Toward nonlinear stability of sources via a modified Burgers equation

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Abstract

Coherent structures are solutions to reaction-diffusion systems that are time-periodic in an appropriate moving frame and spatially asymptotic at $x = \pm\infty$ to spatially periodic travelling waves. This paper is concerned with sources which are coherent structures for which the group velocities in the far field point away from the core. Sources actively select wave numbers and therefore often organize the overall dynamics in a spatially extended system. Determining their nonlinear stability properties is challenging as localized perturbations may lead to a non-localized response even on the linear level due to the outward transport. Using a Burgers-type equation as a model problem that captures some of the essential features of sources, we show how this phenomenon can be analysed and asymptotic nonlinear stability be established in this simpler context.

1 Introduction

In this paper, we analyse the long-time dynamics of solutions to the Burgers-type equation

$$\phi_t + c \tanh\left(\frac{cx}{2}\right)\phi_x = \phi_{xx} + \phi_x^2, \qquad c > 0 \tag{1.1}$$

with small localized initial data, where $x \in \mathbb{R}$, $t > 0$, and $\phi(x,t)$ is a scalar function. The key feature of this equation as opposed to the usual Burgers equation is that the characteristic speeds are $c > 0$ at spatial infinity and $-c < 0$ at spatial minus infinity: hence, transport is always directed away from the shock interface at $x = 0$ and not towards $x = 0$ as would be the case for the Lax shocks of the standard Burgers equation.

We are interested in (1.1) due to its close connection with the dynamics of coherent structures that arise in reaction-diffusion systems

$$u_t = Du_{xx} + f(u), \qquad x \in \mathbb{R}, \qquad u \in \mathbb{R}^n. \tag{1.2}$$

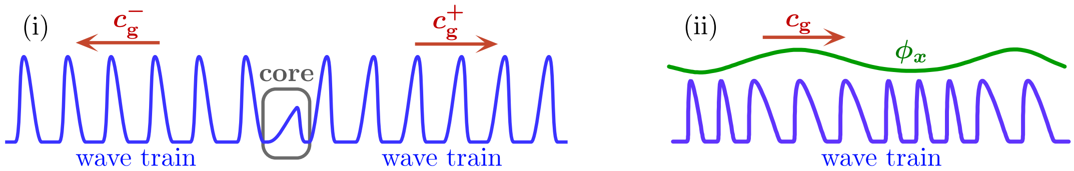

Figure 1: *Panel (i) shows the graph of a source $u^*(x,t)$ as a function of x for fixed time t: the group velocities of the asymptotic wave trains point away from the core of the coherent structure. Panel (ii) shows the behaviour of small phase ϕ or wave number ϕ_x perturbations of a wave train: to leading order, they are transported with speed given by the group velocity c_g without changing their shape [2].*

A coherent structure (or defect) is a solution $u^*(x,t)$ of (1.2) that is time-periodic in an appropriate moving frame $y = x - c^*t$ and spatially asymptotic to wave-train solutions, which are spatially periodic travelling waves of (1.2). Such structures have been observed in many experiments and in various reaction-diffusion models, and we refer to [9] for references and to Figure 1 for an illustration of typical defect profiles. For the sake of simplicity, we shall assume from now on that the speed c^* of the defect we are interested in vanishes, so that the coherent structure is time-periodic. Coherent structures can be classified into several distinct types [3, 8, 9] that have different stability and multiplicity properties. This classification involves the group velocities of the asymptotic wave trains, and we therefore briefly review their definition and features. Wave trains of (1.2) are solutions of the form $u(x,t) = u_\mathrm{wt}(kx - \omega t; k)$, where the profile $u_\mathrm{wt}(y;k)$ is 2π-periodic in the y-variable. Thus, k and ω represent the spatial wave number and the temporal frequency, respectively, of the wave train. Wave trains typically exist as one-parameter families, where the frequency $\omega = \omega_\mathrm{nl}(k)$ is a function, the so-called nonlinear dispersion relation, of the wave number k, which varies in an open interval. The group velocity c_g of the wave train with wave number k is defined as

$$c_\mathrm{g} := \frac{\mathrm{d}\omega_\mathrm{nl}}{\mathrm{d}k}(k).$$

The group velocity is important as it is the speed with which small localized perturbations of a wave train propagate as functions of time t, and we refer to Figure 1(ii) for an illustration and to [2] for a rigorous justification of this statement. The classification of coherent structures mentioned above is based on the group velocities $c_\mathrm{g}^\pm$ of the asymptotic wave trains at $x = \pm\infty$. We are interested in sources for which $c_\mathrm{g}^- < 0 < c_\mathrm{g}^+$ as illustrated in Figure 1(i) so that perturbations are transported away from the defect core towards infinity. Sources are important as they actively select wave numbers in oscillatory media; examples of sources are the Nozaki–Bekki holes of the complex Ginzburg–Landau equation. We note that the profile $u^*(x,t)$ of a source converges exponentially in x towards the asymptotic wave train profiles $u_\mathrm{wt}(kx - \omega t)$, uniformly in t; see [9, Corollary 5.2].

From now on, we focus on a given source and discuss its stability properties with respect to the reaction-diffusion system (1.2). Spectral stability of a source can be investigated through the Floquet spectrum of the period map of the linearization of (1.2) about the time-periodic source. Spectral stability of sources was investigated in [9], and we now summarize their findings. The Floquet spectrum of a spectrally stable source will look as indicated in Figure 2(i). A source $u^*(x,t)$ has two eigenvalues at the origin with eigenfunctions $u_x^*(x,t)$ and $u_t^*(x,t)$; the associated adjoint eigenfunctions are necessarily exponentially localized, so that the source has a well defined spatial position and temporal phase. There will also be two curves of essential spectrum that touch the origin and correspond to phase and wave number modulations

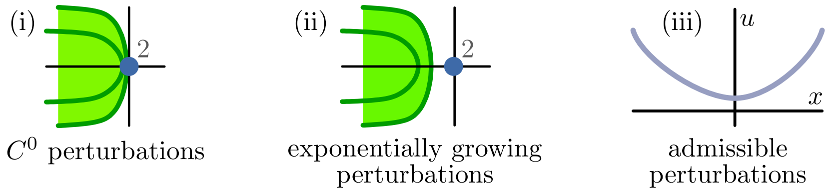

Figure 2: *Panel (i) illustrates the Floquet spectrum of a spectrally stable source posed on C^0: the two eigenvalues at the origin correspond to translations in space and time. If exponentially growing perturbations such as those in panel (iii) are allowed, then the essential spectrum moves into the open left half-plane, while the spatial and temporal translation eigenvalues stay at the origin as indicated in panel (ii).*

of the two asymptotic wave trains. It turns out that the two eigenvalues at the origin cannot be removed by posing the linearized problem in exponentially weighted function spaces; the essential spectrum, on the other hand, can be moved to the left by allowing functions to grow exponentially at infinity as indicated in Figure 2.

The nonlinear stability of spectrally stable sources has not yet been established, and we now outline why this is a challenging problem. From a purely technical viewpoint, an obvious difficulty is related to the fact that there is no spectral gap between the essential spectrum and the imaginary axis. As discussed above, such a gap can be created by posing the linear problem on function spaces that contain exponentially growing functions, but the nonlinear terms will then not even be continuous. To see that these are not just technical obstacles, it is illuminating to discuss the anticipated dynamics near a source from an intuitive perspective. If a source is subjected to a localized perturbation, then one anticipated effect is that the defect core adjusts its position and its temporal phase in response. From its new position, the defect will continue to emit wave trains with the same selected wave number but there will now be a phase difference between the asymptotic wave trains at infinity and those newly emitted near the core. In other words, we expect to see two phase fronts that travel in opposite directions away from the core as illustrated in Figure 3. The resulting phase dynamics can be captured by writing the perturbed solution $u(x,t)$ as

$$u(x,t) = u^*(x, t + \phi(x,t)) + w(x,t), \tag{1.3}$$

where we expect that the perturbation $w(x,t)$ of the defect profile decays in time, while the phase $\phi(x,t)$ resembles an expanding plateau as indicated in Figure 3 whose height depends on the initial perturbation through the spatio-temporal displacement of the defect core.

The preceding heuristic arguments suggest that the overall response of a source to an initial perturbation is organized by the defect core: the spatio-temporal displacement of the core causes a specific phase shift in the emitted wave trains that then spreads into the far field, where its dynamics is governed by the phase dynamics of the asymptotic wave trains. In particular, the height of the anticipated phase plateau shown in Figures 3 and 4 is therefore determined by the dynamics near the core.

As a first step towards a general nonlinear stability result for sources in reaction-diffusion systems, our goal is to identify a simpler model problem that captures the essential features and some of the key difficulties that we outlined above. To simplify the problem, we focus exclusively on the dynamics of the phase function $\phi(x,t)$ that we introduced in (1.3). If $\phi(x,t)$ varies slowly in space and time, then we expect formally that it will satisfy a partial differential equation. For phase perturbations of wave trains, it was indeed established formally in [4] and proved rigorously in [2] that the phase $\phi(x,t)$ satisfies an integrated

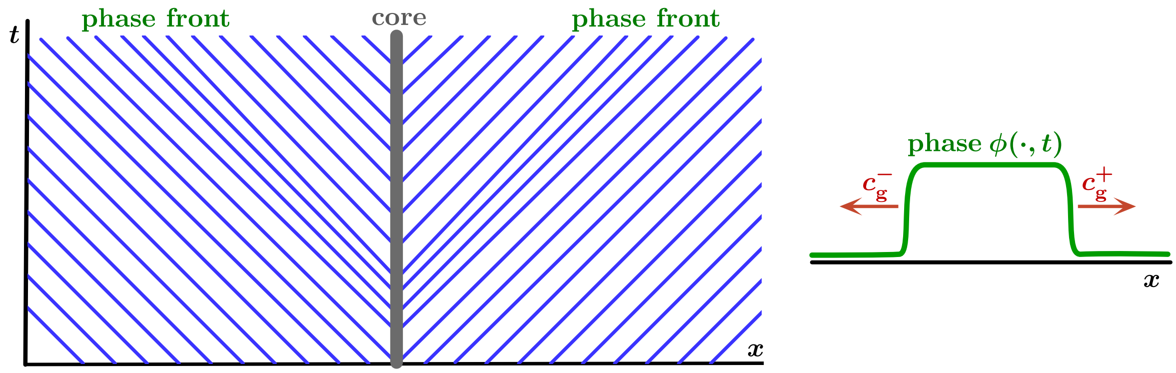

Figure 3: *The left panel contains a sketch of the space-time diagram of a perturbed source. The defect core will adjust in response to an imposed perturbation, and the emitted wave trains, whose maxima are indicated by the lines that emerge from the defect core, will therefore exhibit phase fronts that travel with the group velocities of the asymptotic wave trains away from the core towards* $\pm\infty$*. The right panel illustrates the profile of the anticipated phase function* $\phi(x,t)$ *defined in (1.3).*

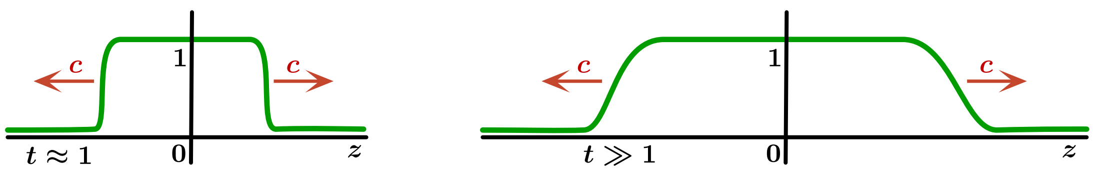

Figure 4: *Shown are the graphs of the function* $\mathrm{errfn}((-z+ct)/\sqrt{4t}) - \mathrm{errfn}((-z-ct)/\sqrt{4t})$ *for smaller and larger values of* t*, which resemble plateaus of height approximately equal to one that spread outwards with speed* $\pm c$*, while the associated interfaces widen like* $\sqrt{t}$*.*

viscous Burgers equation over long time intervals. The effect of the core of the source is that it adjusts the advection terms in the Burgers equations associated with the asymptotic wave trains to account for their outgoing group velocities. These considerations together with the property that the source converges exponentially to the asymptotic wave trains leads us to consider the Burgers-type equation

$$\phi_t + c\tanh\left(\frac{cx}{2}\right)\phi_x = \phi_{xx} + \phi_x^2, \qquad c > 0 \tag{1.4}$$

for the phase function $\phi(x,t)$ as the simplest possible model that incorporates both the dynamical effect of the core and the correct far-field dynamics. We remark that, while the inhomogeneous advection term can be thought of as fixing the position $x = 0$ of the core, equation (1.4) still has a family of constant solutions, which correspond to different temporal phases of the underlying hypothetical sources: in particular, the equation for the phase ϕ should depend only on ϕ_t and ϕ_x but not on ϕ itself. Thus, while (1.4) represents a simplification, we feel that gaining a detailed understanding of its long-time dynamics for small localized initial data will shed significant light on the expected dynamics of sources and on the techniques needed to analyse their stability. We emphasize that even the dynamics of wave trains of reaction-diffusion systems under non-localized phase perturbations was investigated only recently in [6, 7, 10]. We also remark that the maximum principle or the Cole–Hopf transformation can be used to show that solutions of (1.4) stay bounded; however, we are interested in identifying techniques that apply also to sources of reaction-diffusion systems and that give more detailed information about the spatio-temporal decay of perturbations. Finally, we note that, for nonlocalized solutions such as the phase fronts we are interested in, the nonlinear term ϕ_x^2 in (1.4) is not negligible compared to the linear diffusive term; see [2, §2] and references therein.

To obtain insight into (1.4), we linearize it about the stationary solution $\phi = 0$ to get

$$\tilde{\phi}_t = \tilde{\phi}_{xx} - c\tanh\left(\frac{cx}{2}\right)\tilde{\phi}_x. \tag{1.5}$$

The spectrum of the operator on the right-hand side of (1.5) is as shown in Figure 2(i) except that there is only one embedded eigenvalue at the origin: the associated eigenfunction is given by the constant function $\frac{c}{4}$, while the associated adjoint eigenfunction ψ is exponentially localized and given by

$$\psi(y) = \operatorname{sech}^2\left(\frac{cy}{2}\right).$$

The Green's function of (1.5) can be computed explicitly[1] and is given by

$$\begin{aligned} \mathcal{G}(x,y,t) \;&=\; \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{cy}} + \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{-cy}} \\ &\quad + \frac{c}{4}\left[\operatorname{errfn}\left(\frac{y-x+ct}{\sqrt{4t}}\right) - \operatorname{errfn}\left(\frac{y-x-ct}{\sqrt{4t}}\right)\right]\psi(y), \end{aligned} \tag{1.6}$$

where the error function is given by

$$\operatorname{errfn}(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{z}\mathrm{e}^{-s^2}\,\mathrm{d}s.$$

Note that the first two terms in the Green's function are Gaussians that move with speed $\pm c$ away from the core and decay like $1/\sqrt{t}$. The term comprised of the difference of the two error functions, on the other hand, produces a plateau of constant height $\frac{c}{4}$ that spreads outward as indicated in Figure 4; this term arises because of the zero eigenvalue of (1.5) and reflects therefore directly the source that underpins (1.4). It is interesting that the overall contribution of the embedded translation eigenvalue at the origin to the Green's function is not a term of the form $\frac{c}{4}\cdot\psi(y)$, that is, the eigenfunction times its adjoint, but instead a time-dependent phase plateau that captures how perturbations with support near the core spread into the far field. This is, in fact, typical of problems without spectral gaps; see, for instance, [1].

For sufficiently localized initial data $\tilde{\phi}_0(x)$, the solution $\tilde{\phi}(x,t)$ to the linear equation (1.5) is therefore given by

$$\tilde{\phi}(x,t) = \int_{\mathbb{R}}\mathcal{G}(x,y,t)\tilde{\phi}_0(y)\,\mathrm{d}y,$$

and we see that $\tilde{\phi}(x,t)$ converges pointwise to a constant: for each fixed x, we have

$$\tilde{\phi}(x,t) \longrightarrow \frac{c}{4}\int_{\mathbb{R}}\psi(y)\tilde{\phi}_0(y)\,\mathrm{d}y \quad \text{as} \quad t\longrightarrow\infty.$$

In fact, the same is true for the nonlinear equation (1.4): the Cole–Hopf transformation

$$\tilde{\phi}(x,t) = \mathrm{e}^{\phi(x,t)} - 1, \qquad \phi(x,t) = \log\left[1+\tilde{\phi}(x,t)\right] \tag{1.7}$$

relates solutions $\phi(x,t)$ of (1.4) and solutions $\tilde{\phi}(x,t)$ of (1.5), and we conclude that solutions $\phi(x,t)$ of (1.4) with localized initial data $\phi_0(x)$ are given by

$$\phi(x,t) = \log\left[1+\int_{\mathbb{R}}\mathcal{G}(x,y,t)\phi_0(y)\,\mathrm{d}y\right]. \tag{1.8}$$

[1] Note that (1.5) is the formal adjoint of the linearization of the standard viscous Burgers equation $u_t = u_{zz} - 2uu_z$ about the Lax shock $\bar{u}(x) = (c/2)[1-\tanh(cx/2)]$ with $x = z - ct$, whose Green's function can be found via the linearized Cole–Hopf transformation by setting $w(x,t) = \cosh(cx/2)\int_{-\infty}^{x}u(y,t)\,\mathrm{d}y$. The Green's function of (1.5) can then be constructed by reversing the roles of x and y in the Green's function for the Lax shock linearization.

As $t \to \infty$, these solutions converge again pointwise in x to the constant

$$\log\left[1+\frac{c}{4}\int_{\mathbb{R}}\psi(y)(\mathrm{e}^{\phi_0(y)}-1)\,\mathrm{d}y\right].$$

The representation (1.8) of solutions of (1.4) will not extend to more general equations, and we therefore develop here a different approach to prove asymptotic stability that, we hope, will also be useful when investigating the nonlinear stability of sources in general reaction-diffusion systems.

To analyse the long-time behaviour of solutions to (1.4), recall that the height of the anticipated phase plateau shown in Figures 3 and 4 is determined by the dynamics near the core located near $x = 0$ and that the phase plateau converges pointwise but certainly not uniformly in space to an asymptotic value as time goes to infinity. This indicates that it will be important to extract the leading-order phase plateau from the phase $\phi(x,t)$ to ensure that the remaining contributions to $\phi(x,t)$ decay in time. To accomplish this, we define the solution $\mathcal{B}(x,t)$ of the linear problem (1.5) by

$$\mathcal{B}(x,t) := \mathcal{G}(x,0,t+1),$$

where $\mathcal{G}(x,y,t)$ is the Green's function defined in (1.6), and note that

$$\phi^*(x,t,p) := \log\left(1+p\mathcal{B}(x,t)\right) \tag{1.9}$$

is then, by the Cole–Hopf transformation, a solution of (1.4) for each fixed $p \in \mathbb{R}$. We emphasize that we do not need that $\phi^*(x,t,p)$ satisfies (1.4) exactly: our analysis goes through provided $\mathcal{B}(x,t)$ can be chosen such that $\phi^*(x,t,p)$ satisfies (1.4) approximately with an error that is bounded by $(1+t)^{-1}$ times a sum of moving heat kernels: in other words, we only need to be able to find a sufficiently accurate approximation of the Green's function.

Next, we investigate the long-time dynamics of solutions to (1.4) with initial data $\phi(x,0) = \phi_0(x)$ using the ansatz

$$\phi(x,t) = \phi^*(x,t,p(t)) + v(x,t), \tag{1.10}$$

where $p(t)$ is a real-valued function, and $v(x,t)$ is a remainder term; see also Figure 5. At time $t = 0$, we normalize the decomposition in (1.10) by choosing $p(0) = p_0$ such that

$$\int_{\mathbb{R}}\psi(x)[\phi_0(x)-\phi^*(x,0,p_0)]\,\mathrm{d}x = 0.$$

We will prove in §2 that a unique p_0 with this property exists for each sufficiently small localized initial condition ϕ_0. The main result of this paper is as follows.

Theorem 1. *For each $\gamma \in (0,\frac{1}{2})$, there exist constants $\epsilon_0, \eta_0, C_0, M_0 > 0$ such that the following is true. If $\phi_0 \in C^1$ satisfies*

$$\epsilon := \|\mathrm{e}^{x^2/M_0}\phi_0\|_{C^1} \le \epsilon_0 \tag{1.11}$$

then the solution $\phi(x,t)$ of (1.4) with $\phi(\cdot,0) = \phi_0$ exists globally in time and can be written in the form

$$\phi(x,t) = \phi^*(x,t,p(t)) + v(x,t)$$

for appropriate functions $p(t)$ and $v(x,t)$ with ϕ^* as in (1.9). Furthermore, there is a $p_\infty \in \mathbb{R}$ with $|p_\infty| \le C_0$ such that

$$|p(t)-p_\infty| \le \epsilon C_0 \mathrm{e}^{-\eta_0 t},$$

and $v(x,t)$ satisfies

$$|v(x,t)| \le \frac{\epsilon C_0}{(1+t)^\gamma}\left(\mathrm{e}^{-\frac{(x+ct)^2}{M_0(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{M_0(t+1)}}\right), \quad |v_x(x,t)| \le \frac{\epsilon C_0}{(1+t)^{\gamma+1/2}}\left(\mathrm{e}^{-\frac{(x+ct)^2}{M_0(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{M_0(t+1)}}\right)$$

for all $t \ge 0$. In particular, $\|v(\cdot,t)\|_{L^r} \to 0$ as $t \to \infty$ for each fixed $r > \frac{1}{2\gamma}$.

Note that our result assumes that the initial condition is strongly localized in space. We believe that this assumption can be relaxed significantly; for the purposes of this paper, however, phase fronts are created even by highly localized initial data, and the key difficulties are therefore present already in the more specialized situation of Theorem 1.

The exponential convergence of $p(t)$ reflects the intuition that a source has a well-defined position due to the exponential localization of the adjoint eigenfunction ψ, which in turn is a consequence of the property that the group velocities point away from the core. In contrast, the position of Lax shocks, whose group velocities point toward the core, relaxes typically only algebraically. The asymptotics of the perturbation v is given by moving Gaussians that decay only like $(1+t)^{-\gamma}$ for each fixed $\gamma \in (0, \frac{1}{2})$, rather than with $(1+t)^{-\frac{1}{2}}$ as expected from the dynamics of the viscous Burgers equation. This weaker result is due to the form (1.10) of our ansatz, which effectively creates a nonlinear term that is proportional to $g(x)uu_x$ for some function $g(x)$. While this term resembles the term $2uu_x$ in Burgers equation, it does not respect the conservation-law structure. Thus, we only obtain decay at the above rate. Although this may not be optimal, it allows us to avoid terms that grow logarithmically in the nonlinear iteration in §2. We do not know if it is possible to improve this rate to $\gamma = \frac{1}{2}$ by adjusting our ansatz appropriately.

The remainder of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1, while we comment in §3 on extensions of the general approach presented here to sources in Ginzburg–Landau and reaction-diffusion systems.

2 Proof of the main theorem

We proceed as follows to prove Theorem 1. Recall that we decomposed solutions of (1.4) via

$$\phi(x,t) = \phi^*(x,t,p(t)) + v(x,t),$$

where the unknown function $p(t)$ regulates the height of the phase plateau, which is determined implicitly by the initial phase offset induced by the initial condition; see also Figure 5. Of course, this decomposition is not unique: our goal is to evolve $p(t)$ in such a way that the remainder $v(x,t)$ decays in time. Inspecting the Green's function $\mathcal{G}(x,y,t)$ of the linearized equation introduced in (1.6), we see that it consists of two parts, namely two counterpropagating heat kernels and an expanding phase plateau as indicated in Figure 4. Therefore, if we rewrite the partial differential equation (1.4) as an integral equation using the Green's function, we could then collect the terms that involve the contributions coming from the nondecaying phase plateau and use them to evolve $p(t)$. The remaining terms in the integral equation involve only the decaying counterpropagating heat kernels, which we use to evolve $v(x,t)$ in time. Our expectation is then that $v(x,t)$ also behaves, at least to leading order, like two counterpropagating Gaussians, and we will verify this by imposing weight functions on the function $v(x,t)$ when we solve the integral equation.

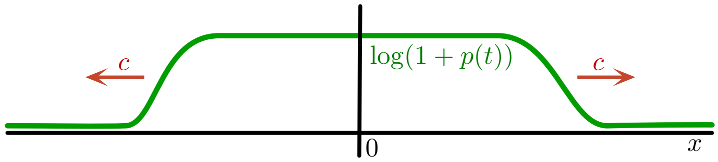

Figure 5: *Shown is the leading-order behaviour of the function* $\phi^*(x,t,p(t))$ *which resembles a plateau with height* $\log(1+p(t))$ *of length* $2ct$ *that spreads outward with speed* $\pm c$.

As already mentioned, the approach outlined above requires knowledge of the nonlinear phase plateau solution $\phi^*(x,t,p)$ for each constant p up to terms that decay at least like $1/t$ at $t\to\infty$. If the Cole–Hopf transformation is not available, we expect that these solutions can be constructed using formal expansions in powers of $1/\sqrt{t}$. This will be discussed in more detail in §3.

Throughout the proof, we denote by C possibly different positive constants that depend only on the underlying equation but not on the initial data or on space or time.

2.1 Derivation of an integral formulation

Substituting the ansatz

$$\phi(x,t)=\log\left(1+p(t)\mathcal{B}(x,t)\right)+v(x,t),\qquad \mathcal{B}(x,t):=\mathcal{G}(x,0,t+1), \tag{2.1}$$

into the Burgers-type equation

$$\phi_t+c\tanh\left(\frac{cx}{2}\right)\phi_x=\phi_{xx}+\phi_x^2, \tag{2.2}$$

we find that (p,v) needs to satisfy the equation

$$v_t=v_{xx}-c\tanh\left(\frac{cx}{2}\right)v_x+v_x^2-\frac{\dot p}{1+\frac{c}{4}p}\mathcal{G}(x,0,t+1)+\mathcal{N}(x,t,p,\dot p,v_x), \tag{2.3}$$

where the nonlinear function $\mathcal{N}$ is given by

$$\mathcal{N}(x,t,p,\dot p,v_x):=\frac{2pv_x\mathcal{B}_x(x,t)}{1+p\mathcal{B}(x,t)}+\dot p\left(\frac{\mathcal{B}(x,t)}{1+\frac{c}{4}p}-\frac{\mathcal{B}(x,t)}{1+p\mathcal{B}(x,t)}\right).$$

In the preceding substitution, we used that the logarithmic term in (2.1) is an exact solution of (2.2): if this term solved (2.2) only up to certain remainder terms, then these would be subsumed in the function $\mathcal{N}$, and we shall see below what that our proof still works provided the remainder terms are small enough. The idea is now to use an appropriate integral representation for v that will allow us to set up a nonlinear iteration argument to show that solutions (p,v) of (2.3) exist and that they satisfy the desired decay estimates in space and time. Recall from (1.6) the expression

$$\begin{aligned}\mathcal{G}(x,y,t) \;&=\; \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{cy}}+\frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{-cy}} \\ &\quad +\frac{c}{4}\left(\operatorname{errfn}\left(\frac{y-x+ct}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{y-x-ct}{\sqrt{4t}}\right)\right)\psi(y)\end{aligned} \tag{2.4}$$

for the Green's function of the linear problem (1.5) and note that it satisfies the identity

$$\int_{\mathbb{R}}\mathcal{G}(x,y,t-s)\mathcal{G}(y,0,s+1)\,\mathrm{d}y=\mathcal{G}(x,0,t+1).$$

Using this identity and the variation-of-constants formula, we can rewrite (2.3) in integral form and obtain[2]

$$\begin{aligned} v(x,t) &= -\frac{4}{c}\mathcal{G}(x,0,t+1)\left(\log\left(1+\frac{cp(t)}{4}\right)-\log\left(1+\frac{cp(0)}{4}\right)\right) \\ &\quad+\int_{\mathbb{R}}\mathcal{G}(x,y,t)v_0(y)\,\mathrm{d}y+\int_0^t\int_{\mathbb{R}}\mathcal{G}(x,y,t-s)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.5}$$

The expression (2.4) of the Green's function $\mathcal{G}(x,y,t)$ shows that the terms involving the error functions do not provide any temporal decay. Since we want the solution $v(x,t)$ to decay in time, the goal is therefore to define $p(t)$ in such a way that the contributions of the nondecaying parts of the Green's function to (2.5) cancel out; the remaining integrals would then contain only the time-decaying parts of the Green's function, which suggests that the solution $v(x,t)$ may also decay in time as desired. To accomplish this, we first write the Green's function $\mathcal{G}(x,y,t)$ as

$$\mathcal{G}(x,y,t)=\mathcal{E}(x,y,t)+\tilde{\mathcal{G}}(x,y,t),$$

where

$$\mathcal{E}(x,y,t)=e(x,t)\psi(y),\qquad e(x,t):=\frac{c}{4}\left(\operatorname{errfn}\left(\frac{x+ct}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{x-ct}{\sqrt{4t}}\right)\right)$$

and

$$\begin{aligned} \tilde{\mathcal{G}}(x,y,t) &:= \mathcal{G}(x,y,t)-\mathcal{E}(x,y,t) \\ &= \frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{cy}}+\frac{1}{\sqrt{4\pi t}}\mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\frac{1}{1+\mathrm{e}^{-cy}} \\ &\quad+\left(\operatorname{errfn}\left(\frac{y-x+ct}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{y-x-ct}{\sqrt{4t}}\right)\right)\frac{c}{4}\operatorname{sech}^2(\frac{cy}{2}) \\ &\quad-\left(\operatorname{errfn}\left(\frac{-x+ct}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{-x-ct}{\sqrt{4t}}\right)\right)\frac{c}{4}\operatorname{sech}^2(\frac{cy}{2}). \end{aligned} \tag{2.6}$$

To verify that $\tilde{G}(x,y,t)$ indeed decays in t, we calculate

$$\left|\operatorname{errfn}\left(\frac{y-x\pm ct}{\sqrt{4t}}\right)-\operatorname{errfn}\left(\frac{-x\pm ct}{\sqrt{4t}}\right)\right|\frac{c}{4}\operatorname{sech}^2(\frac{cy}{2})\le Ct^{-1/2}\left(\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}}+\mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\right)\mathrm{e}^{-c|y|/4}$$

and conclude that

$$|\tilde{\mathcal{G}}(x,y,t)|\le Ct^{-1/2}\left(\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}}+\mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\right). \tag{2.7}$$

Before returning to the integral equation (2.5), we discuss in more detail the initial condition $\phi_0(x)=\phi^*(x,0,p(0))+v_0(x)$ for ϕ. If ϕ_0 is sufficiently small in L^∞, then we claim that $p(0)=p_0$ can be chosen such that

$$\int_{\mathbb{R}}\psi(y)v_0(y)\,\mathrm{d}y=0$$

or, equivalently,

$$\int_{\mathbb{R}}\psi(y)[\phi_0(y)-\phi^*(y,0,p_0)]\,\mathrm{d}y=0. \tag{2.8}$$

To prove this claim, we observe that

$$\phi^*(y,0,p_0)=\log(1+p_0\mathcal{B}(y,0))=p_0\mathcal{G}(y,0,1)+\mathrm{O}(p_0^2). \tag{2.9}$$

[2]We shall often use the notation $[f+g](x,t)$ to denote $f(x,t)+g(x,t)$.

We note that the term $\mathrm{O}(p_0^2)$ in (2.9) is bounded uniformly in $y \in \mathbb{R}$, and substitution into (2.8) therefore gives the equation

$$\int_{\mathbb{R}} \psi(y)\phi_0(y)\,\mathrm{d}y = p_0 \int_{\mathbb{R}} \psi(y)\mathcal{G}(y,0,1)\,\mathrm{d}y + \mathrm{O}(p_0^2),$$

which can be solved uniquely for $p_0 = p(0)$ near zero for each $\phi_0 \in L^\infty$ for which $\|\phi_0\|_{L^\infty}$ is small enough. In particular, there is a constant $C > 0$ such that the resulting initial value $p(0)$ satisfies

$$|p(0)| \le C\|\phi_0\|_{L^\infty} \le C\epsilon.$$

Using this information, the integral equation (2.5) becomes

$$\begin{aligned} v(x,t) &= -\frac{4}{c}\mathcal{G}(x,0,t+1)\left(\log\left(1+\frac{cp(t)}{4}\right) - \log\left(1+\frac{cp(0)}{4}\right)\right) \\ &\quad + \int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t)v_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}} \mathcal{G}(x,y,t-s)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{2.10}$$

We will construct $p(t)$ such that

$$\dot{p}(t) = \left(1+\frac{cp(t)}{4}\right)\int_{\mathbb{R}} \psi(y)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,t)\,\mathrm{d}y \tag{2.11}$$

or, equivalently, that

$$\log\left(1+\frac{cp(t)}{4}\right) = \log\left(1+\frac{cp_0}{4}\right) + \frac{c}{4}\int_0^t\int_{\mathbb{R}} \psi(y)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{2.12}$$

for all $t \ge 0$. Substituting (2.12) into (2.10), we obtain the equation

$$\begin{aligned} v(x,t) &= -\frac{4}{c}\tilde{\mathcal{G}}(x,0,t+1)\left(\log\left(1+\frac{cp(t)}{4}\right) - \log\left(1+\frac{cp_0}{4}\right)\right) + \int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t)v_0(y)\,\mathrm{d}y \\ &\quad + \int_0^t\int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t-s)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{R}} \left(e(x,t-s) - e(x,t+1)\right)\psi(y)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{2.13}$$

for $v(x,t)$. It remains to solve (2.11) and (2.13) for $p(t)$ and $v(x,t)$. To show that the solution $v(x,t)$ decays, it is advantageous to anticipate the expected long-time spatio-temporal behaviour and enforce it using appropriate weight functions. Equation (2.13) contains primarily contributions from $\tilde{G}(x,y,t)$, which, as shown in (2.7), is bounded in norm by two counterpropagating heat kernels. Our expectation is that the nonlinear terms will essentially not change this behaviour, and we therefore expect that $v(x,t)$ can also be bounded by counterpropagating Gaussians. To account for the various interaction terms introduced by the nonlinearity, we will choose slightly weaker weight functions: for each fixed choice of $\gamma \in (0,\frac{1}{2})$ and $M > 0$, define

$$\theta_1(x,t) = \frac{1}{(1+t)^\gamma}\left(\mathrm{e}^{-\frac{(x-ct)^2}{M(t+1)}} + \mathrm{e}^{-\frac{(x+ct)^2}{M(t+1)}}\right), \qquad \theta_2(x,t) = \frac{1}{(1+t)^{\gamma+1/2}}\left(\mathrm{e}^{-\frac{(x-ct)^2}{M(t+1)}} + \mathrm{e}^{-\frac{(x+ct)^2}{M(t+1)}}\right) \tag{2.14}$$

and let

$$h_1(t) := \sup_{x\in\mathbb{R},0\le s\le t}\left[\frac{|v|}{\theta_1} + \frac{|v_x|}{\theta_2}\right](x,s), \qquad h_2(t) := \sup_{0\le s\le t}|\dot{p}(s)|\mathrm{e}^{c^2 s/M}, \qquad h(t) := h_1(t) + h_2(t).$$

We will later pick $M \gg 1$. Note that we expect that $v_x(x,t)$ decays faster than $v(x,t)$ as is the case for the heat kernel.

We remark that we do know existence and smoothness of (v,p) for short times: Indeed, we can solve the original PDE for $\phi(x,t)$ for short times and can substitute the resulting expression into (2.11) upon using our ansatz (2.1). The resulting integral equation has a solution $\dot{p}(t)$ for small times and, using again (2.1), we find a smooth function v that then satisfies (2.13). Furthermore, using that ϕ_0 satisfies (1.11) by assumption, we see that $h(t)$ is well defined and continuous for $0 < t \ll 1$. Finally, standard parabolic theory implies that $h(t)$ retains these properties as long as $h(t)$ stays bounded. The key issue is therefore to show that $h(t)$ stays bounded for all times $t > 0$, and this is what the following proposition asserts.

Proposition 2.1. *For each $\gamma \in (0, \frac{1}{2})$, there exist positive constants ϵ_0, C_0, M such that*

$$h_1(t) \le C_0(\epsilon + h_2(t) + h(t)^2), \qquad h_2(t) \le C_0(\epsilon + h(t)^2). \tag{2.15}$$

for all $t \ge 0$ and all initial data u_0 with $\epsilon := \|\mathrm{e}^{x^2/M}\phi_0\|_{C^1} \le \epsilon_0$.

Using this proposition, we can add the inequalities in (2.15) and eliminate h_2 on the right-hand side to obtain

$$h(t) \le C_0(C_0+1)(\epsilon + h(t)^2).$$

Using this inequality and the continuity of $h(t)$, we find that $h(t) \le 2C_0(C_0+1)\epsilon$ for all $t \ge 0$ provided $0 < \epsilon \le \epsilon_0$ is sufficiently small. Thus, Theorem 1 will be proved once we establish Proposition 2.1. The following sections will be devoted to proving this proposition.

2.2 Estimates of the nonlinear term

We begin by deriving estimates of the nonlinear term

$$\mathcal{N}(x,t,p,\dot{p},v_x) = \frac{2pv_x\mathcal{B}_x(x,t)}{1+p\mathcal{B}(x,t)} + \dot{p}\left(\frac{\mathcal{B}(x,t)}{1+\frac{c}{4}p} - \frac{\mathcal{B}(x,t)}{1+p\mathcal{B}(x,t)}\right)$$

that appears in (2.11) and (2.13). Recalling that

$$e(x,t) = \frac{c}{4}\left(\mathrm{errfn}\left(\frac{x+ct}{\sqrt{4t}}\right) - \mathrm{errfn}\left(\frac{x-ct}{\sqrt{4t}}\right)\right)$$

and using the definition of the error function, we see that

$$\left|e(x,t)\left(\frac{c}{4} - e(x,t)\right)\right| \le C\left(\mathrm{e}^{-\frac{(x+ct)^2}{8t}} + \mathrm{e}^{-\frac{(x-ct)^2}{8t}}\right).$$

Since $\mathcal{B}(x,t) = e(x,t+1) + \tilde{\mathcal{G}}(x,0,t+1)$, we conclude from (2.7) that

$$\left|\mathcal{B}(x,t)\left(\frac{c}{4} - \mathcal{B}(x,t)\right)\right| \le C\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}} + \mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right)$$

and thus

$$\frac{\mathcal{B}(x,t)}{1+p\mathcal{B}(x,t)} - \frac{\mathcal{B}(x,t)}{1+\frac{c}{4}p} \le C|p|\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}} + \mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right).$$

In addition, we have

$$\left|\frac{\mathcal{B}_x(x,t)}{1+p\mathcal{B}(x,t)}\right| \le C(1+t)^{-1/2}\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right).$$

Combining these estimates, we therefore obtain

$$|\mathcal{N}(x,t,p,\dot{p},v_x)| \le C\left((1+t)^{-1/2}|p||v_x|+|p\dot{p}|\right)\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right) \tag{2.16}$$

uniformly in $x\in\mathbb{R}$ and $t\ge 0$, provided p is sufficiently small. In the remainder of the proof, we shall use only the preceding estimate (2.16) but not any other information about the nonlinearity.

2.3 Estimates for $h_2(t)$

To establish the claimed estimate for $h_2(t)$, recall from (2.11) that

$$|\dot{p}(t)| \le C(1+|p(t)|)\int_{\mathbb{R}}\psi(y)\left|\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,t)\right|\mathrm{d}y. \tag{2.17}$$

We shall show that there is a constant $C_1=C_1(M)$ such that

$$|\dot{p}(t)| \le C_1\mathrm{e}^{-c^2t/M}(\epsilon+h^2(t)), \tag{2.18}$$

which then establishes the estimate for $h_2(t)$ stated in Proposition 2.1. To show (2.18), we note that the definitions of $p(t)$ and $h_2(t)$ imply that

$$|p(t)| \le |p(0)|+\int_0^t|\dot{p}(s)|\,\mathrm{d}s \le |p(0)|+\int_0^t \mathrm{e}^{-c^2s/M}h_2(t)\,\mathrm{d}s \le C_1(\epsilon+h_2(t)) \tag{2.19}$$

and therefore

$$|p(t)\dot{p}(t)| \le C_1\mathrm{e}^{-c^2t/M}h_2(t)(\epsilon+h_2(t)) \le C_1\mathrm{e}^{-c^2t/M}(\epsilon+h(t)^2). \tag{2.20}$$

Next, we use the estimate $|\psi(y)|\le 2\mathrm{e}^{-c|y|}$, the bound (2.16) on $\mathcal{N}$, and the inequality

$$\mathrm{e}^{-\frac{c|y|}{2}}\mathrm{e}^{-\frac{(y\pm ct)^2}{M(1+t)}} \le C_1\mathrm{e}^{-\frac{c|y|}{4}}\mathrm{e}^{-\frac{c^2t}{M}},$$

which holds for each $M\ge 8$, to conclude that

$$|\psi(y)v_y^2(y,t)| \le \frac{C_1}{(1+t)^{1+2\gamma}}C\mathrm{e}^{-\frac{c}{2}|y|}\mathrm{e}^{-\frac{2c^2}{M}t}h_1(t)^2 \tag{2.21}$$

and therefore

$$\begin{aligned}
|\psi(y)\mathcal{N}(y,t,p,\dot{p},v_y)| &\le C_1(1+t)^{-1/2}\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right)\psi(y)|v_y(y,t)||p(t)| \\
&\quad +C_1\left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}}+\mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}}\right)\psi(y)|p(t)\dot{p}(t)| \\
&\le C_1(1+t)^{-1/2}\mathrm{e}^{-\frac{c}{2}|y|-\frac{2c^2}{M}t}h_1(t)(\epsilon+h_2(t))+C_1\mathrm{e}^{-\frac{c}{2}|y|-\frac{2c^2}{M}t}(\epsilon+h(t)^2) \\
&\le C_1\mathrm{e}^{-\frac{c}{2}|y|-\frac{2c^2}{M}t}(\epsilon+h(t)^2).
\end{aligned} \tag{2.22}$$

Using these estimates in (2.17), we arrive readily at (2.18), thus proving the estimate for $h_2(t)$ stated in Proposition 2.1.

2.4 Estimates for $v(x,t)$ and $v_x(x,t)$

In this section, we will establish the pointwise bounds

$$
\begin{aligned}
|v(x,t)| &\le C_1(\epsilon + h_2(t) + h(t)^2)\theta_1(x,t) &(2.23)\\
|v_x(x,t)| &\le C_1(\epsilon + h_2(t) + h(t)^2)\theta_2(x,t) &(2.24)
\end{aligned}
$$

for $v(x,t)$ and $v_x(x,t)$, respectively, which taken together prove the inequality for $h_1(t)$ stated in Proposition 2.1. In particular, the proof of Proposition 2.1 is complete once the two estimates above are established. We denote by C_1 possibly different constants that depend only on the choice of γ, M so that $C_1 = C_1(\gamma, M)$.

We focus first on $v(x,t)$. From the integral formulation (2.13) for $v(x,t)$, we find that

$$
\begin{aligned}
|v(x,t)| \le\; & C_1\left(|p_0| + |p(t)|\right)\tilde{\mathcal{G}}(x,0,t+1) + \int_{\mathbb{R}} \tilde{\mathcal{G}}(x,y,t)|v_0(y)|\,\mathrm{d}y &(2.25)\\
& + \int_0^t \int_{\mathbb{R}} \left|\tilde{\mathcal{G}}(x,y,t-s)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\right| \mathrm{d}y\,\mathrm{d}s\\
& + \int_0^t \int_{\mathbb{R}} \left|[e(x,t-s+1) - e(x,t+1)]\psi(y)\left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\right| \mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

In the remainder of this section, we will estimate the right-hand side of (2.25) term by term.

First, we note that (2.7) implies that there is a constant $C_1 = C_1(M)$ so that $\tilde{\mathcal{G}}(x,0,t+1) \le C_1\theta_1(x,t)$. Using (2.19) and the fact that $|p(0)| \le C_1\epsilon$, we therefore obtain

$$
(|p(0)| + |p(t)|)\tilde{\mathcal{G}}(x,0,t+1) \le C_1(\epsilon + h_2(t))\theta_1(x,t),
$$

which is the desired estimate for the first term on the right-hand side of (2.25).

Next, we consider the integral term in (2.25) that involves the initial data v_0. Using (2.7) together with our assumption (1.11) on ϕ_0, and hence on v_0, we see that

$$
\int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t)v_0(y)|\,\mathrm{d}y \le C\epsilon \int_{\mathbb{R}} t^{-1/2}\left(\mathrm{e}^{-\frac{(x-y+ct)^2}{4t}} + \mathrm{e}^{-\frac{(x-y-ct)^2}{4t}}\right)\mathrm{e}^{-\frac{y^2}{M}}\,\mathrm{d}y, \qquad (2.26)
$$

which is clearly bounded by $\epsilon C_1\theta_1(x,t)$ for $t \ge 1$ upon using

$$
\mathrm{e}^{-\frac{(x-y\pm ct)^2}{4t}}\mathrm{e}^{-\frac{y^2}{M}} \le C_1\mathrm{e}^{-\frac{(x\pm ct)^2}{Mt}}\mathrm{e}^{-\frac{y^2}{2M}}.
$$

For $t \le 1$, we can use the estimates

$$
\begin{aligned}
\mathrm{e}^{-\frac{(x-y\pm ct)^2}{8t}}\mathrm{e}^{-\frac{y^2}{M}} \;\le\; 2\mathrm{e}^{-\frac{(x-y)^2}{8t}}\mathrm{e}^{-\frac{y^2}{M}} &\;\le\; C_1\mathrm{e}^{-\frac{x^2}{2M}}\\
\int_{\mathbb{R}} t^{-1/2}\left(\mathrm{e}^{-\frac{(x-y+ct)^2}{8t}} + \mathrm{e}^{-\frac{(x-y-ct)^2}{8t}}\right)\mathrm{d}y &\;\le\; C_1
\end{aligned}
$$

to conclude that the integral in (2.26) is again bounded by $\epsilon C_1\theta_1(x,t)$.

We now consider the remaining two integrals in (2.25). Note that the definition of h_1 gives

$$
|v_y(y,s)| \le \theta_2(y,s)h_1(s).
$$

Using this fact together with the estimates (2.19) and (2.20) and the bound (2.16), we obtain

$$\begin{aligned}
|\mathcal{N}(y,s,p(s),\dot{p}(s),v_y(y,s))| &\leq C_1(1+s)^{-1/2}\left(\mathrm{e}^{-\frac{(y+cs)^2}{8(s+1)}}+\mathrm{e}^{-\frac{(y-cs)^2}{8(s+1)}}\right)|v_y(y,s)||p(s)| \\
&\quad +C_1\left(\mathrm{e}^{-\frac{(y+cs)^2}{8(s+1)}}+\mathrm{e}^{-\frac{(y-cs)^2}{8(s+1)}}\right)|p(s)\dot{p}(s)| \\
&\leq C_1(\epsilon+h_2(t))h_1(t)\theta_2(y,s)(1+s)^{-1/2}\left(\mathrm{e}^{-\frac{(y+cs)^2}{8(s+1)}}+\mathrm{e}^{-\frac{(y-cs)^2}{8(s+1)}}\right) \\
&\quad +C_1(\epsilon+h(t)^2)\mathrm{e}^{-\frac{c^2s}{M}}\left(\mathrm{e}^{-\frac{(y+cs)^2}{8(s+1)}}+\mathrm{e}^{-\frac{(y-cs)^2}{8(s+1)}}\right) \\
&\leq C_1(\epsilon+h(t)^2)\left[(1+s)^{\gamma-1/2}\theta_1(y,s)\theta_2(y,s)+(1+s)^{\gamma}\theta_1(y,s)\mathrm{e}^{-\frac{c^2s}{M}}\right].
\end{aligned}$$

In §2.5, we will prove the following result for our spatio-temporal template functions.

Lemma 2.2. *For each sufficiently large M, there is a constant C_1 so that*

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}}|\tilde{\mathcal{G}}(x,y,t-s)|\left[\theta_2^2+(1+s)^{\gamma-1/2}\theta_1\theta_2+(1+s)^{\gamma}\theta_1\mathrm{e}^{-c^2s/M}\right](y,s)\,\mathrm{d}y\,\mathrm{d}s &\leq C_1\theta_1(x,t) \\
\int_0^t\int_{\mathbb{R}}|\tilde{\mathcal{G}}_x(x,y,t-s)|\left[\theta_2^2+(1+s)^{\gamma-1/2}\theta_1\theta_2+(1+s)^{\gamma}\theta_1\mathrm{e}^{-c^2s/M}\right](y,s)\,\mathrm{d}y\,\mathrm{d}s &\leq C_1\theta_2(x,t).
\end{aligned}$$

Using this lemma and the above estimates for $v_y^2+\mathcal{N}$, we obtain the desired estimate

$$\int_0^t\int_{\mathbb{R}}\tilde{\mathcal{G}}(x,y,t-s)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s\leq C_1(\epsilon+h(t)^2)\theta_1(x,t).$$

Finally, we have the following lemma, whose proof is again given in §2.5, which provides the desired estimate for the last integral in (2.25).

Lemma 2.3. *For each sufficiently large M, there is a constant C_1 so that*

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}}|e(x,t-s+1)-e(x,t+1)|\psi(y)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s &\leq C_1(\epsilon+h(t)^2)\theta_1(x,t) \\
\int_0^t\int_{\mathbb{R}}|e_x(x,t-s+1)-e_x(x,t+1)|\psi(y)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s &\leq C_1(\epsilon+h(t)^2)\theta_2(x,t).
\end{aligned}$$

In summary, combining the estimates obtained above, we have established the claimed estimate (2.23), and it remains to derive the estimate (2.24) to complete the proof of Proposition 2.1. Taking the x-derivative of equation (2.13), we see that

$$\begin{aligned}
|v_x(x,t)| &\leq (|p(0)|+|p(t)|)\,\tilde{\mathcal{G}}_x(x,0,t)+\int_{\mathbb{R}}\tilde{\mathcal{G}}_x(x,y,t)v_0(y)\,\mathrm{d}y \\
&\quad +\int_0^t\int_{\mathbb{R}}\tilde{\mathcal{G}}_x(x,y,t-s)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad +\int_0^t\int_{\mathbb{R}}[e_x(x,t-s+1)-e_x(x,t+1)]\psi(y)\left[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)\right](y,s)\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{2.27}$$

Applying the second estimate in Lemmas 2.2 and 2.3 to (2.27) and using that $\tilde{\mathcal{G}}_x(x,0,t)\leq C\theta_2(x,t)$, we immediately obtain (2.24).

2.5 Proofs of Lemmas 2.2 and 2.3

It remains to prove the lemmas that we used in the preceding section.

Proof of Lemma 2.2. We need to show that for each large M there is a constant C_1 so that

$$\int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)| \left[\theta_2^2 + (1+s)^{\gamma-1/2}\theta_1\theta_2 + (1+s)^{\gamma}\theta_1 \mathrm{e}^{-c^2 s/M}\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \le C_1\theta_1(x,t)$$

for all $t \ge 0$. First, we note that there are constants $C_1, \tilde{C}_1 > 0$ such that

$$\tilde{C}_1 \mathrm{e}^{-y^2/M} \le |\theta_1(y,s)| + |\theta_2(y,s)| \le C_1 \mathrm{e}^{-y^2/M}$$

for all $0 \le s \le t \le 1$. Thus, for some constant C_1 that may change from line to line, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)| \left[\theta_2^2 + (1+s)^{\gamma-1/2}\theta_1\theta_2 + (1+s)^{\gamma}\theta_1 \mathrm{e}^{-\frac{c^2 s}{M}}\right](y,s)\,\mathrm{d}y\,\mathrm{d}s \\
\le\ & C_1 \int_0^t \int_{\mathbb{R}} (t-s)^{-1/2} \mathrm{e}^{-\frac{(x-y)^2}{4(t-s)}} \mathrm{e}^{-\frac{y^2}{M}}\,\mathrm{d}y\,\mathrm{d}s \\
\le\ & C_1 \int_0^t \left[\int_{\{|y|\ge 2|x|\}} (t-s)^{-1/2} \mathrm{e}^{-\frac{(x-y)^2}{8(t-s)}} \mathrm{e}^{-\frac{x^2}{8(t-s)}}\,\mathrm{d}y + \int_{\{|y|\le 2|x|\}} (t-s)^{-1/2} \mathrm{e}^{-\frac{(x-y)^2}{4(t-s)}} \mathrm{e}^{-\frac{4x^2}{M}}\,\mathrm{d}y\right]\mathrm{d}s \\
\le\ & C_1 \int_0^t \left[\mathrm{e}^{-\frac{x^2}{8(t-s)}} + \mathrm{e}^{-\frac{4x^2}{M}}\right]\mathrm{d}s \\
\le\ & C_1 \mathrm{e}^{-\frac{4x^2}{M}} \\
\le\ & \frac{C_1}{\tilde{C}_1}\theta_1(x,t)
\end{aligned}$$

for all $0 \le t \le 1$. An analogous computation can be carried out for the x-derivative since $\int_0^t (t-s)^{-1/2}\,\mathrm{d}s$ is bounded uniformly in $0 \le t \le 1$.

Thus, it remains to estimate the expression

$$\theta_1(x,t)^{-1} \int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)| \left[\theta_2^2 + (1+s)^{\gamma-1/2}\theta_1\theta_2 + (1+s)^{\gamma}\theta_1 \mathrm{e}^{-c^2 s/M}\right](y,s)\,\mathrm{d}y\,\mathrm{d}s$$

for $t \ge 1$. Combining only the exponentials in this expression, we obtain terms that can be bounded by

$$\exp\left(\frac{(x+\alpha_3 ct)^2}{M(1+t)} - \frac{(x-y+\alpha_1 c(t-s))^2}{4(t-s)} - \frac{(y+\alpha_2 cs)^2}{M(1+s)}\right) \tag{2.28}$$

with $\alpha_j = \pm c$. To estimate this expression, we proceed as in [5, Proof of Lemma 7] and complete the square of the last two exponents in (2.28). Written in a slightly more general form, we obtain

$$\begin{aligned}
\frac{(x-y-\alpha_1(t-s))^2}{M_1(t-s)} + \frac{(y-\alpha_2 s)^2}{M_2(1+s)} &= \frac{(x-\alpha_1(t-s)-\alpha_2 s)^2}{M_1(t-s)+M_2(1+s)} \\
&+ \frac{M_1(t-s)+M_2(1+s)}{M_1M_2(1+s)(t-s)}\left(y - \frac{xM_2(1+s) - (\alpha_1 M_2(1+s)+\alpha_2 M_1 s)(t-s)}{M_1(t-s)+M_2(1+s)}\right)^2
\end{aligned}$$

and conclude that the exponent in (2.28) is of the form

$$\begin{aligned}
&\frac{(x+\alpha_3 t)^2}{M(1+t)} - \frac{(x-\alpha_1(t-s)-\alpha_2 s)^2}{4(t-s)+M(1+s)} \\
&\quad - \frac{4(t-s)+M(1+s)}{4M(1+s)(t-s)}\left(y - \frac{xM(1+s)-(\alpha_1 M(1+s)+4\alpha_2 s)(t-s)}{4(t-s)+M(1+s)}\right)^2,
\end{aligned} \tag{2.29}$$

with $\alpha_j = \pm c$. Using that the maximum of the quadratic polynomial $\alpha x^2 + \beta x + \gamma$ is $-\beta^2/(4\alpha) + \gamma$, it is easy to see that the sum of the first two terms in (2.29), which involve only x and not y, is less than or equal to zero. Omitting this term, we therefore obtain the estimate

$$\begin{aligned} & \exp\left(\frac{(x \pm ct)^2}{M(1+t)} - \frac{(x - y\delta_1 c(t-s))^2}{4(t-s)} - \frac{(y - \delta_2 cs)^2}{M(1+s)}\right) \\ \leq\ & \exp\left(-\frac{4(t-s) + Ms}{4M(1+s)(t-s)}\left(y - \frac{xM(1+s) + c(\delta_1 M(1+s) + 4\delta_2 s)(t-s)}{4(t-s) + M(1+s)}\right)^2\right) \end{aligned} \tag{2.30}$$

for $\delta_j = \pm 1$. Using this result, we can now estimate the integral (2.28) term by term using the key assumption that $0 < \gamma < \frac{1}{2}$. The term involving θ_2^2 can be estimated as follows using (2.30):

$$\begin{aligned} & \theta_1(x,t)^{-1} \int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)| \theta_2^2(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^t \frac{1}{\sqrt{t-s}(1+s)^{1+2\gamma}} \\ & \times \int_{\mathbb{R}} \exp\left(-\frac{4(t-s)+M(1+s)}{4M(1+s)(t-s)}\left(y - \frac{[xM(1+s) \pm c(M(1+s)+4s)(t-s)]}{4(t-s)+M(1+s)}\right)^2\right)\mathrm{d}y\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^t \frac{1}{\sqrt{t-s}(1+s)^{1+2\gamma}} \sqrt{\frac{4M(1+s)(t-s)}{4(t-s)+M(1+s)}}\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^{t/2} \frac{1}{(1+s)^{1/2+2\gamma}} \frac{1}{(1+t)^{1/2}}\,\mathrm{d}s + C_1(1+t)^\gamma \int_{t/2}^t \frac{1}{(1+s)^{1+2\gamma}}\,\mathrm{d}s \\ \leq\ & C_1(1+t)^{\gamma-1/2} + C_1(1+t)^{-\gamma}, \end{aligned}$$

which is clearly bounded since $\gamma < \frac{1}{2}$. Similarly, we have

$$\begin{aligned} & \theta_1(x,t)^{-1} \int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)|(1+s)^{\gamma-1/2}\theta_1\theta_2(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^t \frac{1}{\sqrt{t-s}(1+s)^{\gamma+1}} \sqrt{\frac{4M(1+s)(t-s)}{4(t-s)+M(1+s)}}\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^{t/2} \frac{1}{(1+s)^{\gamma+1/2}} \frac{1}{(1+t)^{1/2}}\,\mathrm{d}s + C_1(1+t)^\gamma \int_{t/2}^t \frac{1}{(1+t)^{\gamma+1/2}} \frac{1}{(1+s)^{1/2}}\,\mathrm{d}s \\ \leq\ & C_1(1+t)^{\gamma-1/2} + C_1, \end{aligned}$$

which is again bounded due to $\gamma < \frac{1}{2}$. Finally, we estimate

$$\begin{aligned} & \theta_1(x,t)^{-1} \int_0^t \int_{\mathbb{R}} |\tilde{\mathcal{G}}(x,y,t-s)|(1+s)^\gamma \mathrm{e}^{-\frac{c^2 s}{M}} \theta_1(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^t \frac{\mathrm{e}^{-\frac{c^2 s}{M}}}{\sqrt{t-s}} \sqrt{\frac{4M(1+s)(t-s)}{4(t-s)+M(1+s)}}\,\mathrm{d}s \\ \leq\ & C_1(1+t)^\gamma \int_0^{t/2} \mathrm{e}^{-\frac{c^2 s}{M}} \frac{1}{(1+t)^{1/2}}\,\mathrm{d}s + C_1(1+t)^\gamma \mathrm{e}^{-\frac{c^2 t}{2M}} \int_{t/2}^t \mathrm{d}s \\ \leq\ & C_1(1+t)^{\gamma-1/2} + C_1(1+t)^{\gamma+1}\mathrm{e}^{-\frac{c^2 t}{2M}}, \end{aligned}$$

which is bounded, again due to $\gamma < \frac{1}{2}$.

It remains to verify the second inequality in Lemma 2.2 which involves $\tilde{\mathcal{G}}_x$. We shall check only the term involving $(1+s)^{\gamma-1/2}\theta_1\theta_2$ as the other cases are similar and, in fact, easier. We have shown above that the resulting integrals are bounded for $0 \le t \le 1$ and therefore focus on the case $t \ge 1$. Using that $|\tilde{\mathcal{G}}_x| \le Ct^{-1/2}|\tilde{\mathcal{G}}|$, which follows by inspection, and employing again (2.30), we obtain

$$\begin{aligned}
&\theta_2(x,t)^{-1}\int_0^t\int_{\mathbb{R}}|\tilde{\mathcal{G}}_x(x,y,t-s)|(1+s)^{\gamma-1/2}\theta_1\theta_2(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\le\ C_1(1+t)^{\gamma+1/2}\int_0^t\frac{1}{(t-s)(1+s)^{\gamma+1}}\sqrt{\frac{4M(1+s)(t-s)}{4(t-s)+M(1+s)}}\,\mathrm{d}s \\
&\le\ C_1(1+t)^{\gamma+1/2}\int_0^{t/2}\frac{1}{t^{1/2}(1+s)^{\gamma+1/2}}\frac{1}{(1+t)^{1/2}}\,\mathrm{d}s + C_1\int_{t/2}^t\frac{1}{(t-s)^{1/2}}\frac{1}{(1+t)^{1/2}}\,\mathrm{d}s \\
&\le\ C_1(1+t)^{\gamma}t^{-1/2}+C_1,
\end{aligned}$$

which is bounded for $t \ge 1$. This completes the proof of Lemma 2.2. □

Proof of Lemma 2.3. We need to show that

$$\int_0^t\int_{\mathbb{R}}|e(x,t-s+1)-e(x,t+1)|\psi(y)[v_y^2+\mathcal{N}(\cdot,\cdot,p,\dot{p},v_y)](y,s)\,\mathrm{d}y\,\mathrm{d}s \le C_1(\epsilon+h(t)^2)\theta_1(x,t).$$

Intuitively, this integral should be small for the following reason. The difference $e(x,t-s)-e(x,t+1)$ converges to zero as long as s is not too large, say on the interval $s\in[0,t/2]$. For $s\in[t/2,t]$, on the other hand, we will get exponential decay in s from the localization of $\psi(y)$ in combination with the propagating heat kernels that appear in the nonlinearity and forcing terms. To make this precise, we write

$$e(x,t-s)-e(x,t+1) = \underbrace{e(x,t-s)-e(x,t-s+1)}_{\text{term I}}+\underbrace{e(x,t-s+1)-e(x,t+1)}_{\text{term II}}. \tag{2.31}$$

We focus first on the term I and consider the cases $t \ge 1$ and $0 \le t \le 1$ separately. First, let $t \ge 1$. For $0 \le s \le t-1$, we have $|e(x,t-s)-e(x,t-s+1)| \le C\tilde{\mathcal{G}}(x,0,t-s)$, and we can estimate the resulting integral above in the same way as in the proof of Lemma 2.2; we omit the details. For $t-1 \le s \le t \le 1$, on the other hand, the definition of $e(x,t-s)$ yields

$$|e(x,t-s)-e(x,t-s+1)| \le C\int_{\frac{x^2}{(1+t-s)}}^{\frac{x^2}{(t-s)}}\mathrm{e}^{-z^2}\,\mathrm{d}z \le C\mathrm{e}^{-x^2/2}. \tag{2.32}$$

Using (2.21)-(2.22), namely

$$|\psi(y)\mathcal{N}(y,s,p,\dot{p},v_y)| \le C\mathrm{e}^{-\frac{c}{2}|y|-\frac{2c^2}{M}s}(\epsilon+h(t)^2) \tag{2.33}$$

for all $s \ge 0$, we obtain

$$\begin{aligned}
&\int_{t-1}^t\int_{\mathbb{R}}[e(x,t-s)-e(x,t-s+1)]\psi(y)|\mathcal{N}(y,s,p,\dot{p},v_y)|(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad\le\ C_1(\epsilon+h(t)^2)\int_{t-1}^t\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-\frac{2c^2s}{M}}\,\mathrm{d}s\ \le\ C_1(\epsilon+h(t)^2)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-\frac{2c^2t}{M}},
\end{aligned}$$

which is clearly bounded by $C_1\theta_1(x,t)$ since $\mathrm{e}^{-c^2t/M} \le C_1(1+t)^{-\gamma}$ and

$$\frac{(x+ct)^2}{M(1+t)} \le \frac{2x^2}{M}+\frac{4c^2}{M}+\frac{c^2t}{M}$$

for arbitrary $M \geq 4$. In summary, we have established the desired estimates for the term I in (2.31) for $t \geq 1$. For $t \leq 1$, the estimate (2.32) remains true since $t-s$ is small, and proceeding as above yields

$$\int_0^t \int_{\mathbb{R}} [e(x,t-s) - e(x,t-s+1)]\psi(y)|\mathcal{N}(y,s,p,\dot{p},v_y)|\,\mathrm{d}y\,\mathrm{d}s \leq C(\epsilon + h(t)^2)\mathrm{e}^{-\frac{x^2}{2}}\mathrm{e}^{-\frac{2c^2 t}{M}},$$

which is again bounded by $C_1\theta_1(x,t)$.

It remains to discuss the term II which involves the difference $e(x,t-s+1) - e(x,t+1)$. We have

$$\begin{aligned}
&|e(x,t-s+1) - e(x,t+1)| \\
&\quad = \left| \int_{t+1}^{t-s+1} e_\tau(x,\tau)\,\mathrm{d}\tau \right| \\
&\quad \leq \int_{t-s+1}^{t+1} \left| \frac{c}{\sqrt{4\pi\tau}} \left(\mathrm{e}^{-\frac{(x-c\tau)^2}{4\tau}} + \mathrm{e}^{-\frac{(x+c\tau)^2}{4\tau}} \right) + \frac{1}{\tau\sqrt{4\pi}} \left(\frac{(x-c\tau)}{\sqrt{4\tau}} \mathrm{e}^{-\frac{(x-c\tau)^2}{4\tau}} - \frac{(x+c\tau)}{\sqrt{4\tau}} \mathrm{e}^{-\frac{(x+c\tau)^2}{4\tau}} \right) \right| \mathrm{d}\tau \\
&\quad \leq C \int_{t-s+1}^{t+1} \left(\frac{1}{\sqrt{\tau}} + \frac{1}{\tau} \right) \left(\mathrm{e}^{-\frac{(x-c\tau)^2}{8\tau}} + \mathrm{e}^{-\frac{(x+c\tau)^2}{8\tau}} \right) \mathrm{d}\tau,
\end{aligned}$$

where we used in the last inequality that $z\mathrm{e}^{-z^2}$ is uniformly bounded in z. We now use the preceding expression to estimate $\theta_1^{-1}(x,t)(e(x,t-s+1) - e(x,t+1))$ and focus first on the single exponential term

$$\mathrm{e}^{\frac{(x-ct)^2}{M(1+t)}} \mathrm{e}^{-\frac{(x-c\tau)^2}{8\tau}}.$$

Combining these exponentials and completing the square in x in the resulting exponent, the latter becomes

$$-\frac{[M(t-\tau+1) + (M-8)\tau]}{8M(t+1)\tau} \left[x + \frac{c(8-M)\tau(t+1)}{M(t-\tau) + (M-8)\tau} \right]^2 + \frac{c^2(t-\tau+1)^2}{M(t-\tau+1) + (M-8)\tau}.$$

Using that $\tau \leq t$ and picking $M \geq 8$, we can neglect the exponent resulting from the first expression that involves in x and conclude that

$$\mathrm{e}^{\frac{(x-ct)^2}{M(1+t)}} \mathrm{e}^{-\frac{(x-c\tau)^2}{8\tau}} \leq C_1 \mathrm{e}^{\frac{c^2(t-\tau)}{M}}.$$

The remaining exponentials can be estimated similarly, and we obtain

$$\begin{aligned}
\theta_1^{-1}(x,t)|e(x,t-s+1) - e(x,t+1)| &\leq C_1(1+t)^\gamma \int_{t-s+1}^{t+1} \left(\frac{1}{\sqrt{\tau}} + \frac{1}{\tau} \right) \mathrm{e}^{\frac{c^2(t-\tau)}{M}}\,\mathrm{d}\tau \\
&\leq C_1(1+t)^\gamma (1+t-s)^{-1/2} \mathrm{e}^{\frac{c^2 s}{M}}.
\end{aligned}$$

Using this inequality together with (2.33) finally gives

$$\begin{aligned}
&\theta_1(x,t)^{-1} \int_0^t \int_{\mathbb{R}} [e(x,t-s+1) - e(x,t+1)]\psi(y)|\mathcal{N}(y,s,p,\dot{p},v_y)|(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad \leq C_1(1+t)^\gamma(\epsilon + h(t)^2) \int_0^t (1+t-s)^{-1/2} \mathrm{e}^{\frac{c^2 s}{M}} \mathrm{e}^{-\frac{2c^2 s}{M}}\,\mathrm{d}s \\
&\quad \leq C_1(1+t)^\gamma(\epsilon + h(t)^2) \left[(1+t)^{-1/2} \int_0^{t/2} \mathrm{e}^{-\frac{c^2 s}{M}}\,\mathrm{d}s + \mathrm{e}^{-\frac{c^2 t}{2M}} \int_{t/2}^t (1+t-s)^{-1/2}\,\mathrm{d}s \right] \\
&\quad \leq C_1(\epsilon + h(t)^2).
\end{aligned}$$

for M sufficiently large, which proves the first estimate in Lemma 2.3.

It remains to prove the estimate

$$\int_0^t \int_{\mathbb{R}} |e_x(x,t-s+1) - e_x(x,t+1)| \psi(y) \left[v_y^2 + \mathcal{N}(\cdot,\cdot,p,\dot{p},v_y) \right](y,s) \,\mathrm{d}y \,\mathrm{d}s \\ \leq C_1(\epsilon + h(t)^2)\theta_2(x,t)$$

for the derivative in x. Since the derivative of $e(x,t-s+1) - e(x,t+1)$ with respect to x generates an extra decay term $(1+t)^{-1/2}$, we have

$$\begin{aligned}
&\theta_2(x,t)^{-1} \int_0^t \int_{\mathbb{R}} [e_x(x,t-s+1) - e_x(x,t+1)] \psi(y) |\mathcal{N}(y,s,p,\dot{p},v_y)|(y,s) \,\mathrm{d}y \,\mathrm{d}s \\
\leq\;& C_1(\epsilon^2 + h(t)^2)(1+t)^{\gamma+1/2} \int_0^t (1+t-s)^{-1} \mathrm{e}^{\frac{c^2 s}{M}} \mathrm{e}^{-\frac{2c^2 s}{M}} \,\mathrm{d}s \\
\leq\;& C_1(\epsilon^2 + h(t)^2)(1+t)^{\gamma+1/2} \left[(1+t)^{-1} \int_0^{t/2} \mathrm{e}^{-\frac{c^2 s}{M}} \,\mathrm{d}s + \mathrm{e}^{-\frac{c^2 t}{2M}} \int_{t/2}^t (1+t-s)^{-1} \,\mathrm{d}s \right] \\
\leq\;& C_1(\epsilon^2 + h(t)^2),
\end{aligned}$$

which completes the proof of the lemma. □

3 Discussion and outlook

In this paper, we considered the Burgers-type equation

$$\phi_t + c \tanh\left(\frac{cx}{2}\right) \phi_x = \phi_{xx} + \phi_x^2, \qquad c > 0 \tag{3.1}$$

as a model for the evolution of phase perturbations of a source. We proved that solutions associated with small localized initial data converge pointwise in x to a constant solution as $t \to \infty$. More precisely, since the group velocities point toward infinity, solutions will evolve, up to terms that decay in time, like phase plateaus whose width expands linearly in t and whose interfaces move away from the core toward $x = \pm\infty$, while widening like $\sqrt{t}$. The plateau height is determined by the shape of the perturbation near the core located at $x = 0$, which therefore plays a crucial role in determining the overall dynamics.

To prove this result, we made use of the facts that (i) the Green's function $\mathcal{G}(x,y,t)$ of the linearization of (3.1) about $\phi = 0$ is given explicitly as the sum of two counterpropagating heat kernels plus an expanding phase plateau and that (ii) the Cole–Hopf transformation then gives an explicit phase-plateau solution of (3.1) for each given fixed height. In particular, the first term on the right-hand side of our ansatz

$$\phi(x,t) = \log\left(1 + p(t)\mathcal{G}(x,0,t+1)\right) + v(x,t)$$

for solutions of (3.1) satisfies (3.1) for each constant p, and our expectation is that $v(x,t)$ can be bounded by two counterpropagating heat kernels. Exploiting (i) and (ii) allowed us to show that the remainder terms left in the equation for v after substituting our ansatz into (3.1) can be bounded from above by

$$|\mathcal{N}(x,t,p,\dot{p},v_x)| \leq C|p| \left((1+t)^{-1/2}|v_x| + |\dot{p}| \right) \left(\mathrm{e}^{-\frac{(x+ct)^2}{8(t+1)}} + \mathrm{e}^{-\frac{(x-ct)^2}{8(t+1)}} \right). \tag{3.2}$$

This estimate is again based only on the properties (i) and (ii) stated above. From this point onward, our analysis relied only on the estimate (3.2) and on the decomposition of the Green's function into a phase plateau and terms that can be bounded by moving heat kernels. In particular, the decomposition of the Green's function together with (3.2) allowed us to use weights in the form of moving heat kernels to verify that $v(x,t)$ decays in time like a heat kernel, while $p(t)$ approaches an appropriate finite limit exponentially in time.

Before arguing why we believe that our approach is more broadly applicable, we revisit the properties (i) and (ii) that we relied on in our study of (3.1). Our analysis showed that we can use the weight $\theta_2(t)$ for $v_x(x,t)$. Thus, inspecting (3.2), we see that it suffices that our ansatz for the phase plateau satisfies the underlying PDE up to terms that can be bounded by $(1+t)^{-1}$ times a sum of moving heat kernels. This also shows that adding terms of the form ϕ_x^m with $m \geq 3$ to (3.1) does not affect our arguments, so that we need to account only for the leading-order terms.

We now return to the problem that motivated this paper, namely the nonlinear stability of sources in reaction-diffusion systems

$$u_t = Du_{xx} + f(u), \qquad x \in \mathbb{R}, \qquad u \in \mathbb{R}^n. \tag{3.3}$$

The Green's function associated with the linearization of (3.3) about a time-periodic source $u^*(x,t)$ can be assessed analytically through the methods developed in [1]. In particular, we believe that the leading-order terms in the Green's function will be similar to (1.6). In contrast to (3.1), the linearization of (3.3) about a source has two eigenvalues at the origin, which arise due to invariance under space and time shifts. To account for these independent symmetries, we seek solutions $u(x,t)$ of (3.3) in the form

$$u(x + \phi_1(x,t), t + \phi_2(x,t)) = u^*(x,t) + w(x,t) \tag{3.4}$$

and expect that the evolution of the space and time shifts $\phi_j(x,t)$ is now described by two Burgers-type equations; we remark that there are technical reasons for writing the ansatz in the form (3.4) instead of using (1.3) and refer to [2, §5] for an explanation in the context of wave trains. We do not know whether the resulting Burgers-type equations are exactly of the form (3.1). However, the only difference we expect to encounter is that the coefficients in front of the advection and nonlinear terms may depend on (x,t); since the source converges exponentially in x to the asymptotic wave trains, these coefficients will approach time-independent limits exponentially in x. We therefore feel that it should be possible to derive approximate phase-plateau solutions also in this case up to terms that can be bounded by $(1+t)^{-1}$ times a sum of moving heat kernels. Thus, while there may be numerous technical difficulties that we did not anticipate, we believe that the approach presented here is well suited to address the stability of sources for (3.3).

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References

[1] M. Beck, B. Sandstede and K. Zumbrun. Nonlinear stability of time-periodic viscous shocks. *Arch. Ration. Mech. Anal.* **196** (2010) 1011–1076.

[2] A. Doelman, B. Sandstede, A. Scheel and G. Schneider. The dynamics of modulated wave trains. *Mem. Amer. Math. Soc.* **199/934** (2009).

[3] M. van Hecke. Building blocks of spatiotemporal intermittency. *Phys. Rev. Lett.* **80** (1998) 1896–1899.

[4] L. N. Howard and N. Kopell. Slowly varying waves and shock structures in reaction-diffusion equations. *Studies Appl. Math.* **56** (1976/77) 95–145.

[5] P. Howard and K. Zumbrun. Stability of undercompressive shock profiles. *J. Differ. Eqns.* **225** (2006) 308–360.

[6] M. Johnson, P. Noble, M. Rodrigués and K. Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: nonlinear stability. Preprint (2011).

[7] M. Johnson, P. Noble, M. Rodrigués and K. Zumbrun. Nonlocalized modulation of periodic reaction diffusion waves: the Whitham equation. Preprint (2011).

[8] W. van Saarloos and P. Hohenberg. Fronts, pulses, sources and sinks in generalized complex Ginzburg–Landau equations. *Physica D* **56** (1992) 303–367.

[9] B. Sandstede and A. Scheel. Defects in oscillatory media: toward a classification. *SIAM J. Appl. Dynam. Syst.* **3** (2004) 1–68.

[10] B. Sandstede, A. Scheel, G. Schneider and H. Uecker. Diffusive mixing of wave trains in reaction-diffusion systems with different phases at infinity. Preprint (2011).