate (4.24). Thus, together with a use of the standard interpolation inequality, we have obtained

$$|u_{H_t}|_{L^p(x_1)} \le C\gamma_2 \rho^{-1} |f|_{L^1(x_1)}, \tag{4.77}$$

for all $2 \le p \le \infty$ and γ_2 defined as in (4.75), yielding (4.74) for $\beta = 0$. For $\beta = 1$, we can follow the Kreiss–Kreiss trick as done in the proof of Proposition 4.3.5, completing the proof of Proposition 4.4.1. □

Proof of Theorem 4.1.7. Proposition 4.4.1 is Proposition 3.3.3 in Chapter 3 with an extension to the totally nonglancing cases. Thus, the theorem follows word by word from the proof in the previous chapter. □

Chapter 5

SPECTRAL STABILITY OF ISENTROPIC NAVIER–STOKES LAYERS

In this final chapter, we rigorously establish the Result 4 formally stated in the Introduction. The materials presented below are taken from [9].

5.1 Introduction

Consider the isentropic compressible Navier-Stokes equations

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2)_x + p(\rho)_x &= u_{xx} \end{aligned} \tag{5.1}$$

on the quarter-plane $x, t \geq 0$, where $\rho > 0$, u, p denote density, velocity, and pressure at spatial location x and time t, with γ-law pressure function

$$p(\rho) = a_0 \rho^\gamma, \qquad a_0 > 0,\ \gamma \geq 1, \tag{5.2}$$

and noncharacteristic constant "inflow" or "outflow" boundary conditions

$$(\rho, u)(0, t) \equiv (\rho_0, u_0), \qquad u_0 > 0 \tag{5.3}$$

or

$$u(0,t) \equiv u_0 \qquad u_0 < 0 \tag{5.4}$$

as discussed in [54, 19, 18]. The sign of the velocity at $x = 0$ determines whether characteristics of the hyperbolic transport equation $\rho_t + u\rho_x = f$ enter the domain (considering $f := -\rho u_x$ as a lower-order forcing term), and thus whether $\rho(0,t)$ should be prescribed. The variable-coefficient parabolic equation $\rho u_t - u_{xx} = g$ requires prescription of $u(0,t)$ in either case, with $g := -\rho(u^2/2)_x - p(\rho)_x$.

By comparison, the purely hyperbolic isentropic Euler equations

$$\begin{aligned} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + p(\rho)_x = 0 \end{aligned} \tag{5.5}$$

have characteristic speeds $a = u \pm \sqrt{p'(\rho)}$, hence, depending on the values of $(\rho, u)(0,t)$, may have one, two, or no characteristics entering the domain, hence require one, two, or no prescribed boundary values, respectively. In particular, there is a discrepancy between the number of prescribed boundary values for (5.1) and (5.5) in the case of mild inflow $u_0 > 0$ small (two for (5.1), one for (5.5)) or strong outflow $u_0 < 0$ large (one for (5.1), none for (5.5)), indicating the possibility of *boundary layers*, or asymptotically-constant stationary solutions of (5.1):

$$(\rho, u)(x,t) \equiv (\hat\rho, \hat u)(x), \qquad \lim_{z \to +\infty} (\hat\rho, \hat u)(z) = (\rho_+, u_+). \tag{5.6}$$

Indeed, existence of such solutions is straightforward to verify by direct computations on the (scalar) stationary-wave ODE; see [42, 54, 39, 33, 18, 19] or Section 5.2.3. These may be either of "expansive" type, resembling rarefaction wave solutions on the whole line, or "compressive" type, resembling viscous shock solutions.

A fundamental question is whether or not such boundary layer solutions are *stable* in the sense of PDE. For the expansive inflow case, it has been shown in [39] that *all* boundary layers are stable, independent of amplitude, by energy estimates similar to those used to prove the corresponding result for rarefactions on the whole line. Here, we concentrate on the complementary, *compressive case* (though see discussion, Section 5.1.1).

Linearized and nonlinear stability of general (expansive or compressive) *small-amplitude* noncharacteristic boundary layers of (5.1) have been established in [39, 52, 33, 19]. More generally, it has been shown in [19, 56, 45] that linearized and nonlinear stability are equivalent to spectral stability, or nonexistence of nonstable (nonnegative real part) eigenvalues of the linearized operator about the layer, for boundary layers of arbitrary amplitude. However, up to now the spectral stability of *large-amplitude compressive* boundary layers has remained largely undetermined.[1]

We resolve this question in the present paper by carrying out a systematic global study classifying the stability of all possible compressive boundary-layer solutions of (5.1). Our method of analysis is by a combination of asymptotic ODE techniques and numerical Evans function computations, following a basic approach introduced recently in [3, 24] for the study of the closely related shock wave case. Here, there are interesting complications associated with the richer class of boundary-layer solutions as compared to possible shock solutions, the delicate stability properties of the inflow case, and, in the outflow case, the nonstandard eigenvalue problem arising from reduction to Lagrangian coordinates.

As in [24], our strategy is to carry out rigorous analyses of asymptotic limits in the parameter space, thus truncating the computational domain, then as in [3] carry out an exhaustive numerical study on the remaining compact parameter regime. In the course of the first, analytical, step, we obtain convergence of the Evans function in the shock- and large-amplitude limits, and stability in the large-amplitude limit, for all $\gamma \geq 1$, the first rigorous stability result for other than the nearly-constant case. For a detailed description of our results both analytical and numerical see Section 5.3.

Our ultimate conclusions are, for both inflow and outflow conditions, that compressive boundary layers that are uniformly noncharacteristic in a sense to be made precise later (specifically, v_+ bounded away from 1, in the terminology of Section 5.2.3) are *unconditionally stable*, independent of amplitude, on the physical range $\gamma \in [1, 3]$ considered in our numerical computations. We show by energy estimates that *outflow boundary layers are stable also in the characteristic limit*. The omitted characteristic limit in the inflow case, analogous to the small-amplitude limit for the shock case should be treatable by the singular perturbation methods used in [49, 11] to treat the small-amplitude shock case; however, we do not consider this case here.

In the inflow case, our results, together with those of [39], completely resolve the

[1] See, however, the investigations of [54] on stability index, or parity of the number of nonstable eigenvalues of the linearized operator about the layer.

question of stability of isentropic (expansive or compressive) uniformly noncharacteristic boundary layers for $\gamma \in [1,3]$, yielding *unconditional stability independent of amplitude or type.* In the outflow case, we show stability of all compressive boundary layers without the assumption of uniform noncharacteristicity.

5.1.1 Discussion and open problems

The small-amplitude results obtained in [39, 33, 52, 19] are of "general type", making little use of the specific structure of the equations. Essentially, they all require that the difference between the boundary layer solution and its constant limit at $|x| = \infty$ be small in L^1 (alternatively, as in [39, 52], the more or less equivalent condition that $x\hat{v}'(x)$ be small in L^1; for monotone profiles, $\int_0^{+\infty} |\hat{v} - v_+|dx = \pm \int_0^{+\infty} (\hat{v} - v_+)dx = \mp \int_0^{+\infty} x\hat{v}'dx$). As pointed out in [19], this is the "gap lemma" regime in which standard asymptotic ODE estimates show that behavior is essentially governed by the limiting constant-coefficient equations at infinity, and thus stability may be concluded immediately from stability (computable by exact solution) of the constant layer identically equal to the limiting state. These methods do not suffice to treat either the (small-amplitude) characteristic limit or the large-amplitude case, which require more refined analyses. In particular, up to now, *there was no analysis considering boundary layers approaching a full viscous shock profile, not even a profile of vanishingly small amplitude.* Our analysis of this limit indicates why: the appearance of a small eigenvalue near zero prevents uniform estimates such as would be obtained by usual types of energy estimates.

By contrast, the large-amplitude results obtained here and (for expansive layers) in [39] make use of the specific form of the equations. In particular, both analyses make use of the advantageous structure in Lagrangian coordinates. The possibility to work in Lagrangian coordinates was first pointed out by Matsumura–Nishihara [39] in the inflow case, for which the stationary boundary transforms to a moving boundary with constant speed. Here we show how to convert the outflow problem also to Lagrangian coordinates, by converting the resulting variable-speed boundary problem to a constant-speed one with modified boundary condition. This trick seems of general use. In particular, it might be possible that the energy methods of [39] applied in this framework would yield unconditional stability of expansive boundary-layers, completing the analysis of the outflow case. Alternatively, this case could be attacked by the methods of the present paper. These are two interesting directions for future investigation.

In the outflow case, a further transformation to the "balanced flux form" introduced in [49], in which the equations take the form of the integrated shock equations, allows us to establish stability in the characteristic limit by energy estimates like those of [38] in the shock case. The treatment of the characteristic inflow limit by the methods of [49, 11] seems to be another extremely interesting direction for future study.

Finally, we point to the extension of the present methods to full (nonisentropic) gas dynamics and multidimensions as the two outstanding open problems in this area.

New features of the present analysis as compared to the shock case considered in [3, 24] are the presence of two parameters, strength and displacement, indexing possible boundary layers, vs. the single parameter of strength in the shock case, and the fact that the limiting equations in several asymptotic regimes possess zero eigenvalues, making the limiting stability analysis much more delicate than in the shock case. The latter is seen, for example, in the limit as a compressive boundary layer approaches a full stationary shock solution, which we show to be spectrally equivalent to the situation of unintegrated shock equations on the whole line. As the equations on the line possess always a translational eigenvalue at $\lambda = 0$, we may conclude existence of a zero at $\lambda = 0$ for the limiting equations and thus a zero *near* $\lambda = 0$ as we approach this limit, which could be stable or unstable. Similarly, the Evans function in the inflow case is shown to converge in the large-strength limit to a function with a zero at $\lambda = 0$, with the same conclusions; see Section 5.3 for further details.

To deal with this latter circumstance, we find it necessary to make use also of topological information provided by the stability index of [48, 14, 54], a mod-two index counting the parity of the number of unstable eigenvalues. Together with the information that there is at most one unstable zero, the parity information provided by the stability index is sufficient to determine whether an unstable zero does or does not occur. Remarkably, in the isentropic case we are able to compute explicitly the stability index for all parameter values, recovering results obtained by indirect argument in [54], and thereby completing the stability analysis in the presence of a single possibly unstable zero.

5.2 Preliminaries

We begin by carrying out a number of preliminary steps similar to those carried out in [3, 24] for the shock case, but complicated somewhat by the need to treat the boundary and its different conditions in the inflow and outflow case.

5.2.1 Lagrangian formulation.

The analyses of [24, 3] in the shock wave case were carried out in Lagrangian coordinates, which proved to be particularly convenient. Our first step, therefore, is to convert the Eulerian formulation (5.1) into Lagrangian coordinates similar to those of the shock case. However, standard Lagrangian coordinates in which the spatial variable $\tilde{x}$ is constant on particle paths are not appropriate for the boundary-value problem with inflow/outflow. We therefore introduce instead "psuedo-Lagrangian" coordinates

$$\tilde{x} := \int_0^x \rho(y,t)\,dy, \quad \tilde{t} := t, \tag{5.7}$$

in which the physical boundary $x = 0$ remains fixed at $\tilde{x} = 0$.

Straightforward calculation reveals that in these coordinates (5.1) becomes

$$\begin{aligned} v_t - s v_{\tilde{x}} - u_{\tilde{x}} &= \sigma(t) v_{\tilde{x}} \\ u_t - s u_{\tilde{x}} + p(v)_{\tilde{x}} - \left(\frac{u_{\tilde{x}}}{v}\right)_{\tilde{x}} &= \sigma(t) u_{\tilde{x}} \end{aligned} \tag{5.8}$$

on $\tilde{x} > 0$, where

$$s = -\frac{u_0}{v_0}, \ \sigma(t) = m(t) - s, \ m(t) := -\rho(0,t)u(0,t) = -u(0,t)/v(0,t), \tag{5.9}$$

so that $m(t)$ is the negative of the momentum at the boundary $x = \tilde{x} = 0$. From now on, we drop the tilde, denoting $\tilde{x}$ simply as x.

Inflow case

For the inflow case, $u_0 > 0$ so we may prescribe *two* boundary conditions on (5.8), namely

$$v|_{x=0} = v_0 > 0, \quad u|_{x=0} = u_0 > 0 \tag{5.10}$$

where both u_0, v_0 are constant.

Outflow case

For the outflow case, $u_0 < 0$ so we may prescribe *only one* boundary condition on (5.8), namely

$$u|_{x=0} = u_0 < 0. \tag{5.11}$$

Thus $v(0,t)$ is an unknown in the problem, which makes the analysis of the outflow case more subtle than that of the inflow case.

5.2.2 Rescaled coordinates

Our next step is to rescale the equations in such a way that coefficients remain bounded in the strong boundary-layer limit. Consider the change of variables

$$(x, t, v, u) \to (-\varepsilon s x, \varepsilon s^2 t, v/\varepsilon, -u/(\varepsilon s)), \tag{5.12}$$

where ε is chosen so that

$$0 < v_+ < v_- = 1, \tag{5.13}$$

where v_+ is the limit as $x \to +\infty$ of the boundary layer (stationary solution) $(\hat{v}, \hat{u})$ under consideration and v_- is the limit as $x \to -\infty$ of its continuation into $x < 0$ as a solution of the standing-wave ODE (discussed in more detail just below). Under the rescaling (5.12), (5.8) becomes

$$\begin{aligned} v_t + v_x - u_x &= \sigma(t) v_x, \\ u_t + u_x + (a v^{-\gamma})_x &= \sigma(t) u_x + \left(\frac{u_x}{v}\right)_x, \end{aligned} \tag{5.14}$$

where $a = a_0 \varepsilon^{-\gamma-1} s^{-2}$, $\sigma = -u(0,t)/v(0,t) + 1$, on respective domains

$$x > 0 \,(\text{inflow case}) \qquad x < 0 \,(\text{outflow case}).$$

5.2.3 Stationary boundary layers

Stationary boundary layers

$$(v, u)(x, t) = (\hat{v}, \hat{u})(x)$$

of (5.14) satisfy

$$
\begin{aligned}
&(a) \quad \hat{v}' - \hat{u}' = 0\\
&(b) \quad \hat{u}' + (a\hat{v}^{-\gamma})' = \left(\frac{\hat{u}'}{\hat{v}}\right)'\\
&(c) \quad (\hat{v}, \hat{u})|_{x=0} = (v_0, u_0)\\
&(d) \quad \lim_{x\to\pm\infty} (\hat{v}, \hat{u}) = (v, u)_{\pm},
\end{aligned}
\tag{5.15}
$$

where (d) is imposed at $+\infty$ in the inflow case, $-\infty$ in the outflow case and (imposing $\sigma = 0$) $u_0 = v_0$. Using (5.15)(a) we can reduce this to the study of the scalar ODE,

$$\hat{v}' + (a\hat{v}^{-\gamma})' = \left(\frac{\hat{v}'}{\hat{v}}\right)' \tag{5.16}$$

with the same boundary conditions at $x = 0$ and $x = \pm\infty$ as above. Taking the antiderivative of this equation yields

$$\hat{v}' = \mathcal{H}_C(\hat{v}) = \hat{v}(\hat{v} + a\hat{v}^{-\gamma} + C), \tag{5.17}$$

where C is a constant of integration.

Noting that $\mathcal{H}_C$ is convex, we find that there are precisely two rest points of (5.17) whenever boundary-layer profiles exist, except at the single parameter value on the boundary between existence and nonexistence of solutions, for which there is a degenerate rest point (double root of $\mathcal{H}_C$). Ignoring this degenerate case, we see that boundary layers terminating at rest point v_+ as $x \to +\infty$ must either continue backward into $x < 0$ to terminate at a second rest point v_- as $x \to -\infty$, or else blow up to infinity as $x \to -\infty$. The first case we shall call *compressive*, the second *expansive*.

In the first case, the extended solution on the whole line may be recognized as a standing viscous shock wave; that is, *for isentropic gas dynamics, compressive boundary layers are just restrictions to the half-line $x \geq 0$ [resp. $x \leq 0$] of standing shock waves.* In the second case, as discussed in [39], the boundary layers are somewhat analogous to rarefaction waves on the whole line. From here on, we concentrate exclusively on the compressive case.

With the choice $v_- = 1$, we may carry out the integration of (5.16) once more, this time as a definite integral from $-\infty$ to x, to obtain

$$\hat{v}' = H(\hat{v}) = \hat{v}(\hat{v} - 1 + a(\hat{v}^{-\gamma} - 1)), \tag{5.18}$$

where a is found by letting $x \to +\infty$, yielding

$$a = -\frac{v_+ - 1}{v_+^{-\gamma} - 1} = v_+^\gamma \frac{1 - v_+}{1 - v_+^\gamma}; \tag{5.19}$$

in particular, $a \sim v_+^\gamma$ in the large boundary layer limit $v_+ \to 0$. This is exactly the equation for viscous shock profiles considered in [24].

5.2.4 Eigenvalue equations

Linearizing (5.14) about $(\hat{v}, \hat{u})$, we obtain

$$\begin{aligned}
&\tilde{v}_t + \tilde{v}_x - \tilde{u}_x = \frac{\tilde{v}(0,t)}{v_0}\hat{v}' \\
&\tilde{u}_t + \tilde{u}_x - \left(\frac{h(\hat{v})}{\hat{v}^{\gamma+1}}\tilde{v}\right)_x - \left(\frac{\tilde{u}_x}{\hat{v}}\right)_x = \frac{\tilde{v}(0,t)}{v_0}\hat{u}' \\
&(\tilde{v}, \tilde{u})|_{x=0} = (\tilde{v}_0(t), 0) \\
&\lim_{x \to +\infty} (\tilde{v}, \tilde{u}) = (0, 0),
\end{aligned} \tag{5.20}$$

where $v_0 = \hat{v}(0)$,

$$h(\hat{v}) = -\hat{v}^{\gamma+1} + a(\gamma - 1) + (a+1)\hat{v}^\gamma \tag{5.21}$$

and $\tilde{v}, \tilde{u}$ denote perturbations of $\hat{v}, \hat{u}$.

Inflow case

In the inflow case, $\tilde{u}(0,t) = \tilde{v}(0,t) \equiv 0$, yielding

$$\begin{aligned}
\lambda v + v_x - u_x &= 0 \\
\lambda u + u_x - \left(\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} v\right)_x &= \left(\frac{u_x}{\hat{v}}\right)_x
\end{aligned} \tag{5.22}$$

on $x > 0$, with full Dirichlet conditions $(v, u)|_{x=0} = (0, 0)$.

Outflow case

Letting $\widetilde{U} := (\tilde{v}, \tilde{u})^T$, $\hat{U} := (\hat{v}, \hat{u})^T$, and denoting by $\mathcal{L}$ the operator associated to the linearization about boundary-layer $(\hat{v}, \hat{u})$,

$$\mathcal{L} := -\partial_x A(x) + \partial_x B(x)\partial_x, \tag{5.23}$$

where

$$A(x) = \begin{pmatrix} 1 & -1 \\ -h(\hat{v})/\hat{v}^{\gamma+1} & 1 \end{pmatrix}, \quad B(x) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{v}^{-1} \end{pmatrix}, \tag{5.24}$$

we have $\widetilde{U}_t - \mathcal{L}\widetilde{U} = \frac{\tilde{v}_0(t)}{v_0}\hat{U}'(x)$, with associated eigenvalue equation

$$\lambda\widetilde{U} - \mathcal{L}\widetilde{U} = \frac{\tilde{v}(0,\lambda)}{v_0}\hat{U}'(x), \tag{5.25}$$

where $\bar{U}' = (\hat{v}', \hat{u}')$.

To eliminate the nonstandard inhomogeneous term on the righthand side of (5.25), we introduce a "good unknown" (c.f. [2, 10, 16, 29])

$$U := \widetilde{U} - \lambda^{-1}\frac{\tilde{v}(0,\lambda)}{v_0}\hat{U}'(x). \tag{5.26}$$

Since $\mathcal{L}\bar{U}' = 0$ by differentiation of the boundary-layer equation, the system expressed in the good unknown becomes simply

$$\lambda U - \mathcal{L}U = 0 \quad \text{in } x < 0, \tag{5.27}$$

or, equivalently, (5.22) with boundary conditions

$$\begin{aligned} U|_{x=0} &= \frac{\tilde{v}(0,\lambda)}{v_0}(1 - \lambda^{-1}\hat{v}'(0), \, -\lambda^{-1}\hat{u}'(0))^T \\ \lim_{x\to+\infty} U &= 0. \end{aligned} \tag{5.28}$$

Solving for $u|_{x=0}$ in terms of $v|_{x=0}$ and recalling that $\hat{v}' = \hat{u}'$ by (5.18), we obtain finally

$$u|_{x=0} = \alpha(\lambda)v|_{x=0}, \qquad \alpha(\lambda) := \frac{-\hat{v}'(0)}{\lambda - \hat{v}'(0)}. \tag{5.29}$$

Remark 5.2.1. *Problems* (5.25) *and* (5.27)–(5.22) *are evidently equivalent for all*

$\lambda \neq 0$, but are not equivalent for $\lambda = 0$ (for which the change of coordinates to good unknown becomes singular). For, $U = \hat{U}'$ by inspection is a solution of (5.27), but is not a solution of (5.25). That is, we have introduced by this transformation a spurious eigenvalue at $\lambda = 0$, which we shall have to account for later.

5.2.5 Preliminary estimates

Proposition 5.2.2 ([3]). *For each $\gamma \geq 1$, $0 < v_+ \leq 1/12 < v_0 < 1$, (5.18) has a unique (up to translation) monotone decreasing solution $\hat{v}$ decaying to endstates $v_\pm$ with a uniform exponential rate for v_+ uniformly bounded away from $v_- = 1$. In particular, for $0 < v_+ \leq 1/12$,*

$$|\hat{v}(x) - v_+| \leq Ce^{-\frac{3(x-\delta)}{4}} \quad x \geq \delta, \tag{5.30a}$$

$$|\hat{v}(x) - v_-| \leq Ce^{\frac{(x-\delta)}{2}} \quad x \leq \delta, \tag{5.30b}$$

where δ is defined by $\hat{v}(\delta) = (v_- + v_+)/2$.

Proof. Existence and monotonicity follow trivially by the fact that (5.18) is a scalar first-order ODE with convex righthand side. Exponential convergence as $x \to +\infty$ follows by $H(v, v_+) = (v - v_+)\Big(v - \Big(\frac{1-v_+}{1-v_+^\gamma}\Big)\Big(\frac{1-\left(\frac{v_+}{v}\right)^\gamma}{1-\left(\frac{v_+}{v}\right)}\Big)\Big)$, whence $v - \gamma \leq \frac{H(v,v_+)}{v-v_+} \leq v - (1 - v_+)$ by $1 \leq \frac{1-x^\gamma}{1-x} \leq \gamma$ for $0 \leq x \leq 1$. Exponential convergence as $x \to -\infty$ follows by a similar, but more straightforward calculation, where, in the "centered" coordinate $\tilde{x} := x - \delta$, the constants $C > 0$ are uniform with respect to v_+, v_0. See [3] for details. $\square$

The following estimates are established in Appendices C.1 and C.2.

Proposition 5.2.3. *Nonstable eigenvalues λ of (5.22), i.e., eigenvalues with nonnegative real part, are confined for any $0 < v_+ \leq 1$ to the region*

$$\Lambda := \{\lambda : \Re e(\lambda) + |\Im m(\lambda)| \leq \frac{1}{2}\Big(2\sqrt{\gamma} + 1\Big)^2\} \tag{5.31}$$

for the inflow case, and to the region

$$\Lambda := \{\lambda : \Re e(\lambda) + |\Im m(\lambda)| \leq \max\{\frac{3\sqrt{2}}{2}, 3\gamma + \frac{3}{8}\} \tag{5.32}$$

for the outflow case.

5.2.6 Evans function formulation

Setting $w := \frac{u'}{\hat{v}} + \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}v - u$, we may express (5.22) as a first-order system

$$W' = A(x, \lambda)W, \tag{5.33}$$

where

$$A(x, \lambda) = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ \hat{v} & \hat{v} & f(\hat{v}) - \lambda \end{pmatrix}, \quad W = \begin{pmatrix} w \\ u - v \\ v \end{pmatrix}, \quad ' = \frac{d}{dx}, \tag{5.34}$$

where

$$f(\hat{v}) = \hat{v} - \hat{v}^{-\gamma}h(\hat{v}) = 2\hat{v} - a(\gamma - 1)\hat{v}^{-\gamma} - (a + 1), \tag{5.35}$$

with h as in (5.21) and a as in (5.19), or, equivalently,

$$f(\hat{v}) = 2\hat{v} - (\gamma - 1)\Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)\Big(\frac{v_+}{\hat{v}}\Big)^\gamma - \Big(\frac{1 - v_+}{1 - v_+^\gamma}\Big)v_+^\gamma - 1. \tag{5.36}$$

Remark 5.2.4. *The coefficient matrix A may be recognized as a rescaled version of the coefficient matrix $\mathcal{A}$ appearing in the shock case [3, 24], with*

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{pmatrix}.$$

The choice of variables $(w, u - v, v)^T$ may be recognized as the modified flux form of [49], adapted to the hyperbolic–parabolic case.

Eigenvalues of (5.22) correspond to nontrivial solutions W for which the boundary conditions $W(\pm\infty) = 0$ are satisfied. Because $A(x, \lambda)$ as a function of $\hat{v}$ is asymptotically constant in x, the behavior near $x = \pm\infty$ of solutions of (5.34) is governed by the limiting constant-coefficient systems

$$W' = A_\pm(\lambda)W, \qquad A_\pm(\lambda) := A(\pm\infty, \lambda), \tag{5.37}$$

from which we readily find on the (nonstable) domain $\Re\lambda \geq 0$, $\lambda \neq 0$ of interest that there is a one-dimensional unstable manifold $W_1^-(x)$ of solutions decaying at $x=-\infty$ and a two-dimensional stable manifold $W_2^+(x)\wedge W_3^+(x)$ of solutions decaying at $x=+\infty$, analytic in λ, with asymptotic behavior

$$W_j^{\pm}(x,\lambda)\sim e^{\mu_{\pm}(\lambda)x}V_j^{\pm}(\lambda) \tag{5.38}$$

as $x\to\pm\infty$, where $\mu_{\pm}(\lambda)$ and $V_j^{\pm}(\lambda)$ are eigenvalues and associated analytically chosen eigenvectors of the limiting coefficient matrices $A_{\pm}(\lambda)$. A standard choice of eigenvectors $V_j^{\pm}$ [14, 8, 5, 27], uniquely specifying $W_j^{\pm}$ (up to constant factor) is obtained by Kato's ODE [32], a linear, analytic ODE whose solution can be alternatively characterized by the property that there exist corresponding left eigenvectors $\widetilde{V}_j^{\pm}$ such that

$$(\widetilde{V}_j\cdot V_j)^{\pm}\equiv \text{constant},\quad (\widetilde{V}_j\cdot \dot{V}_j)^{\pm}\equiv 0, \tag{5.39}$$

where " $\dot{}$ " denotes $d/d\lambda$; for further discussion, see [32, 14, 27].

Inflow case

In the inflow case, $0\le x\le +\infty$, we define the *Evans function* D as the analytic function

$$D_{\text{in}}(\lambda):=\det(W_1^0,W_2^+,W_3^+)_{|x=0}, \tag{5.40}$$

where W_j^+ are as defined above, and W_1^0 is a solution satisfying the boundary conditions $(v,u)=(0,0)$ at $x=0$, specifically,

$$W_1^0|_{x=0}=(1,0,0)^T. \tag{5.41}$$

With this definition, eigenvalues of $\mathcal{L}$ correspond to zeroes of D both in location and multiplicity; moreover, the Evans function extends analytically to $\lambda=0$, i.e., to all of $\Re\lambda\ge 0$. See [1, 14, 36, 59] for further details.

Equivalently, following [48, 3], we may express the Evans function as

$$D_{\text{in}}(\lambda)=\big(\widetilde{W}_1^+\cdot W_1^0\big)_{|x=0}, \tag{5.42}$$

where $\widetilde{W}_1^+(x)$ spans the one-dimensional unstable manifold of solutions decaying at $x=+\infty$ (necessarily orthogonal to the span of $W_2^+(x)$ and $W_3^+(x)$) of the adjoint eigenvalue ODE

$$\widetilde{W}' = -A(x,\lambda)^* \widetilde{W}. \tag{5.43}$$

The simpler representation (5.42) is the one that we shall use here.

Outflow case

In the outflow case, $-\infty \le x \le 0$, we define the *Evans function* as

$$D_{\text{out}}(\lambda) := \det(W_1^-, W_2^0, W_3^0)_{|x=0}, \tag{5.44}$$

where W_1^- is as defined above, and W_j^0 are a basis of solutions of (5.33) satisfying the boundary conditions (5.29), specifically,

$$W_2^0|_{x=0} = (1,0,0)^T, \qquad W_3^0|_{x=0} = \Big(0, -\frac{\lambda}{\lambda - \hat{v}'(0)}, 1\Big)^T, \tag{5.45}$$

or, equivalently, as

$$D_{\text{out}}(\lambda) = \big(\widetilde{W}_1^0 \cdot W_1^-\big)_{|x=0}, \tag{5.46}$$

where

$$\widetilde{W}_1^0 = \Big(0, -1, -\frac{\bar{\lambda}}{\bar{\lambda} - \hat{v}'(0)}\Big)^T \tag{5.47}$$

is the solution of the adjoint eigenvalue ODE dual to W_2^0 and W_3^0.

Remark 5.2.5. *As discussed in Remark 5.2.1, D_{out} has a spurious zero at $\lambda = 0$ introduced by the coordinate change to "good unknown".*

5.3 Main results

We can now state precisely our main results.

5.3.1 The strong layer limit

Taking a formal limit as $v_+ \to 0$ of the rescaled equations (5.14) and recalling that $a \sim v_+^\gamma$, we obtain a limiting evolution equation

$$\begin{aligned} v_t + v_x - u_x &= 0, \\ u_t + u_x &= \left(\frac{u_x}{v}\right)_x \end{aligned} \tag{5.48}$$

corresponding to a *pressureless gas*, or $\gamma = 0$.

The associated limiting profile equation $v' = v(v-1)$ has explicit solution

$$\hat{v}^0(x) = \frac{1 - \tanh\left(\frac{x-\delta}{2}\right)}{2}, \tag{5.49}$$

$\hat{v}^0(0) = \frac{1-\tanh(-\delta/2)}{2} = v_0$; the limiting eigenvalue system is

$$W' = A^0(x,\lambda)W, \qquad A^0(x,\lambda) = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ \hat{v}^0 & \hat{v}^0 & f^0(\hat{v}^0) - \lambda \end{pmatrix}, \tag{5.50}$$

where $f^0(\hat{v}^0) = 2\hat{v}^0 - 1 = -\tanh\left(\frac{x-\delta}{2}\right)$.

Convergence of the profile and eigenvalue equations is *uniform* on any interval $\hat{v}^0 \geq \epsilon > 0$, or, equivalently, $x - \delta \leq L$, for L any positive constant, where the sequence of coefficient matrices is therefore a *regular perturbation* of its limit. Following [24], we call $x \leq L + \delta$ the "regular region". For $\hat{v}_0 \to 0$ on the other hand, or $x \to \infty$, the limit is less well-behaved, as may be seen by the fact that $\partial f/\partial \hat{v} \sim \hat{v}^{-1}$ as $\hat{v} \to v_+$, a consequence of the appearance of $\left(\frac{v_+}{\hat{v}}\right)$ in the expression (5.36) for f. Similarly, $A(x,\lambda)$ does not converge to $A_+(\lambda)$ as $x \to +\infty$ with uniform exponential rate independent of v_+, γ, but rather as $C\hat{v}^{-1}e^{-x/2}$. As in the shock case, this makes problematic the treatment of $x \geq L+\delta$. Following [24] we call $x \geq L+\delta$ the "singular region".

To put things in another way, the effects of pressure are not lost as $v_+ \to 0$, but rather pushed to $x = +\infty$, where they must be studied by a careful boundary-layer analysis. (Note: this is not a boundary-layer in the same sense as the background solution, nor is it a singular perturbation in the usual sense, at least as we have framed the problem here.)

Remark 5.3.1. *A significant difference from the shock case of [24] is the appearance of the second parameter v_0 that survives in the $v_+ \to 0$ limit.*

Inflow case

Observe that the limiting coefficient matrix

$$A^0_+(\lambda) := A^0(+\infty, \lambda) = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 0 & 0 & -1-\lambda \end{pmatrix}, \tag{5.51}$$

is nonhyperbolic (in ODE sense) for all λ, having eigenvalues $0, 0, -1-\lambda$; in particular, the stable manifold drops to dimension one in the limit $v_+ \to 0$, and so the prescription of an associated Evans function is *underdetermined*.

This difficulty is resolved by a careful boundary-layer analysis in [24], determining a special "slow stable" mode

$$V_2 := (1, 0, 0)^T$$

augmenting the "fast stable" mode

$$V_3 := (\lambda/\mu)(\lambda/\mu + 1), \lambda/\mu, 1)^T$$

associated with the single stable eigenvalue $\mu = -1 - \lambda$ of A^0_+. This determines a *limiting Evans function* $D^0_{\mathrm{in}}(\lambda)$ by the prescription (5.40), (5.38) of Section 5.2.6, or alternatively via (5.42) as

$$D^0_{\mathrm{in}}(\lambda) = \big(\widetilde{W}^{0+}_1 \cdot W^{00}_1\big)_{|x=0}, \tag{5.52}$$

with $\widetilde{W}^{0+}_1$ defined analogously as a solution of the adjoint limiting system lying asymptotically at $x = +\infty$ in direction

$$\widetilde{V}_1 := (0, -1, \bar\lambda/\bar\mu)^T \tag{5.53}$$

orthogonal to the span of V_2 and V_3, where " $^-$ " denotes complex conjugate, and W^{00}_1 defined as the solution of the limiting eigenvalue equations satisfying boundary condition (5.41), i.e., $(W^{00}_1)_{|x=0} = (1, 0, 0)^T$.

Outflow case

We have no such difficulties in the outflow case, since $A^0_- = A^0(-\infty)$ remains uniformly hyperbolic, and we may define a limiting Evans function D^0_{out} directly by (5.44), (5.38), (5.47), at least so long as v_0 remains bounded from zero. (As perhaps already hinted by Remark 5.3.1, there are complications associated with the double limit $(v_0, v_+) \to (0, 0)$.)

5.3.2 Analytical results

With the above definitions, we have the following main theorems characterizing the strong-layer limit $v_+ \to 0$ as well as the limits $v_0 \to 0$, 1.

Theorem 5.3.2. *For $v_0 \geq \eta > 0$ and λ in any compact subset of $\Re\lambda \geq 0$, $D_{\text{in}}(\lambda)$ and $D_{\text{out}}(\lambda)$ converge uniformly to $D^0_{\text{in}}(\lambda)$ and $D^0_{\text{out}}(\lambda)$ as $v_+ \to 0$.*

Theorem 5.3.3. *For λ in any compact subset of $\Re\lambda \geq 0$ and v_+ bounded from 1, $D_{\text{in}}(\lambda)$, appropriately renormalized by a nonvanishing analytic factor, converges uniformly as $v_0 \to 1$ to the Evans function for the (unintegrated) eigenvalue equations of the associated viscous shock wave connecting $v_- = 1$ to v_+; likewise, $D_{\text{out}}(\lambda)$, appropriately renormalized, converges uniformly as $v_0 \to 0$ to the same limit for λ uniformly bounded away from zero.*

By similar computations, we obtain also the following direct result.

Theorem 5.3.4. *Inflow boundary layers are stable for v_0 sufficiently small.*

We have also the following parity information, obtained by stability-index computations as in [54].[2]

Lemma 5.3.5 (Stability index)**.** *For any $\gamma \geq 1$, v_0, and v_+, $D_{\text{in}}(0) \neq 0$, hence the number of unstable roots of D_{in} is even; on the other hand $D^0_{\text{in}}(0) = 0$ and $\lim_{v_0 \to 0} D^0_{\text{in}}(\lambda) \equiv 0$. Likewise, $(D^0_{\text{in}})'(0)$, $D'_{\text{out}}(0) \neq 0$, $(D^0_{\text{out}})'(0) \neq 0$, hence the number of nonzero unstable roots of D^0_{in}, D_{out}, D^0_{out} is even.*

Finally, we have the following auxiliary results established by energy estimates in Appendices C.3, C.4, C.5, and C.6.

[2] Indeed, these may be deduced from the results of [54], taking account of the difference between Eulerian and Lagrangian coordinates.

Proposition 5.3.6. *The limiting Evans function $D^0_{\rm in}$ is nonzero for $\lambda \neq 0$ on $\Re e\lambda \geq 0$, for all $1 > v_0 > 0$. The limiting Evans function $D^0_{\rm out}$ is nonzero for $\lambda \neq 0$ on $\Re e\lambda \geq 0$, for $1 > v_0 > v_*$, where $v_* \approx 0.0899$ is determined by the functional equation $v_* = e^{-2/(1-v_*)^2}$.*

Proposition 5.3.7. *Compressive outflow boundary layers are stable for v_+ sufficiently close to* 1.

Proposition 5.3.8 ([39]). *Expansive inflow boundary layers are stable for all parameter values.*

Collecting information, we have the following analytical stability results.

Corollary 5.3.9. *For v_0 or v_+ sufficiently small, compressive inflow boundary layers are stable. For v_0 sufficiently small, v_+ sufficiently close to* 1*, or $v_0 > v_* \approx .0899$ and v_+ sufficiently small, compressive outflow layers are stable. Expansive inflow boundary layers are stable for all parameter values.*

Stability of inflow boundary layers in the characteristic limit $v_+ \to 1$ is not treated here, but should be treatable analytically by the asymptotic ODE methods used in [49, 11] to study the small-amplitude (characteristic) shock limit. This would be an interesting direction for future investigation. The characteristic limit is not accessible numerically, since the exponential decay rate of the background profile decays to zero as $|1 - v_+|$, so that the numerical domain of integration needed to resolve the eigenvalue ODE becomes infinitely large as $v_+ \to 1$.

Remark 5.3.10. *Stability in the noncharacteristic weak layer limit $v_0 \to v_+$ [resp.* 1*] in the inflow [outflow] case, for v_+ bounded away from the strong and characteristic limits* 0 *and* 1 *has already been established in [19, 52]. Indeed, it is shown in [19] that the Evans function converges to that for a constant solution, and this is a regular perturbation.*

Remark 5.3.11. *Stability of $D^0_{\rm in}$, $D^0_{\rm out}$ may also be determined numerically, in particular in the region $v_0 \leq v_*$ not covered by Proposition 5.3.6.*

5.3.3 Numerical results

The asymptotic results of Section 5.3.2 reduce the problem of (uniformly noncharacteristic, v_+ bounded away from $v_- = 1$) boundary layer stability to a bounded parameter range on which the Evans function may be efficiently computed numerically in a way that is uniformly well-conditioned; see [8]. Specifically, we may map a semicircle

$$\partial\big(\{\Re\lambda \geq 0\} \cap \{|\lambda| \leq 10\}\big)$$

enclosing Λ for $\gamma \in [1, 3]$ by D^0_{in}, D^0_{out}, D_{in}, D_{out} and compute the winding number of its image about the origin to determine the number of zeroes of the various Evans functions within the semicircle, and thus within Λ. For details of the numerical algorithm, see [3, 8].

In all cases, we obtain results consistent with stability; that is, a winding number of zero or one, depending on the situation. In the case of a single nonzero root, we know from our limiting analysis that this root may be quite near $\lambda = 0$, making delicate the direct determination of its stability; however, in this case we do not attempt to determine the stability numerically, but rely on the analytically computed stability index to conclude stability. See Section 5.6 for further details.

5.3.4 Conclusions

As in the shock case [3, 24], our results indicate *unconditional stability* of uniformly noncharacteristic boundary-layers for isentropic Navier–Stokes equations (and, for outflow layer, in the characteristic limit as well), despite the additional complexity of the boundary-layer case. However, two additional comments are in order, perhaps related. First, we point out that the apparent symmetry of Theorem 5.3.3 in the $v_0 \to 0$ outflow and $v_0 \to 1$ inflow limits is somewhat misleading. For, the limiting, shock Evans function possesses a single zero at $\lambda = 0$, indicating that stability of inflow boundary layers is somewhat delicate as $v_0 \to 1$: specifically, they have an eigenvalue near zero, which, though stable, is (since vanishingly small in the shock limit) not "very" stable. Likewise, the limiting Evans function D^0_{in} as $v_+ \to 0$ possesses a zero at $\lambda = 0$, with the same conclusions.

By contrast, the Evans functions of outflow boundary layers possess a spurious zero at $\lambda = 0$, so that convergence to the shock or strong-layer limit in this case implies the *absence* of any eigenvalues near zero, or "uniform" stability as $v_+ \to 0$. In this sense, strong outflow boundary layers appear to be more stable than inflow boundary layers. One may make interesting comparisons to physical attempts to stabilize laminar flow along an air- or hydro-foil by suction (outflow) along the boundary. See, for example, the interesting treatise [53].

Second, we point out the result of *instability* obtained in [54] for inflow boundary-layers of the full (nonisentropic) ideal-gas equations for appropriate ratio of the coefficients of viscosity and heat conduction. This suggests that the small eigenvalues of the strong inflow-layer limit may in some cases perturb to the unstable side. It would be very interesting to make these connections more precise, as we hope to do in future work.

5.4 Boundary-layer analysis

Since the structure of (5.34) is essentially the same as that of the shock case, we may follow exactly the treatment in [24] analyzing the flow of (5.34) in the singular region $x \to +\infty$. As we shall need the details for further computations (specifically, the proof of Theorem 5.3.4), we repeat the analysis here in full.

Our starting point is the observation that

$$A(x,\lambda) = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ \hat{v} & \hat{v} & f(\hat{v}) - \lambda \end{pmatrix} \tag{5.54}$$

is approximately block upper-triangular for $\hat{v}$ sufficiently small, with diagonal blocks $\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} f(\hat{v}) - \lambda \end{pmatrix}$ that are uniformly spectrally separated on $\Re e\lambda \geq 0$, as follows by

$$f(\hat{v}) \leq \hat{v} - 1 \leq -3/4. \tag{5.55}$$

We exploit this structure by a judicious coordinate change converting (5.34) to a system in exact upper triangular form, for which the decoupled "slow" upper lefthand 2×2 block undergoes a *regular perturbation* that can be analyzed by standard tools introduced in [49]. Meanwhile, the fast, lower righthand 1×1 block, since scalar, may be solved exactly.

5.4.1 Preliminary transformation

We first block upper-triangularize by a static (constant) coordinate transformation the limiting matrix

$$A_+ = A(+\infty, \lambda) = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ v_+ & v_+ & f(v_+) - \lambda \end{pmatrix} \tag{5.56}$$

at $x = +\infty$ using special block lower-triangular transformations

$$R_+ := \begin{pmatrix} I & 0 \\ v_+\theta_+ & 1 \end{pmatrix}, \qquad L_+ := R_+^{-1} = \begin{pmatrix} I & 0 \\ -v_+\theta_+ & 1 \end{pmatrix}, \tag{5.57}$$

where I denotes the 2×2 identity matrix and $\theta_+ \in \mathbb{C}^{1\times 2}$ is a 1×2 row vector.

Lemma 5.4.1. *On any compact subset of $\Re e\lambda \geq 0$, for each $v_+ > 0$ sufficiently small, there exists a unique $\theta_+ = \theta_+(v_+, \lambda)$ such that $\hat{A}_+ := L_+A_+R_+$ is upper block-triangular,*

$$\hat{A}_+ = \begin{pmatrix} \lambda(J + v_+\mathbb{1}\theta_+) & \lambda\mathbb{1} \\ 0 & f(v_+) - \lambda - \lambda v_+\theta_+\mathbb{1} \end{pmatrix}, \tag{5.58}$$

where $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, satisfying a uniform bound

$$|\theta_+| \leq C. \tag{5.59}$$

Proof. Setting the $2-1$ block of $\hat{A}_+$ to zero, we obtain the matrix equation

$$\theta_+(aI - \lambda J) = -\mathbb{1}^T + \lambda v_+\theta_+\mathbb{1}\theta_+,$$

where $a = f(v_+) - \lambda$, or, equivalently, the fixed-point equation

$$\theta_+ = \Big(- \mathbb{1}^T + \lambda v_+\theta_+\mathbb{1}\theta_+\Big)(aI - \lambda J)^{-1}. \tag{5.60}$$

By $\det(aI - \lambda J) = a^2 \neq 0$, $(aI - \lambda J)^{-1}$ is uniformly bounded on compact subsets of $\Re e\lambda \geq 0$ (indeed, it is uniformly bounded on all of $\Re e\lambda \geq 0$), whence, for $|\lambda|$ bounded and v_+ sufficiently small, there exists a unique solution by the Contraction Mapping

Theorem, which, moreover, satisfies (5.59). □

5.4.2 Dynamic triangularization

Defining now $Y := L_+W$ and

$$\hat{A}(x,\lambda) = L_+A(x,\lambda)R_+ =$$
$$\begin{pmatrix} \lambda(J + v_+\mathbb{1}\theta_+) & \lambda\mathbb{1} \\ (\hat{v} - v_+)\mathbb{1}^T - v_+(f(\hat{v}) - f(v_+))\theta_+ & f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1} \end{pmatrix},$$

we have converted (5.34) to an asymptotically block upper-triangular system

$$Y' = \hat{A}(x,\lambda)Y, \tag{5.61}$$

with $\hat{A}_+ = \hat{A}(+\infty,\lambda)$ as in (5.58). Our next step is to choose a *dynamic* transformation of the same form

$$\widetilde{R} := \begin{pmatrix} I & 0 \\ \widetilde{\Theta} & 1 \end{pmatrix}, \qquad \widetilde{L} := \widetilde{R}^{-1} = \begin{pmatrix} I & 0 \\ -\widetilde{\Theta} & 1 \end{pmatrix}, \tag{5.62}$$

converting (5.61) to an exactly block upper-triangular system, with $\widetilde{\Theta}$ uniformly exponentially decaying at $x = +\infty$: that is, a *regular perturbation* of the identity.

Lemma 5.4.2. *On any compact subset of $\Re e\lambda \geq 0$, for L sufficiently large and each $v_+ > 0$ sufficiently small, there exists a unique $\widetilde{\Theta} = \widetilde{\Theta}(x,\lambda,v_+)$ such that $\widetilde{A} := \widetilde{L}\hat{A}(x,\lambda)\widetilde{R} + \widetilde{L}'\widetilde{R}$ is upper block-triangular,*

$$\widetilde{A} = \begin{pmatrix} \lambda(J + v_+\mathbb{1}\theta_+ + \mathbb{1}\widetilde{\Theta}) & \lambda\mathbb{1} \\ 0 & f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1} - \lambda\widetilde{\Theta}\mathbb{1} \end{pmatrix}, \tag{5.63}$$

and $\widetilde{\Theta}(L) = 0$, satisfying a uniform bound

$$|\widetilde{\Theta}(x,\lambda,v_+)| \leq Ce^{-\eta x}, \qquad \eta > 0,\ x \geq L, \tag{5.64}$$

independent of the choice of L, v_+.

Proof. Setting the $2-1$ block of $\widetilde{A}$ to zero and computing

$$\widetilde{L}'\widetilde{R} = \begin{pmatrix} 0 & 0 \\ -\widetilde{\Theta}' & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ \widetilde{\Theta} & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\widetilde{\Theta}' & 0, \end{pmatrix}$$

we obtain the matrix equation

$$\widetilde{\Theta}' - \widetilde{\Theta}\big(aI - \lambda(J + v_+ \mathbb{1}\theta_+)\big) = \zeta + \lambda\widetilde{\Theta}\mathbb{1}\widetilde{\Theta}, \tag{5.65}$$

where $a(x) := f(\hat{v}) - \lambda - \lambda v_+ \theta_+ \mathbb{1}$ and the forcing term

$$\zeta := -(\hat{v} - v_+)\mathbb{1}^T + v_+(f(\hat{v}) - f(v_+))\theta_+$$

by derivative estimate $df/d\hat{v} \le C\hat{v}^{-1}$ together with the Mean Value Theorem is uniformly exponentially decaying:

$$|\zeta| \le C|\hat{v} - v_+| \le C_2 e^{-\eta x}, \qquad \eta > 0. \tag{5.66}$$

Initializing $\widetilde{\Theta}(L) = 0$, we obtain by Duhamel's Principle/Variation of Constants the representation (supressing the argument λ)

$$\widetilde{\Theta}(x) = \int_L^x S^{y\to x}(\zeta + \lambda\widetilde{\Theta}\mathbb{1}\widetilde{\Theta})(y)\, dy, \tag{5.67}$$

where $S^{y\to x}$ is the solution operator for the homogeneous equation

$$\widetilde{\Theta}' - \widetilde{\Theta}\big(aI - \lambda(J + v_+ \mathbb{1}\theta_+)\big) = 0,$$

or, explicitly,

$$S^{y\to x} = e^{\int_y^x a(y)dy} e^{-\lambda(J + v_+ \mathbb{1}\theta_+)(x-y)}.$$

For $|\lambda|$ bounded and v_+ sufficiently small, we have by matrix perturbation theory that the eigenvalues of $-\lambda(J + v_+\mathbb{1}\theta_+)$ are small and the entries are bounded, hence

$$|e^{-\lambda(J+v_+\mathbb{1}\theta_+)z}| \le Ce^{\epsilon z}$$

for $z \ge 0$. Recalling the uniform spectral gap $\Re e(a) = f(\hat{v}) - \Re e\lambda \le -1/2$ for

$\Re e\lambda \geq 0$, we thus have

$$|S^{y\to x}| \leq Ce^{-\eta(x-y)} \tag{5.68}$$

for some C, $\eta > 0$. Combining (5.66) and (5.68), we obtain

$$\begin{aligned} \Big| \int_L^x S^{y\to x}\zeta(y)\, dy \Big| &\leq \int_L^x C_2 e^{-\eta(x-y)} e^{-(\eta/2)y} dy \\ &= C_3 e^{-(\eta/2)x}. \end{aligned} \tag{5.69}$$

Defining $\widetilde{\Theta}(x) =: \tilde{\theta}(x) e^{-(\eta/2)x}$ and recalling (5.67) we thus have

$$\tilde{\theta}(x) = f + e^{(\eta/2)x} \int_L^x S^{y\to x} e^{-\eta y} \lambda \tilde{\theta} \mathbb{1} \tilde{\theta}(y)\, dy, \tag{5.70}$$

where $f := e^{(\eta/2)x} \int_L^x S^{y\to x}\zeta(y)\, dy$ is uniformly bounded, $|f| \leq C_3$, and $e^{(\eta/2)x} \int_L^x S^{y\to x} e^{-\eta y} \lambda \tilde{\theta} \mathbb{1} \tilde{\theta}(y)\, dy$ is contractive with arbitrarily small contraction constant $\epsilon > 0$ in $L^\infty[L, +\infty)$ for $|\tilde{\theta}| \leq 2C_3$ for L sufficiently large, by the calculation

$$\begin{aligned} &\Big| e^{(\eta/2)x} \int_L^x S^{y\to x} e^{-\eta y} \lambda \tilde{\theta}_1 \mathbb{1} \tilde{\theta}_1(y) - e^{(\eta/2)x} \int_L^x S^{y\to x} e^{-\eta y} \lambda \tilde{\theta}_2 \mathbb{1} \tilde{\theta}_2(y) \Big| \\ &\qquad \leq \Big| e^{(\eta/2)x} \int_L^x Ce^{-\eta(x-y)} e^{-\eta y}\, dy \Big| |\lambda| \|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty \\ &\qquad \leq e^{-(\eta/2)L} \Big| \int_L^x Ce^{-(\eta/2)(x-y)}\, dy \Big| |\lambda| \|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty \\ &\qquad = C_3 e^{-(\eta/2)L} |\lambda| \|\tilde{\theta}_1 - \tilde{\theta}_2\|_\infty \max_j \|\tilde{\theta}_j\|_\infty. \end{aligned}$$

It follows by the Contraction Mapping Principle that there exists a unique solution $\tilde{\theta}$ of fixed point equation (5.70) with $|\tilde{\theta}(x)| \leq 2C_3$ for $x \geq L$, or, equivalently (redefining the unspecified constant η), (5.64). □

5.4.3 Fast/Slow dynamics

Making now the further change of coordinates

$$Z = \widetilde{L} Y$$

and computing

$$
\begin{aligned}
(\widetilde{L}Y)' = \widetilde{L}Y' + \widetilde{L}'Y &= (\widetilde{L}A_+ + \widetilde{L}')Y, \\
&= (\widetilde{L}A_+\widetilde{R} + \widetilde{L}'\widetilde{R})Z,
\end{aligned}
$$

we find that we have converted (5.61) to a block-triangular system

$$
Z' = \widetilde{A}Z = \begin{pmatrix} \lambda(J + v_+\mathbb{1}\theta_+ + \mathbb{1}\widetilde{\Theta}) & \lambda\mathbb{1} \\ 0 & f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1} - \lambda\widetilde{\Theta}\mathbb{1} \end{pmatrix} Z, \tag{5.71}
$$

related to the original eigenvalue system (5.34) by

$$
W = LZ, \quad R := R_+R = \begin{pmatrix} I & 0 \\ \Theta & 1 \end{pmatrix}, \quad L := R^{-1} = \begin{pmatrix} I & 0 \\ -\Theta & 1 \end{pmatrix}, \tag{5.72}
$$

where

$$
\Theta = \widetilde{\Theta} + v_+\theta_+. \tag{5.73}
$$

Since it is triangular, (5.71) may be solved completely if we can solve the component systems associated with its diagonal blocks. The *fast system*

$$
z' = \Big(f(\hat{v}) - \lambda - \lambda v_+\theta_+\mathbb{1} - \lambda\widetilde{\Theta}\mathbb{1}\Big)z
$$

associated to the lower righthand block features rapidly-varying coefficients. However, because it is scalar, it can be solved explicitly by exponentiation.

The *slow system*

$$
z' = \lambda(J + v_+\mathbb{1}\theta_+ + \mathbb{1}\widetilde{\Theta})z \tag{5.74}
$$

associated to the upper lefthand block, on the other hand, by (5.64), is an exponentially decaying perturbation of a constant-coefficient system

$$
z' = \lambda(J + v_+\mathbb{1}\theta_+)z \tag{5.75}
$$

that can be explicitly solved by exponentiation, and thus can be well-estimated by comparison with (5.75). A rigorous version of this statement is given by the *conjugation lemma* of [42]:

Proposition 5.4.3 ([42]). *Let $M(x, \lambda) = M_+(\lambda) + \Theta(x, \lambda)$, with M_+ continuous in λ and $|\Theta(x, \lambda)| \leq Ce^{-\eta x}$, for λ in some compact set Λ. Then, there exists a globally* *invertible matrix $P(x,\lambda) = I + Q(x,\lambda)$ such that the coordinate change $z = Pv$ converts the variable-coefficient ODE $z' = M(x,\lambda)z$ to a constant-coefficient equation*

$$v' = M_+(\lambda)v,$$

satisfying for any L, $0 < \hat\eta < \eta$ a uniform bound

$$|Q(x,\lambda)| \le C(L,\hat\eta,\eta,\max|(M_+)_{ij}|,\dim M_+)e^{-\hat\eta x} \quad \textit{for } x \ge L. \tag{5.76}$$

Proof. See [42, 59], or Appendix C, [24]. □

By Proposition 5.4.3, the solution operator for (5.74) is given by

$$P(y,\lambda)e^{\lambda(J+v_+\mathbb{1}\theta_+(\lambda,v_+))(x-y)}P(x,\lambda)^{-1}, \tag{5.77}$$

where P is a uniformly small perturbation of the identity for $x \ge L$ and $L > 0$ sufficiently large.

5.5 Proof of the main theorems

With these preparations, we turn now to the proofs of the main theorems.

5.5.1 Boundary estimate

We begin by recalling the following estimates established in [24] on $\widetilde{W}_1^+(L+\delta)$, that is, the value of the dual mode $\widetilde{W}_1^+$ appearing in (5.42) at the boundary $x = L+\delta$ between regular and singular regions. For completeness, and because we shall need the details in further computations, we repeat the proof in full.

Lemma 5.5.1 ([24])**.** *For λ on any compact subset of $\Re e\lambda \ge 0$, and $L > 0$ sufficiently large, with $\widetilde{W}_1^+$ normalized as in [14, 49, 3],*

$$|\widetilde{W}_1^+(L+\delta) - \widetilde{V}_1| \le Ce^{-\eta L} \tag{5.78}$$

as $v_+ \to 0$, uniformly in λ, where C, $\eta > 0$ are independent of L and

$$\widetilde{V}_1 := (0,-1,\bar\lambda/\bar\mu)^T$$

is the limiting direction vector (5.53) *appearing in the definition of* $D^0_{\rm in}$.

Corollary 5.5.2 ([24]). *Under the hypotheses of Lemma 5.5.1,*

$$|\widetilde{W}_1^{0+}(L+\delta) - \widetilde{V}_1| \le Ce^{-\eta L} \tag{5.79}$$

and

$$|\widetilde{W}_1^{+}(L+\delta) - \widetilde{W}_1^{0+}(L+\delta)| \le Ce^{-\eta L} \tag{5.80}$$

as $v_+ \to 0$*, uniformly in* λ*, where* C*,* $\eta > 0$ *are independent of* L *and* $\widetilde{W}_1^{0+}$ *is the solution of the limiting adjoint eigenvalue system appearing in definition* (5.52) *of* D^0*.*

Proof of Lemma 5.5.1. First, make the independent coordinate change $x \to x-\delta$ normalizing the background wave to match the shock-wave case. Making the dependent coordinate-change

$$\widetilde{Z} := R^*\widetilde{W}, \tag{5.81}$$

R as in (5.72), reduces the adjoint equation $\widetilde{W}' = -A^*\widetilde{W}$ to block lower-triangular form,

$$\widetilde{Z}' = -\widetilde{A}^*\widetilde{Z} = \begin{pmatrix} -\bar\lambda(J^T + v_+\mathbb{1}\theta_+ + \mathbb{1}\widetilde{\Theta})^* & 0 \\ -\bar\lambda\mathbb{1}^T & -f(\hat v) + \bar\lambda + \bar\lambda(v_+\theta_+\mathbb{1} + \widetilde{\Theta}\mathbb{1})^* \end{pmatrix} Z, \tag{5.82}$$

with "$^-$" denoting complex conjugate.

Denoting by $\widetilde{V}_1^+$ a suitably normalized element of the one-dimensional (slow) stable subspace of $-\widetilde{A}^*$, we find readily (see [24] for further discussion) that, without loss of generality,

$$\widetilde{V}_1^+ \to (0, 1, \bar\lambda(\gamma + \bar\lambda)^{-1})^T \tag{5.83}$$

as $v_+ \to 0$, while the associated eigenvalue $\tilde\mu_1^+ \to 0$, uniformly for λ on an compact subset of $\Re e\lambda \ge 0$. The dual mode $\widetilde{Z}_1^+ = R^*\widetilde{W}_1^+$ is uniquely determined by the property that it is asymptotic as $x \to +\infty$ to the corresponding constant-coefficient solution $e^{\tilde\mu_1^+ x}\widetilde{V}_1^+$ (the standard normalization of [14, 49, 3]).

By lower block-triangular form (5.82), the equations for the slow variable $\tilde z^T :=$

$(\widetilde{Z}_1, \widetilde{Z}_2)$ decouples as a slow system

$$\tilde{z}' = -\Big(\lambda(J + v_+ \mathbb{1}\theta_+ + \mathbb{1}\widetilde{\Theta})\Big)^* \tilde{z} \tag{5.84}$$

dual to (5.74), with solution operator

$$P^*(x,\lambda)^{-1} e^{-\bar{\lambda}(J+v_+\mathbb{1}\theta_+)^*)(x-y)} P(y,\lambda)^* \tag{5.85}$$

dual to (5.77), i.e. (fixing $y = L$, say), solutions of general form

$$\tilde{z}(\lambda, x) = P^*(x,\lambda)^{-1} e^{-\bar{\lambda}(J+v_+\mathbb{1}\theta_+)^*)(x-y)} \tilde{v}, \tag{5.86}$$

$\tilde{v} \in \mathbb{C}^2$ arbitrary.

Denoting by

$$\widetilde{Z}_1^+(L) := R^* \widetilde{W}_1^+(L),$$

therefore, the unique (up to constant factor) decaying solution at $+\infty$, and $\tilde{v}_1^+ := ((\widetilde{V}_1^+)_1, (\widetilde{V}_1^+)_2)^T$, we thus have evidently

$$\tilde{z}_1^+(x,\lambda) = P^*(x,\lambda)^{-1} e^{-\bar{\lambda}(J+v_+\mathbb{1}\theta_+)^*)x} \tilde{v}_1^+,$$

which, as $v_+ \to 0$, is uniformly bounded by

$$|\tilde{z}_1^+(x,\lambda)| \le C e^{\epsilon x} \tag{5.87}$$

for arbitrarily small $\epsilon > 0$ and, by (5.83), converges for x less than or equal to $X - \delta$ for any fixed X simply to

$$\lim_{v_+ \to 0} \tilde{z}_1^+(x,\lambda) = P^*(x,\lambda)^{-1}(0,1)^T. \tag{5.88}$$

Defining by $\tilde{q} := (\widetilde{Z}_1^+)_3$ the fast coordinate of $\widetilde{Z}_1^+$, we have, by (5.82),

$$\tilde{q}' + \Big(f(\hat{v}) - \bar{\lambda} - (\lambda v_+ \theta_+ \mathbb{1} + \lambda \widetilde{\Theta}\mathbb{1})^*\Big)\tilde{q} = \bar{\lambda}\mathbb{1}^T \tilde{z}_1^+,$$

whence, by Duhamel's principle, any decaying solution is given by

$$\tilde{q}(x,\lambda) = \int_x^{+\infty} e^{\int_y^x a(z,\lambda,v_+)dz} \bar{\lambda} \mathbb{1}^T z_1^+(y)\, dy,$$

where

$$a(y,\lambda,v_+) := -\Big(f(\hat{v}) - \bar{\lambda} - (\lambda v_+ \theta_+ \mathbb{1} + \lambda \widetilde{\Theta} \mathbb{1})^*\Big).$$

Recalling, for $\Re e\lambda \geq 0$, that $\Re e a \geq 1/2$, combining (5.87) and (5.88), and noting that a converges uniformly on $y \leq Y$ as $v_+ \to 0$ for any $Y > 0$ to

$$\begin{aligned} a_0(y,\lambda) &:= -f_0(\hat{v}) + \bar{\lambda} + (\lambda \widetilde{\Theta}_0 \mathbb{1})^* \\ &= (1+\bar{\lambda}) + O(e^{-\eta y}) \end{aligned}$$

we obtain by the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned} \tilde{q}(L,\lambda) &\to \int_L^{+\infty} e^{\int_y^L a_0(z,\lambda)dz} \bar{\lambda} \mathbb{1}^T (0,1)^T\, dy \\ &= \bar{\lambda} \int_L^{+\infty} e^{(1+\bar{\lambda})(L-y) + \int_y^L O(e^{-\eta z})dz}\, dy \\ &= \bar{\lambda}(1+\bar{\lambda})^{-1}(1 + O(e^{-\eta L})). \end{aligned}$$

Recalling, finally, (5.88), and the fact that

$$|P - Id|(L,\lambda),\ |R - Id|(L,\lambda) \leq C e^{-\eta L}$$

for v_+ sufficiently small, we obtain (5.78) as claimed. □

Proof of Corollary 5.5.2. Again, make the coordinate change $x \to x - \delta$ normalizing the background wave to match the shock-wave case. Applying Proposition 5.4.3 to the limiting adjoint system

$$\widetilde{W}' = -(A^0)^* \widetilde{W} = \begin{pmatrix} 0 & 0 & 0 \\ -\bar{\lambda} & 0 & 0 \\ -\bar{\lambda} & -\bar{\lambda} & 1+\bar{\lambda} \end{pmatrix} \widetilde{W} + O(e^{-\eta x})\widetilde{W},$$

we find that, up to an $Id + O(e^{-\eta x})$ coordinate change, $\widetilde{W}_1^{0+}(x)$ is given by the exact solution $\widetilde{W} \equiv \widetilde{V}_1$ of the limiting, constant-coefficient system

$$\widetilde{W}' = -(A_+^0)^* \widetilde{W} = \begin{pmatrix} 0 & 0 & 0 \\ -\bar{\lambda} & 0 & 0 \\ -\bar{\lambda} & -\bar{\lambda} & 1+\bar{\lambda} \end{pmatrix} \widetilde{W}.$$

This yields immediately (5.79), which, together with (5.78), yields (5.80). □

5.5.2 Convergence to D^0

The rest of our analysis is standard.

Lemma 5.5.3. *On $x \le L - \delta$ for any fixed $L > 0$, there exists a coordinate-change $W = TZ$ conjugating* (5.34) *to the limiting equations* (5.50)*, $T = T(x, \lambda, v_+)$, satisfying a uniform bound*

$$|T - Id| \le C(L) v_+ \tag{5.89}$$

for all $v_+ > 0$ sufficiently small.

Proof. Make the coordinate change $x \to x - \delta$ normalizing the background profile. For $x \in (-\infty, 0]$, this is a consequence of the *Convergence Lemma* of [49], a variation on Proposition 5.4.3, together with uniform convergence of the profile and eigenvalue equations. For $x \in [0, L]$, it is essentially continuous dependence; more precisely, observing that $|A - A^0| \le C_1(L) v_+$ for $x \in [0, L]$, setting $S := T - Id$, and writing the homological equation expressing conjugacy of (5.34) and (5.50), we obtain

$$S' - (AS - SA^0) = (A - A^0),$$

which, considered as an inhomogeneous linear matrix-valued equation, yields an exponential growth bound

$$S(x) \le e^{Cx}(S(0) + C^{-1} C_1(L) v_+)$$

for some $C > 0$, giving the result. □

Proof of Theorem 5.3.2: inflow case. Make the coordinate change $x \to x - \delta$ normalizing the background profile. Lemma 5.5.3, together with convergence as $v_+ \to 0$ of the unstable subspace of A_- to the unstable subspace of A_-^0 at the same rate $O(v_+)$ (as follows by spectral separation of the unstable eigenvalue of A^0 and standard matrix perturbation theory) yields

$$|W_1^0(0,\lambda) - W_1^{00}(0,\lambda)| \le C(L)v_+. \tag{5.90}$$

Likewise, Lemma 5.5.3 gives

$$\begin{aligned}|\widetilde{W}_1^+(0,\lambda) - \widetilde{W}_1^{0+}(0,\lambda)| \le C(L)v_+ |\widetilde{W}_1^+(0,\lambda)| \\ + |S_0^{L\to 0}||\widetilde{W}_1^+(L,\lambda) - \widetilde{W}_1^{0+}(L,\lambda)|,\end{aligned} \tag{5.91}$$

where $S_0^{y\to x}$ denotes the solution operator of the limiting adjoint eigenvalue equation $\widetilde{W}' = -(A_+^0)^*\widetilde{W}$. Applying Proposition 5.4.3 to the limiting system, we obtain

$$|S_0^{L\to 0}| \le C_2|e^{-A_+^0 L}| \le C_2 L|\lambda|$$

by direct computation of $e^{-A_+^0 L}$, where C_2 is independent of $L > 0$. Together with (5.80) and (5.91), this gives

$$|\widetilde{W}_1^+(0,\lambda) - \widetilde{W}_1^{0+}(0,\lambda)| \le C(L)v_+|\widetilde{W}_1^+(0,\lambda)| + L|\lambda|C_2Ce^{-\eta L},$$

hence, for $|\lambda|$ bounded and v_+ sufficiently small relative to $C(L)$,

$$\begin{aligned}|\widetilde{W}_1^+(0,\lambda) - \widetilde{W}_1^{0+}(0,\lambda)| &\le C_3(L)v_+|\widetilde{W}_1^{0+}(0,\lambda)| + LC_4e^{-\eta L} \\ &\le C_5(L)v_+ + LC_4e^{-\eta L}.\end{aligned} \tag{5.92}$$

Taking first $L \to \infty$ and then $v_+ \to 0$, we obtain therefore convergence of $W_1^0(0,\lambda)$ and $\widetilde{W}_1^+(0,\lambda)$ to $W_1^{00}(0,\lambda)$ and $\widetilde{W}_1^{0+}(0,\lambda)$, yielding convergence by definitions (5.42) and (5.52).

This convergence, however, is between Evans functions with profiles shifted by $\delta = \delta(v_+)$. This shift changes the initializing asymptotic behavior at $+\infty$ of $\widetilde{W}_1^+$, modifying the value of the Evans function by a nonvanishing factor $e^{-\delta\tilde{\mu}_1(\lambda)}$, where $\tilde{\mu}_1(\lambda)$ is the decay rate associated with mode $\widetilde{W}_1^+$; for similar computations, see the proof of Theorem 5.3.3. In particular, the value of D^0 is unaffected by a shift, since $\tilde{\mu}_1 \equiv 0$. Noting that $\delta(v_+)$ is uniformly bounded as $v_+ \to 0$ (indeed, it approaches a limit δ^0 as $v_+ \to 0$, determined by $\hat{v}^0(\delta^0) = v_-/2 = 1/2$, as follows by continuous dependence of solutions of ODE), while $\tilde{\mu}_1(\lambda) \to 0$ uniformly on compact subsets of $\Re\lambda \geq 0$, we thus find that both shifted and unshifted versions of $D(\lambda)$ approach $D^0(\lambda)$ as $v_+ \to 0$, uniformly on compact subsets of $\Re\lambda \geq 0$. □

Proof of Theorem 5.3.2: outflow case. Straightforward, following the previous argument in the regular region only. □

5.5.3 Convergence to the shock case

Proof of Theorem 5.3.3: inflow case. First make the coordinate change $x \to x - \delta$ normalizing the background profile location to that of the shock wave case, where $\delta \to +\infty$ as $v_0 \to 1$. By standard duality properties,

$$D_{\text{in}} = \widetilde{W}_1^+ \cdot W_1^0|_{x=x_0}$$

is independent of x_0, so we may evaluate at $x = 0$ as in the shock case. Denote by $\mathcal{W}_1^-$, $\widetilde{\mathcal{W}}_1^+$ the corresponding modes in the shock case, and

$$\mathcal{D} = \widetilde{\mathcal{W}}_1^+ \cdot \mathcal{W}_1^-|_{x=0}$$

the resulting Evans function.

Noting that $\widetilde{\mathcal{W}}_+^1$ and $\widetilde{W}_+^1$ are asymptotic to the unique stable mode at $+\infty$ of the (same) adjoint eigenvalue equation, but with translated decay rates, we see immediately that $\widetilde{\mathcal{W}}_1^+ = \widetilde{W}_+^1 e^{-\delta\tilde{\mu}_1^+}$. On the other hand, W_1^0 is initialized at at $x = -\delta$ (in the new coordinates $\tilde{x} = x - \delta$) as

$$W_1^0(-\delta) = (1, 0, 0)^T,$$

whereas $\mathcal{W}_1^-$ is the unique unstable mode at $-\infty$ decaying as $e^{\mu_1^- x} V_1^-$, where V_1^- is the unstable right eigenvector of

$$A_- = \begin{pmatrix} 0 & \lambda & \lambda \\ 0 & 0 & \lambda \\ 1 & 1 & f(1) - \lambda \end{pmatrix}.$$

Denote by $\widetilde{V}_1^-$ the associated dual unstable left eigenvector and

$$\Pi_1^- := V_1^- (\overline{\widetilde{V}_1^-})^T$$

the eigenprojection onto the stable vector V_1^-. By direct computation,

$$\overline{\widetilde{V}_1^-} = c(\lambda)(1, 1 + \lambda/\mu_1^-, \mu_1^-)^T, \quad c(\lambda) \neq 0,$$

yielding

$$\Pi_1^- W_1^0 =: \beta(\lambda) = c(\lambda) \neq 0 \tag{5.93}$$

for $\Re\lambda \geq 0$, on which $\Re\mu_1^- > 0$.

Once we know (5.93), we may finish by a standard argument, concluding by exponential attraction in the positive x-direction of the unstable mode that other modes decay exponentially as $x \to 0$, leaving the contribution from $\beta(\lambda)V_1^-$ plus a negligible $O(e^{-\eta\delta})$ error, $\eta > 0$, from which we may conclude that $\mathcal{W}_1^-|_{x=0} \sim \beta^{-1}e^{-\delta\mu_1^-}W_1^0|_{x=0}$. Collecting information, we find that

$$\mathcal{D}(\lambda) = \beta(\lambda)^{-1}e^{-\delta(\bar{\mu}_1^- + \tilde{\mu}_1^+)(\lambda)}D_{\text{in}}(\lambda) + O(e^{-\eta\delta}),$$

$\eta > 0$, yielding the claimed convergence $C(\lambda,\delta)D_{\text{in}}(\lambda) \to \mathcal{D}(\lambda)$ as $v_0 \to 1$, $\delta \to +\infty$, with $C(\lambda,\delta) := \beta(\lambda)^{-1}e^{-\delta(\bar{\mu}_1^- + \tilde{\mu}_1^+)(\lambda)} \neq 0$. □

Proof of Theorem 5.3.3: outflow case. For λ uniformly bounded from zero, $\widetilde{W}_1^0 = (0, -1, -\bar{\lambda}/(\bar{\lambda} - \hat{v}'(0)))^T$ converges uniformly as $v_0 \to 0$ to

$$(0, -1, -1)^T,$$

whereas the shock Evans function $\mathcal{D}$ is initiated by $\widetilde{\mathcal{W}}_1^+$ proportional to

$$\widetilde{\mathcal{V}}_1^+ = (0, -1, -1 - \bar{\lambda})^T$$

agreeing in the first two coordinates with $\widetilde{W}_1^0$. By the boundary-layer analysis of Section 5.5.1, the backward (i.e., decreasing x) evolution of the adjoint eigenvalue ODE reduces in the asymptotic limit $v_+ \to 0$ (forced by $v_0 \to 0$) to a decoupled slow flow

$$\tilde{w}' = \begin{pmatrix} 0 & 0 \\ -\bar{\lambda} & 0 \end{pmatrix} \tilde{w}, \qquad \tilde{w} \in \mathbb{C}^2$$

in the first two coordinates, driving an exponentially slaved fast flow in the third coordinate. From this, we may conclude that solutions agreeing in the first two coordinates converge exponentially as x decreases. Performing an appropriate normalization, as in the inflow case just treated, we thus obtain the result. We omit the details, which follow what has already been done in previous cases. □

5.5.4 The stability index

Following [54, 19], we note that $D_{\text{in}}(\lambda)$ is real for real λ, and nonvanishing for real λ sufficiently large, hence $\text{sgnD}_{\text{in}}(+\infty)$ is well-defined and constant on the entire (connected) parameter range. The number of roots of D_{in} on $\Re\lambda \geq 0$ is therefore even or odd depending on the *stability index*

$$\text{sgn}[\text{D}_{\text{in}}(0)\text{D}_{\text{in}}(+\infty)].$$

Similarly, recalling that $D_{\text{out}}(0) \equiv 0$, we find that the number of roots of D_{out} on $\Re\lambda \geq 0$ is even or odd depending on

$$\text{sgn}[\text{D}'_{\text{out}}(0)\text{D}_{\text{out}}(+\infty)].$$

Proof of Lemma 5.3.5: inflow case. Examining the adjoint equation at $\lambda = 0$,

$$\widetilde{W}' = -A^*\widetilde{W}, \qquad -A^*(x,0) = \begin{pmatrix} 0 & 0 & -\hat{v} \\ 0 & 0 & -\hat{v} \\ 0 & 0 & -f(\hat{v}) \end{pmatrix},$$

$-f(v_+) > 0$, we find by explicit computation that the only solutions that are bounded as $x \to +\infty$ are the *constant solutions* $\widetilde{W} \equiv (a, b, 0)^T$. Taking the limit $\widetilde{V}_1^+(0)$ as $\lambda \to 0^+$ along the real axis of the unique stable eigenvector of $-A_+^*(\lambda)$, we find (see, e.g., [59]) that it lies in the direction $(1, 2 + a_j^+, 0)^T$, where $a_j^+ > 0$ is the positive characteristic speed of the hyperbolic convection matrix $\begin{pmatrix} 1 & -1 \\ -h(v_+)/v_+^{\gamma+1} & 1 \end{pmatrix}$, i.e., $\widetilde{V}_1^+ = c(v_0, v_+)(1, 2 + a_j^+, 0)^T$, $c(v_0, v_+) \neq 0$. Thus, $D_{\text{in}}(0) = \widetilde{V}_1^+ \cdot (1, 0, 0)^T = \bar{c}(v_0, v_+) \neq 0$ as claimed. On the other hand, the same computation carried out for $D_{\text{in}}^0(0)$ yields $D_{\text{in}}^0(0) \equiv 0$. (Note: $a_j \sim v_+^{-1/2} \to +\infty$ as $v_+ \to 0$.) Similarly, as $v_0 \to 0$,

$$D_{\text{in}}^0(\lambda) \to (0, -1, *)^T \cdot (1, 0, 0)^T \equiv 0.$$

Finally, note $D_{\rm in}(0) \neq 0$ implies that the stability index, since continuously varying so long as it doesn't vanish and taking discrete values ± 1, must be constant on the connected set of parameter values. Since inflow boundary layers are known to be stable on some part of the parameter regime by energy estimates (Theorem 5.3.4), we may conclude that the stability index is identically one and therefore there are an even number of unstable roots for all $1 > v_0 \geq v_+ > 0$.

To establish that $(D^0_{\rm in})'(0) \neq 0$, we compute

$$D^0_{\rm in}{}'(0) = \widetilde{W}^{0+}_1 \cdot (\partial_\lambda W^{00}_1) + (\partial_{\bar\lambda} \widetilde{W}^{0+}_1) \cdot W^{00}_1. \tag{5.94}$$

Since $W^{00}_1 \equiv (1,0,0)$ is independent of λ, this reduces to

$$D^0_{\rm in}{}'(0) = \overline{\partial_{\bar\lambda} \widetilde{W}^{0+}_{1,1}}|_{x=0}, \tag{5.95}$$

so we need only show that the first component of $\partial_{\bar\lambda}\widetilde{W}^{0+}_1$ is nonzero. Note that $\partial_{\bar\lambda}\widetilde{W}^{0+}_1$ solves the limiting adjoint variational equations

$$(\partial_{\bar\lambda}\widetilde{W}^{0+}_1)'(0) + (A^0)^*(x,0)\partial_{\bar\lambda}\widetilde{W}^{0+}_1 = b(x) \tag{5.96}$$

with $b(x) := -\partial_{\bar\lambda}(A^0)^*(x,0)\widetilde{W}^{0+}_1(x,0)$, $\widetilde{W}^{0+}_1(x,0) = (0,-1,0)^T$,

$$(A^0)^*(x,0) = \begin{pmatrix} 0 & 0 & \hat v^0 \\ 0 & 0 & \hat v^0 \\ 0 & 0 & f^0(\hat v^0) \end{pmatrix}, \qquad \partial_{\bar\lambda}(A^0)^*(x,0) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Thus $b(x) = (0,0,1)^T$. By (5.53), and the fact that $\partial_{\bar\lambda}\tilde\mu^{0+}_1 \equiv 0$, $\partial_{\bar\lambda}\widetilde{W}^{0+}_1(x)$ is chosen so that asymptotically at $x = +\infty$ it lies in the direction of $\partial_{\bar\lambda}\widetilde{V}_1 = (0,0,1)$. Set $\partial_{\bar\lambda}\widetilde{W}^{0+}_1 = (\partial_{\bar\lambda}\widetilde{W}^{0+}_{1,1}, \partial_{\bar\lambda}\widetilde{W}^{0+}_{1,2}, \partial_{\bar\lambda}\widetilde{W}^{0+}_{1,3})^T$. Then the third component solves

$$(\partial_{\bar\lambda}\widetilde{W}^{0+}_{1,3})' + f^0(\hat v^0)\partial_{\bar\lambda}\widetilde{W}^{0+}_{1,3} = 1,$$

where $f^0(\hat v^0) = 2\hat v^0 - 1$. Define $Z(x) := e^{-x}\partial_{\bar\lambda}\widetilde{W}^{0+}_{1,3}(x,0)$. Then Z solves

$$Z' + 2\hat v^0 Z = e^{-x}, \qquad Z(+\infty) = 0,$$

which has solution

$$Z(x) = -\int_x^\infty S_Z^{y\to x} e^{-y} dy$$

where

$$S_Z^{y\to x} = e^{2\int_x^y \hat{v}^0(z)dz},$$

denoting the solution operator of $Z' + 2\hat{v}^0 Z = 0$. Integrating the equation (5.96) for the first component of $\partial_{\bar\lambda}\widetilde{W}_1^{0+}$ with $\partial_{\bar\lambda}\widetilde{W}_{1,1}^{0+}(+\infty) = 0$ yields

$$\begin{aligned}\partial_{\bar\lambda}\widetilde{W}_{1,1}^{0+}(x) &= \partial_{\bar\lambda}\widetilde{W}_{1,1}^{0+}(+\infty) + \int_x^\infty \hat{v}^0(y)\partial_{\bar\lambda}\widetilde{W}_{1,3}^{0+}(y)dy \\ &= -\int_x^\infty \hat{v}^0(y)e^y \int_y^\infty S_Z^{z\to y} e^{-z} dzdy\end{aligned}$$

and thus

$$\partial_{\bar\lambda}\widetilde{W}_{1,1}^{0+}|_{x=0} = -\int_0^\infty \hat{v}^0(y)e^y \int_y^\infty S_Z^{z\to y} e^{-z} dzdy.$$

Finally, note that for all y, $\hat{v}^0(y), S_Z^{z\to y} \geq 0$. Therefore by (5.95),

$$D_{\text{in}}^0{}'(0) = \overline{\partial_{\bar\lambda}\widetilde{W}_{1,1}^{0+}}|_{x=0} \neq 0.$$

□

Remark 5.5.4. *The result $D_{\text{in}}(0) \neq 0$ at first sight appears to contradict that of Theorem 5.3.3, since $\mathcal{D}(0) = 0$ for the shock wave case. This apparent contradiction is explained by the fact that the normalizing factor $e^{-\delta(\bar\mu_1^- + \tilde\mu_1^+)}$ is exponentially decaying in δ for $\lambda = 0$, since $\tilde\mu_1^+(0) = 0$, while $\Re\mu_1^- > 0$. Recalling that $\delta \to +\infty$ as $v_0 \to 1$, we recover the result of Theorem 5.3.3.*

Proof of Lemma 5.3.5: outflow case. Similarly, we compute

$$D'_{\text{out}}(0) = \widetilde{W}_1^0 \cdot \partial_\lambda W_1^- + \partial_{\bar\lambda}\widetilde{W}_1^0 \cdot W_1^-,$$

where $\partial_\lambda W_1^-|_{\lambda=0}$ satisfies the variational equation $L\partial_\lambda U_1^-(0) = \partial_\lambda A(x,0)U_1^-$,or, written as a first-order system,

$$(\partial_\lambda W_1^-)' - A(x,0)\partial_\lambda W_1^- = \begin{pmatrix}\hat{u}_x \\ \hat{v}_x \\ -\hat{v}_x\end{pmatrix}, \qquad A(x,0) = \begin{pmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ \hat{v} & \hat{v} & f(\hat{v})\end{pmatrix},$$

which may be solved exactly for the unique solution decaying at $-\infty$ of

$$W_1^-(0) = \begin{pmatrix} 0 \\ 0 \\ \hat{v}' \end{pmatrix}, \qquad (\partial_\lambda W_1^-)(0) = \begin{pmatrix} \hat{u} - u_- \\ \hat{v} - v_- \\ * \end{pmatrix}.$$

Recalling from (5.47) that $\widetilde{W}_1^0(\lambda) = (0, -1, -\bar{\lambda}/(\bar{\lambda} - \hat{v}'(0)))^T$, hence

$$\widetilde{W}_1^0(0) = (0, -1, 0)^T, \qquad \partial_{\bar{\lambda}} \widetilde{W}_1^0(0) = (0, 0, 1/\hat{v}'(0))^T,$$

we thus find that

$$\begin{aligned} D'_{\text{out}}(0) &= \widetilde{W}_1^0(0) \cdot \partial_\lambda W_1^-(0) + \partial_{\bar{\lambda}} \widetilde{W}_1^0(0) \cdot W_1^-(0) \\ &= -(\hat{v}(0) - 1) + 1 = 2 - v_0 \neq 0 \end{aligned}$$

as claimed. The proof that $(D^0_{\text{out}})'(0) \neq 0$ goes similarly.

Finally, as in the proof of the inflow case, we note that nonvanishing implies that the stability index is constant across the entire (connected) parameter range, hence we may conclude that it is identically one by existence of a stable case (Corollary 5.3.9), and therefore that the number of nonzero unstable roots is even, as claimed. □

5.5.5 Stability in the shock limit

Proof of Corollary 5.3.9: inflow case. By Proposition 5.3.6 we find that D_{in} has at most a single zero in $\Re\lambda \geq 0$. However, by our stability index results, Theorem 5.3.5, the number of eigenvalues in $\Re\lambda \geq 0$ is even. Thus, it must be zero, giving the result. □

Proof of Corollary 5.3.9: outflow case. By Theorem 5.3.3, D_{out}, suitably renormalized, converges as $v_0 \to 0$ to the Evans function for the (unintegrated) shock wave case. But, the shock Evans function by the results of [3, 24] has just a single zero at $\lambda = 0$ on $\Re\lambda \geq 0$, already accounted for in D_{out} by the spurious root at $\lambda = 0$ introduced by recoordinatization to "good unknown". □

5.5.6 Stability for small v_0

Finally, we treat the remaining, "corner case" as v_+, v_0 simultaneously approach zero. The fact (Lemma 5.3.5) that

$$\lim_{v_0 \to 0} \lim_{v_+ \to 0} D_{\rm in}(\lambda) \equiv 0$$

shows that this limit is quite delicate; indeed, this is the most delicate part of our analysis.

Proof of Theorem 5.3.4: inflow case. Consider again the adjoint system

$$\widetilde{W}' = -A^*(x,\lambda)\widetilde{W}, \qquad A^*(x,\lambda) = \begin{pmatrix} 0 & 0 & \hat{v} \\ \bar{\lambda} & 0 & \hat{v} \\ \bar{\lambda} & \bar{\lambda} & f(\hat{v}) - \bar{\lambda} \end{pmatrix}.$$

By the boundary analysis of Section 5.5.1,

$$\widetilde{W} = \Big(\alpha, 1, \frac{\alpha\tilde{\mu} - \bar{\lambda}(\alpha+1)}{-f(\hat{v}) + \bar{\lambda}}\Big)^T + O(e^{-\eta|x-\delta|}),$$

where $\alpha := \frac{\tilde{\mu}_+}{\tilde{\mu}_+ + \bar{\lambda}}$, and $\tilde{\mu}$ is the unique stable eigenvalue of A^*_+, satisfying (by matrix perturbation calculation)

$$\tilde{\mu} = \bar{\lambda}(v_+^{1/2} + O(v_+))$$

and thus $\alpha = v_+^{1/2} + O(v_+)$ as $v_0 \to 0$ (hence $v_+ \to 0$) on bounded subsets of $\Re\lambda \geq 0$. Combining these expansions, we have

$$\widetilde{W}_1(+\infty) = v_+^{1/2}(1 + o(1)), \qquad \widetilde{W}_3 = \frac{-\bar{\lambda}}{-f(\hat{v}) + \bar{\lambda}}(1 + o(1))$$

for v_0 sufficiently small.

From the $\widetilde{W}_1$ equation $\widetilde{W}_1' = \hat{v}\widetilde{W}_3$, we thus obtain

$$\begin{aligned} \widetilde{W}_1(0) &= \widetilde{W}_1(+\infty) - \int_0^{+\infty} \hat{v}\widetilde{W}_3(y)\,dy \\ &= (1 + o(1)) \times \Big(v_+^{1/2} + \int_0^{+\infty} \frac{\bar{\lambda}\hat{v}}{-f(\hat{v}) + \bar{\lambda}}(y)\,dy\Big). \end{aligned}$$

Observing, finally, that, for $\Re\lambda \geq 0$, the ratio of real to imaginary parts of $\frac{\bar{\lambda}\hat{v}}{-f(\hat{v})+\bar{\lambda}}(y)$ is uniformly positive, we find that $\Re\widetilde{W}_1(0) \neq 0$ for v_0 sufficiently small, which yields nonvanishing of $D_{\text{in}}(\lambda)$ on $\Re\lambda \geq 0$ as claimed. □

5.6 Numerical computations

In this section, we show, through a systematic numerical Evans function study, that there are no unstable eigenvalues for

$$(\gamma, v_+) \in [1,3] \times (0,1],$$

in either inflow or outflow cases. As defined in Section 5.2.6, the Evans function is analytic in the right-half plane and reports a value of zero precisely at the eigenvalues of the linearized operator (5.20). Hence we can use the argument principle to determine if there are any unstable eigenvalues for this system. Our approach closely follows that of [3, 24] for the shock case with only two major differences. First, our shooting algorithm is only one sided as we have the boundary conditions (5.41) and (5.47) for the inflow and outflow cases, respectfully. Second, we "correct" for the displacement in the boundary layer when $v_0 \approx 1$ in the inflow case and $v_0 \approx 0$ in the outflow case so that the Evans function converges to the shock case as studied in [3, 24] (see discussion in Section 5.6.3).

The profiles were generated using Matlab's `bvp4c` routine, which is an adaptive Lobatto quadrature scheme. The shooting portion of the Evans function computation was performed using Matlab's `ode45` package, which is the standard 4th order adaptive Runge-Kutta-Fehlberg method (RKF45). The error tolerances for both the profiles and the shooting were set to `AbsTol=1e-6` and `RelTol=1e-8`. We remark that Kato's ODE (see Section 5.2.6 and [32, 27] for details) is used to analytically choose the initial eigenbasis for the stable/unstable manifolds at the numerical values of infinity at $L = \pm 18$. Finally in Section 5.6.4, we carry out a numerical convergence study similar to that in [3].

5.6.1 Winding number computations

The high-frequency estimates in Proposition 5.2.3 restrict the set of admissible unstable eigenvalues to a fixed compact triangle Λ in the right-half plane (see (5.31) and

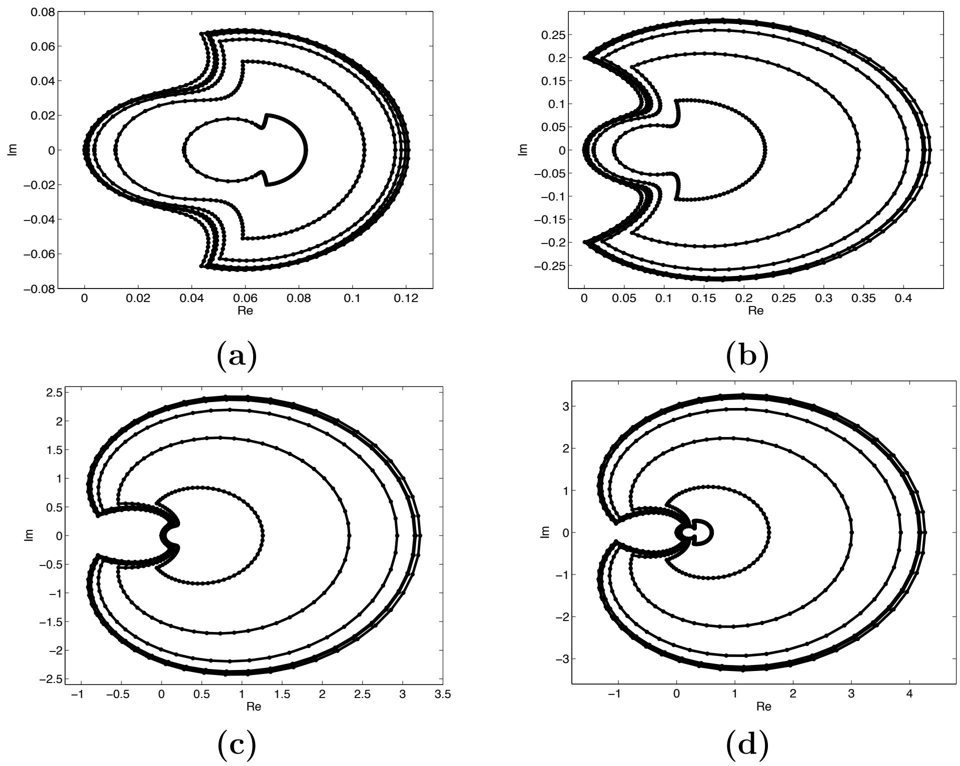


Figure 5.1: Typical examples of the inflow case, showing convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas, $\gamma = 5/3$, with (a) $v_0 = 0.1$, (b) $v_0 = 0.2$, (c) $v_0 = 0.4$, and (d) $v_0 = 0.7$. The contours depicted, going from inner to outer, are images of the semicircle ϕ under D for $v_+ = 1e{-}2, 1e{-}3, 1e{-}4, 1e{-}5, 1e{-}6$, with the outer-most contour given by the image of ϕ under D^0, that is, when $v_+ = 0$. Each contour consists of 60 points in λ.

(5.32) for the inflow and outflow cases, respectively). We reiterate the remarkable property that Λ does not depend on the choice of v_+ or v_0. Hence, to demonstrate stability for a given γ, v_+ and v_0, it suffices to show that the winding number of the Evans function along a contour containing Λ is zero. Note that in our region of interest, $\gamma \in [1, 3]$, the semi-circular contour given by

$$\phi := \partial(\{\lambda \mid \Re e\lambda \geq 0\} \cap \{\lambda \mid |\lambda| \leq 10\}),$$

contains Λ in both the inflow and outflow cases. Hence, for consistency we use this same semicircle for all of our winding number computations.

A remarkable feature of the Evans function for this system, and one that is shared

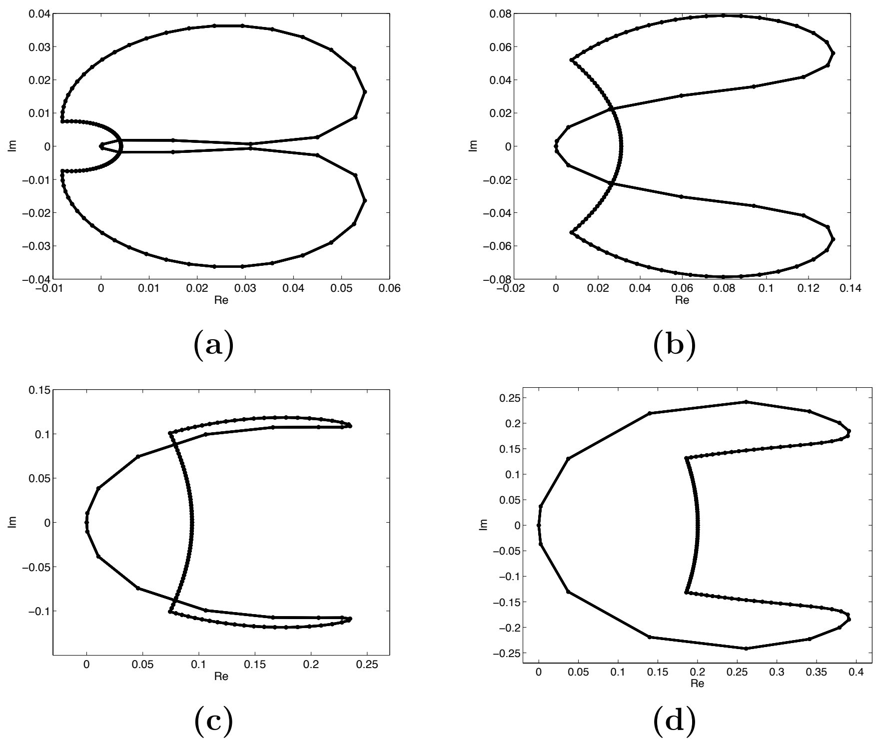
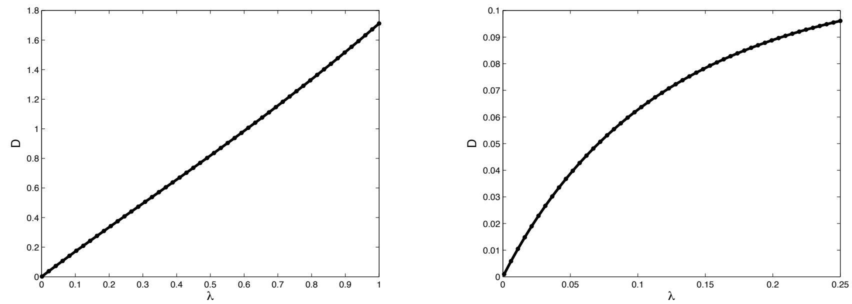


Figure 5.2: Typical examples of the outflow case, showing convergence to the limiting Evans function as $v_+ \to 0$ for a monatomic gas, $\gamma = 5/3$, with (a) $v_0 = 0.2$, (b) $v_0 = 0.4$, (c) $v_0 = 0.6$, and (d) $v_0 = 0.8$. The contours depicted are images of the semicircle ϕ under D for $v_+ = 1e{-}2, 1e{-}3, 1e{-}4, 1e{-}5, 1e{-}6$, and the limiting case $v_+ = 0$. Interestingly the contours are essentially (visually) indistinguishable in this parameter range. Each contour consists of 60 points in λ

with the shock case in [3, 24], is that the Evans function has limiting behavior as the amplitude increases, Section 5.3.2. For the inflow case, we see in Figure 5.1, the mapping of the contour ϕ for the monatomic case ($\gamma = 5/3$), for several different choices of v_0, as $v_+ \to 0$. We remark that the winding numbers for $0 \leq v_+ \leq 1$ are all zero, and the limiting contour touches zero due to the emergence of a zero root in the limit. Note that the limiting case contains the contours of all other amplitudes. Hence, we have spectral stability for all amplitudes.

The outflow case likewise has a limiting behavior, however, all contours cross through zero due to the eigenvalue at the origin. Nonetheless, since the contours only wind around once, we can likewise conclude that these profiles are spectrally stable. We remark that the outflow case converges to the limiting case faster than the inflow



(a) **(b)**

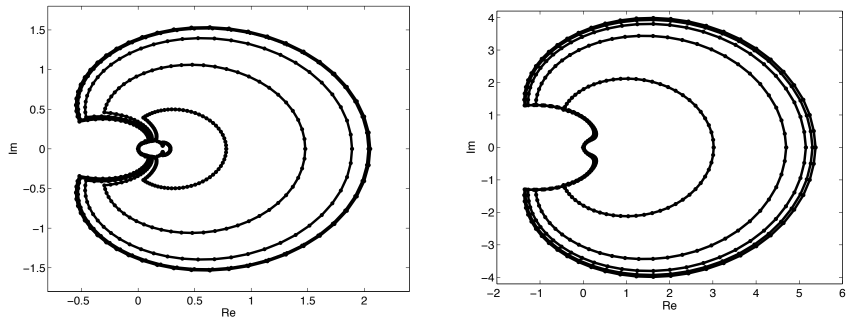
Figure 5.3: Typical examples of the Evans function evaluated along the positive real axis. The (a) inflow case is computed for $v_0 = 0.7$ and $v_0 = 0$ and (b) the outflow case is computed for $v_0 = 0.3$ and $v_+ = 0.001$. Not the transversality at the origin in both cases. Both graphs consist of 50 points in λ.

case as is clear from Figure 5.2. Indeed, $v_+ = 1e{-}2$ and the limiting case $v_+ = 0$, as well as all of the values of v_+ in between, are virtually indistinguishable.

In our study, we systematically varied v_0 in the interval $[.01, .99]$ and took the $v_+ \to 0$ limit at each step, starting from a $v_+ = .9$ (or some other appropriate value, for example when $v_0 < .9$) on the small-amplitude end and decreased v_+ steadily to 10^{-k} for $k = 1, 2, 3, \ldots, 6$, followed by evaluation at $v_+ = 0$. For both inflow and outflow cases, over 2000 contours were computed. We remark that in the $v_+ \to 0$ limit, the system becomes pressureless, and thus all of the contours in the large-amplitude limit look the same regardless of the value of γ chosen.

5.6.2 Nonexistence of unstable real eigenvalues

As an additional verification of stability, we computed the Evans function along the unstable real axis on the interval $[0, 15]$ for varying parameters to show that there are no real unstable eigenvalues. Since the Evans function has a root at the origin in the limiting system for the inflow case, and for all values of v_+ in the outflow case, we can perform in these cases a sort of *numerical stability index analysis* to verify that the Evans function cuts transversely through the origin and is otherwise nonzero, indicating that there are no unstable real eigenvalues as expected. In Figure 5.3, we see a typical example of (a) the inflow and (b) outflow cases. Note that in both images, the Evans function cuts transversally through the origin and is otherwise



(a) **(b)**

Figure 5.4: Shock limit for (a) inflow and (b) outflow cases, both for $\gamma = 5/3$. Note that the images look very similar to those of [3, 24].

nonzero as λ increases.

5.6.3 The shock limit

When v_0 is far from the midpoint $(1 - v_+)/2$ of the end states, the the Evans function of the boundary layer is similar to the Evans function of the shock case evaluated at the displacement point x_0. Hence, when we compute the boundary layer Evans function near the shock limits, $v_0 \approx 1$ for the inflow case and $v_0 \approx 0$ for the outflow case, we multiply for the correction factor $c(\lambda)$ so that our output looks close to that of the shock case studied in [3, 24]. The correction factors are

$$c(\lambda) = e^{(-\mu^{+} - \bar{\mu}^{-})x_0}$$

for the inflow case and

$$c(\lambda) = e^{(-\bar{\mu}^{+} - \mu^{-})x_0},$$

for the outflow case, where μ^- is the growth mode of $A_-(\lambda)$ and μ^+ is the decay mode of $A_+(\lambda)$. In Figure 5.4, we see that these highly displaced profiles appear to be very similar to the shock cases with one notable difference. These images have a small dimple near $\lambda = 0$ to account for the eigenvalue there, whereas those in the shock case [3, 24] were computed in integrated coordinates and thus have no root at the origin.

Inflow Case						
L	$\gamma = 1.2$	$\gamma = 1.4$	$\gamma = 1.666$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
8	7.8(-1)	8.4(-1)	9.2(-1)	1.0(0)	1.2(0)	1.3(0)
10	1.4(-1)	1.2(-1)	9.2(-2)	6.8(-2)	4.4(-2)	2.8(-2)
12	1.4(-2)	7.9(-3)	3.6(-3)	1.3(-3)	3.1(-4)	7.3(-5)
14	1.3(-3)	4.9(-4)	1.3(-4)	2.4(-5)	8.7(-6)	8.2(-6)
16	1.2(-4)	3.0(-5)	4.7(-6)	2.8(-6)	2.7(-6)	2.6(-6)
18	1.1(-5)	5.8(-6)	8.0(-6)	8.1(-6)	8.0(-6)	8.0(-6)
Outflow Case						
L	$\gamma = 1.2$	$\gamma = 1.4$	$\gamma = 1.666$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
8	5.4(-3)	5.4(-3)	5.4(-3)	5.4(-3)	5.4(-3)	5.4(-3)
10	9.2(-4)	9.1(-4)	9.1(-4)	9.1(-4)	9.1(-4)	9.1(-4)
12	1.5(-4)	1.5(-4)	1.5(-4)	1.5(-4)	1.5(-4)	1.5(-4)
14	2.5(-5)	2.7(-5)	2.0(-5)	2.0(-5)	2.0(-5)	2.0(-5)
16	2.3(-6)	2.6(-6)	2.6(-6)	2.5(-6)	2.5(-6)	2.5(-6)
18	6.6(-6)	3.6(-6)	8.7(-6)	8.7(-6)	8.7(-6)	8.7(-6)

Table 5.1: Relative errors in $D(\lambda)$ for the inflow and outflow cases are computed by taking the maximum relative error for 60 contour points evaluated along the semicircle ϕ. Samples were taken for varying L and γ, leaving v_+ fixed at $v_+ = 10^{-4}$ and $v_0 = 0.6$. We used $L = 8, 10, 12, 14, 16, 18, 20$ and $\gamma = 1.2, 1.4, 1.666, 2.0$. Relative errors were computed using the next value of L as the baseline.

5.6.4 Numerical convergence study

As in [3], we carry out a numerical convergence study to show that our results are accurate. We varied the absolute and relative error tolerances, as well as the length of the numerical domain $[-L, L]$. In Tables 1–2, we demonstrate that our choices of $L = 18$, `AbsTol=1e-6` and `RelTol=1e-8` provide accurate results.

Inflow Case						
Abs/Rel	$\gamma = 1.2$	$\gamma = 1.4$	$\gamma = 1.666$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
$10^{-3}/10^{-5}$	5.4(-4)	4.1(-4)	4.0(-4)	5.0(-4)	3.4(-4)	8.6(-4)
$10^{-4}/10^{-6}$	3.1(-5)	4.6(-5)	3.4(-5)	3.3(-5)	3.3(-5)	3.2(-5)
$10^{-5}/10^{-7}$	2.9(-6)	3.6(-6)	3.9(-6)	6.8(-6)	2.7(-6)	2.5(-6)
$10^{-6}/10^{-8}$	4.6(-7)	9.9(-7)	1.1(-6)	6.0(-7)	2.9(-7)	3.2(-7)
Outflow Case						
Abs/Rel	$\gamma = 1.2$	$\gamma = 1.4$	$\gamma = 1.666$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
$10^{-3}/10^{-5}$	9.2(-4)	9.2(-4)	9.1(-4)	9.1(-4)	9.1(-4)	9.2(-4)
$10^{-4}/10^{-6}$	5.3(-5)	4.9(-5)	5.3(-5)	5.3(-5)	5.3(-5)	5.3(-5)
$10^{-5}/10^{-7}$	6.7(-5)	6.7(-5)	6.7(-5)	6.7(-5)	6.7(-5)	6.7(-5)
$10^{-6}/10^{-8}$	2.9(-6)	2.9(-6)	2.9(-6)	2.9(-6)	2.9(-6)	2.9(-6)

Table 5.2: Relative errors in $D(\lambda)$ for the inflow and outflow cases are computed by taking the maximum relative error for 60 contour points evaluated along the semicircle ϕ. Samples were taken for varying the absolute and relative error tolerances and γ in the ODE solver, leaving $L = 18$ and $\gamma = 1.666$, $v_+ = 10^{-4}$, and $v_0 = 0.6$ fixed. Relative errors were computed using the next run as the baseline.

Appendix A

Appendix to Chapter 2

A.1 Profiles

Lemma A.1.1 ([36, 59, 19]). *Given (A1)-(A3) and (H0)-(H2), a standing wave solution* (2.1) *of* (2.2)*, (B) satisfies*

$$\left|(d/dx)^k(\bar{U} - U_+)\right| \le Ce^{-\theta x}, \quad k = 0, ..., 4, \tag{A.1}$$

as $x \to +\infty$. Moreover, a solution, if it exists, is in the inflow or strictly parabolic case unique; in the outflow case it is locally unique.

Proof. As in the shock case [37, 59], (A.1) follows by the observation that, under hypotheses (A1)-(A3) and (H0)-(H2), U_+ is a hyperbolic rest point of the layer profile ODE; see also [19].

Uniqueness follows by the observation [36] that the standing-wave ODE may be integrated from x to $+\infty$ and rearranged to yield

$$\begin{aligned} F^1(U) &\equiv F^1(U_+), \\ (b_1, b_2)(U)U' &= C(U, U_+), \end{aligned} \tag{A.2}$$

and thereby the first-order ODE

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} F^1_u & F^1_v \\ b_1 & b_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ C(U, U_+) \end{pmatrix}. \tag{A.3}$$

In the strictly parabolic or inflow case, $U(0)$ is specified by the boundary conditions at $x = 0$, thus determining a unique solution for all $x \geq 0$ through (A.3). In the outflow case, we observe, comparing U and W equations, that (A.2) can be rewritten alternatively as

$$\begin{aligned} F^1(W) &\equiv F^1(W_+), \\ (w^{II})' &= D(w^I, w^{II}), \end{aligned} \tag{A.4}$$

where the first equation may by the Implicit Function Theorem be *locally* solved for w^I as a function of w^{II}. Substituting in the second equation, and noting that $w^{II}(0)$ is specified by the boundary conditions at $x_1 = 0$, we again obtain uniqueness, this time only local, by uniqueness of solutions of the initial-value problem for ODE $(w^{II})' = D(w^I, w^{II})$. We omit the details. (Local uniqueness is here essentially a remark, as it is a consequence, by Rousset's Lemma [51, 42, 19, 18], of our later assumption (D1) of Evans stability.) □

A.2 Convolution estimates

For sake of completeness, in this section we recall the proof of the convolution estimates given in [23, 22, 50] which were used in Section 2.5.2. First, let us recall notations defined in Chapter 2:

$$\theta(x,t) := \sum_{a_j^+ > 0} (1+t)^{-1/2} e^{-|x - a_j^+ t|^2/Mt}, \tag{A.5}$$

$$\psi_1(x,t) := \chi(x,t) \sum_{a_j^+ > 0} (1 + |x| + t)^{-1/2} (1 + |x - a_j^+ t|)^{-1/2}, \tag{A.6}$$

and

$$\psi_2(x,t) := (1 - \chi(x,t))(1 + |x - a_n^+ t| + t^{1/2})^{-3/2}, \tag{A.7}$$

where $\chi(x,t)=1$ for $x\in[0,a_n^+t]$ and $\chi(x,t)=0$ otherwise and $M>0$ is a sufficiently large constant. Recall also Green's function bounds (2.29):

$$\begin{aligned}
|\partial_x^\gamma\partial_y^\alpha\tilde G(x,t;y)| &\le Ce^{-\eta(|x-y|+t)}\\
&+\quad C(t^{-(|\alpha|+|\gamma|)/2}+|\alpha|e^{-\eta|y|}+|\gamma|e^{-\eta|x|})\Big(\sum_{k=1}^n t^{-1/2}e^{-(x-y-a_k^+t)^2/Mt}\\
&+\sum_{a_k^+<0,\,a_j^+>0}\chi_{\{|a_k^+t|\ge|y|\}}t^{-1/2}e^{-(x-a_j^+(t-|y/a_k^+|))^2/Mt}\Big),
\end{aligned}\tag{A.8}$$

$0\le|\alpha|,|\gamma|\le1$, for some η, C, $M>0$, where indicator function $\chi_{\{|a_k^+t|\ge|y|\}}$ is 1 for $|a_k^+t|\ge|y|$ and 0 otherwise.

Lemma A.2.1 (Linear estimates I). *Under the assumptions of Theorem 2.1.4,*

$$\begin{aligned}
\int_0^{+\infty}|\tilde G(x,t;y)|(1+|y|)^{-3/2}\,dy\le C(\theta+\psi_1+\psi_2)(x,t),\\
\int_0^{+\infty}|\tilde G_x(x,t;y)|(1+|y|)^{-3/2}\,dy\le C(\theta+\psi_1+\psi_2)(x,t),
\end{aligned}\tag{A.9}$$

for $0\le t\le+\infty$ and some $C>0$.

Proof. In view of (A.8), we shall give only a proof of the first estimate in (A.9). In addition, the fast-decaying term $e^{-\eta(|x-y|+t)}$ will be neglected in our computations below.

Convection estimate. We first estimate

$$\int_0^\infty t^{-1/2}e^{-\frac{(x-y-a_k^+t)^2}{Mt}}(1+|y|)^{-3/2}dy.\tag{A.10}$$

In what follows we shall often obtain estimates by first writing

$$x-y-a_k^+t=(x-a_k^+t)-y$$

and then deriving a so–called balance estimate by considering y *linearly close to* or *away from* $|x-a_k^+t|$, i.e., $y\in(\frac12|x-a_k^+t|,\frac32|x-a_k^+t|)$ or otherwise,

$$\begin{aligned}
t^{-1/2}&e^{-\frac{(x-y-a_k^+t)^2}{Mt}}(1+|y|)^{-3/2}\\
&\le Ct^{-1/2}e^{-\frac{|x-a_k^+t|^2}{Mt}}(1+|y|)^{-3/2}+Ct^{-1/2}e^{-\frac{(x-y-a_k^+t)^2}{Mt}}(1+|x-a_k^+t|)^{-3/2}.
\end{aligned}\tag{A.11}$$

For the first term, by integrability of $(1+|y|)^{-3/2}$, we have the integral (A.10) is bounded by

$$C(1+t)^{-1/2} e^{-\frac{|x-a_k^+ t|^2}{Mt}}$$

which is subsumed into $C\theta(x,t)$ in case $a_k^+ > 0$ and into $C(\psi_1+\psi_2)(x,t)$ in case $a_k^+ < 0$. (We remark that the possible blow-up of (A.11) as $t \to 0$ is treated by integrating the Gaussian kernel $e^{-\frac{(x-y-a_k^+ t)^2}{Mt}}$ in y, yielding an extra factor $t^{1/2}$). Meanwhile, for the second term, (noting that $y \in (\frac{1}{2}|x-a_k^+ t|, \frac{3}{2}|x-a_k^+ t|)$), by integrating the Gaussian kernel, we get

$$\int_{\frac{1}{2}|x-a_k^+ t|}^{\frac{3}{2}|x-a_k^+ t|} t^{-1/2} e^{-\frac{(x-y-a_k^+ t)^2}{Mt}} (1+|y|)^{-3/2} dy \le C t^{-1/2} (1+|x-a_k^+ t|)^{-3/2} \min\{|x-a_k^+ t|, t^{1/2}\},$$

which is then bounded by $(1+|x-a_k^+ t|)^{-3/2} \le C(\psi_1+\psi_2)(x,t)$ when $|x-a_k^+ t| \ge C\sqrt{t}$ and by $(1+t)^{-1/2} \le C\theta(x,t)$ when $|x-a_k^+ t| \le C\sqrt{t}$.

Reflection estimate. We estimate

$$\int_0^\infty t^{-1/2} e^{-\frac{(x-\frac{a_j^+}{a_k^+} y - a_j^+ t)^2}{Mt}} (1+|y|)^{-3/2} dy, \tag{A.12}$$

by first applying a change of variable $y := -a_j^+ y / a_k^+$ and then treating the resulting integral as above, yielding the estimate (A.9) as claimed. □

We next give a proof of Lemma 2.5.5 ([23, Lemma 4]). The proof is quite lengthy, and thus we divide the task into three following lemmas.

Lemma A.2.2 (Nonlinearity θ^2). *Under the assumptions of Theorem 2.1.4,*

$$\int_0^t \int_0^{+\infty} |\tilde{G}_y(x,t-s;y)| \theta^2(y,s)\, dyds \le C(\theta+\psi_1+\psi_2)(x,t), \tag{A.13}$$

for $0 \le t \le +\infty$, *some* $C > 0$.

Proof. We shall give an estimate involving the convection term in $\partial_y \tilde{G}$:

$$t^{-1/2}(t^{-1/2} + e^{-\eta y}) e^{-\frac{(x-y-a_k^+ t)^2}{Mt}}. \tag{A.14}$$

The other terms can be estimated similarly. By completing the appropriate square, we first estimate the integral

$$\begin{aligned}
&\int_0^t \int_0^\infty (t-s)^{-1} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} (1+s)^{-1} e^{-\frac{(y-a_j^+ s)^2}{Ms}} dyds, \\
&\quad = \int_0^t \int_0^\infty (t-s)^{-1}(1+s)^{-1} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{Mt}} e^{-\frac{t}{Ms(t-s)}(y-\frac{xs-(a_k^+-a_j^+)s(t-s)}{Mt})^2} dyds \\
&\quad \leq Ct^{-1/2} \int_0^t (t-s)^{-1/2}(1+s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{Mt}} ds.
\end{aligned} \tag{A.15}$$

First observe that for $x > a_n^+ t$, by writing

$$x - a_k^+(t-s) - a_j^+ s = (x - a_k^+ t) + (a_k^+ - a_j^+)s$$

for $a_k^+ \geq a_j^+$ or

$$x - a_k^+(t-s) - a_j^+ s = (x - a_j^+ t) - (a_k^+ - a_j^+)(t-s)$$

for $a_k^+ < a_j^+$, there is no cancellation in these expressions and thus we can estimate the Gaussian kernel by $e^{-(x-a_k^+ t)^2/Mt}$ or $e^{-(x-a_j^+ t)^2/Mt}$. Hence, in this case, the integral (A.15) is estimated by

$$t^{-1/2} e^{-\frac{(x-a_k^+ t)^2}{Mt}} \int_0^t (t-s)^{-1/2}(1+s)^{-1/2} ds \leq C(1+t)^{-1/2} e^{-\frac{(x-a_k^+ t)^2}{Mt}},$$

which is subsumed into $C\theta(x,t)$.

To estimate the integral for $x \leq a_n^+ t$, we divide the analysis into two cases: $s \in [0, t/2]$ and $s \in [t/2, t]$. For $s \in [0, t/2]$, by writing

$$x - a_k^+(t-s) - a_j^+ s = (x - a_k^+ t) + (a_k^+ - a_j^+)s.$$

and thus deriving a balance estimate

$$\begin{aligned}
&(1+s)^{-1/2} e^{-\frac{((x-a_k^+ t)+(a_k^+-a_j^+)s)^2}{Mt}} \\
&\quad \leq C\Big[(1+|x-a_k^+ t|)^{-1/2} e^{-\frac{((x-a_k^+ t)+(a_k^+-a_j^+)s)^2}{Mt}} + (1+s)^{-1/2} e^{-\frac{(x-a_k^+ t)^2}{Mt}}\Big],
\end{aligned}$$

we can estimate

$$
\begin{aligned}
t^{-1}&\int_0^{t/2}(1+s)^{-1/2}e^{-\frac{(x-a_k^+(t-s)-a_j^+s)^2}{Mt}}ds\\
&\le Ct^{-1}\int_0^{t/2}\Big[(1+|x-a_k^+t|)^{-1/2}e^{-\frac{((x-a_k^+t)+(a_k^+-a_j^+)s)^2}{Mt}}+(1+s)^{-1/2}e^{-\frac{(x-a_k^+t)^2}{Mt}}\Big]ds\\
&\le Ct^{-1}\Big[t^{1/2}(1+|x-a_k^+t|)^{-1/2}+(1+t)^{1/2}e^{-\frac{(x-a_k^+t)^2}{Mt}}\Big]\\
&\le C(1+t)^{-1/2}(1+|x-a_k^+t|)^{-1/2}+C(1+t)^{-1/2}e^{-\frac{(x-a_k^+t)^2}{Mt}},
\end{aligned}
$$

where the first term in the last inequality above is bounded by $C\psi_1(x,t)$ and the second term is clearly subsumed in $C\theta(x,t)$.

For $s\in[t/2,t]$, we can argue similarly, beginning with the relation

$$
x-a_k^+(t-s)-a_j^+s=(x-a_j^+t)-(a_k^+-a_j^+)(t-s)
$$

and the balance estimate

$$
\begin{aligned}
(t-s)^{-1/2}&e^{-\frac{((x-a_j^+t)-(a_k^+-a_j^+)(t-s))^2}{Mt}}\\
&\le C\Big[|x-a_j^+t|^{-1/2}e^{-\frac{((x-a_j^+t)-(a_k^+-a_j^+)(t-s))^2}{Mt}}+(t-s)^{-1/2}e^{-\frac{(x-a_j^+t)^2}{Mt}}\Big].
\end{aligned}
$$

Next, we estimate the integral which involves the decaying term $e^{-\eta y}$ in (A.14):

$$
\int_0^t\int_0^\infty(t-s)^{-1/2}e^{-\eta y}e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}}(1+s)^{-1}e^{-\frac{(y-a_j^+s)^2}{Ms}}dyds, \tag{A.16}
$$

for which we observe the inequality

$$
e^{-\eta y}e^{-\frac{(y-a_j^+s)^2}{Ms}}\le Ce^{-\eta_1 y}e^{-\eta_2 s}.
$$

We can now proceed similarly as above to yield a desired estimate for (A.16), taking advantage of the integrability of $e^{-\eta_1 y}$ in y and the integrability of $e^{-\eta_2 s}$ in s. This concludes the analysis of the nonlinearity θ^2. □

Lemma A.2.3 (Nonlinearity ψ_1^2)**.** *Under the assumptions of Theorem 2.1.4,*

$$
\int_0^t\int_0^{+\infty}|\tilde G_y(x,t-s;y)|\psi_1^2(y,s)\,dyds\le C(\theta+\psi_1+\psi_2)(x,t), \tag{A.17}
$$

for $0 \le t \le +\infty$, *some* $C > 0$.

Proof. We consider convolutions of the form

$$\int_0^t \int_0^\infty (t-s)^{-1/2}((t-s)^{-1/2} + e^{-\eta y}) e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} \psi_1^2 dy ds. \tag{A.18}$$

We first estimate

$$\int_0^t \int_0^{a_n^+ s} (t-s)^{-1} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} (1+|y|+s)^{-1}(1+|y-a_j^+ s|)^{-1} dy ds. \tag{A.19}$$

In following estimates, we always integrate $(1+|y-a_j^+ s|)^{-1/2}$ in y in the case $s \in (0, t/2)$ and the Gaussian kernel $e^{-\frac{(y-a)^2}{M(t-s)}}$ in the case $s \in (t/2, t)$, yielding an extra factor $(1+s)^{1/2}$ or $(t-s)^{1/2}$, respectively, and give bounds to other remaining terms.

Let us first consider the case $x > a_n^+ t$. By writing

$$x - y - a_k^+(t-s) = (x - a_n^+ t) + (-(y - a_n^+ s)) + (-(a_k^+ - a_n^+)(t-s)), \tag{A.20}$$

we observe that in this case values in each bracket on the right hand side of (A.20) have the same sign and thus there is no cancellation in this expression. Hence, we get an estimate of (A.18) by

$$\begin{aligned} e^{-\frac{|x-a_n^+ t|^2}{Mt}} \Big[\int_0^{t/2} t^{-1}(1+s)^{-1/2} ds + \int_{t/2}^t (t-s)^{-1/2}(1+t)^{-1} ds \Big] \\ \le C(1+t)^{-1/2} e^{-\frac{|x-a_n^+ t|^2}{Mt}}. \end{aligned}$$

Now consider the case $x \le a_n^+ t$. By writing

$$x - y - a_k^+(t-s) = (x - a_j^+ t) - (a_k^+ - a_j^+)(t-s) - (y - a_j^+ s)$$

and deriving a balance estimate with $(y - a_j^+ s)$ linearly close to or away from $(x -$

$a_j^+t) - (a_k^+ - a_j^+)(t-s)$, we estimate (A.19) by

$$\begin{aligned}\int_0^t \int_0^{a_n^+ s} &(t-s)^{-1}(1+|y|+s)^{-1}(1+|y-a_j^+ s|)^{-1/2} \\ &\times \Big[(1+|x-a_k^+(t-s)-a_j^+ s|)^{-1/2} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} \\ &\quad + (1+|y-a_j^+ s|)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{M(t-s)}} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}}\Big] dyds.\end{aligned} \tag{A.21}$$

Again, as before, we integrate $(1+|y-a_j^- s|)^{-1/2}$ in y on the interval $s \in [0, t/2]$, while on the interval $s \in [t/2, t]$ we integrate the Gaussian kernel. For the second piece of the integral (A.21), we obtain an estimate

$$\begin{aligned}C\int_0^{t/2} t^{-1}(1+s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{M'(t-s)}} ds + C\int_{t/2}^t (1+t)^{-1}(t-s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{M'(t-s)}} ds \\ \le Ct^{-1/2}\int_0^t (t-s)^{-1/2}(1+s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_j^+ s)^2}{M't}} ds,\end{aligned}$$

which is identical to (A.15) whose estimate is given above. Meanwhile, for the first piece of the integral (A.21), we estimate in a same way, yielding

$$\begin{aligned}C\int_0^{t/2} &t^{-1}(1+s)^{-1/2}(1+|x-a_k^+(t-s)-a_j^+ s|)^{-1/2} ds \\ &+ C\int_{t/2}^t (1+t)^{-1}(t-s)^{-1/2}(1+|x-a_k^+(t-s)-a_j^+ s|)^{-1/2} ds.\end{aligned} \tag{A.22}$$

In the case $s \in [0, t/2]$, by writing

$$x - a_k^+(t-s) - a_j^+ s = (x - a_k^+ t) - (a_k^+ - a_j^+)s,$$

and deriving a balance estimate with $(a_k^+ - a_j^+)s$ linearly close to or away from $(x - a_k^+ t)$:

$$\begin{aligned}&(1+s)^{-1/2}(1+|x-a_k^+(t-s)-a_j^+ s|)^{-1/2} \\ &\qquad \le C(1+|x-a_k^+ t|)^{-1/2}\Big[(1+s)^{-1/2} + (1+|x-a_k^+(t-s)-a_j^+ s|)^{-1/2}\Big],\end{aligned}$$

we easily estimate the first term in (A.22) by

$$C(1+t)^{-1/2}(1+|x-a_k^+t|)^{-1/2},$$

which is subsumed into $C\psi_1(x,t)$, noting that we are in the case $x \le a_n^+ t$.
In the case $s \in [t/2, t]$, we write

$$x - a_k^+(t-s) - a_j^+ s = (x - a_j^+ t) - (a_k^+ - a_j^+)(t-s),$$

for which we have a balance estimate

$$\begin{aligned}(t-s)^{-1/2}&(1+|x-a_k^+(t-s)-a_j^+s|)^{-1/2}\\ &\le C\Big[|x-a_j^+t|^{-1/2}(1+|x-a_k^+(t-s)-a_j^+s|)^{-1/2} + (t-s)^{-1/2}(1+|x-a_j^+t|)^{-1/2}\Big]\end{aligned}$$

Thus, we easily estimate the second term in (A.22) by

$$C(1+t)^{-1/2}\Big[|x-a_j^+t|^{-1/2} + (1+|x-a_j^+t|)^{-1/2}\Big],$$

which is again subsumed into $C\psi_1(x,t)$. We remark that the apparent blow-up at $x = a_j^+ t$ is an artifact of the approach and can be removed by the observation that for $|x - a_j^+ t| \le C\sqrt{t}$, we can proceed by alternative estimates to get decay of form $\theta(x,t)$.

Now consider the integral involving the term $e^{-\eta y}$ in (A.18):

$$\int_0^t \int_0^{a_n^+ s} (t-s)^{-1/2} e^{-\eta y} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} (1+|y|+s)^{-1}(1+|y-a_j^+s|)^{-1} dyds. \tag{A.23}$$

We observe here the inequality

$$e^{-\eta y}(1+|y-a_j^+s|)^{-1} \le C\Big[e^{-\eta_1 y}e^{-\eta_2 s} + e^{-\eta y}(1+s)^{-1}\Big].$$

We can now proceed similarly as in the analysis of the above case. This concludes the analysis for the nonlinearity ψ_1^2. □

Lemma A.2.4 (Nonlinearity ψ_2^2)**.** *Under the assumptions of Theorem 2.1.4,*

$$\int_0^t \int_0^{+\infty} |\tilde{G}_y(x,t-s;y)|\psi_2^2(y,s)\, dyds \le C(\theta + \psi_1 + \psi_2)(x,t), \tag{A.24}$$

for $0 \le t \le +\infty$, *some* $C > 0$.

Proof. We consider convolutions of the form

$$\int_0^t \int_0^\infty (t-s)^{-1/2}((t-s)^{-1/2} + e^{-\eta y}) e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} \psi_2^2 dy ds. \tag{A.25}$$

We first estimate

$$\int_0^t \int_{a_n^+ s}^\infty (t-s)^{-1} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} (1 + |y - a_n^+ s| + s^{1/2})^{-3} dy ds. \tag{A.26}$$

We derive an estimate for (A.26) in a same way as done in previous lemma for the case of nonlinearity ψ_1^2. First, the case $x > a_n^+ t$ is now easily analyzed by considering the relation (A.20) with no cancellation, yielding an estimate which will be subsumed into $C\theta(x,t)$.

For the case $x \le a_n^+ t$, by writing

$$x - y - a_k^+(t-s) = (x - a_k^+(t-s) - a_n^+ s) - (y - a_n^+ s),$$

we derive a balance estimate

$$\begin{aligned} &(1 + |y - a_n^+ s| + s^{1/2})^{-3/2} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} \\ &\quad \le C\Big[(1 + |x - a_k^+(t-s) - a_n^+ s| + s^{1/2})^{-3/2} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}} \\ &\qquad + (1 + |y - a_n^+ s| + s^{1/2})^{-3/2} e^{-\frac{(x-a_k^+(t-s)-a_n^+ s)^2}{M'(t-s)}} e^{-\frac{(x-y-a_k^+(t-s))^2}{M(t-s)}}\Big]. \end{aligned} \tag{A.27}$$

Again, we estimate (A.26) by integrating $(1+|y-a_n^+ s|+s^{1/2})^{-3/2}$ in y on the interval $s \in [0, t/2]$ and the Gaussian kernel on the interval $s \in [t/2, t]$. For the second piece of (A.27), we obtain an estimate

$$\begin{aligned} &C\int_0^{t/2} t^{-1}(1+s^{1/2})^{-1/2}(1+s^{1/2})^{-3/2} e^{-\frac{(x-a_k^+(t-s)-a_n^+ s)^2}{M'(t-s)}} ds \\ &\qquad + C\int_{t/2}^t (1+t^{1/2})^{-3/2}(t-s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_n^+ s)^2}{M'(t-s)}} ds \\ &\quad \le Ct^{-1/2}\int_0^t (t-s)^{-1/2}(1+s)^{-1/2} e^{-\frac{(x-a_k^+(t-s)-a_n^+ s)^2}{M'(t-s)}} ds, \end{aligned}$$

which is identical to (A.15) whose estimate is given above. Meanwhile, for the first piece of (A.27), we estimate in a same way, yielding

$$
\begin{aligned}
C\int_0^{t/2} & t^{-1}(1+s^{1/2})^{-1/2}(1+|x-a_k^+(t-s)-a_n^+s|+s^{1/2})^{-3/2}ds \\
&+C\int_{t/2}^{t}(1+t^{1/2})^{-3/2}(t-s)^{-1/2}(1+|x-a_k^+(t-s)-a_n^+s|+s^{1/2})^{-3/2}ds.
\end{aligned}
\tag{A.28}
$$

We further estimate this as follows. In the case $s\in[0,t/2]$, by writing

$$x-a_k^+(t-s)-a_n^+s=(x-a_k^+t)+(a_k^+-a_n^+)s,$$

and thus deriving a balance estimate:

$$
\begin{aligned}
(1+|x-&a_k^+(t-s)-a_n^+s|+s^{1/2})^{-3/2} \\
&\leq C\Big[(1+|x-a_k^+t|+s^{1/2})^{-3/2}+(1+|x-a_k^+t|^{1/2}+s^{1/2})^{-3/2}\Big] \\
&\leq C(1+|x-a_k^+t|^{1/2}+s^{1/2})^{-3/2},
\end{aligned}
$$

we easily give an estimate of the first integral in (A.28) by

$$
\begin{aligned}
Ct^{-1}\int_0^{t/2}&(1+|x-a_k^+t|^{1/2}+s^{1/2})^{-3/2}(1+s^{1/2})^{-1/2}ds \\
&\leq Ct^{-1}(1+|x-a_k^+t|)^{-1/2}\int_0^{t/2}(1+s^{1/2})^{-1/2}(1+s^{1/2})^{-1/2}ds \\
&\leq C(1+t)^{-1/2}(1+|x-a_k^+t|)^{-1/2}
\end{aligned}
$$

For $s\in[t/2,t]$, by writing

$$x-a_k^+(t-s)-a_n^+s=(x-a_n^+t)-(a_k^+-a_n^+)(t-s),$$

and deriving a balance estimate:

$$
\begin{aligned}
(t-s)^{-1/2}&\Big(1+|x-a_k^+(t-s)-a_n^+s|+s^{1/2}\Big)^{-3/2} \\
&\leq C\Big[|x-a_n^+t|^{-1/2}\Big(1+|x-a_k^+(t-s)-a_n^+s|+s^{1/2}\Big)^{-3/2} \\
&+(t-s)^{-1/2}\Big(1+|x-a_n^+t|+s^{1/2}\Big)^{-3/2}\Big].
\end{aligned}
$$

we easily give an estimate of the second integral in (A.28) by

$$C(1+t^{1/2})^{-3/2}\Big[|x-a_n^+t|^{-1/2}\int_{t/2}^{t}(1+|x-a_k^+(t-s)-a_n^+s|+t^{1/2})^{-3/2}ds$$
$$+(1+|x-a_n^+t|+t^{1/2})^{-3/2}\int_{t/2}^{t}(t-s)^{-1/2}ds\Big]$$
$$\le C(1+t)^{-1/2}(1+|x-a_k^+t|)^{-1/2}+C(1+|x-a_k^+t|+t^{1/2})^{-3/2},$$

which can be subsumed into $C\psi_1(x,t)$.

This concludes the analysis for (A.26). For an estimate of (A.19) which involves the term $e^{-\eta y}$, we proceed similarly as in previous lemma (see (A.23)), completing the analysis for the nonlinearity ψ_2^2. □

Lemma A.2.5 (Nonlinear estimates II). *Under the assumptions of Theorem 2.1.4,*

$$\int_{t-1}^{t}\int_0^{+\infty}|\tilde{G}_x(x,t-s;y)|\Upsilon(y,s)\,dyds\le C(\psi_1+\psi_2)(x,t) \tag{A.29}$$

for all $1\le t<+\infty$*, some* $C>0$*, where*

$$\Upsilon(y,s):=s^{-1/4}(\theta+\psi_1+\psi_2)(y,s) \tag{A.30}$$

Proof. We first observe that

$$|\tilde{G}_x(x,t-s;y)|\le C(t-s)^{-1/2}((t-s)^{-1/2}+e^{-\eta x})e^{-\frac{|x-y|^2}{M(t-s)}} \tag{A.31}$$

for $t-1\le s\le t$; indeed, this is clear by (A.8) and

$$|x-y-a_k^+(t-s)|\ge|x-y|-C|t-s|\ge|x-y|-C$$
$$|x-a_j^+((t-s)-|y/a_k^+|)|\ge\Big|x+\frac{|a_j^+|}{|a_k^+|}y\Big|-C|t-s|\ge\min\{1,|a_j^+/a_k^+|\}|x-y|-C$$

for all s such that $t-1\le s\le t$.

Now, using (A.31) and following the treatments as were done in the previous lemmas corresponding to nonlinearities $\theta(x,t),\psi_1(x,t),\psi_2(x,t)$, we easily obtain the lemma, omitting further details of the proof. □

Appendix B

Appendix to Chapter 3

B.1 Physical discussion in the isentropic case

In this appendix, we revisit in slightly more detail the drag-reduction problem sketched in Examples 1.2.1–1.2.2, in the simplified context of the two-dimensional isentropic case. Following the notation of [19], consider the two-dimensional isentropic compressible Navier–Stokes equations

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \tag{B.1}$$

$$(\rho u)_t + (\rho u^2)_x + (\rho uv)_y + p_x = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy}, \tag{B.2}$$

$$(\rho v)_t + (\rho uv)_x + (\rho v^2)_y + p_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx} \tag{B.3}$$

on the half-space $y > 0$, where ρ is density, u and v are velocities in x and y directions, and $p = p(\rho)$ is pressure, and $\mu > |\eta| \geq 0$ are coefficients of first ("dynamic") and second viscosity, making the standard monotone pressure assumption $p'(\rho) > 0$.

We imagine a porous airfoil lying along the x-axis, with constant imposed normal velocity $v(0) = V$ and zero transverse relative velocity $u(0) = 0$ imposed at the airfoil surface, and seek a laminar boundary-layer flow $(\rho, u, v)(y)$ with transverse relative velocity u_∞ a short distance away the airfoil, with $|V|$ much less than the sound speed c_∞ and $|u_\infty|$ of an order roughly comparable to c_∞.

B.1.1 Existence

The possible boundary-layer solutions have been completely categorized in this case in Section 5.1 of [19]. We here cite the relevant conclusions, referring to [19] for the (straightforward) justifying computations.

Outflow case $(V < 0)$

In the outflow case, the scenario described above corresponds to case (5.15) of [19], in which it is found that the only solutions are purely *transverse* flows

$$(\rho, v) \equiv (\rho_0, V), \quad u(y) = u_\infty(1 - e^{\rho_0 V y/\mu}), \tag{B.4}$$

varying only in the tranverse velocity u. The drag force per unit length at the airfoil, by Newton's law of viscosity, is

$$\mu \bar{u}_y|_{y=0} = u_\infty \rho_\infty |V|, \tag{B.5}$$

since momentum $m := \rho_0 V = \rho_\infty V$ is constant throughout the layer, so that (ρ_∞, u_∞ being imposed by ambient conditions away from the wing) *drag is proportional to the speed* $|V|$ *of the imposed normal velocity.*

Inflow case $(V > 0)$

Consulting again [19] (p. 61), we find for $V > 0$ with specified $(\rho, u, v)(0)$ of the orders described above, the only solutions are purely *normal* flows,

$$u \equiv u(0), \quad (\rho, v) = (\rho, v)(y), \tag{B.6}$$

varying only in the normal velocity v. Thus, it is not possible to reconcile the velocity $u(0)$ at the airfoil with the velocity $u_\infty >> c$ some distance away.

As discussed in [38], the expected behavior in such a case consists rather of a combination of a boundary-layer at $y = 0$ and one or more elementary planar shock, rarefaction, or contact waves moving away from $y = 0$: in this case a shear wave moving with normal fluid velocity V into the half-space, across which the transverse velocity changes from zero to u_∞. That is, a characteristic layer analogous to the solid-boundary case *detaches* from the airfoil and travels outward into the flow field.

In this case, one would not expect drag reduction compared to the solid-boundary case, but rather some increase.

B.1.2 Stability

If we consider one-dimensional stability, or stability with respect to perturbations depending only on y, we find that the linearized eigenvalue equations decouple into the constant-coefficient linearized eigenvalue equations for (ρ, v) about a constant layer $(\rho, v) \equiv (\rho_0, V)$, and the scalar linearized eigenvalue equation

$$\lambda \bar{\rho} u + m u_y = \mu u_{yy} \tag{B.7}$$

associated with the constant-coefficient convection-diffusion equation $\bar{\rho} u_t + mu = \mu u_{yy}$, $m := \bar{\rho}\bar{v} \equiv \rho_0 V$, $\bar{\rho} \equiv \rho_0$. As the constant layer (ρ_0, V) is stable by Corollary 3.1.2 or direct calculation (Fourier transform), and (B.7) is stable by direct calculation, we may thus conclude that purely transverse layers are *one-dimensionally stable.*

Considered with respect to general perturbations, the equations do not decouple, nor do they reduce to constant-coefficient form, but to a second order system whose coefficients are quadratic polynomials in $e^{\rho_0 V y}$. It would be very interesting to try to resolve the question of spectral stability by direct solution using this special form, or, alternatively, to perform a numerical study as done in [25] for the multi-dimensional shock wave case.

Remark B.1.1. For general laminar boundary layers $(\bar{\rho}, \bar{u}, \bar{v})(y)$, the one-dimensional stability problem, now variable-coefficient, does not completely decouple, but has triangular form, breaking into a system in (ρ, v) alone and an equation in u forced by (ρ, v). Stability with respect to general perturbations, therefore, is equivalent to stability with respect to perturbations of form $(\rho, 0, v)$ or $(0, u, 0)$. For perturbations $(\rho, u, v) = (0, u, 0)$, the u equation again becomes (B.7), with μ, m still constant, but $\bar{\rho}$ varying in y. Taking the real part of the complex L^2 inner product of u against (B.7) gives

$$\Re\lambda \|u\|_{L^2}^2 + \|u_y\|_{L^2}^2 = 0,$$

hence for $\Re\lambda \geq 0$, $u \equiv \text{constant} = 0$. Thus, the layer is one-dimensionally stable if and only if the normal part $(\bar{\rho}, \bar{v})$ is stable with respect to perturbations (ρ, v). Stability of normal layers was studied in [9] for a γ-law gas $p(\rho) = a\rho^\gamma$, $1 \leq \gamma \leq 3$, with the conclusion that *all layers are one-dimensionally stable*, independent of amplitude, in the general inflow and compressive outflow cases. Hence, we can make the same conclusion for full layers $(\bar{\rho}, \bar{u}, \bar{v})$. In the present context, this includes all cases except for suction with supersonic velocity $|V| > c_\infty$, which in the notation of [9] is of *expansive outflow* type, since $|\bar{v}|$ is decreasing with y, so that density $\bar{\rho}$ (since $m = \bar{\rho}\bar{v} \equiv$ constant) is increasing.

B.1.3 Discussion

Note that we do not achieve by subsonic boundary suction an exact laminar flow connecting the values $(u, v) = (0, V)$ at the wing to the values $(u_\infty, 0)$ of the ambient flow at infinity, but rather to an intermediate value (u_∞, V). That is, we trade a large variation u_∞ in shear for a possibly small variation V in normal velocity, which appears now as a boundary condition for the outer, approximately Euler flow away from the boundary layer. Whether the full solution is stable appears to be a question concerning also nonstationary Euler flow. It is not clear either what is the optimal outflux velocity V. From (B.5) and the discussion just above, it appears desirable to minimize $|V|$, since this minimizes both drag and the imbalance between flow v_∞ just outside the boundary layer and the ambient flow at infinity. On the other hand, we expect that stability becomes more delicate in the characteristic limit $V \to 0^-$, in the sense that the size of the basin of attraction of the boundary layer shrinks to zero (recall, we have ignored throughout our analysis the size of the basin of attraction, taking perturbations as small as needed without keeping track of constants). These would be quite interesting issues for further investigation.

Appendix C

Appendix to Chapter 5

C.1 Proof of preliminary estimate: inflow case

Our starting point is Remark 5.2.4, in which we observed that the first-order eigensystem (5.34) in variable $W = (w, u - v, v)^T$ may be converted by the rescaling $W \to \widetilde{W} := (w, u - v, \lambda v)^T$ to a system identical to that of the integrated equations in the shock case; see [49]. Artificially defining $(\tilde{u}, \tilde{v}, \tilde{v}')^T := \widetilde{W}$, we obtain a system

$$\lambda\tilde{v} + \tilde{v}' - \tilde{u}' = 0, \tag{C.1a}$$

$$\lambda\tilde{u} + \tilde{u}' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}\tilde{v}' = \frac{\tilde{u}''}{\hat{v}}. \tag{C.1b}$$

identical to that in the integrated shock case [3], but with boundary conditions

$$\tilde{v}(0) = \tilde{v}'(0) = \tilde{u}'(0) = 0 \tag{C.2}$$

imposed at $x = 0$. This new eigenvalue problem differs spectrally from (5.22) only at $\lambda = 0$, hence spectral stability of (5.22) is implied by spectral stability of (C.1). Hereafter, we drop the tildes, and refer simply to u, v.

With these coordinates, we may establish (5.31) by exactly the same argument used in the shock case in [3, 24], for completeness reproduced here.

Lemma C.1.1. *The following inequality holds for $\Re e\lambda \geq 0$:*

$$(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^+} \hat{v}|u|^2 + \int_{\mathbb{R}^+} |u'|^2 \\ \leq \sqrt{2} \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^\gamma} |v'||u| + \sqrt{2} \int_{\mathbb{R}^+} \hat{v}|u'||u|. \tag{C.3}$$

Proof. We multiply (C.1b) by $\hat{v}\bar{u}$ and integrate along x. This yields

$$\lambda \int_{\mathbb{R}^+} \hat{v}|u|^2 + \int_{\mathbb{R}^+} \hat{v}u'\bar{u} + \int_{\mathbb{R}^+} |u'|^2 = \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^\gamma} v'\bar{u}.$$

We get (C.3) by taking the real and imaginary parts and adding them together, and noting that $|\Re e(z)| + |\Im m(z)| \leq \sqrt{2}|z|$. □

Lemma C.1.2. *The following identity holds for $\Re e\lambda \geq 0$:*

$$\int_{\mathbb{R}^+} |u'|^2 = 2\Re e(\lambda)^2 \int_{\mathbb{R}^+} |v|^2 + \Re e(\lambda) \int_{\mathbb{R}^+} \frac{|v'|^2}{\hat{v}} + \frac{1}{2} \int_{\mathbb{R}^+} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}} \right] |v'|^2 \tag{C.4}$$

Proof. We multiply (C.1b) by $\bar{v}'$ and integrate along x. This yields

$$\lambda \int_{\mathbb{R}^+} u\bar{v}' + \int_{\mathbb{R}^+} u'\bar{v}' - \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} |v'|^2 = \int_{\mathbb{R}^+} \frac{1}{\hat{v}} u''\bar{v}' = \int_{\mathbb{R}^+} \frac{1}{\hat{v}} (\lambda v' + v'')\bar{v}'.$$

Using (C.1a) on the right-hand side, integrating by parts, and taking the real part gives

$$\Re e \left[\lambda \int_{\mathbb{R}^+} u\bar{v}' + \int_{\mathbb{R}^+} u'\bar{v}' \right] = \int_{\mathbb{R}^+} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{\hat{v}_x}{2\hat{v}^2} \right] |v'|^2 + \Re e(\lambda) \int_{\mathbb{R}^+} \frac{|v'|^2}{\hat{v}}.$$

The right hand side can be rewritten as

$$\Re e \left[\lambda \int_{\mathbb{R}^+} u\bar{v}' + \int_{\mathbb{R}^+} u'\bar{v}' \right] = \frac{1}{2} \int_{\mathbb{R}^+} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}} \right] |v'|^2 + \Re e(\lambda) \int_{\mathbb{R}^+} \frac{|v'|^2}{\hat{v}}. \tag{C.5}$$

Now we manipulate the left-hand side. Note that

$$\begin{aligned}\lambda \int_{\mathbb{R}^+} u\bar{v}' + \int_{\mathbb{R}^+} u'\bar{v}' &= (\lambda + \bar{\lambda}) \int_{\mathbb{R}^+} u\bar{v}' - \int_{\mathbb{R}^+} u(\bar{\lambda}\bar{v}' + \bar{v}'') \\ &= -2\Re e(\lambda) \int_{\mathbb{R}^+} u'\bar{v} - \int_{\mathbb{R}^+} u\bar{u}'' \\ &= -2\Re e(\lambda) \int_{\mathbb{R}^+} (\lambda v + v')\bar{v} + \int_{\mathbb{R}^+} |u'|^2.\end{aligned}$$

Hence, by taking the real part we get

$$\Re e \left[\lambda \int_{\mathbb{R}^+} u\bar{v}' + \int_{\mathbb{R}^+} u'\bar{v}'\right] = \int_{\mathbb{R}^+} |u'|^2 - 2\Re e(\lambda)^2 \int_{\mathbb{R}^+} |v|^2.$$

This combines with (C.5) to give (C.4). □

Lemma C.1.3 ([3]). *For $h(\hat{v})$ as in* (5.21)*, we have*

$$\sup_{\hat{v}} \left| \frac{h(\hat{v})}{\hat{v}^\gamma} \right| = \gamma \frac{1 - v_+}{1 - v_+^\gamma} \leq \gamma, \tag{C.6}$$

where $\hat{v}$ is the profile solution to (5.18)*.*

Proof. Defining

$$g(\hat{v}) := h(\hat{v})\hat{v}^{-\gamma} = -\hat{v} + a(\gamma - 1)\hat{v}^{-\gamma} + (a + 1), \tag{C.7}$$

we have $g'(\hat{v}) = -1 - a\gamma(\gamma - 1)\hat{v}^{-\gamma-1} < 0$ for $0 < v_+ \leq \hat{v} \leq v_- = 1$, hence the maximum of g on $\hat{v} \in [v_+, v_-]$ is achieved at $\hat{v} = v_+$. Substituting (5.19) into (C.7) and simplifying yields (C.6). □

Proof of Proposition 5.2.3. Using Young's inequality twice on right-hand side of (C.3) together with (C.6), we get

$$
\begin{aligned}
(\Re e(\lambda) + |\Im m(\lambda)|) &\int_{\mathbb{R}^+} \hat{v}|u|^2 + \int_{\mathbb{R}^+} |u'|^2 \\
&\leq \sqrt{2} \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^\gamma} |v'||u| + \sqrt{2} \int_{\mathbb{R}^+} \hat{v}|u'||u| \\
&\leq \theta \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} |v'|^2 + \frac{(\sqrt{2})^2}{4\theta} \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^\gamma} \hat{v}|u|^2 + \epsilon \int_{\mathbb{R}^+} \hat{v}|u'|^2 + \frac{1}{4\epsilon} \int_{\mathbb{R}^+} \hat{v}|u|^2 \\
&< \theta \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} |v'|^2 + \epsilon \int_{\mathbb{R}^+} |u'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{2\epsilon} \right] \int_{\mathbb{R}^+} \hat{v}|u|^2.
\end{aligned}
$$

Assuming that $0 < \epsilon < 1$ and $\theta = (1 - \epsilon)/2$, this simplifies to

$$
\begin{aligned}
(\Re e(\lambda) + |\Im m(\lambda)|) &\int_{\mathbb{R}^+} \hat{v}|u|^2 + (1 - \epsilon) \int_{\mathbb{R}^+} |u'|^2 \\
&< \frac{1 - \epsilon}{2} \int_{\mathbb{R}^+} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} |v'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{2\epsilon} \right] \int_{\mathbb{R}^+} \hat{v}|u|^2.
\end{aligned}
$$

Applying (C.4) yields

$$
(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^+} \hat{v}|u|^2 < \left[\frac{\gamma}{1 - \epsilon} + \frac{1}{2\epsilon} \right] \int_{\mathbb{R}^+} \hat{v}|u|^2,
$$

or equivalently,

$$
(\Re e(\lambda) + |\Im m(\lambda)|) < \frac{(2\gamma - 1)\epsilon + 1}{2\epsilon(1 - \epsilon)}.
$$

Setting $\epsilon = 1/(2\sqrt{\gamma} + 1)$ gives (5.31). □

C.2 Proof of preliminary estimate: outflow case

Similarly as in the inflow case, we can convert the eigenvalue equations into the integrated equations as in the shock case; see [49]. Artificially defining $(\tilde{u}, \tilde{v}, \tilde{v}')^T := \widetilde{W}$, we obtain a system

$$
\lambda \tilde{v} + \tilde{v}' - \tilde{u}' = 0, \tag{C.8a}
$$

$$
\lambda \tilde{u} + \tilde{u}' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} \tilde{v}' = \frac{\tilde{u}''}{\hat{v}}. \tag{C.8b}
$$

identical to that in the integrated shock case [3], but with boundary conditions

$$\tilde{v}'(0) = \frac{\lambda}{\alpha - 1}\tilde{v}(0), \quad \tilde{u}'(0) = \alpha\tilde{v}'(0) \tag{C.9}$$

imposed at $x = 0$. We shall write w_0 for $w(0)$, for any function w. This new eigenvalue problem differs spectrally from (5.22) only at $\lambda = 0$, hence spectral stability of (5.22) is implied by spectral stability of (C.8). Hereafter, we drop the tildes, and refer simply to u, v.

Lemma C.2.1. *The following inequality holds for $\Re e\lambda \geq 0$:*

$$\begin{aligned}(\Re e(\lambda) + |\Im m(\lambda)|)\int_{\mathbb{R}^-} \hat{v}|u|^2 - \frac{1}{2}\int_{\mathbb{R}^-} \hat{v}_x|u|^2 + \int_{\mathbb{R}^-} |u'|^2 + \frac{1}{2}\hat{v}_0|u_0|^2 \\ \leq \sqrt{2}\int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^\gamma}|v'||u| + \int_{\mathbb{R}^-} \hat{v}|u'||u| + \sqrt{2}|\alpha||v_0'||u_0|.\end{aligned} \tag{C.10}$$

Proof. We multiply (C.8b) by $\hat{v}\bar{u}$ and integrate along x. This yields

$$\lambda\int_{\mathbb{R}^-} \hat{v}|u|^2 + \int_{\mathbb{R}^-} \hat{v}u'\bar{u} + \int_{\mathbb{R}^-} |u'|^2 = \int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^\gamma}v'\bar{u} + u_0'\bar{u}_0.$$

We get (C.10) by taking the real and imaginary parts and adding them together, and noting that $|\Re e(z)| + |\Im m(z)| \leq \sqrt{2}|z|$. □

Lemma C.2.2. *The following inequality holds for $\Re e\lambda \geq 0$:*

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^-} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}^-} \frac{|v'|^2}{\hat{v}} + \frac{|v_0'|^2}{4\hat{v}_0} + 2\Re e(\lambda)^2\int_{\mathbb{R}^-} |v|^2 \\ \leq \int_{\mathbb{R}^-} |u'|^2 + \hat{v}_0|u_0|^2.\end{aligned} \tag{C.11}$$

Proof. We multiply (C.8b) by $\bar{v}'$ and integrate along x. This yields

$$\lambda\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-} u'\bar{v}' - \int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 = \int_{\mathbb{R}^-} \frac{1}{\hat{v}}u''\bar{v}' = \int_{\mathbb{R}^-} \frac{1}{\hat{v}}(\lambda v' + v'')\bar{v}'.$$

Using (C.8a) on the right-hand side, integrating by parts, and taking the real part gives

$$\Re e\left[\lambda\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-} u'\bar{v}'\right] = \int_{\mathbb{R}^-} \left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{\hat{v}_x}{2\hat{v}^2}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}^-} \frac{|v'|^2}{\hat{v}} + \frac{|v_0'|^2}{2\hat{v}_0}.$$

The right hand side can be rewritten as

$$\Re e\left[\lambda\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-} u'\bar{v}'\right] = \frac{1}{2}\int_{\mathbb{R}^-}\left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}^-}\frac{|v'|^2}{\hat{v}} + \frac{|v_0'|^2}{2\hat{v}_0}. \tag{C.12}$$

Now we manipulate the left-hand side. Note that

$$\begin{aligned}\lambda\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-} u'\bar{v}' &= (\lambda+\bar{\lambda})\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-}(u'\bar{v}' - \bar{\lambda}u\bar{v}') \\ &= -2\Re e(\lambda)\int_{\mathbb{R}^-} u'\bar{v} + 2\Re e\lambda u_0\bar{v}_0 + \int_{\mathbb{R}^-} u'(\bar{v}' + \bar{\lambda}\bar{v}) - \bar{\lambda}u_0\bar{v}_0 \\ &= -2\Re e(\lambda)\int_{\mathbb{R}^-}(\lambda v + v')\bar{v} + \int_{\mathbb{R}^-}|u'|^2 + 2\Re e\lambda u_0\bar{v}_0 - \bar{\lambda}u_0\bar{v}_0.\end{aligned}$$

Hence, by taking the real part and noting that

$$\Re e(2\Re e\lambda u_0\bar{v}_0 - \bar{\lambda}u_0\bar{v}_0) = \Re e\lambda\Re e(u_0\bar{v}_0) - \Im m\lambda\Im m(u_0\bar{v}_0) = \Re e(\lambda u_0\bar{v}_0)$$

we get

$$\Re e\left[\lambda\int_{\mathbb{R}^-} u\bar{v}' + \int_{\mathbb{R}^-} u'\bar{v}'\right] = \int_{\mathbb{R}^-}|u'|^2 - 2\Re e(\lambda)^2\int_{\mathbb{R}^-}|v|^2 - \Re e\lambda|v_0|^2 + \Re e(\lambda u_0\bar{v}_0).$$

This combines with (C.12) to give

$$\frac{1}{2}\int_{\mathbb{R}^-}\left[\frac{h(\hat{v})}{\hat{v}^{\gamma+1}} + \frac{a\gamma}{\hat{v}^{\gamma+1}}\right]|v'|^2 + \Re e(\lambda)\int_{\mathbb{R}^-}\frac{|v'|^2}{\hat{v}} + \frac{|v_0'|^2}{2\hat{v}_0} + 2\Re e(\lambda)^2\int_{\mathbb{R}^-}|v|^2 + \Re e\lambda|v_0|^2 = \int_{\mathbb{R}^-}|u'|^2 + \Re e(\lambda u_0\bar{v}_0).$$

We get (C.11) by observing that (C.9) and Young's inequality yield

$$|\Re e(\lambda u_0\bar{v}_0)| \le |\alpha-1||v_0'v_0| \le |v_0'v_0| \le \frac{|v_0'|^2}{4\hat{v}_0} + \hat{v}_0|u_0|^2.$$

Here we used $|\alpha-1| = \frac{|\lambda|}{|\lambda-\hat{v}_0'|} \le 1$. Note that $\Re e\lambda \ge 0$ and $\hat{v}_0' \le 0$. □

Proof of Proposition 5.2.3. Using Young's inequality twice on right-hand side of (C.10) together with (C.6), and denoting the boundary term on the right by I_b, we get

$$\begin{aligned}
&(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^-} \hat{v}|u|^2 - \frac{1}{2}\int_{\mathbb{R}^-} \hat{v}_x |u|^2 + \int_{\mathbb{R}^-} |u'|^2 + \frac{1}{2}\hat{v}_0|u_0|^2 \\
&\quad \le \sqrt{2}\int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^\gamma}|v'||u| + \int_{\mathbb{R}^-} \hat{v}|u'||u| + I_b \\
&\quad \le \theta \int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \frac{1}{2\theta}\int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^\gamma}\hat{v}|u|^2 + \epsilon \int_{\mathbb{R}^-} \hat{v}|u'|^2 + \frac{1}{4\epsilon}\int_{\mathbb{R}^-} \hat{v}|u|^2 + I_b \\
&\quad < \theta \int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \epsilon \int_{\mathbb{R}^-} |u'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2 + I_b.
\end{aligned}$$

Here we treat the boundary term by

$$I_b \le \sqrt{2}|\alpha||v_0'||u_0| \le \frac{\theta}{2}\frac{|v_0'|^2}{\hat{v}_0} + \frac{1}{\theta}|\alpha|^2\hat{v}_0|u_0|^2.$$

Therefore using (C.11), we simply obtain from the above estimates

$$\begin{aligned}
&(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^-} \hat{v}|u|^2 + (1-\epsilon)\int_{\mathbb{R}^-} |u'|^2 + \frac{1}{2}\hat{v}_0|u_0|^2 \\
&\qquad < \theta \int_{\mathbb{R}^-} \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}|v'|^2 + \frac{\theta}{2}\frac{|v_0'|^2}{\hat{v}_0} + \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2 + \frac{1}{\theta}|\alpha|^2\hat{v}_0|u_0|^2 \\
&\qquad < 2\theta \int_{\mathbb{R}^-} |u'|^2 + \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2 + J_b
\end{aligned}$$

where $J_b := (\frac{1}{\theta}|\alpha|^2 + 2\theta)\hat{v}_0|u_0|^2$. Assuming that $\epsilon + 2\theta \le 1$, this simplifies to

$$(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^-} \hat{v}|u|^2 + \frac{1}{2}\hat{v}_0|u_0|^2 < \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2 + J_b.$$

Note that $|\alpha| \le \frac{-\hat{v}_0'}{|\lambda|} \le \frac{1}{4|\lambda|}$. Therefore for $|\lambda| \ge \frac{1}{4\theta}$, we get $|\alpha| \le \theta$ and $J_b \le 3\theta\hat{v}_0|u_0|^2$. For sake of simplicity, choose $\theta = 1/6$ and $\epsilon = 2/3$. This shows that J_b can be absorbed into the left by the term $\frac{1}{2}\hat{v}_0|u_0|^2$ and thus we get

$$(\Re e(\lambda) + |\Im m(\lambda)|) \int_{\mathbb{R}^-} \hat{v}|u|^2 < \left[\frac{\gamma}{2\theta} + \frac{1}{4\epsilon}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2 = \left[3\gamma + \frac{3}{8}\right]\int_{\mathbb{R}^-} \hat{v}|u|^2,$$

provided that $|\lambda| \ge 1/(4\theta) = 3/2$.

This shows

$$(\Re e(\lambda) + |\Im m(\lambda)|) < \max\{\frac{3\sqrt{2}}{2}, 3\gamma + \frac{3}{8}\}.$$

□

C.3 Nonvanishing of D^0_{in}

Working in $(\tilde{v}, \tilde{u})$ variables as in (C.1), the limiting eigenvalue system and boundary conditions take the form

$$\lambda\tilde{v} + \tilde{v}' - \tilde{u}' = 0, \tag{C.13a}$$

$$\lambda\tilde{u} + \tilde{u}' - \frac{1-\hat{v}}{\hat{v}}\tilde{v}' = \frac{\tilde{u}''}{\hat{v}} \tag{C.13b}$$

corresponding to a pressureless gas, $\gamma = 0$, with

$$(\tilde{u}, \tilde{u}', \tilde{v}, \tilde{v}')(0) = (d, 0, 0, 0), \ \ (\tilde{u}, \tilde{u}', \tilde{v}, \tilde{v}')(+\infty) = (c, 0, 0, 0). \tag{C.14}$$

Hereafter, we drop the tildes.

Proof of Proposition 5.3.6. Multiplying (C.13b) by $\hat{v}\bar{u}/(1-\hat{v})$ and integrating on $[0, b] \subset \mathbb{R}^+$, we obtain

$$\lambda \int_0^b \frac{\hat{v}}{1-\hat{v}}|u|^2 dx + \int_0^b \frac{\hat{v}}{1-\hat{v}} u'\bar{u} dx - \int_0^b v'\bar{u} dx = \int_0^b \frac{u''\bar{u}}{1-\hat{v}} dx.$$

Integrating the third and fourth terms by parts yields

$$\begin{aligned}\lambda \int_0^b \frac{\hat{v}}{1-\hat{v}}|u|^2 dx &+ \int_0^b \left[\frac{\hat{v}}{1-\hat{v}} + \left(\frac{1}{1-\hat{v}}\right)'\right] u'\bar{u} dx \\ &+ \int_0^b \frac{|u'|^2}{1-\hat{v}} dx + \int_0^b v(\overline{\lambda v + v'}) dx \\ &= \left[v\bar{u} + \frac{u'\bar{u}}{1-\hat{v}}\right]\Big|_0^b.\end{aligned}$$

Integrating the second term by parts and taking the real part, we have

$$\Re e(\lambda)\int_0^b \left(\frac{\hat{v}}{1-\hat{v}}|u|^2+|v|^2\right)dx+\int_0^b g(\hat{v})|u|^2dx+\int_0^b \frac{|u'|^2}{1-\hat{v}}dx$$
$$=\Re e\left[v\bar{u}+\frac{u'\bar{u}}{1-\hat{v}}-\frac{1}{2}\left[\frac{\hat{v}}{1-\hat{v}}+\left(\frac{1}{1-\hat{v}}\right)'\right]|u|^2-\frac{|v|^2}{2}\right]\Big|_0^b, \tag{C.15}$$

where

$$g(\hat{v})=-\frac{1}{2}\left[\left(\frac{\hat{v}}{1-\hat{v}}\right)'+\left(\frac{1}{1-\hat{v}}\right)''\right].$$

Note that

$$\frac{d}{dx}\left(\frac{1}{1-\hat{v}}\right)=-\frac{(1-\hat{v})'}{(1-\hat{v})^2}=\frac{\hat{v}_x}{(1-\hat{v})^2}=\frac{\hat{v}(\hat{v}-1)}{(1-\hat{v})^2}=-\frac{\hat{v}}{1-\hat{v}}.$$

Thus, $g(\hat{v})\equiv 0$ and the third term on the right-hand side vanishes, leaving

$$\Re e(\lambda)\int_0^b \left(\frac{\hat{v}}{1-\hat{v}}|u|^2+|v|^2\right)dx+\int_0^b \frac{|u'|^2}{1-\hat{v}}dx$$
$$=\left[\Re e(v\bar{u})+\frac{\Re e(u'\bar{u})}{1-\hat{v}}-\frac{|v|^2}{2}\right]\Big|_0^b$$
$$=\left[\Re e(v\bar{u})+\frac{\Re e(u'\bar{u})}{1-\hat{v}}-\frac{|v|^2}{2}\right](b).$$

We show finally that the right-hand side goes to zero in the limit as $b\to\infty$. By Proposition 5.4.3, the behavior of u, v near $\pm\infty$ is governed by the limiting constant–coefficient systems $W'=A^0_{\pm}(\lambda)W$, where $W=(u,v,v')^T$ and $A^0_{\pm}=A^0(\pm\infty,\lambda)$. In particular, solutions W asymptotic to $(1,0,0)$ at $x=+\infty$ decay exponentially in (u',v,v') and are bounded in coordinate u as $x\to+\infty$. Observing that $1-\hat{v}\to 1$ as $x\to+\infty$, we thus see immediately that the boundary contribution at b vanishes as $b\to+\infty$.

Thus, in the limit as $b\to+\infty$,

$$\Re e(\lambda)\int_0^{+\infty} \left(\frac{\hat{v}}{1-\hat{v}}|u|^2+|v|^2\right)dx+\int_0^{+\infty} \frac{|u'|^2}{1-\hat{v}}dx=0. \tag{C.16}$$

But, for $\Re e\lambda\geq 0$, this implies $u'\equiv 0$, or $u\equiv$ constant, which, by $u(0)=1$, implies

$u \equiv 1$. This reduces (C.13a) to $v' = \lambda v$, yielding the explicit solution $v = Ce^{\lambda x}$. By $v(0) = 0$, therefore, $v \equiv 0$ for $\Re e\lambda \geq 0$. Substituting into (C.13b), we obtain $\lambda = 0$. It follows that there are no nontrivial solutions of (C.13), (C.14) for $\Re e\lambda \geq 0$ except at $\lambda = 0$. □

Remark C.3.1. *The above energy estimate is essentially identical to that used in [24] to treat the limiting shock case.*

C.4 Nonvanishing of D^0_{out}

Working in $(\tilde{v}, \tilde{u})$ variables as in (C.1), the limiting eigenvalue system and boundary conditions take the form

$$\lambda\tilde{v} + \tilde{v}' - \tilde{u}' = 0, \tag{C.17a}$$

$$\lambda\tilde{u} + \tilde{u}' - \frac{1-\hat{v}}{\hat{v}}\tilde{v}' = \frac{\tilde{u}''}{\hat{v}} \tag{C.17b}$$

corresponding to a pressureless gas, $\gamma = 0$, with

$$(\tilde{u}, \tilde{u}', \tilde{v}, \tilde{v}')(-\infty) = (0, 0, 0, 0), \tag{C.18}$$

$$\tilde{v}'(0) = \frac{\lambda}{\alpha - 1}\tilde{v}(0), \quad \tilde{u}'(0) = \alpha\tilde{v}'(0). \tag{C.19}$$

In particular,

$$\tilde{u}'(0) = \frac{\lambda\alpha}{\alpha - 1}\tilde{v}(0) = \hat{v}'(0)\tilde{v}(0) = (v_0 - 1)\hat{v}_0\tilde{v}(0). \tag{C.20}$$

Hereafter, we drop the tildes.

Proof of Proposition 5.3.6. Multiplying (C.17b) by $\hat{v}\bar{u}/(1-\hat{v})$ and integrating on $[a, 0] \subset \mathbb{R}^-$, we obtain

$$\lambda \int_a^0 \frac{\hat{v}}{1-\hat{v}}|u|^2 dx + \int_a^0 \frac{\hat{v}}{1-\hat{v}} u'\bar{u} dx - \int_a^0 v'\bar{u} dx = \int_a^0 \frac{u''\bar{u}}{1-\hat{v}} dx.$$

Integrating the third and fourth terms by parts yields

$$
\begin{aligned}
\lambda \int_a^0 \frac{\hat{v}}{1-\hat{v}} |u|^2 dx + \int_a^0 \left[\frac{\hat{v}}{1-\hat{v}} + \left(\frac{1}{1-\hat{v}} \right)' \right] u' \bar{u} dx \\
+ \int_a^0 \frac{|u'|^2}{1-\hat{v}} dx + \int_a^0 v(\overline{\lambda v + v'}) dx \\
= \left[v\bar{u} + \frac{u'\bar{u}}{1-\hat{v}} \right] \Big|_a^0 .
\end{aligned}
$$

Taking the real part, we have

$$
\begin{aligned}
\Re e(\lambda) \int_a^0 \left(\frac{\hat{v}}{1-\hat{v}} |u|^2 + |v|^2 \right) dx + \int_a^0 g(\hat{v}) |u|^2 dx + \int_a^0 \frac{|u'|^2}{1-\hat{v}} dx \\
= \Re e \left[v\bar{u} + \frac{u'\bar{u}}{1-\hat{v}} - \frac{1}{2} \left[\frac{\hat{v}}{1-\hat{v}} + \left(\frac{1}{1-\hat{v}} \right)' \right] |u|^2 - \frac{|v|^2}{2} \right] \Big|_a^0 ,
\end{aligned}
\tag{C.21}
$$

where

$$
g(\hat{v}) = -\frac{1}{2} \left[\left(\frac{\hat{v}}{1-\hat{v}} \right)' + \left(\frac{1}{1-\hat{v}} \right)'' \right] \equiv 0
$$

and the third term on the right-hand side vanishes, as shown in Section C.3, leaving

$$
\begin{aligned}
\Re e(\lambda) \int_a^0 \left(\frac{\hat{v}}{1-\hat{v}} |u|^2 + |v|^2 \right) dx + \int_a^0 \frac{|u'|^2}{1-\hat{v}} dx \\
= \left[\Re e(v\bar{u}) + \frac{\Re e(u'\bar{u})}{1-\hat{v}} - \frac{|v|^2}{2} \right] \Big|_a^0 .
\end{aligned}
$$

A boundary analysis similar to that of Section C.3 shows that the contribution at a on the righthand side vanishes as $a \to -\infty$; see [24] for details. Thus, in the limit as $a \to -\infty$ we obtain

$$\begin{aligned}
\Re e(\lambda) \int_{-\infty}^{0} \left(\frac{\hat{v}}{1-\hat{v}} |u|^2 + |v|^2 \right) dx + \int_{-\infty}^{0} \frac{|u'|^2}{1-\hat{v}} dx \\
&= \left[\Re e(v\bar{u}) + \frac{\Re e(u'\bar{u})}{1-\hat{v}} - \frac{|v|^2}{2} \right](0) \\
&= \left[(1-v_0)\Re e(v\bar{u}) - \frac{|v|^2}{2} \right](0), \\
&\leq \left[(1-v_0)|v||u| - \frac{|v|^2}{2} \right](0) \\
&\leq (1-v_0)^2 \frac{|u(0)|^2}{2},
\end{aligned}$$

where the second equality follows by (C.20) and the final line by Young's inequality.

Next, observe the Sobolev-type bound

$$|u(0)|^2 \leq \Big(\int_{-\infty}^{0} |u'(x)| dx \Big)^2 \leq \int_{-\infty}^{0} \frac{|u'|^2}{1-\hat{v}}(x) dx \int_{-\infty}^{0} (1-\hat{v})(x) dx,$$

together with

$$\int_{-\infty}^{0} (1-\hat{v})(x) dx = \int_{-\infty}^{0} -\frac{\hat{v}'}{\hat{v}}(x) dx = \int_{-\infty}^{0} (\log \hat{v}^{-1})'(x) dx = \log v_0^{-1},$$

hence $\int_{-\infty}^{0} (1-\hat{v})(x)dx < \frac{2}{(1-v_0)^2}$ for $v_0 > v_*$, where $v_* < e^{-2}$ is the unique solution of

$$v_* = e^{-2/(1-v_*)^2}. \tag{C.22}$$

Thus, for $v_0 > v_*$,

$$\Re e(\lambda) \int_{-\infty}^{0} \left(\frac{\hat{v}}{1-\hat{v}} |u|^2 + |v|^2 \right) dx + \epsilon \int_{-\infty}^{0} \frac{|u'|^2}{1-\hat{v}} dx \leq 0, \tag{C.23}$$

for $\epsilon := 1 - \frac{(1-v_0)^2}{2} \int_{-\infty}^{0} (1-\hat{v})(x) dx > 0$. For $\Re e\lambda \geq 0$, this implies $u' \equiv 0$, or $u \equiv$ constant, which, by $u(-\infty) = 0$, implies $u \equiv 0$. This reduces (C.17a) to $v' = \lambda v$, yielding the explicit solution $v = Ce^{\lambda x}$. By $v(0) = 0$, therefore, $v \equiv 0$ for $\Re e\lambda \geq 0$. It follows that there are no nontrivial solutions of (C.17), (C.18) for $\Re e\lambda \geq 0$ except at $\lambda = 0$.

By iteration, starting with $v_* \approx 0$, we obtain first $v_* < e^{-2} \approx 0.14$ then $v_* >$

$e^{2/(1-.14)^2} \approx .067$, then $v_* < e^{2/(1-.067)^2} \approx .10$, then $v_* > e^{2/(1-.10)^2} \approx .085$, then $v_* < e^{2/(1-.085)} \approx .091$ and $v_* > e^{2/(1-.091)} \approx .0889$, terminating with $v_* \approx .0899$. $\square$

Remark C.4.1. *Our Evans function results show that the case v_0 small not treated corresponds to the shock limit for which stability is already known by [24]. This suggests that a more sophisticated energy estimate combining the above with a boundary-layer analysis from $x = 0$ back to $x = L + \delta$ might yield nonvanishing for all $1 > v_0 > 0$.*

C.5 The characteristic limit: outflow case

We now show stability of compressive outflow boundary layers in the characteristic limit $v_+ \to 1$, by essentially the same energy estimate used in [38] to show stability of small-amplitude shock waves.

As in the above section on the outflow case, we obtain a system

$$\lambda \tilde{v} + \tilde{v}' - \tilde{u}' = 0, \tag{C.24a}$$

$$\lambda \tilde{u} + \tilde{u}' - \frac{h(\hat{v})}{\hat{v}^{\gamma+1}} \tilde{v}' = \frac{\tilde{u}''}{\hat{v}} \tag{C.24b}$$

identical to that in the integrated shock case [3], but with boundary conditions

$$\tilde{v}'(0) = \frac{\lambda}{\alpha - 1} \tilde{v}(0), \quad \tilde{u}'(0) = \alpha \tilde{v}'(0). \tag{C.25}$$

In particular,

$$\tilde{u}'(0) = \frac{\lambda \alpha}{\alpha - 1} \tilde{v}(0) = \hat{v}'(0) \tilde{v}(0). \tag{C.26}$$

This new eigenvalue problem differs spectrally from (5.22) only at $\lambda = 0$, hence spectral stability of (5.22) is implied by spectral stability of (C.24). Hereafter, we drop the tildes, and refer simply to u, v.

Proof of Proposition 5.3.7. We note that $h(\hat{v}) > 0$. By multiplying (C.24b) by both the conjugate $\bar{u}$ and $\hat{v}^{\gamma+1}/h(\hat{v})$ and integrating along x from $-\infty$ to 0, we have

$$\int_{-\infty}^{0} \frac{\lambda u \bar{u} \hat{v}^{\gamma+1}}{h(\hat{v})} dx + \int_{-\infty}^{0} \frac{u' \bar{u} \hat{v}^{\gamma+1}}{h(\hat{v})} dx - \int_{-\infty}^{0} v' \bar{u} dx = \int_{-\infty}^{0} \frac{u'' \bar{u} \hat{v}^{\gamma}}{h(\hat{v})} dx.$$

Integrating the last two terms by parts and appropriately using (C.24a) to substitute for u' in the third term gives us

$$\int_{-\infty}^{0} \frac{\lambda|u|^2\hat{v}^{\gamma+1}}{h(\hat{v})}dx + \int_{-\infty}^{0} \frac{u'\bar{u}\hat{v}^{\gamma+1}}{h(\hat{v})}dx + \int_{-\infty}^{0} v(\overline{\lambda v + v'})dx + \int_{-\infty}^{0} \frac{\hat{v}^{\gamma}|u'|^2}{h(\hat{v})}dx$$
$$= -\int_{-\infty}^{0} \left(\frac{\hat{v}^{\gamma}}{h(\hat{v})}\right)' u'\bar{u}dx + \left[v\bar{u} + \frac{v^{\gamma}u'\bar{u}}{h(\hat{v})}\right]\Big|_{x=0}.$$

We take the real part and appropriately integrate by parts to get

$$\Re e(\lambda)\int_{-\infty}^{0} \left[\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}|u|^2 + |v|^2\right]dx + \int_{-\infty}^{0} g(\hat{v})|u|^2dx + \int_{-\infty}^{0} \frac{\hat{v}^{\gamma}}{h(\hat{v})}|u'|^2dx = G(0), \quad \text{(C.27)}$$

where

$$g(\hat{v}) = -\frac{1}{2}\left[\left(\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}\right)' + \left(\frac{\hat{v}^{\gamma}}{h(\hat{v})}\right)''\right]$$

and

$$G(0) = -\frac{1}{2}\left[\frac{\hat{v}^{\gamma+1}}{h(\hat{v})} + \left(\frac{\hat{v}^{\gamma}}{h(\hat{v})}\right)'\right]|u|^2 + \Re e\left[v\bar{u} + \frac{v^{\gamma}u'\bar{u}}{h(\hat{v})}\right] - \frac{|v|^2}{2}$$

evaluated at $x = 0$. Here, the boundary term appearing on the righthand side is the only difference from the corresponding estimate appearing in the treatment of the shock case in [38, 3]. We shall show that as $v_+ \to 1$, the boundary term $G(0)$ is nonpositive. Observe that boundary conditions yield

$$\left[v\bar{u} + \frac{v^{\gamma}u'\bar{u}}{h(\hat{v})}\right]\Big|_{x=0} = \Re e(v(0)\bar{u}(0))\left[1 + \frac{\hat{v}^{\gamma}\hat{v}'}{h(\hat{v})}\right]\Big|_{x=0}.$$

We first note, as established in [38, 3], that $g(\hat{v}) \geq 0$ on $[v_+, 1]$, under certain conditions including the case $v_+ \to 1$. Straightforward computation gives identities:

$$\gamma h(\hat{v}) - \hat{v}h'(\hat{v}) = a\gamma(\gamma-1) + \hat{v}^{\gamma+1} \quad \text{and} \qquad \text{(C.28)}$$
$$\hat{v}^{\gamma-1}\hat{v}_x = a\gamma - h(\hat{v}). \qquad \text{(C.29)}$$

Using (C.28) and (C.29), we abbreviate a few intermediate steps below:

$$
\begin{aligned}
g(\hat{v}) &= -\frac{\hat{v}_x}{2}\left[\frac{(\gamma+1)\hat{v}^{\gamma}h(\hat{v})-\hat{v}^{\gamma+1}h'(\hat{v})}{h(\hat{v})^2}+\frac{d}{d\hat{v}}\left[\frac{\gamma\hat{v}^{\gamma-1}h(\hat{v})-\hat{v}^{\gamma}h'(\hat{v})}{h(\hat{v})^2}\hat{v}_x\right]\right] \\
&= -\frac{\hat{v}_x}{2}\left[\frac{\hat{v}^{\gamma}\left((\gamma+1)h(\hat{v})-\hat{v}h'(\hat{v})\right)}{h(\hat{v})^2}+\frac{d}{d\hat{v}}\left[\frac{\gamma h(\hat{v})-\hat{v}h'(\hat{v})}{h(\hat{v})^2}(a\gamma-h(\hat{v}))\right]\right] \\
&= -\frac{a\hat{v}_x\hat{v}^{\gamma-1}}{2h(\hat{v})^3}\times \\
&\quad \left[\gamma^2(\gamma+1)\hat{v}^{\gamma+2}-2(a+1)\gamma(\gamma^2-1)\hat{v}^{\gamma+1}+(a+1)^2\gamma^2(\gamma-1)\hat{v}^{\gamma}\right. \\
&\qquad \left.+\ a\gamma(\gamma+2)(\gamma^2-1)\hat{v}-a(a+1)\gamma^2(\gamma^2-1)\right] \\
&= -\frac{a\hat{v}_x\hat{v}^{\gamma-1}}{2h(\hat{v})^3}[(\gamma+1)\hat{v}^{\gamma+2}+\hat{v}^{\gamma}(\gamma-1)\left((\gamma+1)\hat{v}-(a+1)\gamma\right)^2 \qquad (C.30) \\
&\quad +a\gamma(\gamma^2-1)(\gamma+2)\hat{v}-a(a+1)\gamma^2(\gamma^2-1)] \\
&\geq -\frac{a\hat{v}_x\hat{v}^{\gamma-1}}{2h(\hat{v})^3}[(\gamma+1)\hat{v}^{\gamma+2}+a\gamma(\gamma^2-1)(\gamma+2)\hat{v}-a(a+1)\gamma^2(\gamma^2-1)] \\
&\geq -\frac{\gamma^2a^3\hat{v}_x(\gamma+1)}{2h(\hat{v})^3v_+}\left[\left(\frac{v_+^{\gamma+1}}{a\gamma}\right)^2+2(\gamma-1)\left(\frac{v_+^{\gamma+1}}{a\gamma}\right)-(\gamma-1)\right]. \qquad (C.31)
\end{aligned}
$$

This verifies $g(\hat{v}) \geq 0$ as $v_+ \to 1$.

Second, examine

$$
G(0) = -\frac{1}{2}\left[\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}+\left(\frac{\hat{v}^{\gamma}}{h(\hat{v})}\right)'\right]|u(0)|^2+\left[1+\frac{\hat{v}^{\gamma}\hat{v}'}{h(\hat{v})}\right]\Re e(v(0)\bar{u}(0))-\frac{|v(0)|^2}{2}.
$$

Applying Young's inequality to the middle term, we easily get

$$
G(0) \leq -\frac{1}{2}\left[\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}+\left(\frac{\hat{v}^{\gamma}}{h(\hat{v})}\right)'-\left(1+\frac{\hat{v}^{\gamma}\hat{v}'}{h(\hat{v})}\right)^2\right]|u(0)|^2 =: -\frac{1}{2}I|u(0)|^2.
$$

Now observe that I can be written as

$$
I = \frac{\hat{v}^{\gamma+1}}{h(\hat{v})}-1+\left[\frac{\gamma\hat{v}^{\gamma-1}}{h(\hat{v})}-\frac{2\hat{v}^{\gamma}}{h(\hat{v})}-\frac{\hat{v}^{2\gamma}\hat{v}'}{h^2(\hat{v})}\right]\hat{v}'-\frac{\hat{v}^{\gamma}h'(\hat{v})}{h^2(\hat{v})}.
$$

Using (C.28) and (C.29), we get

$$
\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}-1 = -\frac{(\gamma-1)\hat{v}^{\gamma-1}\hat{v}'+\hat{v}h'(\hat{v})}{h(\hat{v})}
$$

and thus

$$I = -\frac{(\gamma-1)\hat{v}^{\gamma-1}\hat{v}' + \hat{v}h'(\hat{v})}{h(\hat{v})} + \left[\frac{\gamma\hat{v}^{\gamma-1}}{h(\hat{v})} - 2\frac{\hat{v}^{\gamma}}{h(\hat{v})} - \frac{\hat{v}^{2\gamma}\hat{v}'}{h^2(\hat{v})}\right]\hat{v}' - \frac{\hat{v}^{\gamma}h'(\hat{v})}{h^2(\hat{v})}.$$

Now since $h'(\hat{v}) = -(\gamma+1)\hat{v}^{\gamma}\hat{v}' + (a+1)\gamma\hat{v}^{\gamma-1}\hat{v}'$, as $v_+ \to 1$, $I \sim -\hat{v}' \geq 0$. Therefore, as v_+ is close to 1, $G(0) \leq \frac{1}{4}\hat{v}'(0)|u(0)|^2 \leq 0$. This, $g(\hat{v}) \geq 0$, and (C.27) give, as v_+ is close enough to 1,

$$\Re e(\lambda)\int_{-\infty}^{0}\left[\frac{\hat{v}^{\gamma+1}}{h(\hat{v})}|u|^2 + |v|^2\right]dx + \int_{-\infty}^{0}\frac{\hat{v}^{\gamma}}{h(\hat{v})}|u'|^2 dx \leq 0, \tag{C.32}$$

which evidently gives stability as claimed. □

C.6 Nonvanishing of D_{in}: expansive inflow case

For completeness, we recall the argument of [39] in the expansive inflow case.

Profile equation. Note that, in the expansive inflow case, we assume $v_0 < v_+$. Therefore we can still follow the scaling (5.12) to get

$$0 < v_0 < v_+ = 1.$$

Then the stationary boundary layer $(\hat{v}, \hat{u})$ satisfies (5.15) with $v_0 < v_+ = 1$. Now by integrating (5.16) from x to $+\infty$ with noting that $\hat{v}(+\infty) = 1$ and $\hat{v}'(+\infty) = 0$, we get the profile equation

$$\hat{v}' = \hat{v}(\hat{v} - 1 + a(\hat{v}^{-\gamma} - 1)).$$

Note that $\hat{v}' > 0$. We now follow the same method for compressive inflow case to get the following eigenvalue system

$$\lambda v + v' - u' = 0, \tag{C.33a}$$

$$\lambda u + u' - (fv)' = \left(\frac{u'}{\hat{v}}\right)' \tag{C.33b}$$

with boundary conditions

$$u(0) = v(0) = 0, \tag{C.34}$$

where $f(\hat{v}) = \frac{h(\hat{v})}{\hat{v}^{\gamma+1}}$.

Proof of Proposition 5.3.8. Multiply the equation (C.33b) by $\bar{u}$ and integrate along x. By integration by parts, we get

$$\lambda \int_0^\infty |u|^2 dx + \int_0^\infty u'\bar{u} + fv\bar{u}' + \frac{|u'|^2}{\hat{v}} dx = 0.$$

Using (C.33a) and taking the real part of the above yield

$$\Re e\lambda \int_0^\infty |u|^2 + f|v|^2 dx - \frac{1}{2}\int_0^\infty f'|v|^2 dx + \int_0^\infty \frac{|u'|^2}{\hat{v}} dx = 0. \tag{C.35}$$

Note that

$$f' = \left(1 + a + \frac{a(\gamma^2 - 1)}{\hat{v}^\gamma}\right) \frac{-\hat{v}'}{\hat{v}^2} \leq 0$$

which together with (C.35) gives $\Re e\lambda < 0$, the proposition is proved. $\square$

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