LONG-TIME STABILITY OF NONCHARACTERISTIC VISCOUS BOUNDARY LAYERS

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ABSTRACT. We report our results on long-time stability of multi–dimensional noncharacteristic boundary layers of a class of hyperbolic–parabolic systems including the compressible Navier–Stokes equations with inflow [outflow] boundary conditions, under the assumption of strong spectral, or uniform Evans, stability. Evans stability has been verified for small-amplitude layers by Guès, Métivier, Williams, and Zumbrun. For large–amplitudes, it may be checked numerically, as done in one–dimensional case for isentropic gas by Costanzino, Humpherys, Nguyen, and Zumbrun.

1. INTRODUCTION

We consider a boundary layer, or stationary solution,

$$\tilde U = \bar U(x_1), \quad \lim_{x_1\to+\infty} \bar U(x_1) = U_+, \quad \bar U(0) = \bar U_0 \tag{1.1}$$

of a system of conservation laws on the quarter-space

$$\tilde U_t + \sum_j F^j(\tilde U)_{x_j} = \sum_{jk} (B^{jk}(\tilde U)\tilde U_{x_k})_{x_j}, \quad x \in \mathbb{R}^d_+ = \{x_1 > 0\}, \quad t > 0, \tag{1.2}$$

$\tilde U, F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n\times n}$, with initial data $\tilde U(x,0) = \tilde U_0(x)$ and Dirichlet type boundary conditions specified in (1.5), (1.6) below.

Our studies to boundary layers are restricted to the case that the layers are assumed to be *noncharacteristic*, that is, the matrix dF^1_{11} in the hyperbolic equations of $\tilde u$ is either strictly positive (inflow case) or strictly negative (outflow case). Roughly speaking, the noncharacteristicity limits the signals to be transmitted into or out of but not along the boundary. In the context of gas dynamics or MHD, this corresponds to the situation of a porous boundary with prescribed inflow or outflow conditions accomplished by suction or blowing, a scenario that has been suggested as a means to reduce drag along an airfoil by stabilizing laminar flow; see Example 1.1 below.

A fundamental question is whether or not such boundary layer solutions are *stable* in the sense of PDE, i.e., whether or not a sufficiently small (initial and boundary) perturbation of $\bar U$ remains close to $\bar U$, or converges time-asymptotically to $\bar U$, under the evolution of (1.2). Purpose of this note is to report our results in [NZ2], addressing this time–asymptotic stability question.

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1.1. **Equations and assumptions.** We consider the general hyperbolic-parabolic system of conservation laws (1.2) in conserved variable $\tilde U$, with

$$\tilde U = \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},$$

$\tilde u \in \mathbb{R}^{n-r}$, and $\tilde v \in \mathbb{R}^r$, where

$$\Re\sigma \sum_{jk} b_2^{jk}\xi_j\xi_k \geq \theta|\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^n\backslash\{0\}.$$

Following [MaZ4, Z3, Z4], we assume that equations (1.2) can be written, alternatively, after a triangular change of coordinates

$$\tilde W := \tilde W(\tilde U) = \begin{pmatrix} \tilde w^I(\tilde u) \\ \tilde w^{II}(\tilde u, \tilde v) \end{pmatrix}, \tag{1.3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde A^0 \tilde W_t + \sum_j \tilde A^j \tilde W_{x_j} = \sum_{jk} (\tilde B^{jk}\tilde W_{x_k})_{x_j} + \tilde G, \tag{1.4}$$

where

$$\tilde A^0 = \begin{pmatrix} \tilde A^0_{11} & 0 \\ 0 & \tilde A^0_{22} \end{pmatrix}, \qquad \tilde A = \begin{pmatrix} \tilde A^{11} & \tilde A^{12} \\ \tilde A^{21} & \tilde A^{22} \end{pmatrix}, \qquad \tilde B = \begin{pmatrix} 0 & 0 \\ 0 & \tilde b \end{pmatrix}, \qquad \tilde G = \begin{pmatrix} 0 \\ \tilde g \end{pmatrix}$$

and, defining $\tilde W_+ := \tilde W(U_+)$,

(A1) $\tilde A^j(\tilde W_+), \tilde A^0, \tilde A^1_{11}$ are symmetric, $\tilde A^0$ block diagonal, $\tilde A^0 \geq \theta_0 > 0$,

(A2) for each $\xi \in \mathbb{R}^d \setminus \{0\}$, no eigenvector of $\sum_j \xi_j \tilde A^j (\tilde A^0)^{-1}(\tilde W_+)$ lies in the kernel of $\sum_{jk} \xi_j\xi_k \tilde B^{jk}(\tilde A^0)^{-1}(\tilde W_+)$,

(A3) $\sum \tilde b^{jk}\xi_j\xi_k \geq \theta|\xi|^2$, and $\tilde g(\tilde W_x, \tilde W_x) = \mathcal{O}(|\tilde W_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F^j, B^{jk}, \tilde A^0, \tilde A^j, \tilde B^{jk}, \tilde W(\cdot), \tilde g(\cdot,\cdot) \in C^{s+1}$, with $s \geq [(d-1)/2] + 4$ in our analysis of linearized stability, and $s \geq s(d) := [(d-1)/2] + 7$ in our analysis of nonlinear stability.

(H1) $\tilde A^{11}_1$ is either strictly positive or strictly negative, that is, either $\tilde A^{11}_1 \geq \theta_1 > 0$, or $\tilde A^{11}_1 \leq -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF^1(U_+)$ are distinct and nonzero.

(H3) The eigenvalues of $\sum_j dF^j_+ \xi_j$ have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

(H4) The set of branch points of the eigenvalues of $(\tilde A^1)^{-1}(i\tau \tilde A^0 + \sum_{j\neq 1} i\xi_j \tilde A^j)_+$, $\tau \in \mathbb{R}$, $\tilde\xi \in \mathbb{R}^{d-1}$ is the (possibly intersecting) union of finitely many smooth curves $\tau = \eta_q^+(\tilde\xi)$, on which the branching eigenvalue has constant multiplicity s_q (by definition ≥ 2).

Condition (H1) corresponds to hyperbolic–parabolic noncharacteristicity, while (H2) is the condition for the hyperbolicity at U_+ of the associated first-order hyperbolic system obtained by dropping second-order terms. The assumptions (A1)-(A3) and (H0)-(H2) are satisfied for gas dynamics and MHD with van der Waals equation of state under inflow or outflow conditions; see discussions in [MaZ4, CHNZ, GMWZ5, GMWZ6]. Condition (H3) holds always for gas dynamics, but fails always for MHD in dimension $d \geq 2$. Condition (H4) is a technical requirement of the analysis introduced in [Z2]. It is satisfied always in dimension $d = 2$ or for rotationally invariant systems in dimensions $d \geq 2$, for which it serves only to define notation; in particular, it holds always for gas dynamics.

We also assume:

(B) Dirichlet boundary conditions in $\tilde{W}$-coordinates:

$$(\tilde{w}^I, \tilde{w}^{II})(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) := (\tilde{h}_1, \tilde{h}_2)(\tilde{x}, t) \tag{1.5}$$

for the inflow case, and

$$\tilde{w}^{II}(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) \tag{1.6}$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

This is sufficient for the main physical applications; the situation of more general, Neumann and mixed-type boundary conditions on the parabolic variable v can be treated as discussed in [GMWZ5, GMWZ6].

Example 1.1. The main example we have in mind consists of *laminar solutions* $(\rho, u, e)(x_1, t)$ of the compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \text{div}(\rho u) = 0 \\ \partial_t(\rho u) + \text{div}(\rho u^t u) + \nabla p = \varepsilon \mu \Delta u + \varepsilon(\mu + \eta)\nabla \text{div} u \\ \partial_t(\rho E) + \text{div}\big((\rho E + p)u\big) = \varepsilon \kappa \Delta T + \varepsilon \mu \text{div}\big((u \cdot \nabla)u\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon(\mu + \eta)\nabla(u \cdot \text{div} u), \end{cases} \tag{1.7}$$

$x \in \mathbb{R}^d$, on a half-space $x_1 > 0$, where ρ denotes density, $u \in \mathbb{R}^d$ velocity, e specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho, e)$ pressure, $T = T(\rho, e)$ temperature, $\mu > 0$ and $|\eta| \leq \mu$ first and second coefficients of viscosity, $\kappa > 0$ the coefficient of heat conduction, and $\varepsilon > 0$ (typically small) the reciprocal of the Reynolds number, with no-slip *suction-type* boundary conditions on the velocity,

$$u_j(0, x_2, \dots, x_d) = 0,\ j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) < 0,$$

and prescribed temperature, $T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x})$. Under the standard assumptions p_ρ, $T_e > 0$, this can be seen to satisfy all of the hypotheses (A1)–(A3), (H0)–(H4), (B) in the *outflow case* (1.6); indeed these are satisfied also under much weaker van der Waals gas assumptions [MaZ4, Z3, CHNZ, GMWZ5, GMWZ6]. In particular, boundary-layer solutions are of noncharacteristic type, scaling as $(\rho, u, e) = (\bar{\rho}, \bar{u}, \bar{e})(x_1/\varepsilon)$, with layer thickness $\sim \varepsilon$ as compared to the $\sim \sqrt{\varepsilon}$ thickness of the characteristic type found for an impermeable boundary.

This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow; see [S, Bra]. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds [Bra].[1] Possible mechanisms for this reduction are smaller thickness $\sim \varepsilon << \sqrt{\varepsilon}$ of noncharacteristic boundary layers as compared to characteristic type, and greater stability, delaying the transition from laminar to turbulent flow. In particular, stability properties appear to be quite important for the understanding of this phenomenon. For further discussion, including the related issues of matched asymptotic expansion, multi-dimensional effects, and more general boundary configurations, see [GMWZ5].

Example 1.2. Alternatively, we may consider the compressible Navier–Stokes equations (1.7) with *blowing-type* boundary conditions

$$u_j(0, x_2, \dots, x_d) = 0,\ j \neq 1 \quad \text{and} \quad u_1(0, x_2, \dots, x_d) = V(x) > 0,$$

and prescribed temperature and pressure

$$T(0, x_2, \dots, x_d) = T_{wall}(\tilde{x}), \quad p(0, x_2, \dots, x_d) = p_{wall}(\tilde{x})$$

(equivalently, prescribed temperature and density). Under the standard assumptions p_ρ, $T_e > 0$ on the equation of state (alternatively, van der Waals gas assumptions), this can be seen to satisfy hypotheses (A1)–(A3), (H0)–(H4), (B) in the *inflow case* (1.5).

1.2. **The Evans condition and strong spectral stability.** The linearized equations of (1.2), (B) about $\bar{U}$ are

$$U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j} \tag{1.8}$$

with initial data $U(0) = U_0$ and boundary conditions in (linearized) $\tilde{W}$-coordinates of

$$W(0, \tilde{x}, t) := (w^I, w^{II})^T(0, \tilde{x}, t) = h$$

for the inflow case, and

$$w^{II}(0, \tilde{x}, t) = h$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$, where $W := (\partial \tilde{W}/\partial U)(\bar{U})U$. Here, $B^{jk} := B^{jk}(\bar{U}(x_1))$ and $A^j U := dF^j(\bar{U}(x_1))U - [dB^{j1}(\bar{U}(x_1))U]\bar{U}_{x_1}(x_1)$.

A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re\lambda > 0$ of the linearized operator L about the wave. This is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re\lambda > 0$ of the *Evans function*

$$D_L(\tilde{\xi}, \lambda)$$

a Wronskian associated with the Fourier-transformed eigenvalue ODE.

Definition 1.3. We define *strong spectral stability* as *uniform Evans stability*:

$$|D_L(\tilde{\xi}, \lambda)| \geq \theta(C) > 0 \tag{D}$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$.

[1] See also NASA site http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html

For the class of equations we consider, this is equivalent to the uniform Evans condition of [GMWZ5, GMWZ6], which includes an additional high-frequency condition that for these equations is always satisfied (see Proposition 3.8, [GMWZ5]). A fundamental result proved in [GMWZ5] is that small-amplitude noncharacteristic boundary-layers are always strongly spectrally stable.[2]

Proposition 1.4 ([GMWZ5])**.** *Assuming (A1)-(A3), (H0)-(H3), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small satisfy the strong spectral stability condition (D).

As demonstrated in [SZ], stability of large-amplitude boundary layers may fail for the class of equations considered here, even in a single space dimension, so there is no such general theorem in the large-amplitude case. Stability of large-amplitude boundary-layers may be checked efficiently by numerical Evans computations as in [BDG, Br1, Br2, BrZ, HuZ, BHRZ, HLZ, CHNZ, HLyZ1, HLyZ2].

1.3. **Main results.** Our main results are as follows.

Theorem 1.5 (Linearized stability)**.** *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^{[(d-1)/2]+5} \to L^p$ stability of* (1.8) *in dimension $d \geq 2$, and any $2 \leq p \leq \infty$, with rate of decay*

$$\begin{aligned} &|U(t)|_{L^2} \leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^3},\\ &|U(t)|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}|U_0|_{L^1\cap H^{[(d-1)/2]+5}}, \end{aligned} \tag{1.9}$$

provided that the initial perturbations U_0 are in $L^1 \cap H^3$ for $p = 2$, or in $L^1 \cap H^{[(d-1)/2]+5}$ for $p > 2$, and zero boundary perturbations $h = 0$.

Theorem 1.6 (Nonlinear stability)**.** *Assuming (A1)-(A3), (H0)-(H4), (B), and strong spectral stability (D), we obtain asymptotic $L^1 \cap H^s \to L^p \cap H^s$ stability of $\bar{U}$ as a solution of* (1.2) *in dimension $d \geq 2$, for $s \geq s(d)$ as defined in (H0), and any $2 \leq p \leq \infty$, with rate of decay*

$$\begin{aligned} &|\tilde{U}(t) - \bar{U}|_{L^p} \leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}|U_0|_{L^1\cap H^s}\\ &|\tilde{U}(t) - \bar{U}|_{H^s} \leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^s}, \end{aligned} \tag{1.10}$$

provided that the initial perturbations $U_0 := \tilde{U}_0 - \bar{U}$ are sufficiently small in $L^1 \cap H^s$ and zero boundary perturbations $h = 0$.

Remark 1.7. Nonzero boundary perturbations are also treated in [NZ2]. However, for simplicity, we only report here the case of zero boundary perturbations.

Combining Theorem 1.6 and Proposition 1.4, we obtain the following small-amplitude stability result, applying in particular to the motivating situation of Example 1.1.

[2]The result of [GMWZ5] applies also to more general types of boundary conditions and in some situations to systems with variable multiplicity characteristics, including, in some parameter ranges, MHD.

Corollary 1.8. *Assuming (A1)-(A3), (H0)-(H4), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 1.5 and 1.6.

Remark 1.9. The obtained rate of decay in L^2 may be recognized as that of a $(d-1)$-dimensional heat kernel, and the obtained rate of decay in L^∞ as that of a d-dimensional heat kernel. We believe that the sharp rate of decay in L^2 is rather that of a d-dimensional heat kernel and the sharp rate of decay in L^∞ dependent on the characteristic structure of the associated inviscid equations, as in the constant-coefficient case [HoZ1, HoZ2].

Remark 1.10. In one dimension, strong spectral stability is necessary for linearized asymptotic stability; see Theorem 1.6, [NZ1]. However, in multi-dimensions, it appears likely that, as in the shock case [Z3], there are intermediate possibilities between strong and weak spectral stability for which linearized stability might hold with degraded rates of decay. In any case, the gap between the necessary weak spectral and the sufficient strong spectral stability conditions concerns only pure imaginary spectra $\Re\lambda = 0$ on the boundary between strictly stable and unstable half-planes, so this should not interfere with investigation of physical stability regions.

1.4. **Discussion and open problems.** Asymptotic stability, without rates of decay, has been shown for small amplitude noncharacteristic "normal" boundary layers of the isentropic compressible Navier–Stokes equations with outflow boundary conditions and vanishing transverse velocity in [KK], using energy estimates. Corollary 1.8 recovers this existing result and extends it to the general arbitrary transverse velocity, outflow or inflow, and isentropic or nonisentropic (full compressible Navier–Stokes) case, in addition giving asymptotic rates of decay. Also, the type of boundary layer relevant to the drag-reduction strategy discussed in Examples 1.1–1.2 is a noncharacteristic "transverse" type with constant normal velocity, complementary to the normal type considered in [KK].

The large-amplitude asymptotic stability result of Theorem 1.6 extends to multi dimensions corresponding one-dimensional results of [YZ, NZ1], reducing the problem of stability to verification of a numerically checkable Evans condition. See also the related, but technically rather different, work on the small viscosity limit in [MZ, GMWZ5, GMWZ6]. By a combination of numerical Evans function computations and asymptotic ODE estimates, spectral stability has been checked for *arbitrary amplitude* noncharacteristic boundary layers of the one-dimensional isentropic compressible Navier–Stokes equations in [CHNZ]. Extensions to the nonisentropic and multi-dimensional case should be possible by the methods used in [HLyZ1] and [HLyZ2] respectively to treat the related shock stability problem.

This (investigation of large-amplitude spectral stability) would be a very interesting direction for further investigation. In particular, note that it is large-amplitude stability that is relevant to drag-reduction at flight speeds, since the transverse relative velocity (i.e., velocity parallel to the airfoil) is zero at the wing surface and flight speed outside a thin boundary layer, so that variation across the boundary layer is substantial.

Our method of analysis follows the basic approach introduced in [Z2, Z3, Z4] for the study of multi-dimensional shock stability and we are able to make use of much of that analysis without modification. However, there are some new difficulties to be overcome in the boundary-layer case.

The main new difficulty is that the boundary-layer case is analogous to the *undercompressive shock* case rather than the more favorable *Lax shock* case emphasized in [Z3], in that $G_{y_1} \not\sim t^{-1/2}G$ as in the Lax shock case but rather $G_{y_1} \sim (e^{-\theta|y_1|} + t^{-1/2})G$, $\theta > 0$, as in the undercompressive case. This is a significant difficulty; indeed, for this reason, the undercompressive shock analysis was carried out in [Z3] only in nonphysical dimensions $d \geq 4$. On the other hand, there is no translational invariance in the boundary layer problem, so no zero-eigenvalue and no pole of the resolvent kernel at the origin for the one-dimensional operator, and in this sense G is somewhat better in the boundary layer than in the shock case.

Thus, the difficulty of the present problem is roughly intermediate to that of the Lax and undercompressive shock cases. Though the undercompressive shock case is still open in multi-dimensions for $d \leq 3$, the slight advantage afforded by lack of pole terms allows us to close the argument in the boundary-layer case. Specifically, thanks to the absence of pole terms, we are able to get a slightly improved rate of decay in $L^\infty(x_1)$ norms, though our $L^2(x_1)$ estimates remain the same as in the shock case. By keeping track of these improved sup norm bounds throughout the proof, we are able to close the argument without using detailed pointwise bounds as in the one-dimensional analyses of [HZ, RZ].

Other difficulties include the appearance of boundary terms in integrations by parts, which makes the auxiliary energy estimates by which we control high-frequency effects considerably more difficult in the boundary-layer than in the shock-layer case, and the treatment of boundary perturbations. In terms of the homogeneous Green function G, boundary perturbations lead by a standard duality argument to contributions consisting of integrals on the boundary of perturbations against various derivatives of G, and these are a bit too singular as time goes to zero to be absolutely integrable. Following the strategy introduced in [YZ, NZ1], we instead use duality to convert these to less singular integrals over the whole space, that *are* absolutely integrable in time. However, we make a key improvement here over the treatment in [YZ, NZ1], integrating against an exponentially decaying test function to obtain terms of exactly the same form already treated for the homogeneous problem. This is necessary for us in the multi-dimensional case, for which we have insufficient information about individual parts of the solution operator to estimate them separately as in [YZ, NZ1], but makes things much more transparent also in the one-dimensional case.

Among physical systems, our hypotheses appear to apply to and essentially only to the case of compressible Navier–Stokes equations with inflow or outflow boundary conditions. However, the method of analysis should apply, with suitable modifications, to more general situations such as MHD; see an extension to MHD case via a different approach [N2], and see also the recent results on the related small-viscosity problem in [GMWZ5, GMWZ6].

Finally, as pointed out in Remark 1.10, the strong spectral stability condition does not appear to be necessary for asymptotic stability. It would be interesting to develop a refined stability condition similarly as was done in [SZ, Z2, Z3, Z4] for the shock case.

2. Linearized estimates

We first establish estimates on the linearized inhomogeneous problem

$$U_t - LU = f \tag{2.1}$$

with initial data $U(0) = U_0$ and Dirichlet boundary conditions as usual in $\tilde{W}$-coordinates:

$$W(0,\tilde{x},t) := (w^I, w^{II})^T(0,\tilde{x},t) = h \tag{2.2}$$

for the inflow case, and

$$w^{II}(0,\tilde{x},t) = h \tag{2.3}$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

Let us define low- and high-frequency parts of the linearized solution operator $\mathcal{S}(t)$ of the linearized problem with homogeneous boundary and forcing data, f, $h \equiv 0$, as

$$\mathcal{S}_1(t) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}|\le r} \oint_{\Gamma^{\tilde{\xi}} \cap \{|\lambda|\le r\}} e^{\lambda t + i\tilde{\xi}\cdot\tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi} \tag{2.4}$$

and $\mathcal{S}_2(t) := e^{Lt} - \mathcal{S}_1(t)$.

Then we obtain the following linearized estimates.

Proposition 2.1 (Low-frequency estimate)**.** *Under the hypotheses of Theorem 1.6, for* $\beta = (\beta_1, \beta')$ *with* $\beta_1 = 0, 1$,

$$\begin{aligned}
|\mathcal{S}_1(t)\partial_x^\beta f|_{L^2_x} &\le C(1+t)^{-(d-1)/4-|\beta|/2}|f|_{L^1_x} + C\beta_1(1+t)^{-(d-1)/4}|f|_{L^{1,\infty}_{\tilde{x},x_1}},\\
|\mathcal{S}_1(t)\partial_x^\beta f|_{L^{2,\infty}_{\tilde{x},x_1}} &\le C(1+t)^{-(d+1)/4-|\beta|/2}|f|_{L^1_x} + C\beta_1(1+t)^{-(d+1)/4}|f|_{L^{1,\infty}_{\tilde{x},x_1}},\\
|\mathcal{S}_1(t)\partial_x^\beta f|_{L^{\infty}_{\tilde{x},x_1}} &\le C(1+t)^{-d/2-|\beta|/2}|f|_{L^1_x} + C\beta_1(1+t)^{-d/2}|f|_{L^{1,\infty}_{\tilde{x},x_1}},
\end{aligned} \tag{2.5}$$

where $|\cdot|_{L^{p,q}_{\tilde{x},x_1}}$ *denotes the norm in* $L^p(\tilde{x}; L^q(x_1))$*.*

According to [Z4, Corollary 4.11], we can write

$$\begin{aligned}
\mathcal{S}_2(t)f = \frac{1}{(2\pi i)^d}\text{P.V.}\int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{\xi}|^2 + |\Im m\lambda|^2 \ge \theta_1 + \theta_2}\\
\times e^{i\tilde{\xi}\cdot\tilde{x} + \lambda t}(\lambda - L_{\tilde{\xi}})^{-1}\hat{f}(x_1, \tilde{\xi}) d\tilde{\xi} d\lambda,
\end{aligned} \tag{2.6}$$

and we obtain the following.

Proposition 2.2 (High-frequency estimate)**.** *Given (A1)-(A2), (H0)-(H2), (D), and homogeneous boundary conditions (B), for* $0 \le |\alpha| \le s-3$, s *as in (H0),*

$$\begin{aligned}
|\mathcal{S}_2(t)f|_{L^2_x} &\le Ce^{-\theta_1 t}|f|_{H^3_x},\\
|\partial_x^\alpha \mathcal{S}_2(t)f|_{L^2_x} &\le Ce^{-\theta_1 t}|f|_{H^{|\alpha|+3}_x}.
\end{aligned} \tag{2.7}$$

2.1. **Resolvent bounds.** Our first step of proving the linearized estimates is to estimate solutions of the resolvent equation with homogeneous boundary data $\hat{h} \equiv 0$.

Proposition 2.3 (High-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and homogeneous boundary conditions (B), for some R, C sufficiently large and $\theta > 0$ sufficiently small,*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\hat{f}|_{\hat{H}^1(x_1)} \leq C|\hat{f}|_{\hat{H}^1(x_1)}, \tag{2.8}$$

and

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\hat{f}|_{L^2(x_1)} \leq \frac{C}{|\lambda|^{1/2}}|\hat{f}|_{\hat{H}^1(x_1)}, \tag{2.9}$$

for all $|(\tilde{\xi}, \lambda)| \geq R$ and $\Re e\lambda \geq -\theta$, where $\hat{f}$ is the Fourier transform of f in variable $\tilde{x}$ and $|\hat{f}|_{\hat{H}^1(x_1)} := |(1 + |\partial_{x_1}| + |\tilde{\xi}|)\hat{f}|_{L^2(x_1)}$.

Proof. The proposition follows easily by applying a Laplace-Fourier transformed version with respect to variables $(\lambda, \tilde{x})$ of the nonlinear energy estimate in Section 3.1 with $s = 1$, carried out on the linearized equations written in W-coordinates. See [NZ2] for all the details. □

We next have the following:

Proposition 2.4 (Mid-frequency bounds). *Given (A1)-(A2), (H0)-(H2), and strong spectral stability (D),*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}|_{\hat{H}^1(x_1)} \leq C, \quad \text{for } R^{-1} \leq |(\tilde{\xi}, \lambda)| \leq R \text{ and } \Re e\lambda \geq -\theta, \tag{2.10}$$

for any R and $C = C(R)$ sufficiently large and $\theta = \theta(R) > 0$ sufficiently small, where $|\hat{f}|_{\hat{H}^1(x_1)}$ is defined as in Proposition 2.3.

Proof. Immediate, by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - L_{\tilde{\xi}})^{-1}$ is analytic with respect to H^1 in $(\tilde{\xi}, \lambda)$; see Proposition 4.8, [Z4]. □

We next obtain the following resolvent bound for low-frequency regions as a direct consequence of pointwise bounds on the resolvent kernel.

Proposition 2.5 (Low-frequency bounds). *Under the hypotheses of Theorem 1.6, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\partial_{x_1}^{\beta}\hat{f}|_{L^p(x_1)} \leq C\gamma_2\rho^{-2/p}\Big[\rho^{\beta}|\hat{f}|_{L^1(x_1)} + \beta|\hat{f}|_{L^\infty(x_1)}\Big] \tag{2.11}$$

for all $2 \leq p \leq \infty$, $\beta = 0, 1$, where

$$\gamma_2 := 1 + \sum_j \Big[\rho^{-1}|\Im m\lambda - \eta_j^+(\tilde{\xi})| + \rho\Big]^{1/s_j - 1}, \tag{2.12}$$

and $s_j, \eta_j^+(\tilde{\xi})$ are as defined in (H4).

Proof. Applying the following pointwise bounds on the resolvent kernel deliberately constructed in [Z3] and recalled in [NZ2], Proposition 2.5,

$$|\partial_{y_1}^{\beta} G_{\tilde\xi,\lambda}(x_1,y_1)| \le C\gamma_2(\rho^\beta + \beta e^{-\theta y_1})e^{-\theta\rho^2|x_1-y_1|},$$

and using the convolution inequality $|g*h|_{L^p} \le |g|_{L^p}|h|_{L^1}$, we obtain

$$\begin{aligned}
&|(L_{\tilde\xi}-\lambda)^{-1}\partial_{x_1}^{\beta}\hat f|_{L^p(x_1)}\\
&\quad= \Big|\int \partial_{y_1}^{\beta} G_{\tilde\xi,\lambda}(x_1,y_1)\hat f(y_1,\tilde\xi)\,dy_1\Big|_{L^p(x_1)} + \beta|G_{\tilde\xi,\lambda}(x_1,0)\hat f(0,\tilde\xi)|_{L^p(x_1)}\\
&\quad\le \Big|\int C\gamma_2(\rho^\beta+\beta e^{-\theta y_1})e^{-\theta\rho^2|x_1-y_1|}|\hat f(y_1,\tilde\xi)|\,dy_1\Big|_{L^p} + C\gamma_2\beta|\hat f(0,\tilde\xi)||e^{-\theta\rho^2 x_1}|_{L^p(x_1)}\\
&\quad\le C\gamma_2\rho^{-2/p}\Big[\rho^\beta|\hat f|_{L^1(x_1)} + \beta|\hat f|_{L^\infty(x_1)}\Big]
\end{aligned}$$

as claimed. □

Remark 2.6. The above L^p bounds may alternatively be obtained directly by the argument of Section 12, [GMWZ1], using quite different Kreiss symmetrizer techniques, again omitting pole terms arising from vanishing of the Evans function at the origin, and also the auxiliary problem construction of Section 12.6 used to obtain sharpened bounds in the Lax or overcompressive shock case (not relevant here). See also [N2] in this direction with treatment of the boundary layer case.

2.2. **Estimates on homogeneous solution operators.** We sketch the proof of Propositions 2.1 and 2.2.

Proof of Proposition 2.1. The proof will follow closely the treatment of the shock case in [Z3]. Let $\hat u(x_1,\tilde\xi,\lambda)$ denote the solution of $(L_{\tilde\xi}-\lambda)\hat u = \hat f$, where $\hat f(x_1,\tilde\xi)$ denotes Fourier transform of f, and

$$u(x,t) := \mathcal{S}_1(t)f = \frac{1}{(2\pi i)^d}\int_{|\tilde\xi|\le r}\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t + i\tilde\xi\cdot\tilde x}(L_{\tilde\xi}-\lambda)^{-1}\hat f(x_1,\tilde\xi)d\lambda d\tilde\xi.$$

Recalling the resolvent estimates in Proposition 2.5, we have

$$|\hat u(x_1,\tilde\xi,\lambda)|_{L^p(x_1)} \le C\gamma_2\rho^{-2/p}|\hat f|_{L^1(x_1)} \le C\gamma_2\rho^{-2/p}|f|_{L^1(x)}$$

where γ_2 is as defined in (2.12).

Therefore, using Parseval's identity, Fubini's theorem, and the triangle inequality, we may estimate

$$\begin{aligned}
|u|^2_{L^2(x_1,\tilde x)}(t) &= \frac{1}{(2\pi)^{2d}}\int_{x_1}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t}\hat u(x_1,\tilde\xi,\lambda)d\lambda\Big|^2 d\tilde\xi dx_1\\
&= \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\lambda t}\hat u(x_1,\tilde\xi,\lambda)d\lambda\Big|^2_{L^2(x_1)} d\tilde\xi\\
&\le \frac{1}{(2\pi)^{2d}}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^2(x_1)}d\lambda\Big|^2 d\tilde\xi\\
&\le C|f|^2_{L^1(x)}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2\rho^{-1}d\lambda\Big|^2 d\tilde\xi.
\end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde\xi}$ by

$$\lambda(\tilde\xi, k) = ik - \theta_1(k^2 + |\tilde\xi|^2), \quad k \in \mathbb{R},$$

and observing that by (2.12),

$$\begin{aligned}\gamma_2\rho^{-1} &\le (|k| + |\tilde\xi|)^{-1}\Big[1 + \sum_j \Big(\frac{|k-\tau_j(\tilde\xi)|}{\rho}\Big)^{1/s_j - 1}\Big] \\ &\le (|k| + |\tilde\xi|)^{-1}\Big[1 + \sum_j \Big(\frac{|k-\tau_j(\tilde\xi)|}{\rho}\Big)^{\epsilon - 1}\Big],\end{aligned} \tag{2.13}$$

where $\epsilon := \frac{1}{\max_j s_j}$ ($0 < \epsilon < 1$ chosen arbitrarily if there are no singularities), we estimate

$$\begin{aligned}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}\cap\{|\lambda|\le r\}} e^{\Re e\lambda t}\gamma_2\rho^{-1}d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi}\Big|\int_{\mathbb{R}} e^{-\theta_1(k^2+|\tilde\xi|^2)t}\gamma_2\rho^{-1}dk\Big|^2 d\tilde\xi \\ &\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi \\ &\quad + \sum_j \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k-\tau_j(\tilde\xi)|^{\epsilon-1}dk\Big|^2 d\tilde\xi \\ &\le \int_{\tilde\xi} e^{-2\theta_1|\tilde\xi|^2 t}|\tilde\xi|^{-2\epsilon}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi \\ &\le Ct^{-(d-1)/2}.\end{aligned}$$

In the same way as above, we also obtain similar estimates for $|u|^2_{L^{2,\infty}_{\tilde x, x_1}}$ and $|u|_{L^\infty_{\tilde x, x_1}}$. The x_1-derivative bounds follow similarly by using the resolvent bounds in Proposition 2.5 with $\beta_1 = 1$. The $\tilde x$-derivative bounds are straightforward by the fact that $\widehat{\partial_{\tilde x}^{\tilde\beta} f} = (i\tilde\xi)^{\tilde\beta}\hat f$.

Finally, each of the above integrals is bounded by $C|f|_{L^1(x)}$ as the product of $|f|_{L^1(x)}$ times the integral quantities $\gamma_2\rho^{-1}$, γ_2 over a bounded domain, hence we may replace t by $(1+t)$ in the above estimates. □

Proof of Proposition 2.2. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(L_{\tilde\xi})$ of the resolvent $(\lambda - L_{\tilde\xi})^{-1}$, for all $f \in \mathcal{D}(L_{\tilde\xi})$,

$$(\lambda - L_{\tilde\xi})^{-1} f = \lambda^{-1}(\lambda - L_{\tilde\xi})^{-1}L_{\tilde\xi} f + \lambda^{-1} f. \tag{2.14}$$

Using this identity and (2.6), we estimate

$$\begin{aligned}\mathcal{S}_2(t)f &= \frac{1}{(2\pi i)^d}\mathrm{P.V.}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\int_{\mathbb{R}^{d-1}}\chi_{|\tilde\xi|^2+|\Im m\lambda|^2\geq\theta_1+\theta_2}\\ &\qquad\times e^{i\tilde\xi\cdot\tilde x+\lambda t}\lambda^{-1}(\lambda-L_{\tilde\xi})^{-1}L_{\tilde\xi}\hat f(x_1,\tilde\xi)d\tilde\xi d\lambda\\ &\quad+\frac{1}{(2\pi i)^d}\mathrm{P.V.}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\int_{\mathbb{R}^{d-1}}\chi_{|\tilde\xi|^2+|\Im m\lambda|^2\geq\theta_1+\theta_2}\\ &\qquad\times e^{i\tilde\xi\cdot\tilde x+\lambda t}\lambda^{-1}\hat f(x_1,\tilde\xi)d\tilde\xi d\lambda\\ &=: S_1+S_2,\end{aligned}\tag{2.15}$$

where, by Plancherel's identity and Propositions 2.2 and 2.4, we have

$$\begin{aligned}|S_1|_{L^2(\tilde x,x_1)} &\leq C\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}|\lambda|^{-1}|e^{\lambda t}||(\lambda-L_{\tilde\xi})^{-1}L_{\tilde\xi}\hat f|_{L^2(\tilde\xi,x_1)}|d\lambda|\\ &\leq Ce^{-\theta_1 t}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}|\lambda|^{-3/2}\Big|(1+|\tilde\xi|)|L_{\tilde\xi}\hat f|_{H^1(x_1)}\Big|_{L^2(\tilde\xi)}|d\lambda|\\ &\leq Ce^{-\theta_1 t}|f|_{H^3_x}\end{aligned}$$

and

$$\begin{aligned}|S_2|_{L^2_x} &\leq \frac{1}{(2\pi)^d}\Big|\mathrm{P.V.}\int_{-\theta_1-i\infty}^{-\theta_1+i\infty}\lambda^{-1}e^{\lambda t}d\lambda\int_{\mathbb{R}^{d-1}}e^{i\tilde x\cdot\tilde\xi}\hat f(x_1,\tilde\xi)d\tilde\xi\Big|_{L^2}\\ &\quad+\frac{1}{(2\pi)^d}\Big|\mathrm{P.V.}\int_{-\theta_1-ir}^{-\theta_1+ir}\lambda^{-1}e^{\lambda t}d\lambda\int_{\mathbb{R}^{d-1}}e^{i\tilde x\cdot\tilde\xi}\hat f(x_1,\tilde\xi)d\tilde\xi\Big|_{L^2}\\ &\leq Ce^{-\theta_1 t}|f|_{L^2_x},\end{aligned}\tag{2.16}$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the first inequality stated in the proposition. Derivative bounds follow similarly. □

2.3. **Proof of linearized stability.**

Proof of Theorem 1.5. Applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, we obtain

$$\begin{aligned}|U(t)|_{L^2} &\leq |\mathcal{S}_1(t)U_0|_{L^2}+|\mathcal{S}_2(t)U_0|_{L^2}\\ &\leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1}+Ce^{-\eta t}|U_0|_{H^3}\\ &\leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^3}\end{aligned}\tag{2.17}$$

and

$$\begin{aligned}|U(t)|_{L^\infty} &\le |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty}\\ &\le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + C|\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}}\\ &\le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1} + Ce^{-\eta t}|U_0|_{H^{[(d-1)/2]+5}}\\ &\le C(1+t)^{-\frac{d}{2}}|U_0|_{L^1\cap H^{[(d-1)/2]+5}}.\end{aligned} \tag{2.18}$$

These prove the bounds as stated in the theorem for $p = 2$ and $p = \infty$. For $2 < p < \infty$, we use the interpolation inequality between L^2 and L^∞. □

3. Nonlinear stability

3.1. **Auxiliary energy estimates.** For the analysis of nonlinear stability, we need the following energy estimate adapted from [MaZ4, NZ1, Z4]. Define the nonlinear perturbation variables $U = (u, v)$ by

$$U(x,t) := \tilde{U}(x,t) - \bar{U}(x_1). \tag{3.1}$$

Proposition 3.1. *Under the hypotheses of Theorem 1.6, let $U_0 \in H^s$ and $U = (u,v)^T$ be a solution of* (1.2) *and* (3.1)*. Suppose that, for $0 \le t \le T$, the $W^{2,\infty}_x$ norm of the solution U remains bounded by a sufficiently small constant $\zeta > 0$. Then*

$$|U(t)|^2_{H^s} \le Ce^{-\theta t}|U_0|^2_{H^s} + C\int_0^t e^{-\theta(t-\tau)}|U(\tau)|^2_{L^2}d\tau \tag{3.2}$$

for all $0 \le t \le T$.

Proof. The proof uses the Goodmann-weighted and Kawashima-type energy estimates adapted from [MaZ4, Z4] for the shock case. See [NZ2] for details. □

3.2. **Proof of nonlinear stability.** Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$U_t - LU = \sum_j Q^j(U, U_x)_{x_j}, \tag{3.3}$$

where $Q^j(U,U_x) = \mathcal{O}(|U||U_x| + |U|^2)$ so long as $|U|$ remains bounded.

Applying the Duhamel formula (see Lemma 3.9, [NZ2]) to (3.3), we obtain

$$U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)\sum_j \partial_{x_j} Q^j(U,U_x)ds \tag{3.4}$$

where $U(x,0) = U_0(x)$.

Proof of Theorem 1.6. Define

$$\begin{aligned}\zeta(t) := \sup_s\Big(&|U(s)|_{L^2_x}(1+s)^{\frac{d-1}{4}} + |U(s)|_{L^\infty_x}(1+s)^{\frac{d}{2}}\\ &+ (|U(s)| + |U_x(s)| + |\partial^2_{\tilde{x}}U(s)|)_{L^{2,\infty}_{\tilde{x},x_1}}(1+s)^{\frac{d+1}{4}}\Big).\end{aligned} \tag{3.5}$$

We shall prove here that for all $t \geq 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \leq C(|U_0|_{L^1\cap H^s} + E_0 + \zeta(t)^2). \tag{3.6}$$

This bound together with continuity of $\zeta(t)$ implies that

$$\zeta(t) \leq 2C(|U_0|_{L^1\cap H^s} + E_0) \tag{3.7}$$

for $t \geq 0$, provided that $|U_0|_{L^1\cap H^s} + E_0 < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s, and the standard principle of continuation, there exists a solution $U \in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0, T)$ be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta > 0$. By Proposition 3.1, and the Sobolev embeding inequality $|U|_{W^{2,\infty}} \leq C|U|_{H^s}$, we have

$$\begin{aligned} |U(t)|_{H^s}^2 &\leq Ce^{-\theta t}|U_0|_{H^s}^2 + C\int_0^t e^{-\theta(t-\tau)}|U(\tau)|_{L^2}^2 d\tau \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2)(1+t)^{-(d-1)/2}. \end{aligned} \tag{3.8}$$

and so the solution continues so long as ζ remains small, with bound (3.7), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (3.6). First by (3.4), we obtain

$$\begin{aligned} |U(t)|_{L^2} \leq& |\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds + \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ \leq& I_1 + I_2 + I_3 \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} I_1 :=& |\mathcal{S}(t)U_0|_{L^2} \leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^3}, \\ I_2 :=& \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ &\leq C\int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac12}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d-1}{4}}|Q^j(s)|_{L^{1,\infty}_{\tilde x, x_1}}ds \\ &\leq C\int_0^t (1+t-s)^{-\frac{d-1}{4}-\frac12}|U|_{H^1}^2 + (1+t-s)^{-\frac{d-1}{4}}\Big(|U|^2_{L^{2,\infty}_{\tilde x, x_1}} + |U_x|^2_{L^{2,\infty}_{\tilde x, x_1}}\Big)ds \\ &\leq C(|U_0|_{H^s}^2 + \zeta(t)^2)\int_0^t \Big[(1+t-s)^{-\frac{d-1}{4}-\frac12}(1+s)^{-\frac{d-1}{2}} \\ &\qquad + (1+t-s)^{-\frac{d-1}{4}}(1+s)^{-\frac{d+1}{2}}\Big]ds \\ &\leq C(1+t)^{-\frac{d-1}{4}}(|U_0|_{H^s}^2 + \zeta(t)^2) \end{aligned}$$

and

$$\begin{aligned} I_3 := \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j} Q^j(s)|_{L^2} ds &\le \int_0^t e^{-\theta(t-s)} |\partial_{x_j} Q^j(s)|_{H^3} ds \\ &\le C\int_0^t e^{-\theta(t-s)}(|U|_{L^\infty} + |U_x|_{L^\infty})|U|_{H^5} ds \le C\int_0^t e^{-\theta(t-s)}|U|^2_{H^s} ds \\ &\le C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{d-1}{2}} ds \\ &\le C(1+t)^{-\frac{d-1}{2}}(|U_0|^2_{H^s} + \zeta(t)^2). \end{aligned}$$

Combining these above estimates yields

$$|U(t)|_{L^2}(1+t)^{\frac{d-1}{4}} \le C(|U_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{3.10}$$

Similarly, we can obtain estimates for $|U(t)|_{L^{2,\infty}_{\tilde{x},x_1}}$, $|U_x(t)|_{L^{2,\infty}_{\tilde{x},x_1}}$, $|U_{\tilde{x}\tilde{x}}|_{L^{2,\infty}}$, and $|U(t)|_{L^\infty_x}$, completing the proof of claim (3.6), and the main theorem. □

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