REMARKS ON THE ILL-POSEDNESS
OF THE PRANDTL EQUATION

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ABSTRACT. In the lines of the recent paper [4], we establish various ill-posedness results for the Prandtl equation. By considering perturbations of stationary shear flows, we show that for some linearizations of the Prandtl equation and some C^∞ initial data, local in time C^∞ solutions do not exist. At the nonlinear level, we prove that if a flow exists in the Sobolev setting, it cannot be Lipschitz continuous. Besides ill-posedness in time, we also establish some ill-posedness in space, that casts some light on the results obtained by Oleinik for monotonic data.

1. INTRODUCTION

Our concern in this paper is the famous Prandtl equation:

$$(1.1) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x P &=& f, & y > 0, \\ \partial_x u + \partial_y v &=& 0, & y > 0, \\ u = v &=& 0, & y = 0, \\ \lim_{y \to +\infty} u &=& U(t, x), \end{cases}$$

that was proposed by Ludwig Prandtl [12] in 1904 as a model for fluids with low viscosity near a solid boundary. This model is obtained formally as a singular limit of the Navier-Stokes equations, in the limit of vanishing viscosity. In this asymptotics, $y = 0$ is the boundary, x is a curvilinear coordinate along the boundary, whereas $u = u(t, x, y)$ and $v = v(t, x, y)$ are the tangential and normal components of the velocity in the so-called boundary layer. The pressure $P = P(t, x)$ and tangential velocity $U = U(t, x)$ are given: they describe the flow just outside the boundary layer, and satisfy the Bernoulli equation

$$\partial_t U + U\partial_x U + \partial_x P = 0.$$

Finally, the source term $f = f(t, x, y)$ accounts for possible additional forcings. We refer to [7] for the formal asymptotic derivation and all necessary physical background. We shall restrict here to two settings:

- *the initial value problem (IVP)*:

$$(1.2) \qquad (t, x, y) \text{ in } [0, T) \times \mathbb{T} \times \mathbb{R}_+, \ u|_{t=0} = u_0(x, y).$$

- *the boundary value problem (BVP)*:

$$(1.3) \qquad (t, x, y) \text{ in } \mathbb{T} \times [0, X) \times \mathbb{R}_+, \ u|_{x=0} = u_1(t, y).$$

Although (1.1) is the cornerstone of the boundary layer theory, the well-posedness of the equation and its rigorous derivation from Navier-Stokes are far from being established. The reason is that the boundary layer undergoes many instabilities, the impact of which on the relevance of the Prandtl model is not clear. One popular instability is the so-called boundary layer separation, which is created by an adverse pressure gradient ($\partial_x P > 0$) and a loss of monotonicity in y of the tangential velocity: see [7].

Roughly, only two frameworks have led to positive mathematical results:

- the analytic framework: under analyticity of the initial data and the Euler flow, Sammartino and Caflish [13] showed the well-posedness of the initial value problem, and successfully justified the asymptotics locally in time. See also [9]. Up to our knowledge it is the only setting in which the Prandtl model is fully justified.
- the monotonic framework: under a main assumption of monotonicity in y of the initial data, Oleinik and Samokhin [11] proved the local well-posedness of both the boundary value and initial value problems. The latter result was extended to be global in time by a work of Xin and Zhang [14] when $f = 0$ and $\partial_x P \leq 0$.

We refer to the review article [3] for precise statements and ideas of proofs. We stress that in all the aforementioned works, either analyticity or monotonicity of the initial data is assumed. When such assumptions are no longer satisfied, instabilities develop, and the Prandtl model is unlikely to be valid, at least globally in space time. For instance, it was shown by E and Engquist [2] that C^∞ solutions of (1.1) do not always exist globally in time. As regards the asymptotic derivation of Prandtl, some counterexamples due to Grenier [5] have shown that the asymptotics does not hold in the Sobolev space $W^{1,\infty}$. Finally, negative results have culminated in the recent paper [4] by the first author and Dormy, that establishes some linear ill-posedness for the initial value problem in a Sobolev setting. We shall come back to this article in due course. Broadly, the authors consider the linearized Prandtl equation around a non-monotonic shear flow, and construct $\mathcal{O}(k^{-\infty})$ approximate solutions that grow like $e^{\sqrt{k}t}$ for high frequencies k in x. *Our aim in this note is to ponder on this construction to establish further ill-posedness results for the Prandtl equation.*

Let us now present our results. We only treat the case of constant $U \geq 0$, and shall consider perturbations of some steady shear flow solutions:

$$(u, v) = (u_s(y), 0), \quad u_s(0) = 0, \quad \lim_{y \to +\infty} u_s = U.$$

Note that only non-trivial source terms f yield non-trivial solutions of this form. But up to minor changes, our results adapt to some unsteady shear flows $(u(t, y), 0)$ satisfying $\partial_t u - \partial_y^2 u = 0$. Thus, the important special case $f = 0$ can be treated as well.

The system satisfied by the perturbation (u, v) of $(u_s, 0)$ reads:

$$(1.4) \quad \begin{cases} \partial_t u + u_s \partial_x u + u_s' v + u \partial_x u + v \partial_y u - \partial_y^2 u &=& 0, \quad y > 0, \\ \partial_x u + \partial_y v &=& 0, \quad y > 0, \\ u = v &=& 0, \quad y = 0, \end{cases}$$

plus the condition $\lim_{y \to +\infty} u = 0$, that will be encoded in the functional spaces.

Our first result is related to the linearized version of this system, that is:

(1.5)
$$\begin{cases} \partial_t u + u_s \partial_x u + u_s' \, v - \partial_y^2 u & = & 0, & y > 0, \\ \partial_x u + \partial_y v & = & 0, & y > 0, \\ u = v & = & 0, & y = 0. \end{cases}$$

We state a strong ill-posedness result for the initial value problem, namely:

Theorem 1.1 (Non-existence of solutions for linearized Prandtl). *There exists a shear flow u_s with $u_s - U \in C_c^\infty(\mathbb{R}_+)$ such that : for all $T > 0$, there exists an initial data u_0 satisfying*
i) $e^y u_0 \in H^\infty(\mathbb{T} \times \mathbb{R}_+)$.
ii) The IVP (1.5)-(1.2) has no distributional solution u with

$$u \in L^\infty(]0, T[; L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(]0, T[\times \mathbb{T} \times \mathbb{R}_+).$$

We quote that a solution u of (1.5) with the above regularity satisfies

$$\partial_t u \in L^2(]0, T[; H^{-1}(\mathbb{T} \times \mathbb{R}_+)),$$

so that in turn,

$$u \in C([0, T], H^{-1}(\mathbb{T} \times \mathbb{R}_+)) \cap C_w([0, T]; L^2(\mathbb{T} \times \mathbb{R}_+)).$$

This gives a meaning to the initial condition.

Theorem 1.1 shows that the linearized Prandtl system (1.5) is ill-posed in any reasonable sense, which strengthens the result of [4]. A difficult open problem is the extension of such theorem to the nonlinear setting (1.4). For the time being, we are unable to disprove the existence of a flow for the nonlinear equation. In short, we are only able to show that if the flow exists, it is not Lipschitz continuous from H^m to H^1, for any m. More precisely, we introduce the

Definition 1.2 (Lipschitz well-posedness for Prandtl equation near a shear flow). *For any smooth u_s and $m \geq 0$, we say the IVP (1.4)–(1.2) is locally (H^m, H^1) Lipschitz well-posed if there are constants C, δ_0, T, and a subspace $\mathcal{X}$ of $L^\infty(]0, T[; H^1(\mathbb{T} \times \mathbb{R}_+))$ s.t.*
for any initial data u_0 with $e^y u_0 \in H^m(\mathbb{T} \times \mathbb{R}_+)$ and $\|e^y u_0\|_{H^m} \leq \delta_0$, there exists a unique distributional solution u of (1.4)-(1.2) in $\mathcal{X}$, and there holds

(1.6)
$$\underset{0 \leq t \leq T}{ess \sup} \|u(t)\|_{H^1_{x,y}} \leq C\|e^y u_0\|_{H^m_{x,y}}$$

Let us quote that if u belongs to $\mathcal{X}$, all terms in equation (1.5) are well-defined as distributions, including the nonlinear term

$$u \partial_x u + v \partial_y u = \partial_x(u^2) + \partial_y(v \, u) \in L^\infty(0, T; W_{loc}^{-1,p}(\mathbb{T} \times \mathbb{R}_+)), \quad \forall p < 2.$$

Again, the solution u satisfies

$$\partial_t u \in L^\infty(0, T; W_{loc}^{-1,p}(\mathbb{T} \times \mathbb{R}_+)), \quad u \in C([0, T]; W_{loc}^{-1,p}(\mathbb{T} \times \mathbb{R}_+)), \quad \forall p < 2,$$

which gives a meaning to the initial condition. We can now state

Theorem 1.3 (No Lipschitz continuity of the flow). *There exists a shear flow u_s with $u_s - U \in C_c^\infty(\mathbb{R}_+)$ such that: for all $m \geq 0$, the Cauchy problem (1.4)-(1.2) is not locally (H^m, H^1) Lipschitz well-posed.*

The proof of the linear ill-posedness result, Theorem 1.1, is based on the previous construction ([4]) of a strong unstable quasimode for the linearized Prandtl operator, together with a use of the standard closed graph theorem. For Theorem 1.3, we make a simple use of an idea of Guo and Tice ([6]) on deriving the ill-posedness of the flow from a strong linearized instability result.

We shall end this introduction with space instability results, related to the boundary value problem. Let us stress that space instability is very natural in boundary layer theory. Indeed, many works on boundary layers are related to steady problems for flows around obstacles. In this context, x can be seen as the evolution variable, $x = 0$ corresponding to the leading edge of the obstacle. Boundary layer separation, that takes place upstream from the leading edge, can also be seen as a blow up phenomenon in space. One can also refer to the paper by Métivier [10], for a mathematical study of spacial instabilities in the context of Zakharov equations.

Precisely, at the linear level, we prove the following:

Theorem 1.4 (Spacial ill-posedness for linearized Prandtl equation)**.** *Let $U > 0$. There exists a shear flow u_s with $u_s(y) > 0$ for $y > 0$, $u_s - U \in C_c^\infty(\mathbb{R}_+)$ and such that : for all $X > 0$, there exists an initial data u_1 satisfying*

i) $e^y u_1 \in H^\infty(\mathbb{T} \times \mathbb{R}_+)$.

ii) The BVP (1.4)-(1.3) has no weak solution u with

$$u_s \, u \in L_t^2(\mathbb{T}; C_x([0, X]; H_y^2(\mathbb{R}_+))), \quad u \in L_{t,x}^2(\mathbb{T} \times (0, X); H_y^2(\mathbb{R}_+)).$$

This is of course an analogue of Theorem 1.1. From there, one could also obtain an analogue of Theorem 1.3 : we skip it for the sake of brevity.

The proof of these spacial results are again based on construction of an unstable quasimode for the linearized operator. It turns out that the instability mechanism introduced in [4] to construct these unstable modes can be modified in such a way that it yields the ill-posedness. One should note that in course of deriving the non-existence result, we are obliged to prove a *uniqueness* result, and as it turns out, obtaining such a result is not as straightforward as in the case for the IVP problem. The difficulty lies in the fact that we now view the equation as an evolution in x and it is not at all obvious for one to obtain certain *energy or a priori estimates* for solutions of the BVP problem. Nevertheless, we present a proof of the uniqueness result in the last section of the paper.

The outline of the paper is as follows: section 2 details the non-existence result stated in Theorem 1.1. The nonlinear result is explained in section 3. We show in section 4 how to adapt the arguments of [4] to get spacial ill-posedness. We shall conclude the paper with some comments on this spacial instability, and how it relates to the well-known results of Oleinik.

2. The linearized IVP

This section is devoted to the proof of Theorem 1.1. We start by recalling some key elements of article [4]. It deals with the ill-posedness of the linearized Prandtl system (1.5) in the Sobolev setting. The high frequencies in x are investigated. The main point in the article is the construction of a strong unstable quasimode for the linearized Prandtl

operator $\mathcal{L}_s u := u_s \partial_x u + v \partial_y u_s - \partial_y^2 u$. It is achieved under the main assumption that u_s has a non-degenerate critical point $a > 0$: $u_s'(a) = 0$, $u_s''(a) < 0$. More precisely, if u_s takes the form

$$(2.1) \qquad u_s = u_s(a) - \frac{1}{2}(y-a)^2 \quad \text{in the vicinity of } a,$$

one can build accurate approximate solutions $(u_\varepsilon^n, v_\varepsilon^n)$ of (1.4) *that have x-frequency ε^{-1} and grow exponentially at rate $\varepsilon^{-1/2}$.* Namely,

$$u_\varepsilon^n(t,x,y) = i\, e^{i\varepsilon^{-1}x}\, e^{i\varepsilon^{-1}\omega(\varepsilon)t}\, U_\varepsilon^n(y), \quad v_\varepsilon^n(t,x,y) = \varepsilon^{-1}\, e^{i\varepsilon^{-1}x}\, e^{i\varepsilon^{-1}\omega(\varepsilon)t}\, V_\varepsilon^n(y)$$

where

$$\omega(\varepsilon) = -u_s(a) + \varepsilon^{1/2}\tau, \quad \text{for some } \tau \text{ with } \Im m\, \tau < 0,$$

and U_ε^n, V_ε^n are smooth functions of y. These functions are expansions of boundary layer type, made of $\mathcal{O}(n)$ terms, with both a regular and a singular part in ε. In particular, one has

$$\|U_\varepsilon^n\|_{L^2(\mathbb{R}_+)} \geq c_n, \quad \|e^y U_\varepsilon^n\|_{H^k(\mathbb{R}_+)} \leq C_{n,k}\,(1 + \varepsilon^{-(k-1)/4}), \quad \forall k \in \mathbb{N}$$

the loss in ε being due to the singular part. By accurate approximate solutions, we mean that $\partial_t u_\varepsilon^n + \mathcal{L}_s u_\varepsilon^n = r_\varepsilon^n$, with

$$r_\varepsilon^n(t,x,y) = e^{i\varepsilon^{-1}x}\, e^{i\varepsilon^{-1}\omega(\varepsilon)t}\, R_\varepsilon^n, \quad \|e^y R_\varepsilon^n\|_{H^k(\mathbb{R}_+)} \leq C_{n,k}\, \varepsilon^n, \quad \forall k.$$

Again, we refer to [4] for all necessary details. Actually, the construction of [4], that deals with a time dependent u_s, can be much simplified in the case of our steady flow u_s. A similar construction will be described at the end of the paper, when dealing with spacial instability.

We can now turn to the proof of the theorem. It ponders on the previous construction and on the use of the closed graph theorem. We refer to Lax [8] for similar arguments in the context of geometric optics. We argue by contradiction. Let us assume that for some $T > 0$, and for any data u_0 with $e^y u_0 \in H^\infty(\mathbb{T} \times \mathbb{R}_+)$, there is a unique solution of (1.5)

$$u \in L^\infty(]0,T[;L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(]0,T[\times \mathbb{T} \times \mathbb{R}_+).$$

We can then define

$$\mathcal{T} \ : \ e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+) \mapsto L^\infty(]0,T[;L^2(\mathbb{T} \times \mathbb{R}_+)) \times L^2(]0,T[\times \mathbb{T} \times \mathbb{R}_+), \quad u_0 \mapsto (u, \partial_y u).$$

It is a linear map between Fréchet spaces, and it is easy to check that it has closed graph. By the closed graph theorem, we deduce that $\mathcal{T}$ is bounded. This means that for some $K \in \mathbb{N}$, for all $u_0 \in e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+)$,

$$\sup_{t \in [0,T]} \|u(t)\|_{L^2_{x,y}} + \|\partial_y u\|_{L^2_{t,x,y}} \leq \mathcal{C}\, \|e^y u_0\|_{H^K_{x,y}}.$$

We quote that this bounds hold for the supremum in time and not only for the essential supremum, thanks to the weak continuity in time of u with values in L^2.

Let us define the family of linear operators

$$S(t) \ : \ e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+) \mapsto L^2(\mathbb{T} \times \mathbb{R}_+), \quad u_0 \mapsto (\mathcal{T}u_0)_1(s), \quad t \in [0,T],$$

that is $S(t)u_0 = u(t)$, where u is the unique solution of (1.5) with initial data u_0. The previous inequality shows that $S(t)$ extends into a bounded linear operator from $e^{-y}H^K$ to L^2, with a bound $\mathcal{C}$ independent of t.

We now introduce $u(t) := S(t)u_\varepsilon^n(0)$, and $v = u - u_\varepsilon^n$, where u_ε^n is the growing function defined above. It satisfies

(2.2) $$\partial_t v + \mathcal{L}_s v = -r_\varepsilon^n, \quad v|_{t=0} = 0.$$

We claim that v has the Duhamel representation

$$v(t) = -\int_0^t S(t-s)r_\varepsilon^n(s)\, ds$$

Indeed, as r_ε^n is continuous with values in $e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+)$, the integral is well-defined, and straightforward differentiation with respect to t shows that it defines another solution $\tilde{v}$ of (2.2). Thus, the difference $w = v - \tilde{v}$ satisfies $\partial_t w + \mathcal{L}_s w = 0$, $w|_{t=0} = 0$, with regularity

$$w \in L^\infty(]0,T[; L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y w \in L^2(0, T \times \mathbb{T} \times \mathbb{R}_+).$$

By our uniqueness assumption for this equation, we obtain $w = 0$, that is the Duhamel formula.

On one hand, thanks to the bound on the $S(t)$ and the remainder r_ε^n, we get that

$$\|u(t)\|_{L^2_{x,y}} \le \mathcal{C}\|e^y\, u_\varepsilon^n(0)\|_{H^K_{x,y}} \le C_K\, \varepsilon^{-K},$$

as well as

$$\|v(t)\|_{L^2_{x,y}} \le \mathcal{C} \int_0^t \|e^y r_\varepsilon^n(s)\|_{H^K_{x,y}}(s)ds \le C_{n,K}\, \varepsilon^{n+\frac{1}{2}-K}\, e^{\frac{|\Im\tau|t}{\sqrt{\varepsilon}}}, \quad \forall\, n, K \ge 0.$$

On the other hand, one has:

$$\|u_\varepsilon^n(t)\|_{L^2_{x,y}} \ge c_n\, e^{\frac{|\Im\tau|t}{\sqrt{\varepsilon}}}$$

by the properties of u_ε^n recalled below. Combining the last three inequalities, we deduce

$$C_K\, \varepsilon^{-K} \ge \|u(t)\|_{L^2_{x,y}} \ge \|u_\varepsilon^n(t)\|_{L^2_{x,y}} - \|v(t)\|_{L^2_{x,y}} \ge \left(c_n - C_{n,K}\, \varepsilon^{n+\frac{1}{2}-K}\right) e^{\frac{|\Im\tau|t}{\sqrt{\varepsilon}}}.$$

This yields a contradiction for small enough ε, if we take n and t such that

$$n + \frac{1}{2} > K, \quad t > \frac{(\ln(C_{n,K}/c_n) + K|\ln\varepsilon|)\sqrt{\varepsilon}}{|\Im\tau|}.$$

So far, we have proved that under assumption (2.1), there is for any $T > 0$ an initial data $u_0 \in e^y H^\infty(\mathbb{T} \times \mathbb{R}_+)$ for which either existence or uniqueness of a solution of (1.5) on $[0,T]$ fails. To rule out a possible lack of uniqueness, we further assume that

(2.3) $$\sup_{t\ge 0}\left(\sup_{y\ge 0}|u_s| + \int_0^\infty y|\partial_y u_s|^2 dy\right) < +\infty.$$

Theorem 1.1 is then a consequence of the following uniqueness result:

Proposition 2.1. *Let u_s be a smooth shear flow satisfying* (2.3). *Let*

$$w \in L^\infty(]0,T[; L^2_x(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y w \in L^2(0, T \times \mathbb{T} \times \mathbb{R}_+).$$

a solution of (1.5) *with $w|_{t=0} = 0$. Then, $w \equiv 0$.*

Proof. Let us define $\hat{w}_k(t,y)$, $k \in \mathbb{Z}$, the Fourier transform of $w(t,x,y)$ in x variable. We observe that for each k, $\hat{w}_k$ solves

(2.4)
$$\begin{cases} \partial_t \hat{w}_k + iku_s\hat{w}_k - ik\partial_y u_s \int_0^y \hat{w}_k(y')dy' - \partial_y^2 \hat{w}_k &= 0 \\ \hat{w}_k(t,0) &= 0 \\ \hat{w}_k(0,y) &= 0. \end{cases}$$

Taking the standard inner product of the equation (2.4) against the complex conjugate of $\hat{w}_k$ and using the standard Cauchy–Schwartz inequality to the term $\int_0^y \hat{w}_k dy'$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\hat{w}_k\|_{L^2(\mathbb{R}_+)}^2 + \|\partial_y \hat{w}_k\|_{L^2(\mathbb{R}_+)}^2 \leq |k|\int_0^\infty |u_s||\hat{w}_k|^2 dy + |k|\int_0^\infty |\partial_y u_s|y^{1/2}|\hat{w}_k|\|\hat{w}_k\|_{L^2(\mathbb{R}_+)}dy$$

$$\leq |k|\Big(\sup_{t,y}|u_s| + \int_0^\infty y|\partial_y u_s|^2 dy\Big)\|\hat{w}_k\|_{L^2(\mathbb{R}_+)}^2.$$

Applying the Gronwall lemma into the last inequality yields

$$\|\hat{w}_k(t)\|_{L^2(\mathbb{R}_+)} \leq Ce^{C|k|t}\|\hat{w}_k(0)\|_{L^2(\mathbb{R}_+)},$$

for some constant C. Thus, $\hat{w}_k(t) \equiv 0$ for each $k \in \mathbb{Z}$ since $\hat{w}_k(0) \equiv 0$. That is, $w \equiv 0$, and the theorem is proved. $\qquad\square$

3. The nonlinear IVP

With Theorem 1.1 at hand, we can turn to the nonlinear statement of Theorem 1.3. We shall make use of an idea of Guo and Tice [6] on deriving a nonlinear instability from a strong linearized one. Note that the nonlinear equation in (1.4) reads $\partial_t u + \mathcal{L}_s u = N(u)$, with $\mathcal{L}_s$ as in the previous section, and nonlinearity $N(u) := -u\partial_x u - v\partial_y u$.

We now prove the theorem by contradiction. Let u_s be as in Theorem 1.1. Assume that the IVP (1.4)-(1.2) is (H^m, H^1) locally Lipschitz well-posed for some $m \geq 0$. Let C, δ_0, T be the constants given in the definition of Lipschitz well-posedness. By Theorem 1.1, there is an initial data $u_0 \in e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+)$ that does not generate any solution u of the linearized equation (1.5) with

$$u \in L^\infty(]0,T[; L_x^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(]0,T[\times\mathbb{T} \times \mathbb{R}_+).$$

Up to multiplication by a constant, we can assume $\|e^y u_0\|_{H^m} = 1$. Let us take $v_0^\delta := \delta u_0$, with δ a small parameter less than δ_0. By the Lipschitz well-posedness hypothesis, there is a solution v^δ of (1.4) on $[0,T]$ with initial data v_0^δ. Moreover,

$$\underset{t\in[0,T]}{\text{ess sup}} \|v^\delta(t)\|_{H^1_{x,y}} \leq C\delta.$$

In other words, $u^\delta = v^\delta/\delta$ is bounded in $L^\infty(0,T;H^1)$ uniformly with respect to δ, and moreover

(3.1)
$$\partial_t u^\delta + \mathcal{L}_s u^\delta = \delta N(u^\delta), \qquad u^\delta(0,x,y) = u_0.$$

From the bound on u^δ, we deduce that, up to a subsequence,

$$u^\delta \to u \quad L^\infty(0,T;H^1(\mathbb{T}\times\mathbb{R}_+)) \text{ weak} * \text{ as } \delta \to 0.$$

Furthermore, the nonlinearity $\delta N(u^\delta)$ goes to zero strongly in $L^\infty(0,T;W^{-1,p}_{loc})$, for all $p < 2$. We end up with

$$\partial_t u + \mathcal{L}_s u = 0, \quad u|_{t=0} = u_0.$$

As

$$u \in L^\infty(]0,T[;L^2(\mathbb{T} \times \mathbb{R}_+)), \quad \partial_y u \in L^2(]0,T[\times\mathbb{T} \times \mathbb{R}_+).$$

this contradicts the result of non-existence of solutions starting from u_0.

4. SPACIAL INSTABILITY

This section is devoted to the boundary value problem for the linearized Prandtl equation. We assume $U > 0$, and consider some shear flow u_s with $u_s(y) > 0$ for $y > 0$. As before, we assume $u_s - U \in C_c^\infty(\mathbb{R}_+)$, and

$$u_s = u_s(a) + u_s''(a)\frac{(y-a)^2}{2} \quad \text{in the vicinity of some } a > 0, \quad \text{with } u_s''(a) < 0.$$

As the time t and the space x are somewhat symmetric in the Prandtl equation, we may adapt the construction of the unstable quasimode performed in [4]. We sketch this construction in the next paragraph, and then turn to the proof of Theorem 1.4.

4.1. **The unstable quasimode.** The aim of this paragraph is to construct an approximate solution of (1.5), that has high time frequency ε^{-1} and grows exponentially for positive x at rate $\sqrt{\varepsilon}^{-1}$. We look for growing solutions in the form

$$(4.1) \quad u(t,x,y) = e^{-it/\varepsilon - i\omega(\varepsilon)x/\varepsilon}u_\varepsilon(y), \qquad v(t,x,y) = \varepsilon^{-1}e^{-it/\varepsilon - i\omega(\varepsilon)x/\varepsilon}v_\varepsilon(y), \qquad \varepsilon > 0.$$

We plug the Ansatz into (1.5), and eliminate u_ε by the divergence free condition $u_\varepsilon = -\frac{iv'_\varepsilon}{\omega(\varepsilon)}$. We end up with

$$(4.2) \quad \begin{cases} (1 + \omega(\varepsilon)u_s)v'_\varepsilon - \omega(\varepsilon)u'_s v_\varepsilon - i\varepsilon v_\varepsilon^{(3)} = 0, & y > 0 \\ v_\varepsilon|_{y=0} = v'_\varepsilon|_{y=0} = 0. \end{cases}$$

Introducing the notations

$$\tilde{\omega}(\varepsilon) := \omega(\varepsilon)^{-1}, \quad \tilde{\varepsilon} := -\omega(\varepsilon)^{-1}\varepsilon,$$

it reads

$$(4.3) \quad \begin{cases} (\tilde{\omega}(\varepsilon) + u_s)v'_\varepsilon - u'_s v_\varepsilon + i\tilde{\varepsilon}v_\varepsilon^{(3)} = 0, & y > 0 \\ v_\varepsilon|_{y=0} = v'_\varepsilon|_{y=0} = 0. \end{cases}$$

Thus, the equation gets formally close to the equation

$$(4.4) \quad \begin{cases} (\omega(\varepsilon) + u_s)v'_\varepsilon - u'_s v_\varepsilon + i\varepsilon v_\varepsilon^{(3)} = 0, & y > 0 \\ v_\varepsilon|_{y=0} = v'_\varepsilon|_{y=0} = 0, \end{cases}$$

which has been studied in [4], in connection to the IVP for (1.5). More precisely, the authors build an approximate solution of (4.4) under the form

$$(4.5) \quad \begin{cases} \omega^{app}(\varepsilon) = -u_s(a) + \varepsilon^{1/2}\tau, \\ v_\varepsilon^{app}(y) = H(y-a)(u_s + \omega(\varepsilon)) + \varepsilon^{1/2}V\left(\dfrac{y-a}{\varepsilon^{1/4}}\right). \end{cases}$$

The streamfunction v_ε^{app} divides into two parts: a regular part

$$v_\varepsilon^{reg}(y) \; := \; H(y-a)(u_s + \omega(\varepsilon)),$$

and a "shear layer part" $v_\varepsilon^{sl}(y) \; := \; \varepsilon^{1/2} V\left(\dfrac{y-a}{\varepsilon^{1/4}}\right)$. Note that $(\omega^{app}(\varepsilon), v_\varepsilon^{reg})$ solves (1.5) except for the $\mathcal{O}(\varepsilon)$ term coming from diffusion. However, both v_ε^{reg} and its second derivative have discontinuities at $y = a$. This explains the introduction of v_ε^{sl}. The shear layer profile $V = V(z)$ satisfies the system

(4.6)
$$\begin{cases} \left(\tau + u_s''(a)\dfrac{z^2}{2}\right) V' \; - \; u_s''(a)\, z\, V \; + \; i\, V^{(3)} \; = \; 0, \quad z \neq 0, \\[2mm] [V]_{|z=0} \; = \; -\tau, \quad [V']_{|z=0} \; = \; 0, \quad [V'']_{|z=0} = -u''(a), \\[2mm] \lim\limits_{\pm\infty} V \; = \; 0. \end{cases}$$

The jump conditions on V compensate those of the regular part. The parameter τ allows the system not to be overdetermined. Indeed, one shows that for some appropriate τ with $\Im\tau < 0$, (4.6) has a solution, with rapid decay to zero as $z \to \pm\infty$. This singular perturbation is responsible for the instability, through the eigenvalue perturbation τ. We refer to [4] for all necessary details. Note that writing

$$\tilde{V}(z) \; = \; V(z) \; + \; \mathbf{1}_{\mathbb{R}_+}\left(\tau + u_s''(a)\dfrac{z^2}{2}\right),$$

one gets rid of the jump conditions:

(4.7)
$$\begin{cases} \left(\tau + u_s''(a)\dfrac{z^2}{2}\right) \tilde{V}' \; - \; u_s''(a)\, z\, \tilde{V} \; + \; i\, \tilde{V}^{(3)} \; = \; 0, \quad z \in \mathbb{R}, \\[2mm] \lim\limits_{z\to-\infty} \tilde{V} \; = \; 0, \qquad \tilde{V} \; \sim \; \tau + u_s''(a)\dfrac{z^2}{2} \;\; \text{as} \;\; z \to +\infty \end{cases}$$

Accordingly, the approximate solution (4.5) takes the slightly simpler form

(4.8)
$$\begin{cases} \omega^{app}(\varepsilon) \; = \; -u_s(a) \; + \; \varepsilon^{1/2}\tau, \\[2mm] v_\varepsilon^{app}(y) \; = \; H(y-a)\left(u_s - u_s(a) - u_s''(a)\dfrac{(y-a)^2}{2}\right) \; + \; \varepsilon^{1/2}\tilde{V}\left(\dfrac{y-a}{\varepsilon^{1/4}}\right). \end{cases}$$

On the basis of this former analysis, it is tempting to consider for our system (4.3) the following expansion:

(4.9)
$$\begin{cases} \tilde{\omega}^{app}(\varepsilon) \; = \; -u_s(a) \; + \; \tilde{\varepsilon}^{1/2}\tau, \\[2mm] v_\varepsilon^{app}(y) \; = \; H(y-a)\left(u_s - u_s(a) - u_s''(a)\dfrac{(y-a)^2}{2}\right) \; + \; \tilde{\varepsilon}^{1/2}\tilde{V}\left(\dfrac{y-a}{\tilde{\varepsilon}^{1/4}}\right). \end{cases}$$

However, it is not so straightforward:

- Back to the initial notations, the first relation reads

(4.10)
$$\omega^{app}(\varepsilon)^{-1} \; = \; -u_s(a) \; + \; (-\omega^{app}(\varepsilon)^{-1}\varepsilon)^{1/2}\tau.$$

     In particular, it is no longer an *explicit* definition of $\omega^{app}(\varepsilon)$. It is an equation for $\omega^{app}(\varepsilon)$, and we must check that the equation defines it *implicitly*.

- Assuming that $\omega^{app}(\varepsilon)$ is defined, $\tilde{\varepsilon}$ is some unreal complex number. Hence, v_ε^{app} is not *a priori* properly defined, because $\tilde{V}$ no longer has a real argument.

Fortunately, as we will now show, these difficulties can be solved, leading to the desired unstable quasimode.

We start with equation (4.10). A standard application of the implicit function theorem implies that for ε small enough, the complex equation

$$F(z) = 0, \quad F(z) := z + u_s(a) - (-\varepsilon z)^{1/2}\tau$$

(where $z^{1/2}$ is the principal value of the square root) has a unique solution near $-u_s(a)$. This allows to define $\omega^{app}(\varepsilon)^{-1}$, so $\omega^{app}(\varepsilon)$. Furthermore, easy computations yield

$$(4.11) \qquad \omega^{app}(\varepsilon) = -\frac{1}{u_s(a)} - \varepsilon^{1/2}\frac{\tau}{u_s(a)^{3/2}} + o(\varepsilon^{1/2}).$$

Note that injecting this expression in (4.1) gives some exponential growth in x at rate $\varepsilon^{-1/2}$.

     It remains to clarify the definition of v_ε in (4.9). Of course, the complex powers $\tilde{\varepsilon}^{1/2}$, $\tilde{\varepsilon}^{1/4}$ can still be defined. The problem lies in the profile $\tilde{V} = \tilde{V}(z)$, which is a function over $\mathbb{R}$, instead of $(-\omega^{app}(\varepsilon))^{-1/4}\mathbb{R}$. To overcome this problem, we will show that $\tilde{V}$ extends holomorphically to the neighborhood of the real line

$$U_\tau := \mathbb{C}\setminus\left(i(-\tau)^{1/2}[-\infty, -1]\cup i(-\tau)^{1/2}[1, +\infty]\right).$$

The extension will satisfy the ODE in (4.7) over U_τ, and will have the following asymptotic behaviour, for small $\theta \geq 0$:

$$\lim_{z\in e^{i\theta}\mathbb{R},\, z\to -e^{i\theta}\infty} \tilde{V} = 0, \qquad V \sim \tau + u_s''(a)\frac{z^2}{2} \text{ as } z \in e^{i\theta}\mathbb{R},\ z \to +e^{i\theta}\infty.$$

In particular, $\tilde{V}$ will satisfy all the requirements along the line $(-\omega(\varepsilon))^{1/4}\mathbb{R}$, for ε small enough.

     To perform our extension, we write $\tilde{V}$ as

$$\tilde{V}(z) := \left(\tau + u_s''(a)\frac{z^2}{2}\right)W(z)$$

where W satisfies

$$\begin{cases} \left(\tau + u_s''(a)\, z^2/2\right)^2 \dfrac{d}{dz}W \, + \, i\,\dfrac{d^3}{dz^3}\left(\left(\tau + u_s''(a)\, z^2/2\right)W\right) \, = \, 0, \\[2mm] \lim_{-\infty} W \, = \, 0, \quad \lim_{+\infty} W \, = \, 1. \end{cases}$$

Then, performing the changes of variables

$$\tau = \frac{1}{\sqrt{2}}|u_s''(a)|^{1/2}\,\tilde{\tau}, \quad z = 2^{1/4}\,|u_s''(a)|^{-1/4}\tilde{z}$$

leaves us with

$$
\begin{cases}
(\tilde{\tau} - \tilde{z}^2)^2 \dfrac{d}{d\tilde{z}}\tilde{W} \;+\; i\,\dfrac{d^3}{d\tilde{z}^3}\left((\tilde{\tau} - \tilde{z}^2)\,\tilde{W}\right) \;=\; 0, \\[2mm]
\displaystyle\lim_{-\infty}\tilde{W} \;=\; 0, \quad \lim_{+\infty}\tilde{W} \;=\; 1.
\end{cases}
$$

This system already appears in [4]. It is shown that for some appropriate $\tilde{\tau}$ with negative imaginary part, there exists some solution $\tilde{W}$. Actually, much more is known on $\tilde{W}$. Indeed, $X = \tilde{W}'$ satisfies the ODE with holomorphic coefficients

$$
i(\tilde{\tau} - \tilde{z}^2)\,X'' - 6i\,\tilde{z}\,X'' + (\tau - \tilde{z}^2)^2 - 6i\,X = 0,
$$

In particular, the coefficient of the leading order term does not vanish in U_τ. Thus, X can be extended into a holomorphic solution in U_τ. Moreover, the previous equation on X can be put into the form of a first order system:

$$
(4.12) \qquad \frac{d}{d\tilde{z}}\mathcal{X} \;=\; \tilde{z}\mathcal{B}(\tilde{z})\mathcal{X}, \quad \mathcal{X} = \begin{pmatrix} X \\ \tilde{z}^{-1}\frac{d}{d\tilde{z}}X \end{pmatrix}, \quad \mathcal{B}(\tilde{z}) = \begin{pmatrix} 0 & 1 \\ \frac{6+i(\tau - \tilde{z}^2)^2}{\tilde{z}^2(\tau - \tilde{z}^2)} & \frac{6}{\tau - \tilde{z}^2} - \frac{1}{\tilde{z}^2} \end{pmatrix}.
$$

In particular, $\mathcal{B}$ is holomorphic at infinity, with $\mathcal{B}(\infty) = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$. It has the two distinct eigenvalues $\pm i e^{i\pi/4}$. From there, it is possible to obtain explicit asymptotic expansions for X at infinity. Following [1, Theorem 5.1 p163], in any closed sector $\overline{S}_{\theta,\theta'}$ inside which $\Re e(i e^{i\pi/4}z^2)$ does not cancel, there are solutions $\mathcal{X}_\pm$ (depending *a priori* on θ, θ') with the following asymptotic behaviour:

$$
(4.13) \qquad \mathcal{X}_\pm \;\sim\; \left(\sum_{i\geq 0}\mathcal{X}_\pm^i\,\tilde{z}^{\alpha_\pm}\right) e^{P_\pm(\tilde{z})}, \quad |\tilde{z}| \to +\infty, \quad \tilde{z} \in \overline{S}_{\theta,\theta'},
$$

where $\alpha_\pm$ is a complex constant, and $P_\pm(\tilde{z})$ is a polynomial of degree 2. Moreover, the leading term of $P_\pm$ is $\pm\frac{i e^{i\pi/4}\tilde{z}^2}{2}$. In the present case, tedious computations yield

$$
\mathcal{X}_\pm(\tilde{z}) \;\sim\; C_\pm\,\tilde{z}^{\alpha_\pm}\,\exp\left(\pm i e^{i\pi/4}\frac{\tilde{z}^2}{2}\right), \quad \text{for some } C_+, \alpha_+ \in \mathbb{C},
$$

as $|\tilde{z}| \to +\infty$, in any closed sector that is strictly inside $S_{-\frac{\pi}{8}+\frac{k\pi}{2}, \frac{3\pi}{8}+\frac{k\pi}{2}}$ for some $k \in \mathbb{Z}$. As X decays to zero in $\pm\infty$, we deduce that: for all $\delta > 0$,

$$
|X(\tilde{z})| \;\leq\; C\exp(-\alpha|\tilde{z}|^2), \text{ for some } \alpha > 0, \quad z \in \overline{S}_{-\frac{\pi}{8}+\delta, \frac{3\pi}{8}-\delta} \cup \overline{S}_{\frac{7\pi}{8}+\delta, \frac{11\pi}{8}-\delta}.
$$

This implies easily that the antiderivative $\tilde{W}$ (now defined in U_τ) satisfies

$$
|\tilde{W}(\tilde{z}) - 1| \leq C\exp(-\alpha'|\tilde{z}|^2), \text{ uniformly in } \overline{S}_{-\frac{\pi}{8}+\delta, \frac{3\pi}{8}-\delta},
$$

and similarly

$$
|\tilde{W}(\tilde{z})| \leq C\exp(-\alpha'|\tilde{z}|^2), \text{ uniformly in } \overline{S}_{\frac{7\pi}{8}+\delta, \frac{11\pi}{8}-\delta}.
$$

From there, one can go back to W, and then to $\tilde{V}$.

In this way, we obtain an appropriate approximate solution of (4.3). Exactly as in [4], we can refine this approximation by the addition of higher order "regular" terms in v_ε^{app}.

Namely, we consider an expansion of the form

$$v_\varepsilon^{app}(y) \;=\; H(y-a)\left(u_s - u_s(a) - u_s''(a)\frac{(y-a)^2}{2}\right) \;+\; \tilde\varepsilon^{1/2}\tilde V\left(\frac{y-a}{\tilde\varepsilon^{1/4}}\right) + \sum_{i=2}^{n} \tilde\varepsilon^{i/2} v_{i,reg}(y)$$

Each additional term $v_{i,reg}$ satisfies a first order equation of the type

$$(\tilde\omega(\varepsilon) + u_s)v_{i,reg}' - u_s' v_{i,reg} \;=\; f^i,$$

where f^i comes from lower order terms. For example, $f^2 := H(y-a)\partial_y^3 u_s$. As explained in [4], due to the quadratic structure of u_s near a, one can show by an easy induction that f^i is identically zero near a, so that

$$v_{i,reg} \;=\; H(y-a)\Big(u_s(y) + \tilde\omega(\varepsilon)\Big)\int_a^y \frac{f^i}{(u_s(z) + \tilde\omega(\varepsilon))^2}\, dz,$$

is smooth across $y = a$.

Back to (4.1), we obtain some approximate solution of (1.5), with frequency ε^{-1} in t, that grows exponentially with $x > 0$ at rate $\varepsilon^{-1/2}\sigma$,

$$\sigma \;:=\; \frac{|\Im m(\tau)|}{u_s(a)^{3/2}}.$$

This approximation solves (1.5) up to some $\mathcal{O}\left(\varepsilon^{n-k} e^{\sigma x/\varepsilon^{1/2}}\right)$ error term in $e^{-y}H^k$ for all k. Again, we refer to [4] for all details.

4.2. **Proof of spacial ill-posedness.** Thanks to the spacially unstable quasimode from the last paragraph, one can try to mimic the proof of Theorem 1.1, to derive Theorem 1.4. Indeed, all arguments related to the closed graph theorem adapt straightforwardly, as well as those disproving the continuity of the flow if it exists. In other words, we leave to the reader to check the following fact: for any $X > 0$, there exists some boundary data $u_1 \in e^{-y}H^\infty(\mathbb{T} \times \mathbb{R}_+)$, for which either existence or uniqueness of a solution u with

$$u_s\, u \in L_t^2(\mathbb{T}; C_x([0,X]; H_y^2(\mathbb{R}_+))), \quad u \in L_{t,x}^2(\mathbb{T} \times (0,X); H_y^2(\mathbb{R}_+)).$$

fails.

To complete the proof of the Theorem, it remains to show some uniqueness result for the BVP in this functional setting. It turns out to be more difficult than for the IVP. Therefore, we strengthen the assumptions on u_s: besides all assumptions needed for the construction of the quasimode, we make the following hypothesis: the function u_s, which belongs to $U + C_c^\infty(\mathbb{R}_+)$, satisfies

$$u_s(y) = y, \quad \text{in the vicinity of } y = 0.$$

Under this additional hypothesis, we are able to state

Proposition 4.1 (Uniqueness for the BVP). *Let u be a weak solution of* (1.5) *such that*

$$u_s\, u \in L_t^2(\mathbb{T}; C_x([0,X]; H_y^2(\mathbb{R}_+))), \quad u \in L_{t,x}^2(\mathbb{T} \times (0,X); H_y^2(\mathbb{R}_+)),$$

and $u|_{x=0} = 0$. *Then* $u \equiv 0$.

Proof. Up to a replacement of u by $P_N u$, where P_N is the projector on temporal Fourier modes less than N, we can assume that

$$(4.14) \qquad |\partial_t^s \partial_x^\alpha \partial_y^\beta u(t,x,y)| \;\leq\; C_s \, |\partial_x^\alpha \partial_y^\beta u(t,x,y)|, \quad \forall t,x,y, \quad \forall \alpha, \beta.$$

Our key idea to get uniqueness for equation (1.5) is to write it as

$$\partial_x Lu = -\partial_t u + \partial_y^2 u, \quad \text{where } \; Lu \; := \; u_s u \; - \; u_s' \int_0^y u,$$

and to use Lu and some variants of it as multipliers. Denoting by $(\,|\,)$ the scalar product in $L^2(\mathbb{R}_+)$, we have for $k = 0, 1, 2$:

$$(4.15) \qquad \frac{1}{2}\partial_x \|\partial_y^k Lu\|_{L^2(\mathbb{R}_+)}^2 = (-\partial_t \partial_y^k u | \partial_y^k Lu) \; + \; (\partial_y^{k+2} u | \partial_y^k Lu) \; := \; I_{k,1} \; + \; I_{k,2}.$$

We estimate those two terms separately, starting with $I_{k,1} = I_{k,1}(t,x)$. Let M be a constant such that $u_s = U$ for all $y \geq M$. We claim that : $\forall \varepsilon > 0$, $\forall k = 0, 1, 2$,

$$(4.16) \quad \int_{\mathbb{T}} I_{k,1}(t)\, dt \leq C_\varepsilon \int_{\mathbb{T}} \left(\|u(t)\|_{L^2(0,M)}^2 \; + \; \|\partial_y u(t)\|_{L^2(0,M)}^2 \right) dt + \varepsilon \int_{\mathbb{T}} \|\partial_y^2 u(t)\|_{L^2(0,M)}^2 dt.$$

Note that, for clarity, we omit to indicate the dependance on x of the various quantities. For brevity, we just prove our claim in the case $k = 2$, which is the most involved: one has

$$I_{2,1} \; = \; -(\partial_t \partial_y^2 u | \partial_y^2 Lu) \; = \; -\int_{\mathbb{R}_+} \partial_t \partial_y^2 u \left(u_s \partial_y^2 u + u_s' \partial_y u - u_s^{(3)} \int_0^y u - u_s'' u \right) dy$$

$$= \; -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} u_s |\partial_y^2 u_s|^2 dy - \int_{\mathbb{R}_+} \partial_t \partial_y^2 u \left(u_s' \partial_y u - u_s^{(3)} \int_0^y u - u_s'' u \right) dy.$$

Integration in time makes the first term at the r.h.s. vanishes. The other terms involve derivatives of u_s, so that the integrals over $\mathbb{R}_+$ can be replaced by integrals over $(0, M)$. Using Cauchy-Schwarz inequality, we end up with

$$\int_{\mathbb{T}} I_{2,1} \leq C \int_{\mathbb{T}} \|\partial_t \partial_y^2 u\|_{L^2(0,M)} \left(\|\partial_y u\|_{L^2(0,M)} \; + \; \|\int_0^y u\|_{L^2(0,M)} \; + \; \|u\|_{L^2(0,M)} \right)$$

$$\leq C \int_{\mathbb{T}} \|\partial_t \partial_y^2 u\|_{L^2(0,M)} \left(\|\partial_y u\|_{L^2(0,M)} \; + \; (1 + M^2/2)\|u\|_{L^2(0,M)} \right)$$

$$\leq \varepsilon \int_{\mathbb{T}} \|\partial_y^2 u\|_{L^2(0,M)}^2 \; + \; C_\varepsilon \int_{\mathbb{T}} \left(\|\partial_y u\|_{L^2(0,M)} \; + \; \|u\|_{L^2(0,M)} \right)^2$$

where Young's inequality and (4.14) were used for the last inequality. This proves the claim.

We now turn to the estimate of $I_{k,2}$. Let $m > 0$ such that $u_s(y) = y$ for $y \leq m$. We claim that: for all $k = 0, 1, 2$,

$$(4.17) \qquad I_{k,2} \; \leq \; C \left(\|u\|_{L^2(0,M)}^2 \; + \; \|\partial_y u\|_{L^2(0,M)}^2 \; + \; \|\partial_y^2 u\|_{L^2(m,M)}^2 \right).$$

Again, we omitted the t, x dependence in the notations. These inequalities follow from simple integration by parts. As before, we consider the case $k = 2$, the other ones being

easiest. We have

$$
I_{2,2} \;=\; \left(\partial_y^4 u | \partial_y^2 Lu\right) \;=\; \int_{\mathbb{R}_+} \partial_y^4 u \left(u_s \partial_y^2 u + u_s' \partial_y u - u_s^{(3)} \int_0^y u - u_s'' u \right) dy
$$

$$
=\; -\int_{\mathbb{R}_+} \partial_y^3 u \, \partial_y \left(u_s \partial_y^2 u + u_s' \partial_y u \right) dy \;-\; \int_{\mathbb{R}_+} \partial_y^2 u \, \partial_y^2 \left(u_s^{(3)} \int_0^y u + u_s'' u \right) dy
$$

through integrations by parts. The last term at the r.h.s involves derivatives of u_s', so is for $m \leq y \leq M$. It also involves up to second order derivatives of u only. Thus

$$
I_{2,2} \;\leq\; -\int_{\mathbb{R}_+} \partial_y^3 u \partial_y \left(u_s \partial_y^2 u + u_s' \partial_y u \right) dy \;+\; C \sum_{k=0}^2 \|\partial_y^k u\|_{L^2(m,M)}^2.
$$

As regards the first term, we have

$$
-\int_{\mathbb{R}_+} \partial_y^3 u \, \partial_y \left(u_s \partial_y^2 u + u_s' \partial_y u \right) dy \;=\; -\int_{\mathbb{R}_+} \partial_y^3 u \left(u_s \partial_y^3 u + 2u_s' \partial_y^2 u + u_s'' \partial_y u \right)
$$

$$
\leq\; -\int_{\mathbb{R}_+} \partial_y^3 u \left(2u_s' \partial_y^2 u + u_s'' \partial_y u \right) \;=\; -\int_{\mathbb{R}_+} u_s' \partial_y (\partial_y^2 u)^2 + \int_{\mathbb{R}_+} \partial_y^2 u \partial_y (u_s'' \partial_y u)
$$

$$
=\; \int_{\mathbb{R}_+} u_s'' (\partial_y^2 u)^2 + \int_{\mathbb{R}_+} \partial_y^2 u \partial_y (u_s'' \partial_y u) \;\leq\; C \sum_{k=1}^2 \|\partial_y^k u\|_{L^2(m,M)}^2.
$$

This proves our claim.

We must now link the Sobolev norms of u and Lu. This is the purpose of the

Lemma 4.2. *Let* $u = u(y) \in H^2(0,M)$, $u|_{y=0} = 0$. *Then,*

$$
\|u\|_{H^1(0,M)} \;\leq\; C \, \|Lu\|_{H^2(0,M)}
$$

and

$$
\|u\|_{H^2(m,M)} \;\leq\; C_m \, \|Lu\|_{H^2(0,M)}
$$

Proof of the lemma. Note that by our assumptions on u_s, Lu belongs to $H^2(0,M)$, and satisfies $Lu|_{y=0} = \partial_y Lu|_{y=0} = 0$. Denoting $f := Lu$, we have an explicit representation of u in terms of f, by solving the first order ODE $Lu = f$:

$$
u(y) \;=\; u_s'(y) \int_0^y \frac{f(t)}{u_s(t)^2} dt \;+\; \frac{f(y)}{u_s(y)}
$$

$$
=\; u_s'(y) \int_0^m \frac{f(t)}{t^2} dt \;+\; u_s'(y) \int_m^y \frac{f(t)}{u_s(t)^2} dt + \frac{f(y)}{u_s(y)} \;:=\; u_1(y) \;+\; u_2(y) \;+\; u_3(y).
$$

Clearly, $\|u_2 + u_3\|_{H^2(m,M)} \leq C_m \|f\|_{H^2(0,M)}$. As regards u_1, we compute

$$
\int_0^m \frac{f(t)}{t^2} dt \;=\; \int_0^m \frac{f'(t)}{t} dt - \frac{f(m)}{m}
$$

so that

$$
\|u_1\|_{H^2(0,M)} \leq C \left| \int_0^m \frac{f(t)}{t^2} dt \right| \leq C' \left(\left(\int_0^m \left| \frac{f'(t)}{t} \right|^2 \right)^{1/2} + \|f\|_{L^\infty(0,M)} \right) \leq C'' \|f\|_{H^2(0,M)},
$$

using Hardy and Sobolev inequalities. We thus obtain the second inequality. For the proof of the first inequality, it remains to control u on the interval $(0, m)$. It satisfies there the equation $yu - \int_0^y u = f$, which gives $yu' = f'$, and from Hardy's inequality, we deduce that $\|u'\|_{L^2(0,m)} \lesssim \|f\|_{H^2(0,m)}$. This ends the proof of the lemma.

Combining this lemma with (4.15), (4.16) and (4.17), we obtain that the solution u of (1.5) with boundary data $u|_{x=0} = 0$ satisfies

$$(4.18) \quad \|Lu(x)\|^2_{L^2_t(\mathbb{T};H^2_y(\mathbb{R}_+))} \leq C_\varepsilon \int_0^x \|Lu(x')\|^2_{L^2_t(\mathbb{T};H^2_y(\mathbb{R}_+))} dx' + \varepsilon \|\partial^2_y u\|^2_{L^2((0,x)\times\mathbb{T}\times\mathbb{R}_+)}$$

for all $\varepsilon > 0$ ($\varepsilon = 1$ will be enough here). We still need to control the L^2 norm of $\partial^2_y u$. Therefore, we differentiate (1.5) with respect to y, and write the resulting equation as

$$u_s \partial_x L'u = \partial^3_y u - \partial_t \partial_y u, \quad L'u := \partial_y u - \frac{u''_s}{u_s} \int_0^y u.$$

Note that the second term defining $L'u$ has no singularity at zero, as u''_s vanishes identically for $y \leq m$. Multiplying this equation by $L'u$, and integrating in t, x, y we obtain

$$\frac{1}{2} \int_{\mathbb{T}\times\mathbb{R}_+} u_s |L'u(x)|^2 = \int_{(0,x)\times\mathbb{T}\times\mathbb{R}_+} (\partial^3_y u - \partial_t \partial_y u) L'u.$$

One has easily, still with (4.14)

$$\int_{(0,x)\times\mathbb{T}\times\mathbb{R}_+} (-\partial_t \partial_y u) L'u \leq C \sum_{k=0}^{1} \|\partial^k_y u\|^2_{L^2((0,x)\times\mathbb{T}\times(0,M))},$$

whereas a simple integration by parts yield

$$\int_{(0,x)\times\mathbb{T}\times\mathbb{R}_+} \partial^3_y u\, L'u = -\int_{(0,x)\times\mathbb{T}\times\mathbb{R}_+} |\partial^2_y u|^2 + \sum_{k=0}^{2} \|\partial^k_y u\|^2_{L^2((0,x)\times\mathbb{T}\times(m,M))}$$

Hence,

$$(4.19) \quad \frac{1}{2} \int_{\mathbb{T}\times\mathbb{R}_+} u_s |L'u(x)|^2 dy dt + \|\partial^2_y u\|_{L^2((0,x)\times)\mathbb{T}\times\mathbb{R}_+)}$$

$$\leq C \left(\sum_{k=0}^{1} \|\partial^k_y u\|^2_{L^2((0,x)\times\mathbb{T}\times(0,M))} + \|\partial^2_y u\|^2_{L^2((0,x)\times\mathbb{T}\times(m,M))} \right).$$

Combining this inequality with Lemma 4.2 implies that

$$\|\partial^2_y u\|^2_{L^2((0,x)\times\mathbb{T}\times\mathbb{R}_+)} \leq C \|Lu\|^2_{L^2((0,x)\times\mathbb{T};H^2_y(\mathbb{R}_+))}$$

Together with (4.18), this leads to

$$\|Lu(x)\|^2_{H^2_{t,y}(\mathbb{T}\times\mathbb{R}_+)} \leq C_\varepsilon \int_0^x \|Lu(x')\|^2_{L^2_t(\mathbb{T};H^2_y(\mathbb{R}_+))} dx'.$$

By Gronwall lemma, we deduce that $Lu = 0$, and from there that $u = 0$. This concludes the proof of the proposition, and in turn the proof of Theorem 1.4. $\qquad\square$

4.3. **Final comments.** The ill-posedness results discussed in this paper may give some insight into the classical results of Oleinik on boundary layer theory. These results, collected in the book [11], establish various well-posedness theorems for the Prandtl equation, under some monotonicity properties for the data with respect to y. Interestingly, the monotonicity assumptions used by Oleinik are different whether the steady case or the unsteady one is considered. Roughly:

- In the steady case, the main hypothesis is that $u|_{x=0}$ be monotonic near 0, that is $\partial_y u|_{x=0}(0) > 0$.
- But in the unsteady case, the monotonicity hypothesis is stronger. The data $u|_{x=0}$ and $u|_{t=0}$ are assumed to be monotonic everywhere, that is $\partial_y u|_{t=0}(x,y) > 0$ and $\partial_y u|_{x=0}(t,y) > 0$ for all $y > 0$.

In Oleinik's work, these assumptions appear as technical, connected to some special changes of variables that are different in the steady and unsteady case (these are the so-called Von Mises and Crocco transforms).

From our results, one may believe that this difference in the steady and unsteady assumptions is more than technical. Indeed, we have proved that as soon as time is involved in the Prandtl system, either through an IVP or a BVP with time periodicity, a lack of monotonicity somewhere in the flow ($u_s'(a) > 0$ for some $a > 0$) is enough to trigger a strong instability (at least at the linearized level). On the contrary, as soon as a steady situation is considered, this instability mechanism dissapears. For instance, the space instability developped in this section relies on high frequency time oscillations, and is not there when the steady BVP is considered. Actually, the BVP for the steady version of (1.5):

$$\begin{cases} u_s \partial_x u + u_s' v - \partial_y^2 u &=& 0, & y > 0, \\ \partial_x u + \partial_y v &=& 0, & y > 0, \\ u = v &=& 0, & y = 0. \end{cases}$$

can be shown to be well-posed for the shear flow u_s considered in Proposition 4.1. Indeed, the estimates established there to prove uniqueness can be used in the same way to establish some *a priori estimates*, and from there a well-posedness result (in appropriate functional spaces). Again, this is coherent with the works of Oleinik, as we assumed in Proposition 4.1 that $u_s(y) = y$ for small y. Still in the spirit of this proposition, it would be very interesting to recover the well-posedness results of Oleinik without special changes of variables, that is simply through linearization and/or energy estimates.

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