Stability of radiative shock profiles for hyperbolic-elliptic coupled systems

Toan Nguyen[a], Ramón G. Plaza[b,*], Kevin Zumbrun[a]

[a]*Department of Mathematics, Indiana University, Bloomington, IN 47405 (U.S.A.)*
[b]*Departamento de Matemáticas y Mecánica, IIMAS-UNAM, Apartado Postal 20-726, C.P. 01000 México D.F. (México)*

Abstract

Extending previous work with Lattanzio and Mascia on the scalar (in fluid-dynamical variables) Hamer model for a radiative gas, we show nonlinear orbital asymptotic stability of small-amplitude shock profiles of general systems of coupled hyperbolic–eliptic equations of the type modeling a radiative gas, that is, systems of conservation laws coupled with an elliptic equation for the radiation flux, including in particular the standard Euler–Poisson model for a radiating gas. The method is based on the derivation of pointwise Green function bounds and description of the linearized solution operator, with the main difficulty being the construction of the resolvent kernel in the case of an eigenvalue system of equations of degenerate type. Nonlinear stability then follows in standard fashion by linear estimates derived from these pointwise bounds, combined with nonlinear-damping type energy estimates.

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Hyperbolic-elliptic systems, shock profiles, nonlinear stability, pointwise Green function bounds, radiating gases
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1. Introduction

In the theory of non-equilibrium radiative hydrodynamics, it is often assumed that an inviscid compressible fluid interacts with radiation through energy exchanges. One widely accepted model [38] considers the one dimensional Euler system of equations coupled with an elliptic equation for the radiative energy, or *Euler–Poisson equation*. With this system in mind, this paper considers

*Corresponding author. Tel.: +52 (55) 5622-3567. Fax: +52 (55) 5622-3564.
Email addresses: `nguyentt@indiana.edu` (Toan Nguyen), `plaza@mym.iimas.unam.mx` (Ramón G. Plaza), `kzumbrun@indiana.edu` (Kevin Zumbrun)

general hyperbolic-elliptic coupled systems of the form,

$$\begin{aligned} u_t + f(u)_x + Lq_x &= 0, \\ -q_{xx} + q + g(u)_x &= 0, \end{aligned} \tag{1}$$

with $(x,t) \in \mathbb{R} \times [0, +\infty)$ denoting space and time, respectively, and where the unknowns $u \in \mathcal{U} \subseteq \mathbb{R}^n$, $n \geq 1$, play the role of state variables, whereas $q \in \mathbb{R}$ represents a general heat flux. In addition, $L \in \mathbb{R}^{n\times 1}$ is a constant vector, and $f \in C^2(\mathcal{U};\mathbb{R}^n)$ and $g \in C^2(\mathcal{U};\mathbb{R})$ are nonlinear vector- and scalar-valued flux functions, respectively.

The study of general systems like (1) has been the subject of active research in recent years [10, 11, 13, 17]. There exist, however, more complete results regarding the simplified model of a radiating gas, also known as the *Hamer model* [6], consisting of a scalar velocity equation (usually endowed with a Burgers' flux function which approximates the Euler system), coupled with a scalar elliptic equation for the heat flux. Following the authors' concurrent analysis with Lattanzio and Mascia of the reduced scalar model [16], this work studies the asymptotic stability of *general radiative shock profiles*, which are traveling wave solutions to system (1) of the form

$$u(x,t) = U(x - st), \qquad q(x,t) = Q(x - st), \tag{2}$$

with asymptotic limits

$$U(x) \to u_\pm, \qquad Q(x) \to 0, \qquad \text{as } \ x \to \pm\infty,$$

being $u_\pm \in \mathcal{U} \subseteq \mathbb{R}^n$ constant states and $s \in \mathbb{R}$ the shock speed. The main assumption is that the triple (u_+, u_-, s) constitutes a shock front [19] for the underlying "inviscid" system of conservation laws

$$u_t + f(u)_x = 0, \tag{3}$$

satisfying canonical jump conditions of Rankine-Hugoniot type,

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0, \tag{4}$$

plus classical Lax entropy conditions. In the sequel we denote the jacobians of the nonlinear flux functions as

$$A(u) := Df(u) \in \mathbb{R}^{n\times n}, \qquad B(u) := Dg(u) \in \mathbb{R}^{1\times n}, \qquad u \in \mathcal{U}.$$

Right and left eigenvectors of A will be denoted as $r \in \mathbb{R}^{n\times 1}$ and $l \in \mathbb{R}^{1\times n}$, and we suppose that system (3) is hyperbolic, so that A has real eigenvalues $a_1 \leq \cdots \leq a_n$.

It is assumed that system (1) represents some sort of regularization of the inviscid system (3) in the following sense. Formally, if we eliminate the q variable, then we end up with a system of form

$$u_t + f(u)_x = (LB(u)u_x)_x + (u_t + f(u)_x)_{xx},$$

which requires a nondegeneracy hypothesis

$$l_p \cdot (B \otimes L^\top r_p) > 0, \tag{5}$$

for some $1 \le p \le n$, in order to provide a good dissipation term along the p-th characteristic field in its Chapman-Enskog expansion [35].

More precisely, we make the following structural assumptions:

$$f, g \in C^2 \quad \text{(regularity)}, \tag{S0}$$

For all $u \in \mathcal{U}$ there exists A_0 symmetric, positive definite such that $A_0 A$ is symmetric, and $A_0 LB$ is symmetric, positive semi-definite of rank one (symmetric dissipativity $\Rightarrow$ non-strict hyperbolicity). Moreover, we assume that the principal eigenvalue a_p of A is simple. (S1)

No eigenvector of A lies in $\ker LB$ (genuine coupling). (S2)

Remark 1.1. Assumption (S1) assures non-strict hyperbolicty of the system, with simple principal characteristic field. Notice that (S1) also implies that $(A_0)^{1/2} A (A_0)^{-1/2}$ is symmetric, with real and semi-simple spectrum, and that, likewise, $(A_0)^{1/2} B (A_0)^{-1/2}$ preserves symmetric positive semi-definiteness with rank one. Assumption (S2) defines a general class of hyperbolic-elliptic equations analogous to the class defined by Kawashima and Shizuta [9, 14, 37] and compatible with (5). Moreover, there is an equivalent condition to (S2) given by the following

Lemma 1.2 (Shizuta–Kawashima [14, 37])**.** *Under* (S0) - (S1)*, assumption* (S2) *is equivalent to the existence of a skew-symmetric matrix valued function* $K : \mathcal{U} \to \mathbb{R}^{n \times n}$ *such that*

$$\mathrm{Re}\,(KA + A_0 LB) > 0, \tag{6}$$

for all $u \in \mathcal{U}$*.*

Proof. See, e.g., [8]. □

As usual, we can reduce the problem to the analysis of a stationary profile with $s = 0$, by introducing a convenient change of variable and relabeling the flux function f accordingly. Therefore, we end up with a stationary solution $(U, Q)(x)$ of the system

$$\begin{aligned} f(U)_x + LQ_x &= 0, \\ -Q_{xx} + Q + g(U)_x &= 0. \end{aligned} \tag{7}$$

After such normalizations and under (S0) - (S2), we make the following assumptions about the shock:

$$f(u_+) = f(u_-), \quad \text{(Rankine-Hugoniot jump conditions)}, \tag{H0}$$

$$\begin{aligned} a_p(u_+) < 0 < a_{p+1}(u_+), \\ a_{p-1}(u_-) < 0 < a_p(u_-), \end{aligned} \quad \text{(Lax entropy conditions)}, \tag{H1}$$

$$(\nabla a_p)^\top r_p \neq 0, \ \text{ for all } \ u \in \mathcal{U}, \quad \text{(genuine nonlinearity)}, \tag{H2}$$

$$l_p(u_\pm) L B(u_\pm) r_p(u_\pm) > 0, \qquad \text{(positive diffusion)}. \tag{H3}$$

Remark 1.3. Systems of form (1) arise in the study of radiative hydrodynamics, for which the paradigmatic system has the form

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ \Big(\rho(e + \tfrac{1}{2}u^2)\Big)_t + \Big(\rho u(e + \tfrac{1}{2}u^2) + pu + q\Big)_x &= 0, \\ -q_{xx} + aq + b(\theta^4)_x &= 0, \end{aligned} \tag{8}$$

which corresponds to the one dimensional Euler system coupled with an elliptic equation describing radiations in a stationary diffusion regime. In (8), u is the velocity of the fluid, ρ is the mass density and θ denotes the temperature. Likewise, $p = p(\rho, \theta)$ is the pressure and $e = e(\rho, \theta)$ is the internal energy. Both p and e are assumed to be smooth functions of $\rho > 0$, $\theta > 0$ satisfying

$$p_\rho > 0, \quad p_\theta \neq 0, \quad e_\theta > 0.$$

Finally, $q = \rho \chi_x$ is the radiative heat flux, where χ represents the radiative energy, and $a, b > 0$ are positive constants related to absorption. System (8) can be (formally) derived from a more complete system involving a kinetic equation for the specific intensity of radiation. For this derivation and further physical considerations on (8) the reader is referred to [38, 20, 11].

The existence and regularity of traveling wave type solutions of (1) under hypotheses (S0) - (S2), (H0) - (H3) is known, even in the more general case of non-convex velocity fluxes (assumption (H2) does not hold). For details of existence, as well as further properties of the profiles such as monotonicity and regularity under small-amplitude assumption (features which will be used throughout the analysis), the reader is referred to [17, 18].

1.1. Main results

In the spirit of [42, 23, 25, 26], we first consider the linearized equations of (1) about the profile (U, Q):

$$\begin{aligned} u_t + (A(U)u)_x + Lq_x &= 0, \\ -q_{xx} + q + (B(U)u)_x &= 0, \end{aligned} \tag{9}$$

with initial data $u(0) = u_0$. Hence, the Laplace transform applied to system (9) gives

$$\begin{aligned} \lambda u + (A(U)\, u)_x + L q_x &= S, \\ -q_{xx} + q + (B(U)u)_x &= 0, \end{aligned} \tag{10}$$

where source S is the initial data u_0.

As it is customary in related nonlinear wave stability analyses (see, e.g., [1, 34, 42, 39]), we start by studying the underlying spectral problem, namely, the homogeneous version of system (10):

$$\begin{aligned} \lambda u + (A(U)\, u)_x + L q_x &= 0, \\ -q_{xx} + q + (B(U)u)_x &= 0. \end{aligned} \tag{11}$$

An evident necessary condition for orbital stability is the absence of L^2 solutions to (11) for values of λ in $\{\operatorname{Re}\lambda \geq 0\}\backslash\{0\}$, being $\lambda = 0$ the eigenvalue associated to translation invariance. This spectral stability condition can be expressed in terms of the *Evans function*, an analytic function playing a role for differential operators analogous to that played by the characteristic polynomial for finite-dimensional operators (see [1, 34, 3, 42, 23] and the references therein). The main property of the Evans function is that, on the resolvent set of a certain operator $\mathcal{L}$, its zeroes coincide in both location and multiplicity with the eigenvalues of $\mathcal{L}$. Thence, we express the spectral stability condition as follows:

There exists no zero of the Evans function D on $\{\operatorname{Re}\lambda \geq 0\} \setminus \{0\}$; equivalently, there exist no nonzero eigenvalues of $\mathcal{L}$ with $\operatorname{Re}\lambda \geq 0$. (SS)

Like in previous analyses [42, 39, 41], we define the following *stability condition* (or *Evans function condition*) as follows:

There exists precisely one zero (necessarily at $\lambda = 0$; see Lemmas 2.5 - 2.6) of the Evans function on the nonstable half plane $\{\operatorname{Re}\lambda \geq 0\}$, (D)

which implies the spectral stability condition (SS) plus the condition that D vanishes at $\lambda = 0$ at order one. Notice that just like in the scalar case [16], due to the degenerate nature of system (11) (observe that $A(U)$ vanishes at $x = 0$) the number of decaying modes at $\pm\infty$, spanning possible eigenfunctions, depends on the region of space around the singularity. Therefore the definition of D is given in terms of the Evans functions $D_\pm$ in regions $x \gtrless 0$, with same regularity and spectral properties (see its definition in (43) and Lemmas 2.5 - 2.6 below).

Our main result is then as follows.

Theorem 1.4. *Assuming* (5), (S0)–(S2), (H0)–(H3)*, and the spectral stability condition* (D)*, then the Lax radiative shock profile* (U, Q) *with sufficiently small amplitude is asymptotically orbitally stable. More precisely, the solution* $(\tilde{u}, \tilde{q})$ *of* (1) *with initial data* $\tilde{u}_0$ *satisfies*

$$\begin{aligned} |\tilde{u}(x,t) - U(x-\alpha(t))|_{L^p} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4} \\ |\tilde{u}(x,t) - U(x-\alpha(t))|_{H^4} &\le C(1+t)^{-1/4}|u_0|_{L^1\cap H^4} \end{aligned} \tag{12}$$

and

$$\begin{aligned} |\tilde{q}(x,t) - Q(x-\alpha(t))|_{W^{1,p}} &\le C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4} \\ |\tilde{q}(x,t) - Q(x-\alpha(t))|_{H^5} &\le C(1+t)^{-1/4}|u_0|_{L^1\cap H^4} \end{aligned} \tag{13}$$

for initial perturbation $u_0 := \tilde{u}_0 - U$ *that are sufficiently small in* $L^1 \cap H^4$*, for all* $p \ge 2$*, for some* $\alpha(t)$ *satisfying* $\alpha(0) = 0$ *and*

$$\begin{aligned} |\alpha(t)| &\le C|u_0|_{L^1\cap H^4} \\ |\dot{\alpha}(t)| &\le C(1+t)^{-1/2}|u_0|_{L^1\cap H^4}. \end{aligned} \tag{14}$$

Remark 1.5. The time-decay rate of q is not optimal. In fact, it can be improved as we observe that $|q(t)|_{L^2} \le C|u_x(t)|_{L^2}$ and $|u_x(t)|_{L^2}$ is expected to decay like $t^{-1/2}$; however, we omit the detail of carrying this out. Likewise, assuming in addition a small L^1 first moment on the initial perturbation, we could obtain by the approach of [33] the sharpened bounds $|\dot{\alpha}| \le C(1+t)^{\sigma-1}$, and $|\alpha - \alpha(+\infty)| \le C(1+t)^{\sigma-1/2}$, for $\sigma > 0$ arbitrary, including in particular the information that α converges to a specific limit (phase-asymptotic orbital stability); however, we omit this again in favor of simplicity.

We shall prove the following result in appendix Appendix A, verifying Evans condition (D).

Theorem 1.6. *For* $\epsilon := |u_+ - u_-|$ *sufficiently small, radiative shock profiles are spectrally stable.*

Corollary 1.7. *The condition (D) is satisfied for small amplitudes.*

Proof. In Lemmas 2.5 - 2.6 below, we show that $D(\lambda)$ has a single zero at $\lambda = 0$. Together with Theorem 1.6, this gives the result. □

1.1.1. Discussion

Prior to [16], asymptotic stability of radiative shock profiles has been studied in the scalar case in [12] for the particular case of Burgers velocity flux and for linear $g(u) = Mu$, with constant M. Another scalar result is the partial analysis of Serre [36] for the exact Rosenau model (see also [22]). In the case of systems, we mention the stability result of [21] for the full Euler radiating system under special *zero-mass* perturbations, based on an adaptation of the classical energy method of Goodman-Matsumura-Nishihara [4, 28]. Here, we recover for systems, under general (not necessarily zero-mass) perturbations, the sharp rates of decay established in [12] for the scalar case.

We mention that works [12, 16] in the scalar case concerned also *large-amplitude* shock profiles (under the Evans condition (D), automatically satisfied in the Burgers case [12]). At the expense of further effort book-keeping– specifically in the resolution of flow near the singular point and construction of the resolvent– we could obtain by our methods a large-amplitude result similar to that of [16]. However, we greatly simplify the exposition by the small-amplitude assumption allowing us to approximately diagonalize *before* carrying out these steps. As the existence theory is only for small-amplitude shocks, with upper bounds on the amplitudes for which existence holds, known to occur, and since the domain of our hypotheses in [16] does not cover the whole domain of existence in the scalar case (in contrast to [12], which does address the entire domain of existence), we have chosen here for clarity to restrict to the small-amplitude setting. It would be interesting to carry out a large-amplitude analysis valid on the whole domain of existence in the system case.

1.2. Abstract framework

Before beginning the analysis, we orient ourselves with a few simple observations framing the problem in a more standard way. Consider now the inhomogeneous version

$$\begin{aligned} u_t + (A(U)\,u)_x + Lq_x &= g, \\ -q_{xx} + q + (B(U)\,u)_x &= h, \end{aligned} \tag{15}$$

of (9), with initial data $u(x,0) = u_0$. Defining the operator $\mathcal{K} := (-\partial_x^2 + 1)^{-1}$ of order -1, locally compact from L^2 to H^2, and the bounded operator

$$\mathcal{J} := \partial_x L \mathcal{K} \partial_x B(U)$$

of order 0, we may rewrite this as a nonlocal equation

$$\begin{aligned} u_t + (A(U)\,u)_x + \mathcal{J}u &= \partial_x L\mathcal{K}h + g, \\ u(x,0) &= u_0(x) \end{aligned} \tag{16}$$

in u alone, recovering q by

$$q = -\mathcal{K}\partial_x B(U)u + \mathcal{K}h. \tag{17}$$

The generator $\mathcal{L} := -(A(U)\,u)_x - \mathcal{J}u$ of (16) is a zero-order perturbation of the generator $-A(U)u_x$ of a hyperbolic equation, so generates a C^0 semigroup $e^{\mathcal{L}t}$ and an associated Green distribution $G(x,t;y) := e^{\mathcal{L}t}\delta_y(x)$. Moreover, $e^{\mathcal{L}t}$ and G may be expressed through the inverse Laplace transform formulae

$$\begin{aligned} e^{\mathcal{L}t} &= \frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t}(\lambda - \mathcal{L})^{-1} d\lambda, \\ G(x,t;y) &= \frac{1}{2\pi i}\int_{\eta - i\infty}^{\eta + i\infty} e^{\lambda t} G_\lambda(x,y) d\lambda, \end{aligned} \tag{18}$$

for all $\eta \geq \eta_0$, where $G_\lambda(x,y) := (\lambda - \mathcal{L})^{-1}\delta_y(x)$ is the resolvent kernel of $\mathcal{L}$.

Collecting information, we may write the solution of (15) using Duhamel's principle/variation of constants as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} G(x,t;y)u_0(y)dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)(\partial_x L\mathcal{K}h + g)(y,s)\,dy\,ds, \\ q(x,t) &= \Big((-\mathcal{K}\partial_x B(U))u + \mathcal{K}h\Big)(x,t), \end{aligned} \tag{19}$$

where G is determined through (18).

That is, the solution of the linearized problem reduces to finding the Green kernel for the u-equation alone, which in turn amounts to solving the resolvent equation for $\mathcal{L}$ with delta-function data, or, equivalently, solving the differential equation (10) with source $S = \delta_y(x)$. This we shall do in standard fashion by writing (10) as a first-order system and solving appropriate jump conditions at y obtained by the requirement that G_λ be a distributional solution of the resolvent equations.

This procedure is greatly complicated by the circumstance that the resulting $(n+2)\times(n+2)$ first-order system

$$\Theta(x)W_x = \mathbb{A}(x,\lambda)W \tag{20}$$

is *singular* at the special point where $A(U)$ vanishes (see Remark 1.9 below), with Θ dropping to rank $n+1$. However, in the end we find as usual that G_λ is uniquely determined by these criteria, not only for the values $\operatorname{Re}\lambda \geq \eta_0 > 0$ guaranteed by C^0-semigroup theory/energy estimates, but, as in the usual nonsingular case [7], on the *set of consistent splitting* for the first-order system (20), which includes all of $\{\operatorname{Re}\lambda \geq 0\} \setminus \{0\}$. This has the implication that the essential spectrum of $\mathcal{L}$ is confined to $\{\operatorname{Re}\lambda < 0\} \cup \{0\}$.

Remark 1.8. The fact (obtained by energy-based resolvent estimates) that $\mathcal{L}-\lambda$ is coercive for $\operatorname{Re}\lambda \geq \eta_0$ shows by elliptic theory that the resolvent is well-defined and unique in class of distributions for $\operatorname{Re}\lambda$ large, and thus the resolvent kernel may be determined by the usual construction using appropriate jump conditions. That is, from standard considerations, we see that the construction *must* work, despite the apparent wrong dimensions of decaying manifolds (which happens for any $\operatorname{Re}\lambda > 0$).

Remark 1.9. Recall that Lax entropy condition reads $a_p^+ < 0 < a_{p+1}^+$ and $a_{p-1}^- < 0 < a_p^-$. This, together with continuity of $a_p(x) = a_p(U(x))$ (from regularity of profile $U(x)$ of the existence theory [17, 18]), shows that $a_p(x_0) = 0$ at some point x_0 which, without loss of generality, we take as $x_0 = 0$. Thus, the p-Lax eigenvalue $a_p(x)$ connects $a_p^- > 0$ with $a_p^+ < 0$ at $x = \pm\infty$. The degeneracy occurs naturally as the coefficient matrix $A(U)$ of the highest order derivative of u with respect to x in (16) becomes singular at that point. When the associated spectral system is written in first order form (20), this degeneracy shows up in the drop of rank of the coefficient matrix Θ. To deal with the singularity of the first-order system is the most delicate and novel part of the present analysis (see also the analysis of the scalar model [16]). It is our hope that the methods we use here may be of use also in other situations where the resolvent equation becomes singular, for example in the closely related situation of relaxation systems discussed in [23, 26].

Remark 1.10. Comparing to the concurent analysis in the scalar model [16], the main difference when constructing the Green function in the system case is that we now have n slow modes at each far field (i.e., at $x = \pm\infty$), rather than only one in scalar case, and one of these modes will have to connect to the mode having the degeneracy at $x = 0$ in the above sense (the vanishing p-Lax eigenvalue). A priori, we were not sure how to treat such situations. However, as it turns out, we can diagonalize the system with acceptable error of the same order zero as terms coming from $\mathcal{J}u$, and thus, are able to treat the slow mode with degeneracy as in the scalar case. We emphasize that the wrong number of decaying modes clearly remains in the system case, but the cure for it remains as well: construction of full sets of decaying modes at each side of the singularity.

Another significant difference in the analysis is that the verification of the spectral stability condition is not straightforward as for the scalar problem; indeed, here we have to use a combination of Kawashima- and weighted Goodman-type energy estimates (see appendix Appendix A), which work (only) for small amplitudes; we observe that in [16], we (and co-authors) were also able to verify the condition for arbitrary amplitudes in the existence region of the profile, provided that b is linear. In addition, the fact that LB is not strictly positive required special attention while obtaining the damping nonlinear energy estimates, in contrast with the scalar case.

Plan of the paper

This work is structured as follows. Section 2 pertains to the construction of the resolvent kernel, based on the study of the solutions to the eigenvalue equations both near and away from the singularity. In section 3 we establish bounds for the resolvent kernel in low-frequency regions. Section 4 contains the analysis towards pointwise bounds for the low-frequency Green function, based on the spectral resolution formulae. The auxiliary damping energy estimate and the high-frequency estimate are the content of section 5. The final section 6 blends all the previous estimations into the proof of the main nonlinear Theorem 1.4. Appendix Appendix A contains the verification of the spectral stability condition in the small amplitude regime, whereas appendix Appendix B contains a pointwise extension of the tracking lemma of [24].

2. Construction of the resolvent kernel

2.1. Outline

In what follows we shall denote $' = \partial_x$ for simplicity; we also write $A(x) = A(U)$ and $B(x) = B(U)$. Let us now construct the resolvent kernel for $\mathcal{L}$, or equivalently, the solution of (10) with delta-function source in the u component. The novelty in the present case is the extension of this standard method to a situation in which the spectral problem can only be written as a *degenerate* first order ODE. Unlike the real viscosity and relaxation cases [23, 24, 25, 26] (where the operator L, although degenerate, yields a non-degenerate first order ODE in an appropriate reduced space), here we deal with a system of form

$$\Theta W' = \mathbb{A}(x, \lambda) W,$$

where

$$\Theta = \begin{pmatrix} A & \\ & I_2 \end{pmatrix},$$

is degenerate at $x = 0$.

To construct the resolvent kernel we solve

$$(\Theta\partial_x - \mathbb{A}(x, \lambda))\mathcal{G}_\lambda(x, y) = \delta_y(x), \tag{21}$$

in the distributional sense, so that

$$(\Theta\partial_x - \mathbb{A}(x, \lambda))\mathcal{G}_\lambda(x, y) = 0, \tag{22}$$

in the distributional sense for all $x \neq y$ with appropriate jump conditions (to be determined) at $x = y$. The first entry of the three-vector $\mathcal{G}_\lambda$ is the resolvent kernel G_λ of $\mathcal{L}$ that we seek.

Namely $\mathcal{G}_\lambda$, is the solution in the sense of distribution of system (10) (written in conservation form):

$$\begin{aligned} (Au)' &= -\left(\lambda + LB\right) u + Lp + \delta_y(x) \\ q' &= Bu - p \\ p' &= -q. \end{aligned} \tag{23}$$

2.2. Asymptotic behavior

First, we study at the asymptotic behavior of solutions to the spectral system

$$\begin{aligned} (A(x)u)' &= -(\lambda + LB(x))u + Lp, \\ q' &= B(x)u - p, \\ p' &= -q, \end{aligned} \tag{24}$$

away from the singularity at $x = 0$, and for values of $\lambda \neq 0$, $\operatorname{Re}\lambda \geq 0$. We pay special attention to the small frequency regime, $\lambda \sim 0$. First, we diagonalize A as

$$\tilde{A} := L_p A R_p = \begin{pmatrix} A_1^- & & 0 \\ & a_p & \\ 0 & & A_2^+ \end{pmatrix} \tag{25}$$

where $A_1^- \leq -\theta < 0$, $A_2^+ \geq \theta > 0$, and $a_p \in \mathbb{R}$, satisfying $a_p(+\infty) < 0 < a_p(-\infty)$. Here, L_p, R_p are bounded matrices and $L_pR_p = I$. Defining $v := L_pu$, we rewrite (24) as

$$\begin{aligned}(\tilde{A}(x)v)' &= -(\lambda + \tilde{L}\tilde{B} + L_p'AR_p)v + \tilde{L}p,\\ q' &= \tilde{B}v - p,\\ p' &= -q,\end{aligned} \tag{26}$$

where $\tilde{L} := L_pL$ and $\tilde{B} := BR_p$. Denote the limits of the coefficient as

$$\tilde{A}_\pm := \lim_{x\to\pm\infty} \tilde{A}(x), \qquad \tilde{B}_\pm := \lim_{x\to\pm\infty} B(x)R_p. \tag{27}$$

The asymptotic system thus can be written as

$$W' = \mathbb{A}_\pm(\lambda)W, \tag{28}$$

where $W = (v, q, p)^\top$, and

$$\mathbb{A}_\pm(\lambda) = \begin{pmatrix} -\tilde{A}_\pm^{-1}(\lambda + \tilde{L}_\pm\tilde{B}_\pm) & 0 & \tilde{A}_\pm^{-1}\tilde{L}_\pm \\ \tilde{B}_\pm & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{29}$$

To determine the dimensions of the stable/unstable eigenspaces, let $\lambda \in \mathbb{R}^+$ and $\lambda \to 0, +\infty$, respectively. The 2×2 lower right-corner matrix clearly gives one strictly positive and one strictly negative eigenvalues (this later will give one fast-decaying and one fast-growing modes). In the "slow" system (as $|\lambda| \to 0$), eigenvalues are

$$\mu_j^\pm(\lambda) = -\lambda/a_j^\pm + \mathcal{O}(\lambda^2), \tag{30}$$

where $a_j^\pm$ are eigenvalues of $A_\pm = A(\pm\infty)$. Thus, we readily conclude that at $x = +\infty$, there are $p+1$ unstable eigenvalues and $n-p+1$ stable eigenvalues. The stable $S^+(\lambda)$ and unstable $U^+(\lambda)$ manifolds (solutions which decay, respectively, grow at $+\infty$) have, thus, dimensions

$$\begin{aligned}\dim U^+(\lambda) &= p + 1,\\ \dim S^+(\lambda) &= n - p + 1,\end{aligned} \tag{31}$$

in $\operatorname{Re}\lambda > 0$. Likewise, there exist $n - p + 2$ unstable eigenvalues and p stable eigenvalues so that the stable (solutions which grow at $-\infty$) and unstable (solutions which decay at $-\infty$) manifolds have dimensions

$$\begin{aligned}\dim U^-(\lambda) &= p,\\ \dim S^-(\lambda) &= n - p + 2.\end{aligned} \tag{32}$$

Remark 2.1. Notice that, unlike customary situations in the Evans function literature [1, 42, 3, 23, 24, 34], here the dimensions of the stable (resp. unstable) manifolds S^+ and S^- (resp. U^+ and U^-) *do not agree.* Under these considerations, we look at the dispersion relation

$$\pi_\pm(i\xi) = \det\big(-\tilde{A}_\pm(\lambda + \tilde{L}_\pm\tilde{B}_\pm)\xi^2 - i\xi(1+\xi^2)I - \lambda\tilde{A}_\pm^{-1}\big) = 0.$$

For each $\xi \in \mathbb{R}$, the λ-roots of the last equation define algebraic curves

$$\lambda_j^{\pm}(\xi) \in \sigma\big((1+\xi^2)^{-1}(-\xi^2 \tilde{L}_{\pm}\tilde{B}_{\pm} - i\xi(1+\xi^2)\tilde{A}_{\pm})\big),$$

touching the origin at $\xi = 0$. Denote Λ as the open connected subset of $\mathbb{C}$ bounded on the left by the rightmost envelope of the curves $\lambda_j^{\pm}(\xi)$, $\xi \in \mathbb{R}$. Note that the set $\{\operatorname{Re}\lambda \geq 0, \lambda \neq 0\}$ is properly contained in Λ. By connectedness the dimensions of $U^{\pm}(\lambda)$ and $S^{\pm}(\lambda)$ do not change in $\lambda \in \Lambda$. We define Λ as the set of *(not so) consistent splitting* [1], in which the matrices $\mathbb{A}_{\pm}(\lambda)$ remain hyperbolic, with not necessarily agreeing dimensions of stable (resp. unstable) manifolds.

Notably, the degeneracy of the spectral system shows that there is one slowly growing mode defined on $x > 0$ which will vanish as it travels pass the singularity point $x = 0$. The same happens for the slow-decaying mode at $x = -\infty$. It might be useful to the reader to think of it as some loss of information when passing the singularity point. This phenomenon yields the non-agreeing dimensions of the stable (resp. unstable) manifolds S^+ and S^- (resp. U^+ and U^-).

Lemma 2.2. *For each $\lambda \in \Lambda$, the spectral system* (28) *associated to the limiting, constant coefficients asymptotic behavior of* (24)*, has a basis of solutions*

$$e^{\mu_j^{\pm}(\lambda)x} V_j^{\pm}(\lambda), \quad x \gtrless 0,\ j = 1, ..., n+2.$$

Moreover, for $|\lambda| \sim 0$, we can find analytic representations for $\mu_j^{\pm}$ and $V_j^{\pm}$, which consist of $2n$ slow modes

$$\mu_j^{\pm}(\lambda) = -\lambda/a_j^{\pm} + \mathcal{O}(\lambda^2), \qquad j = 2, ..., n+1,$$

and four fast modes,

$$\begin{aligned} \mu_1^{\pm}(\lambda) &= \pm\theta_1^{\pm} + \mathcal{O}(\lambda), \\ \mu_{n+2}^{\pm}(\lambda) &= \mp\theta_{n+2}^{\pm} + \mathcal{O}(\lambda). \end{aligned}$$

where $\theta_1^{\pm}$ and $\theta_{n+2}^{\pm}$ are positive constants.

In view of the structure of the asymptotic systems, we are able to conclude that for each initial condition $x_0 > 0$, the solutions to (24) in $x \geq x_0$ are spanned by decaying/growing modes

$$\begin{aligned} \Phi^+ &: = \{\phi_1^+, ..., \phi_{n-p+1}^+\}, \\ \Psi^+ &: = \{\psi_{n-p+2}^+, ..., \psi_{n+2}^+\}, \end{aligned} \tag{33}$$

as $x \to +\infty$, whereas for each initial condition $x_0 < 0$, the solutions to (24) are spanned in $x < x_0$ by growing/decaying modes

$$\begin{aligned} \Psi^- &: = \{\psi_1^-, ..., \psi_{n-p+2}^-\}, \\ \Phi^- &: = \{\phi_{n-p+3}^-, ..., \phi_{n+2}^-\}, \end{aligned} \tag{34}$$

as $x \to -\infty$.

We rely on the conjugation lemma of [30] to link such modes to those of the limiting constant coefficient system (28).

Lemma 2.3. *For $|\lambda|$ sufficiently small, there exist growing and decaying solutions $\psi_j^\pm(x,\lambda), \phi_j^\pm(x,\lambda)$, in $x \gtrless 0$, of class C^1 in x and analytic in λ, satisfying*

$$\begin{aligned} \psi_j^\pm(x,\lambda) &= e^{\mu_j^\pm(\lambda)x} V_j^\pm(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \\ \phi_j^\pm(x,\lambda) &= e^{\mu_j^\pm(\lambda)x} V_j^\pm(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \end{aligned} \tag{35}$$

where $0 < \eta$ is the decay rate of the traveling wave, and $\mu_j^\pm$ and $V_j^\pm$ are as in Lemma 2.2 above.

Proof. This a direct application of the conjugation lemma of [30] (see also the related gap lemma in [3, 42, 23, 24]). ☐

2.3. Solutions near $x \sim 0$

Our goal now is to analyze system (24) close to the singularity $x = 0$. To fix ideas, let us again stick to the case $x > 0$, the case $x < 0$ being equivalent. We introduce a "stretched" variable ξ as follows:

$$\xi = \int_1^x \frac{dz}{a_p(z)},$$

so that $\xi(1) = 0$, and $\xi \to +\infty$ as $x \to 0^+$. Under this change of variables we get

$$u' = \frac{du}{dx} = \frac{1}{a_p(x)} \frac{du}{d\xi} = \frac{1}{a_p(x)} \dot{u},$$

and denoting $\dot{} = d/d\xi$. In the stretched variables, making some further changes of variables if necessary, the system (26) becomes a block-diagonalized system at leading order of the form

$$\dot{Z} = \begin{pmatrix} -\alpha I & 0 \\ 0 & 0 \end{pmatrix} Z + a_p(\xi)\check{\Theta}(\xi) Z, \tag{36}$$

where $\check{\Theta}(\xi)$ is some bounded matrix and α is the (p,p) entry of the matrix $\lambda + \tilde{L}\tilde{B} + L_p' A R_p + \tilde{A}'$, noting that

$$\alpha(\xi) \geq \delta_0 > 0,$$

for some δ_0 and any ξ sufficiently large or x sufficiently near zero.

The blocks $-\alpha I$ and 0 are clearly spectrally separated and the error is of order $\mathcal{O}(|a_p(\xi)|) \to 0$ as $\xi \to +\infty$. By the pointwise reduction lemma (see Lemma Appendix B.1 and Remark Appendix B.2 below), we can separate the flow into slow and fast coordinates. Indeed, after proper transformations we separate the flows on the reduced manifolds of form

$$\dot{Z}_1 = -\alpha Z_1 + \mathcal{O}(a_p) Z_1, \tag{37}$$

$$\dot{Z}_2 = \mathcal{O}(a_p) Z_2. \tag{38}$$

Since $-\alpha \leq -\delta_0 < 0$ for $\lambda \sim 0$ and $\xi \geq 1/\epsilon$, with $\epsilon > 0$ sufficiently small, and since $a_p(\xi) \to 0$ as $\xi \to +\infty$, the Z_1 mode decay to zero as $\xi \to +\infty$, in view of

$$e^{-\int_0^\xi \alpha(z)\,dz} \lesssim e^{-(\mathrm{Re}\,\lambda + \frac{1}{2}\delta_0)\xi}.$$

These fast decaying modes correspond to fast decaying to zero solutions when $x \to 0^+$ in the original u-variable. The Z_2 modes comprise slow dynamics of the flow as $x \to 0^+$. Hence we have the following

Proposition 2.4. *There exists $0 < \epsilon_0 \ll 1$ sufficiently small, such that, in the small frequency regime $\lambda \sim 0$, the solutions to the spectral system* (24) *in $(-\epsilon_0, 0) \cup (0, \epsilon_0)$ are spanned by fast modes*

$$w_{k_p}^{\pm}(x,\lambda) = \begin{pmatrix} \tilde{u}_{k_p}^{\pm} \\ \tilde{q}_{k_p}^{\pm} \\ \tilde{p}_{k_p}^{\pm} \end{pmatrix} \qquad \pm\,\epsilon_0 \gtrless x \gtrless 0, \tag{39}$$

decaying to zero as $x \to 0^{\pm}$ with $k_p := n - p + 2$, and slowly varying modes

$$z_j^{\pm}(x,\lambda) = \begin{pmatrix} \tilde{u}_j^{\pm} \\ \tilde{q}_j^{\pm} \\ \tilde{p}_j^{\pm} \end{pmatrix}, \qquad \pm\epsilon_0 \gtrless x \gtrless 0, \;\; j \neq k_p, \tag{40}$$

with bounded limits as $x \to 0^{\pm}$.

Moreover, the fast modes (39) *decay as*

$$\tilde{u}_{k_p p}^{\pm} \sim |x|^{\alpha_0} \to 0 \tag{41}$$

and

$$\begin{pmatrix} \tilde{u}_{k_p j}^{\pm} \\ \tilde{q}_{k_p}^{\pm} \\ \tilde{p}_{k_p}^{\pm} \end{pmatrix} \sim \mathcal{O}(|x|^{\alpha_0} a_p(x)) \to 0, \qquad j \neq p, \tag{42}$$

as $x \to 0^{\pm}$; here, α_0 is some positive constant and $u_{k_p} = (u_{k_p 1}, ..., u_{k_p p}, ..., u_{k_p n})^\top$.

2.4. Two Evans functions

We first define the following related Evans functions

$$D_{\pm}(y,\lambda) := \det(\Phi^+ \; W_{k_p}^{\mp} \; \Phi^-)(y,\lambda), \qquad \text{for } y \gtrless 0, \tag{43}$$

where $\Phi^{\pm}$ are defined as in (33), (34), and $W_{k_p}^{\pm} = (u_{k_p}^{\pm}, q_{k_p}^{\pm}, p_{k_p}^{\pm})^\top$ are defined as in (39). Note that k_p here is always fixed and equals to $n - p + 2$.

We first observe the following simple properties of $D_{\pm}$.

Lemma 2.5. *For λ sufficiently small, we have*

$$D_{\pm}(y,\lambda) = (\det A)^{-1} \gamma_{\pm}(y) \Delta\lambda + \mathcal{O}(|\lambda|^2), \tag{44}$$

where

$$
\begin{aligned}
\Delta &:= \det \begin{pmatrix} r_2^+ & \cdots & r_{k_p-1}^+ & r_{k_p+1}^- & \cdots & r_{n+1}^- & -[u] \end{pmatrix} \\
\gamma_\pm(y) &:= \det \begin{pmatrix} q_1^+ & q_{k_p}^\mp \\ p_1^+ & p_{k_p}^\mp \end{pmatrix}_{|\lambda=0}
\end{aligned}
\tag{45}
$$

with $[u] = u_+ - u_-$ *and* $r_j^\pm$ *eigenvectors of* $(A_\pm)^{-1}(LB)_\pm$*, spanning the stable/unstable subspaces at* $\pm\infty$*, respectively.*

Proof. By our choice, at $\lambda = 0$, we can take

$$
\phi_1^+(x,0) = \phi_{n+2}^-(x,0) = \bar{W}_x(x) \tag{46}
$$

where $\bar{W}$ is the shock profile. By Leibnitz' rule and using (46), we compute

$$
\begin{aligned}
\partial_\lambda D_-(y,0) = \det &\left(\partial_\lambda \phi_1^+, ..., \phi_{k_p-1}^+, W_{k_p}^+, \phi_{k_p+1}^-, ..., \phi_{n+2}^-\right)_{|\lambda=0} + \cdots \\
&\cdots + \det\left(\phi_1^+, ..., \phi_{k_p-1}^+, W_{k_p}^+, \phi_{k_p+1}^-, ..., \partial_\lambda \phi_{n+2}^-\right)_{|\lambda=0},
\end{aligned}
$$

where, by using (46), only the first and third terms are possibly nonvanishing and thus grouped together, yielding

$$
\partial_\lambda D_-(y,0) = \det\left(\phi_1^+, ..., \phi_{k_p-1}^+, W_{k_p}^+, \phi_{k_p+1}^-, ..., \phi_{n+1}^-, \partial_\lambda \phi_{n+2}^- - \partial_\lambda \phi_1^+\right)_{|\lambda=0}. \tag{47}
$$

Recall that $W_{k_p}^+, \phi_j^\pm$ satisfy

$$
\Theta W_x = \mathbb{A}(x,\lambda) W, \tag{48}
$$

where $W = (u,q,p)$ and

$$
\Theta = \begin{pmatrix} A & \\ & I_2 \end{pmatrix}.
$$

Thus, $\partial_\lambda \phi_1^+(x,\lambda)$ satisfies

$$
\Theta(\partial_\lambda \phi_1^+)_x = \mathbb{A}(x,0)\partial_\lambda \phi_1^+(x,0) + \partial_\lambda \mathbb{A}(x,0)\phi_1^+(x,0),
$$

which directly gives

$$
(a\partial_\lambda u_1^+)_x = -L(\partial_\lambda q_1^+)_x - \bar{u}_x. \tag{49}
$$

Likewise, $\partial_\lambda \phi_{n+2}^-(x,\lambda) = (\partial_\lambda u_{n+2}^-, \partial_\lambda q_{n+2}^-, \partial_\lambda p_{n+2}^-)$ satisfies

$$
(a\partial_\lambda u_{n+2}^-)_x = -L(\partial_\lambda q_{n+2}^-)_x - \bar{u}_x. \tag{50}
$$

Integrating equations (49) and (50) from $+\infty$ and $-\infty$, respectively, with use of boundary conditions $\partial_\lambda \phi_1^+(+\infty) = \partial_\lambda \phi_{n+2}^-(-\infty) = 0$, we obtain

$$
\begin{aligned}
A\partial_\lambda u_1^+ &= -L\partial_\lambda q_1^+ - \bar{u} + u_+ \\
A\partial_\lambda u_{n+2}^- &= -L\partial_\lambda q_{n+2}^- - \bar{u} + u_-.
\end{aligned}
\tag{51}
$$

and thus

$$A(\partial_\lambda u_{n+2}^- - \partial_\lambda u_1^+) = -L(\partial_\lambda q_{n+2}^- - \partial_\lambda q_1^+) - [u]. \tag{52}$$

In addition, we note that $W_{k_p}^+, \phi_j^\pm$ satisfy the equation (48) and thus $(Au)' = -Lq'$ with $W_{k_p}^+(+\infty) = \phi_1^+(+\infty) = 0$, $\phi_{n+2}^-(-\infty) = 0$, $q_j^\pm(\pm\infty) = 0$, and

$$\begin{aligned} u_j^+(+\infty) &= (A_+)^{-1} r_j^+, \qquad j = 2, ..., k_p - 1 \\ u_j^-(-\infty) &= (A_-)^{-1} r_j^-, \qquad j = k_p + 1, ..., n + 1. \end{aligned}$$

Thus, we integrate the equation $(Au)' = -Lq'$, yielding

$$\begin{aligned} Au_j^+ &= -Lq_j^+, \qquad \text{for } j = 1, k_p, \\ Au_j^+ &= -Lq_j^+ + r_j^+, \qquad \text{for } j = 2, ..., k_p - 1, \\ Au_j^- &= -Lq_j^- + r_j^-, \qquad \text{for } j = k_p + 1, ..., n + 1 \\ Au_j^- &= -Lq_j^-, \qquad \text{for } j = n + 2. \end{aligned} \tag{53}$$

Using estimates (53) and (52), we can now compute the λ-derivative (47) of $D_\pm$ at $\lambda = 0$ as

$$\begin{aligned} \partial_\lambda D_-(y, 0) &= \det \begin{pmatrix} u_1^+ & \cdots & u_j^+ & \cdots & u_{k_p}^+ & \cdots & u_j^- & \cdots & \partial_\lambda u_{n+2}^- - \partial_\lambda u_1^+ \\ q_1^+ & \cdots & q_j^+ & \cdots & q_{k_p}^+ & \cdots & q_j^- & \cdots & \partial_\lambda q_{n+2}^- - \partial_\lambda q_1^+ \\ p_1^+ & \cdots & p_j^+ & \cdots & p_{k_p}^+ & \cdots & p_j^- & \cdots & \partial_\lambda p_{n+2}^- - \partial_\lambda p_1^+ \end{pmatrix} \\ &= (\det A)^{-1} \det \begin{pmatrix} 0 & \cdots & r_j^+ & \cdots & 0 & \cdots & r_j^- & \cdots & -[u] \\ q_1^+ & \cdots & q_j^+ & \cdots & q_{k_p}^+ & \cdots & q_j^- & \cdots & \partial_\lambda q_{n+2}^- - \partial_\lambda q_1^+ \\ p_1^+ & \cdots & p_j^+ & \cdots & p_{k_p}^+ & \cdots & p_j^- & \cdots & \partial_\lambda p_{n+2}^- - \partial_\lambda p_1^+ \end{pmatrix} \\ &= (\det A)^{-1} \det \begin{pmatrix} q_1^+ & q_{k_p}^+ \\ p_1^+ & p_{k_p}^+ \end{pmatrix} \det \begin{pmatrix} r_2^+ & \cdots & r_{k_p-1}^+ & r_{k_p+1}^- & \cdots & r_{n+1}^- & -[u] \end{pmatrix} \end{aligned} \tag{54}$$

which proves (44). The proof for D_+ follows similarly. ◻

Lemma 2.6. *Defining the Evans functions*

$$D_\pm(\lambda) := D_\pm(\pm 1, \lambda), \tag{55}$$

we then have

$$D_+(\lambda) = mD_-(\lambda) + \mathcal{O}(|\lambda|^2) \tag{56}$$

where m is some nonzero factor.

Proof. Proposition 2.4 gives

$$w_{k_p}^\pm(x) = \begin{pmatrix} \tilde{u}_{k_p}^\pm \\ \tilde{q}_{k_p}^\pm \\ \tilde{p}_{k_p}^\pm \end{pmatrix} = \mathcal{O}(|x|^{\alpha_0}), \tag{57}$$

as $x \to 0$, where α_0 is defined as in Proposition 2.4, which guarantees an existence of positive constants ϵ_1, ϵ_2 near zero such that

$$w^+_{k_p}(-\epsilon_1) = w^-_{k_p}(+\epsilon_2).$$

Thus, this together with the fact that $w^\pm_{k_p}$ are solutions of the ODE (48) yields

$$w^+_{k_p}(-1) = m_{k_p} w^-_{k_p}(+1)$$

for some nonzero constant m_{k_p}. Putting these estimates into (44) and using analyticity of $D_\pm$ in λ near zero, we easily obtain the conclusion. □

Remark 2.7. Since at both sides of the singularity the Evans functions $D_\pm$ are constructed by means of a Wronskian of a full set of decaying modes, we are able to evaluate each function at any point on each side, say, $y = \pm 1$; note that the number of λ-zeroes of $D_\pm(\cdot, y)$ are the same at each side of the singularity, for any $y \gtrless 0$.

3. Resolvent kernel bounds in low–frequency regions

In this section, we shall derive pointwise bounds on the resolvent kernel $G_\lambda(x, y)$ in low-frequency regimes, that is, $|\lambda| \to 0$. For definiteness, throughout this section, we consider only the case $y < 0$. The case $y > 0$ is completely analogous by symmetry.

We solve (23) with the jump conditions at $x = y$:

$$[\mathcal{G}_\lambda(., y)] = \begin{pmatrix} A(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{58}$$

where, working on diagonalized coordinates (see (26)), we can assume that A is of diagonal form as in (25),

$$A = \begin{pmatrix} A_1^- & & 0 \\ & a_p & \\ 0 & & A_2^+ \end{pmatrix},$$

with $A_1^- \le -\theta < 0, A_2^+(y) \ge \theta > 0$. Meanwhile, we can write $\mathcal{G}_\lambda(x, y)$ in terms of decaying solutions at $\pm\infty$ as follows

$$\mathcal{G}_\lambda(x, y) = \begin{cases} \Phi^+(x, \lambda) C^+(y, \lambda) + W^+_{k_p}(x, \lambda) C^+_{k_p}(y, \lambda), & x > y, \\ -\Phi^-(x, \lambda) C^-(y, \lambda), & x < y. \end{cases} \tag{59}$$

where $C_j^\pm$ are row vectors. We compute the coefficients $C_j^\pm$ by means of the transmission conditions (58) at y. Therefore, solving by Cramer's rule the system

$$\begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix} \begin{pmatrix} C^+ \\ C^+_{k_p} \\ C^- \end{pmatrix}_{|(y, \lambda)} = \begin{pmatrix} A(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{60}$$

we readily obtain,

$$\begin{pmatrix} C^+ \\ C^+_{k_p} \\ C^- \end{pmatrix} (y,\lambda) = D_-(y,\lambda)^{-1} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{adj} \begin{pmatrix} A(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{61}$$

where M^{adj} denotes the adjugate matrix of a matrix M. Note that

$$C^\pm_{jp}(y,\lambda) = a_p(y)^{-1} D_-(y,\lambda)^{-1} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{pj} (y,\lambda), \tag{62}$$

$$C^\pm_{jl}(y,\lambda) = \sum_k D_-(y,\lambda)^{-1} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{kj} (y,\lambda)(A(y)^{-1})_{kl}, \qquad l \neq p, \tag{63}$$

where $()^{ij}$ is the determinant of the (i,j) minor, and $(A(y)^{-1})_{kl}$, $l \neq p$, are bounded in y.

We then easily obtain the following.

Lemma 3.1. *For y near zero, we have*

$$\begin{aligned} C^+_1(y,\lambda) &= \frac{1}{\lambda} v_0([u]) + \mathcal{O}(1), \\ C^-_{n+2}(y,\lambda) &= -\frac{1}{\lambda} v_0([u]) + \mathcal{O}(1), \end{aligned} \tag{64}$$

where $v_0([u])$ is some constant vector depending only on $[u]$ and

$$\begin{aligned} C^+_{k_p}(y,\lambda) &= a_p(y)^{-1}|y|^{-\alpha_0}\mathcal{O}(1), \\ C^+_j(y,\lambda) &= \mathcal{O}(1) \qquad 1 < j < k_p, \\ C^-_j(y,\lambda) &= \mathcal{O}(1) \quad k_p < j < n+2, \end{aligned} \tag{65}$$

where $k_p = n-p+2$, α_0 is defined as in Proposition 2.4 and $\mathcal{O}(1)$ is a uniformly bounded function, probably depending on y and λ.

Proof. We shall first estimate $C^-_{n+2,p}(y,\lambda)$. Observe that

$$\begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{p,n+2} (y,\lambda) = \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{p,n+2} (y,0) + \mathcal{O}(\lambda)$$

where by the same way as done in Lemma 2.5 we obtain an estimate

$$\begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{p,n+2} (y,0) = a_p(\det A)^{-1}\gamma_-(y)\Delta^{p,n+2},$$

where $\gamma_-(y)$ and Δ are defined as in (45), and $\Delta^{p,n+2}$ denotes the minor determinant. Thus, recalling (44) and (62), we can estimate $C^-_{n+2,p}(y,\lambda)$ as

$$\begin{aligned} C^-_{n+2,p}(y,\lambda) &= a_p(y)^{-1} D_-(y,\lambda)^{-1} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{p,n+2} (y,\lambda) \\ &= -\frac{1}{\lambda}\Delta^{-1}\Delta^{p,n+2} + \mathcal{O}(1), \end{aligned}$$

where $\mathcal{O}(1)$ is uniformly bounded since $a_p(y)D_-(y,\lambda)$ and normal modes $\phi_j^\pm$ are all bounded uniformly in y near zero. Similar computations can be done for $C^-_{n+2,l}(y,\lambda)$. Thus, we obtain the bound for C^-_{n+2} as claimed. The bound for C_1^+ follows similarly, noting that $\phi^-_{n+2} \equiv \phi_1^+$ at $\lambda = 0$.

For the estimate on $C^+_{k_p}$, we first observe that by view of (44), with noting that $\det(A) \sim a_p(y)$ as $|y| \to 0$, and the estimate (39) on $w^+_{k_p p}$,

$$|D_-(y,\lambda)| \geq \theta|\lambda||y|^{\alpha_0}, \tag{66}$$

for some $\theta > 0$. This together with the fact that $\phi^-_{n+2} \equiv \phi_1^+$ at $\lambda = 0$ yields the estimate for $C^+_{k_p}$ as claimed.

We next estimate C_j^+ (resp. C_j^-) for $1 < j < k_p$ (resp. $k_p < j < n+2$). We note that by view of estimate (39) on W_{k_p},

$$\begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{pj} = \mathcal{O}(\lambda)\mathcal{O}(|y|^{\alpha_0} a_p(y))$$

and for $k \neq p$,

$$\begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{kj} = \mathcal{O}(\lambda)\mathcal{O}(|y|^{\alpha_0})$$

These estimates together with (66) and (63),(62) immediately yield estimates for $C_j^\pm$ as claimed. □

Proposition 3.2 (Resolvent kernel bounds as $|y| \to 0$)**.** *For y near zero, there hold*

$$G_\lambda(x,y) = \lambda^{-1}\bar{W}_x v_0([u]) + \mathcal{O}(1)\sum_{a_j^+>0} e^{-(\lambda/a_j^+ + \mathcal{O}(\lambda^2))x} + \mathcal{O}(e^{-\theta|x|}) \tag{67}$$

for $y < 0 < x$, and

$$G_\lambda(x,y) = \lambda^{-1}\bar{W}_x v_0([u]) + \mathcal{O}(1)\Big(1 + \frac{|x|^\alpha}{a_p(y)|y|^\alpha}\Big) \tag{68}$$

for $y < x < 0$, and

$$G_\lambda(x,y) = \lambda^{-1}\bar{W}_x v_0([u]) + \mathcal{O}(1)\sum_{a_j^-<0} e^{-(\lambda/a_j^- + \mathcal{O}(\lambda^2))x} + \mathcal{O}(e^{-\theta|x|}) \tag{69}$$

for $x < y < 0$.

Similar bounds can be obtained for the case $y > 0$.

Proof. For the case $y < 0 < x$, using (64) and recalling that $\phi_1^+(x) = \bar{W}_x + \mathcal{O}(\lambda)e^{-\theta|x|}$ and $W^+_{k_p}(x) \equiv 0$, we have

$$\begin{aligned} G_\lambda(x,y) = \Phi^+(x)C^+(y) &= \sum_{j=1}^{k_p-1} \phi_j^+(x)C_j^+(y) \\ &= \Big(\bar{W}_x + \mathcal{O}(\lambda)e^{-\theta|x|}\Big)\Big(\frac{1}{\lambda}v_0([u]) + \mathcal{O}(1)\Big) + \mathcal{O}(1)\sum_{j=2}^{k_p-1} e^{\mu_j^+ x}, \end{aligned}$$

yielding (67); here, we recall that

$$\mu_j^\pm = -\lambda/a_j^\pm + \mathcal{O}(\lambda^2)$$

with $a_j^+ > 0$ for $j = 2, ..., k_p - 1$ and $a_j^- < 0$ for $j = k_p + 1, ..., n + 1$ ($a_j^\pm$ are necessarily eigenvalues of $A_\pm$). In the second case $y < x < 0$, from the formula (59), we have

$$G_\lambda(x, y) = \Phi^+(x, \lambda)C^+(y, \lambda) + W_{k_p}^+(x, \lambda)C_{k_p}^+(y, \lambda)$$

where the first term contributes $\lambda^{-1}v_0([u])\bar{W}_x + \mathcal{O}(1)$ as in the first case, and the second term is estimated by (65) and (41).

Finally, we estimate the last case $x < y < 0$ in a same way as done in the first case, noting that y is still near zero and $W_{n+2}^-(x) = \bar{W}_x + \mathcal{O}(\lambda)e^{-\theta|x|}$. $\square$

Next, we estimate the kernel $G_\lambda(x, y)$ for y away from zero. Note however that the representations (59) and above estimates fail to be useful in the $y \to -\infty$ limit, since we actually need precise decay rates in order to get an estimate of form

$$|G_\lambda(x, y)| \leq Ce^{-\eta|x-y|},$$

which are unavailable from ϕ_j^+ in the $y \to -\infty$ regime. Thus, we need to express the (+)-bases in terms of the growing modes ψ_j^- at $-\infty$, and the decaying mode ϕ_j^- where ψ_j^-, ϕ_j^- are defined as in Lemma 2.3. Expressing such solutions in the basis for $y < 0$, away from zero, there exist *analytic* coefficients $d_{jk}(\lambda), e_{jk}(\lambda)$ such that

$$\begin{aligned} \phi_j^+(x, \lambda) &= \sum d_{jk}(\lambda)\phi_k^-(x, \lambda) + \sum e_{jk}(\lambda)\psi_k^-(x, \lambda) \\ W_{k_p}^+(x, \lambda) &= \sum d_{k_pk}(\lambda)\phi_k^-(x, \lambda) + \sum e_{k_pk}(\lambda)\psi_k^-(x, \lambda). \end{aligned} \tag{70}$$

Furthermore, for our convenience, we define the following adjoint normal modes

$$\begin{pmatrix} \tilde{\Psi}^- & \tilde{\Phi}^- \end{pmatrix} := \begin{pmatrix} \Psi^- & \Phi^- \end{pmatrix}^{-1} \Theta^{-1}. \tag{71}$$

We then obtain the following estimates.

Lemma 3.3. *For $|\lambda|$ sufficiently small and $|x|$ sufficiently large,*

$$\begin{aligned} \tilde{\psi}_j^-(x, \lambda) &= \mathcal{O}(e^{-\mu_j^-(\lambda)x})\tilde{V}_j^-(\lambda)(I + \mathcal{O}(e^{-\theta|x|})), \\ \tilde{\phi}_j^-(x, \lambda) &= \mathcal{O}(e^{-\mu_j^-(\lambda)x})\tilde{V}_j^-(\lambda)(I + \mathcal{O}(e^{-\theta|x|})) \end{aligned} \tag{72}$$

where μ_j^- are defined as in Lemma 2.3.

Proof. The proof is clear from the estimates of ψ_j^-, ϕ_j^- in (35). $\square$

Lemma 3.4. *We have*

$$C_j^+(y,\lambda) = \sum c_{jk}^+(\lambda)\tilde{\psi}_k^-(y,\lambda)^* \tag{73}$$
$$C_j^-(y,\lambda) = \sum c_{jk}^-(\lambda)\tilde{\psi}_k^-(y,\lambda)^* + \tilde{\phi}_j^-(y,\lambda)^*, \tag{74}$$

for meromorphic coefficients $c_{jk}^{\pm}$ in λ.

Proof. The proof follows by using (70), definition (71), and property of computing determinants. □

We then have the following representation for $G_\lambda(x,y)$, for y large.

Proposition 3.5. *Under the assumptions of Theorem 1.4, for $|\lambda|$ sufficiently small and $|y|$ sufficiently large, we have*

$$G_\lambda(x,y) = \sum_{j,k} c_{jk}^+(\lambda)\phi_j^+(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^*, \tag{75}$$

for $y < 0 < x$, and

$$G_\lambda(x,y) = \sum_{j,k} d_{jk}^+(\lambda)\phi_j^-(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^* - \sum_k \psi_k^-(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^*, \tag{76}$$

for $y < x < 0$, and

$$G_\lambda(x,y) = \sum_{j,k} d_{jk}^-(\lambda)\phi_j^-(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^* + \sum_k \phi_k^-(x,\lambda)\tilde{\phi}_k^-(y,\lambda)^*, \tag{77}$$

for $x < y < 0$, where $c_{jk}^+(\lambda), d_{jk}^{\pm}(\lambda)$ are scalar meromorphic functions satisfying

$$c^+ = \begin{pmatrix} -I_{k_p} & 0 \end{pmatrix} \begin{pmatrix} \Phi^+ & W_{k_p}^+ & \Phi^- \end{pmatrix}^{-1} \Psi^-$$

and

$$d^{\pm} = \begin{pmatrix} 0 & -I_{n-k_p} \end{pmatrix} \begin{pmatrix} \Phi^+ & W_{k_p}^+ & \Phi^- \end{pmatrix}^{-1} \Psi^-.$$

Proof. Using representation (59) of $G_\lambda(x,y)$ together with (73) and (74), we easily obtain the expansions (75) and (77), respectively. For (76), again, using (73), (70), and (59), we can write

$$\begin{aligned} G_\lambda(x,y) &= \sum_{j,k} d_{jk}^+(\lambda)\phi_j^-(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^* + \sum_{j,k} e_{jk}^+\psi_j^-(x,\lambda)\tilde{\psi}_k^-(y,\lambda)^* \\ &= \begin{pmatrix} \Psi^- & \Phi^- \end{pmatrix}(x) \begin{pmatrix} e^+ \\ d^+ \end{pmatrix} \tilde{\Psi}^-(y)^* \end{aligned} \tag{78}$$

Meanwhile, by (59) and (61),

$$G_\lambda(x,y) = \begin{pmatrix} \Phi^+ & W_{k_p}^+ & 0 \end{pmatrix}(x) \begin{pmatrix} \Phi^+ & W_{k_p}^+ & \Phi^- \end{pmatrix}^{-1}(y)\Theta^{-1}(y) \tag{79}$$

In view of the definition (71) of $\tilde{\Psi}^-, \tilde{\Phi}^-$, (78) and (79) yield

$$\begin{aligned}\begin{pmatrix} e^+ \\ d^+ \end{pmatrix} &= \begin{pmatrix} \tilde{\Psi}^- & \tilde{\Phi}^- \end{pmatrix} \Theta \begin{pmatrix} \Phi^+ & W^+_{k_p} & 0 \end{pmatrix} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{-1} \Psi^- \\ &= \begin{pmatrix} \Psi^- & \Phi^- \end{pmatrix}^{-1} \Big[I - \begin{pmatrix} 0 & \Phi^- \end{pmatrix} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{-1} \Big] \Psi^- \\ &= \begin{pmatrix} I_{k_p} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & I_{n-k_p} \end{pmatrix} \begin{pmatrix} \Phi^+ & W^+_{k_p} & \Phi^- \end{pmatrix}^{-1} \Psi^-, \end{aligned}$$

which proves the proposition. ☐

Proposition 3.6 (Resolvent kernel bounds as $|y| \to +\infty$)**.** *Make the assumptions of Theorem 1.4. Then, for $|y|$ large, defining*

$$E_\lambda(x,y) = \lambda^{-1} \sum_{a_j^- > 0} \tilde{V}^-_{j,0} e^{(\lambda/a_j^- + \mathcal{O}(\lambda^2))y} \bar{W}_x(x),$$

there hold

$$\begin{aligned} G_\lambda(x,y) &= E_\lambda(x,y) \\ &+ \mathcal{O}(1) \Big(\sum_{a_j^- > 0} e^{(\lambda/a_j^- + \mathcal{O}(\lambda^2))y} + \mathcal{O}(e^{-\theta|y|}) \Big) \Big(\sum_{a_k^- > 0} e^{(-\lambda/a_k^- + \mathcal{O}(\lambda^2))x} + \mathcal{O}(e^{-\theta|x|}) \Big) \end{aligned} \tag{80}$$

for $y < 0 < x$, and

$$\begin{aligned} G_\lambda(x,y) =& E_\lambda(x,y) + \mathcal{O}(1) \sum_{a_j^- > 0} e^{(-\lambda/a_j^- + \mathcal{O}(\lambda^2))(x-y)} \\ &+ \mathcal{O}(1) \sum_{a_j^- > 0,\ a_k^- < 0} e^{(\lambda/a_j^- + \mathcal{O}(\lambda^2))y} e^{(-\lambda/a_k^- + \mathcal{O}(\lambda^2))x} + \mathcal{O}(e^{-\theta(|x-y|)}) \end{aligned} \tag{81}$$

for $y < x < 0$, and

$$\begin{aligned} G_\lambda(x,y) =& E_\lambda(x,y) + \mathcal{O}(1) \sum_{a_j^- < 0} e^{(-\lambda/a_j^- + \mathcal{O}(\lambda^2))(x-y)} \\ &+ \mathcal{O}(1) \sum_{a_j^- > 0,\ a_k^- < 0} e^{(\lambda/a_j^- + \mathcal{O}(\lambda^2))y} e^{(-\lambda/a_k^- + \mathcal{O}(\lambda^2))x} + \mathcal{O}(e^{-\theta(|x-y|)}) \end{aligned} \tag{82}$$

for $x < y < 0$.

Similar bounds can be obtained for the case $y > 0$.

Proof. The proof follows directly from the representations of $G_\lambda(x,y)$ derived in Proposition 3.5 and the corresponding estimates on normal modes, noting that

$$|c^+_{jk}|, |d^\pm_{jk}| = \begin{cases} \mathcal{O}(\lambda^{-1}) & j = 1, \\ \mathcal{O}(1) & \text{otherwise.} \end{cases}$$

Indeed, we recall, for instance, that

$$c_{jk}^{+} = D_{-}^{-1}\begin{pmatrix}-I_{k_p} & 0\end{pmatrix}\begin{pmatrix}\Phi^{+} & W_{k_p}^{+} & \Phi^{-}\end{pmatrix}^{kj}\Psi^{-},$$

where $()^{kj}$ denotes the determinant of the (k,j) minors. For the case $j \neq 1$, by using the fact that we choose $\phi_1^+ \equiv \phi_{n+2}^- \equiv \bar{W}_x$ at $\lambda = 0$, determinant of the (k,j) minor therefore has the order one in λ, which cancels out the λ^{-1} term coming from our spectral stability condition: $|D_{-}^{-1}| \leq \mathcal{O}(\lambda^{-1})$.

□

4. Pointwise bounds and low-frequency estimates

In this section, using the previous pointwise bounds (Propositions 3.2 and 3.6) for the resolvent kernel in low-frequency regions, we derive pointwise bounds for the "low-frequency" Green function:

$$G^I(x,t;y) := \frac{1}{2\pi i}\int_{\Gamma\bigcap\{|\lambda|\leq r\}} e^{\lambda t}G_\lambda(x,y)d\lambda \tag{83}$$

where Γ is any contour near zero, but away from the essential spectrum.

Proposition 4.1. *Under the assumptions of Theorem 1.4, defining the effective diffusion $\beta_\pm := (L_pLBR_p)_\pm$ (see (25)), the low-frequency Green distribution $G^I(x,t;y)$ associated with the linearized evolution equations may be decomposed as*

$$G^I(x,t;y) = E + \widetilde{G}^I + R, \tag{84}$$

where, for $y < 0$:

$$E(x,t;y) := \sum_{a_k^- > 0} \bar{U}_x(x)\tilde{V}_{k,0}^{-} e_k(y,t), \tag{85}$$

$$e_k(y,t) := \left(errfn\left(\frac{y + a_k^- t}{\sqrt{4\beta_- t}}\right) - errfn\left(\frac{y - a_k^- t}{\sqrt{4\beta_- t}}\right)\right); \tag{86}$$

$$\begin{aligned}|\partial_x^\gamma \partial_y^\beta \widetilde{G}^I(x,t;y)| \leq Ct^{-(|\alpha|+|\gamma|)/2}\Big(&\sum_{k=1}^n t^{-1/2}e^{-(x-y-a_k^- t)^2/Mt} \\ &+ \sum_{a_k^- < 0,\, a_j^- > 0} \chi_{\{|a_k^- t|\geq|y|\}} t^{-1/2} e^{-(x-a_j^-(t-|y/a_k^-|))^2/Mt}\Big),\end{aligned} \tag{87}$$

$$R(x,t;y) = \mathcal{O}(e^{-\eta(|x-y|+t)}) + \mathcal{O}(e^{-\eta t})\chi(x,y)\Big[1 + \frac{1}{a_p(y)}(x/y)^\alpha\Big], \tag{88}$$

for some η, C, $M > 0$, where $0 \leq |\beta|, |\gamma| \leq 1$, $\alpha = \frac{LB(0)+a_p'(0)}{|a_p'(0)|}$ and

$$\chi(x,y) = \begin{cases} 1, & -1 < y < x < 0 \\ 0, & \textit{otherwise.}\end{cases}$$

Symmetric bounds hold for $y \geq 0$.

Proof. Having the resolvent kernel estimates in Propositions 3.2 and 3.6, we can now follow the previous analyses of [42, 23, 24]. Indeed, the claimed bound for E precisely comes from the λ^{-1} term. Likewise, estimates of $\widetilde{G}^I$ are due to bounds in Proposition 3.6 for y away from zero and those in Proposition 3.2 for y near zero but x away from zero. The singularity occurs only in the case $-1 < y < x < 0$, as reported in Proposition 3.2. In this case, using the estimate (68) and moving the contour Γ in (83) into the stable half-plane $\{\mathrm{Re}\,\lambda < 0\}$, we have

$$\int_\Gamma e^{\lambda t}\Big(1 + \frac{|x|^\alpha}{a_p(y)|y|^\alpha}\Big)d\lambda = \mathcal{O}(e^{-\eta t})\Big(1 + \frac{|x|^\alpha}{a_p(y)|y|^\alpha}\Big),$$

which precisely contributes to the second term in $R(x,t;y)$. The first term in $R(x,t;y)$ is as usual the fast decaying term. □

With the above pointwise estimates on the (low-frequency) Green function, we have the following from [23, 24].

Lemma 4.2 ([23, 24])**.** *Under the assumptions of Theorem 1.4, $\widetilde{G}^I$ satisfies*

$$\Big|\int_{-\infty}^{+\infty}\partial_y^\beta \widetilde{G}^I(\cdot,t;y)f(y)dy\Big|_{L^p} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)-|\beta|/2}|f|_{L^q}, \tag{89}$$

for all $t \ge 0$, some $C > 0$, for any $1 \le q \le p$.

We recall the following fact from [40].

Lemma 4.3 ([40])**.** *The kernel e satisfies*

$$\begin{aligned} &|e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p}, \le Ct^{-\frac{1}{2}(1-1/p)},\\ &|e_{yt}(\cdot,t)|_{L^p} \le Ct^{-\frac{1}{2}(1-1/p)-1/2}. \end{aligned} \tag{90}$$

for all $t > 0$, some $C > 0$, for any $p \ge 1$.

Finally, we have the following estimate on R term.

Lemma 4.4. *Under the assumptions of Theorem 1.4, $R(x,t;y)$ satisfies*

$$\Big|\int_{-\infty}^{+\infty} R(\cdot,t;y)f(y)dy\Big|_{L^p} \le Ce^{-\eta t}(|f|_{L^p} + |f|_{L^\infty}), \tag{91}$$

for all $t \ge 0$, some $C, \eta > 0$, for any $1 \le p \le \infty$.

Proof. The estimate clearly holds for the fast decaying term $e^{-\eta(|x-y|+t)}$ in R. Whereas, to estimate the second term, first notice that it is only nonzero precisely when $-1 < y < x < 0$ or $0 < x < y < 1$. Thus, for instance, when $-1 < x < 0$, we estimate

$$\begin{aligned} \Big|\int_{-\infty}^{+\infty}\chi(x,y)\Big[1 + \frac{1}{a_p(y)}(x/y)^\alpha\Big]f(y)dy\Big| &= \Big|\int_{-1}^{x}\Big[1 + \frac{1}{a_p(y)}(x/y)^\alpha\Big]f(y)dy\Big| \\ &\le C|f|_{L^\infty}\Big[1 + \int_{-1}^{x}\frac{1}{|a_p(y)|}(x/y)^\alpha dy\Big] \\ &\le C|f|_{L^\infty}, \end{aligned}$$

where the last integral is bounded by that fact that $a_p(x) \sim x$ as $|x| \to 0$. From this, we easily obtain

$$\Big| \int_{-\infty}^{+\infty} e^{-\eta t} \chi(x,y) \Big[1 + \frac{1}{a_p(y)} (x/y)^\alpha \Big] f(y) dy \Big|_{L^p(-1,0)} \le C e^{-\eta t} |f|_{L^\infty},$$

which proves the lemma. $\square$

Remark 4.5. We note here that the singular term $a_p^{-1}(y)(x/y)^\alpha$ appearing in (68) and (88) contributes in the time-exponential decaying term. This thus agrees with the resolvent kernel for the scalar convected-damped equation $u_t + a_p u_x = -LBu$, for which we can find explicitly the Green function as a convected time-exponential decaying delta function similar as in the relaxation or real viscosity case.

5. Nonlinear damping estimate and high–frequency estimate

In this section, we establish an auxiliary damping energy estimate. We first recall the nonlinear perturbation equations with (u, q) perturbation variables

$$\begin{aligned} u_t + (A(u)u)_x + Lq_x &= \dot\alpha(U_x + u_x), \\ -q_{xx} + q + (B(u)u)_x &= 0, \end{aligned} \tag{92}$$

where we now denote

$$A(u) := Df(U+u), \qquad B(u) := Dg(U+u). \tag{93}$$

We prove the following:

Proposition 5.1. *Under the assumptions of Theorem 1.4, so long as $\|u\|_{W^{2,\infty}}$ and $|\dot\alpha|$ remain smaller than a small constant ζ and the amplitude $|U_x|$ is sufficiently small, there holds*

$$\|u\|_{H^k}^2(t) \le Ce^{-\theta t}\|u\|_{H^k}^2(0) + C\int_0^t e^{-\theta(t-s)}(\|u\|_{L^2}^2 + |\dot\alpha|^2)(s)ds, \qquad \theta > 0, \tag{94}$$

for $k = 1, ..., 4$.

Proof. Let us work for the case $\dot\alpha \equiv 0$. The general case will be seen as a straightforward extension. We first observe that

$$|A_{0x}|, |A_{0t}|, |A_x|, |A_t|, |B_x|, |B_t| = \mathcal{O}(|U_x| + \zeta) \tag{95}$$

where A, B are defined as in (93) and A_0 the symmetrizer matrix as in (S1).

We note that from the second equation of (92) we easily obtain

$$\|q\|_{H^k} \le C\|u\|_{H^{k-1}}, \tag{96}$$

for $k \geq 1$. Meanwhile, from the first equation, we estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A_0 u, u\rangle &= \langle A_0 u_t, u\rangle + \frac{1}{2}\langle A_{0t} u, u\rangle \\ &= -\langle A_0 A_x u + A_0 A u_x + Lq_x, u\rangle + \frac{1}{2}\langle A_{0t} u, u\rangle \\ &= -\langle A_0 A_x u - \frac{1}{2}(A_0 A)_x u + Lq_x, u\rangle + \frac{1}{2}\langle A_{0t} u, u\rangle,\end{aligned}$$

which, by (95) and (96), yields

$$\frac{1}{2}\frac{d}{dt}\langle A_0 u, u\rangle \leq C\|u\|_{L^2}^2. \tag{97}$$

Now, to obtain the estimates (94) in the case of $k = 1$, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A_0 u_x, u_x\rangle &= \langle (A_0 u_t)_x, u_x\rangle + \frac{1}{2}\langle A_{0t} u_x, u_x\rangle - \langle A_{0x} u_t, u_x\rangle \\ &= -\langle (A_0 A u_x + A_0 L q_x)_x, u_x\rangle + \frac{1}{2}\langle A_{0t} u_x, u_x\rangle - \langle A_{0x} u_t, u_x\rangle \\ &= -\langle A_0 L q_{xx}, u_x\rangle - \langle A_0 A u_{xx}, u_x\rangle, + \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle + \|q\|_{H^1}^2 \\ &= -\langle A_0 L q_{xx}, u_x\rangle + \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle + \mathcal{O}(1)\|u\|_{L^2}^2 \\ &= -\langle A_0 L B u_x, u_x\rangle + \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle + \mathcal{O}(1)\|u\|_{L^2}^2,\end{aligned} \tag{98}$$

noting that since $A_0 A$ is symmetric, we have

$$-\langle A_0 A u_{xx}, u_x\rangle = \frac{1}{2}\langle (A_0 A)_x u_x, u_x\rangle = \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle.$$

Likewise, in spirit of Kawashima-type estimates, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle K u, u_x\rangle &= \frac{1}{2}\langle K_t u, u_x\rangle + \frac{1}{2}\langle K u_t, u_x\rangle + \frac{1}{2}\langle K u, u_{xt}\rangle \\ &= \frac{1}{2}\langle K_t u, u_x\rangle + \frac{1}{2}\langle K u_t, u_x\rangle - \frac{1}{2}\langle K u_x, u_t\rangle - \frac{1}{2}\langle K_x u, u_t\rangle \\ &= \langle K u_t, u_x\rangle + \frac{1}{2}\langle K_t u, u_x\rangle - \frac{1}{2}\langle K_x u, u_t\rangle \\ &= -\langle K A u_x + K A_x u + K L q_x, u_x\rangle + \frac{1}{2}\langle K_t u, u_x\rangle - \frac{1}{2}\langle K_x u, u_t\rangle \\ &= -\langle K A u_x, u_x\rangle + \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle + \mathcal{O}(1)\|u\|_{L^2}^2.\end{aligned} \tag{99}$$

Adding (98) and (99) together, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\langle K u, u_x\rangle + \langle A_0 u_x, u_x\rangle\Big) \\ &\qquad = -\langle (KA + A_0 L B) u_x, u_x\rangle + \langle \mathcal{O}(|U_x| + \zeta) u_x, u_x\rangle + \mathcal{O}(1)\|u\|_{L^2}^2\end{aligned} \tag{100}$$

which, by the Kawashima-type condition (6): $KA + A_0LB \geq \theta$ and the fact that $\mathcal{O}(|U_x| + \zeta)$ is sufficiently small, yields

$$\frac{1}{2}\frac{d}{dt}\Big(\langle Ku, u_x\rangle + \langle A_0 u_x, u_x\rangle\Big) \leq -\frac{1}{2}\theta\langle u_x, u_x\rangle + \mathcal{O}(1)\|u\|^2_{L^2} \tag{101}$$

Similarly, for $k \geq 1$, paying attention to the leading terms, we can compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle A_0\partial_x^k u, \partial_x^k u\rangle &= \langle A_0\partial_x^k u_t, \partial_x^k u\rangle + \frac{1}{2}\langle A_{0t}\partial_x^k u, \partial_x^k u\rangle \\ &= \langle \partial_x^k(A_0u_t), \partial_x^k u\rangle + \langle \mathcal{O}(|U_x| + \zeta)\partial_x^k u, \partial_x^k u\rangle + \mathcal{O}(1)\|u\|^2_{H^{k-1}},\end{aligned}$$

where by using the first equation and then the second one, we obtain

$$\begin{aligned}\langle \partial_x^k(A_0u_t), \partial_x^k u\rangle &= -\langle \partial_x^k(A_0Au_x + A_0A_xu + A_0Lq_x), \partial_x^k u\rangle \\ &= -\langle A_0L\partial_x^{k-1}q_{xx}, \partial_x^k u\rangle - \langle A_0A\partial_x^{k+1}u, \partial_x^k u\rangle + \cdots \\ &= -\langle A_0L\partial_x^k(Bu), \partial_x^k u\rangle + \frac{1}{2}\langle (A_0A)_x\partial_x^k u, \partial_x^k u\rangle + \cdots .\end{aligned}$$

Thus, we have obtained

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\langle A_0\partial_x^k u, \partial_x^k u\rangle \\ &\quad = -\langle A_0LB\partial_x^k u, \partial_x^k u\rangle + \langle \mathcal{O}(|U_x| + \zeta)\partial_x^k u, \partial_x^k u\rangle + \mathcal{O}(1)\|u\|^2_{H^{k-1}}.\end{aligned} \tag{102}$$

Meanwhile, we have the following k^{th}-order Kawashima-type energy estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle K\partial_x^{k-1}u, \partial_x^k u\rangle &= \langle K\partial_x^{k-1}u_t, \partial_x^k u\rangle + \frac{1}{2}\langle K_t\partial_x^{k-1}u, \partial_x^k u\rangle - \frac{1}{2}\langle K_x\partial_x^{k-1}u, \partial_x^{k-1}u_t\rangle \\ &= -\langle KA\partial_x^k u, \partial_x^k u\rangle + \langle \mathcal{O}(|U_x| + \zeta)\partial_x^k u, \partial_x^k u\rangle + \mathcal{O}(1)\|u\|^2_{H^{k-1}}.\end{aligned} \tag{103}$$

Hence, as before, adding (102) and (103) together and using the Kawashima-type condition (6): $KA + A_0LB \geq \theta$ and the fact that $\mathcal{O}(|U_x| + \zeta)$ is sufficiently small, we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\langle K\partial_x^{k-1}u, \partial_x^k u\rangle + \langle A_0\partial_x^k u, \partial_x^k u\rangle\Big) \leq -\frac{1}{2}\theta\langle \partial_x^k u, \partial_x^k u\rangle + \mathcal{O}(1)\|u\|^2_{H^{k-1}}. \tag{104}$$

Now, for $\delta > 0$, let us define

$$\mathcal{E}(t) := \sum_{k=0}^{s}\delta^k\Big(\langle K\partial_x^{k-1}u, \partial_x^k u\rangle + \langle A_0\partial_x^k u, \partial_x^k u\rangle\Big).$$

By applying the standard Cauchy's inequality on $\langle K\partial_x^{k-1}u, \partial_x^k u\rangle$ and using the positive definiteness of A_0, we observe that $\mathcal{E}(t) \sim \|u\|^2_{H^k}$. We then use the above estimates (97),(101), (104), and take δ sufficiently small to derive

$$\frac{d}{dt}\mathcal{E}(t) \leq -\theta_3\mathcal{E}(t) + C\|u\|^2_{L^2}(t) \tag{105}$$

for some $\theta_3 > 0$, from which (94) follows by the standard Gronwall's inequality. □

With the damping nonlinear energy estimates in hands, we immediately obtain the following estimates for high-frequency part of the solution operator $e^{\mathcal{L}t}$:

$$\mathcal{S}_2(t) = \frac{1}{2\pi i}\int_{-\theta_1 - i\infty}^{-\theta_1 + i\infty} \chi_{\{|\operatorname{Im}\lambda|\geq\theta_2\}} e^{\lambda t}(\lambda - \mathcal{L})^{-1} d\lambda, \tag{106}$$

for small positive numbers θ_1, θ_2; see (18). Here, $\chi_{\{|\operatorname{Im}\lambda|\geq\theta_2\}}$ equals to 1 for $|\operatorname{Im}\lambda| \geq \theta_2$ and zero otherwise.

Proposition 5.2 (High-frequency estimate)**.** *Under the assumptions of Theorem 1.4,*

$$\begin{aligned} \|\mathcal{S}_2(t)f\|_{L^2} &\leq Ce^{-\theta_1 t}\|f\|_{H^2}, \\ \|\partial_x^\alpha \mathcal{S}_2(t)f\|_{L^2} &\leq Ce^{-\theta_1 t}\|f\|_{H^{\alpha+2}}, \end{aligned} \tag{107}$$

for some $\theta_1 > 0$.

Proof of the proposition follows exactly in a same way as done in our companion paper [16] for the scalar case. We recall it here for sake of completeness. The first step is to estimate the solution of the resolvent system

$$\begin{aligned} \lambda u + (A\,u)_x + Lq_x &= \varphi, \\ -q_{xx} + q + (B\,u)_x &= \psi, \end{aligned}$$

where $A(x) = Df(U)$ and $B(x) = Dg(U)$ as before.

Proposition 5.3 (High-frequency bounds)**.** *Under the assumptions of Theorem 1.4, for some R, C sufficiently large and $\gamma > 0$ sufficiently small, we obtain*

$$\begin{aligned} |(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{H^1} &\leq C\Big(|\varphi|_{H^1}^2 + |\psi|_{L^2}^2\Big), \\ |(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{L^2} &\leq \frac{C}{|\lambda|^{1/2}}\Big(|\varphi|_{H^1}^2 + |\psi|_{L^2}^2\Big), \end{aligned}$$

for all $|\lambda| \geq R$ and $\operatorname{Re}\lambda \geq -\gamma$, where $\mathcal{K} := (-\partial_x^2 + 1)^{-1}$.

Proof. A Laplace transformed version of the nonlinear energy estimates (94) in Section 5 with $k = 1$ (see [41], pp. 272–273, proof of Proposition 4.7 for further details) yields

$$\Big(\operatorname{Re}\lambda + \frac{\gamma_1}{2}\Big)|u|_{H^1}^2 \leq C\Big(|u|_{L^2}^2 + |\varphi|_{H^1}^2 + |\psi|_{L^2}^2\Big). \tag{108}$$

On the other hand, taking the imaginary part of the L^2 inner product of U against $\lambda u = \mathcal{L}u + \partial_x L\mathcal{K}h + f$ and applying the Young's inequality, we also obtain the standard estimate

$$\begin{aligned} |\operatorname{Im}\lambda||u|_{L^2}^2 &\leq |\langle \mathcal{L}u, u\rangle| + |\langle L\mathcal{K}\psi, u_x\rangle| + |\langle \varphi, u\rangle| \\ &\leq C\Big(|u|_{H^1}^2 + |\psi|_{L^2}^2 + |\varphi|_{L^2}^2\Big), \end{aligned} \tag{109}$$

noting the fact that $\mathcal{L}$ is a bounded operator from H^1 to L^2 and $\mathcal{K}$ is bounded from L^2 to H^1.

Therefore, taking $\gamma = \gamma_1/4$, we obtain from (108) and (109)

$$|\lambda||u|_{L^2}^2 + |u|_{H^1}^2 \le C\Big(|u|_{L^2}^2 + |\psi|_{L^2}^2 + |\varphi|_{H^1}^2\Big),$$

for any $\operatorname{Re}\lambda \ge -\gamma$. Now take R sufficiently large such that $|u|_{L^2}^2$ on the right hand side of the above can be absorbed into the left hand side for $|\lambda| \ge R$, thus yielding

$$|\lambda||u|_{L^2}^2 + |u|_{H^1}^2 \le C\Big(|\psi|_{L^2}^2 + |\varphi|_{H^1}^2\Big),$$

for some large $C > 0$, which gives the result as claimed. □

Next, we have the following

Proposition 5.4 (Mid-frequency bounds)**.** *Under the assumptions of Theorem 1.4,*

$$|(\lambda - \mathcal{L})^{-1}\varphi|_{L^2} \le C\,|\varphi|_{H^1} \quad \textit{for } \; R^{-1} \le |\lambda| \le R \textit{ and } \operatorname{Re}\lambda \ge -\gamma,$$

for any R and $C = C(R)$ sufficiently large and $\gamma = \gamma(R) > 0$ sufficiently small.

Proof. Immediate, by compactness of the set of frequency under consideration together with the fact that the resolvent $(\lambda - \mathcal{L})^{-1}$ is analytic with respect to H^1 in λ; see, for instance, [40]. □

With Propositions 5.3 and 5.4 in hand, we are now ready to give:

Proof of Proposition 5.2. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(\mathcal{L})$ of the resolvent $(\lambda - \mathcal{L})^{-1}$, for all $\varphi \in \mathcal{D}(\mathcal{L})$,

$$(\lambda - \mathcal{L})^{-1}\varphi = \lambda^{-1}(\lambda - \mathcal{L})^{-1}\mathcal{L}\varphi + \lambda^{-1}\varphi.$$

Using this identity and (106), we estimate

$$\begin{aligned}
\mathcal{S}_2(t)\varphi &= \frac{1}{2\pi i}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\operatorname{Im}\lambda| \ge \gamma_2\}} e^{\lambda t}\lambda^{-1}(\lambda - \mathcal{L})^{-1}\mathcal{L}\,\varphi\, d\lambda \\
&\quad + \frac{1}{2\pi i}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\operatorname{Im}\lambda| \ge \gamma_2\}} e^{\lambda t}\lambda^{-1}\varphi\, d\lambda \\
&=: S_1 + S_2,
\end{aligned}$$

where, by Propositions 5.2 and 5.4, we have

$$\begin{aligned}
|S_1|_{L^2} &\le C\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} |\lambda|^{-1} e^{\operatorname{Re}\lambda t}|(\lambda - \mathcal{L})^{-1}\mathcal{L}\varphi|_{L^2}|d\lambda| \\
&\le Ce^{-\gamma_1 t}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} |\lambda|^{-3/2}|\mathcal{L}\varphi|_{H^1}|d\lambda| \\
&\le Ce^{-\gamma_1 t}|\varphi|_{H^2}
\end{aligned}$$

and

$$|S_2|_{L^2} \le \frac{1}{2\pi}\Big|\varphi \int_{-\gamma_1 - i\infty}^{-\gamma_1+i\infty} \lambda^{-1} e^{\lambda t} d\lambda\Big|_{L^2} + \frac{1}{2\pi}\Big|\varphi \int_{-\gamma_1 - ir}^{-\gamma_1+ir} \lambda^{-1} e^{\lambda t} d\lambda\Big|_{L^2}$$
$$\le Ce^{-\gamma_1 t}|\varphi|_{L^2},$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the bound for the term involving φ as stated in the proposition. The estimate involving ψ follows by observing that $L\,\partial_x\mathcal{K}$ is bounded from H^s to H^s. Derivative bounds can be obtained similarly. $\square$

Remark 5.5. We note that in our treating the high-frequency terms by energy estimates (as also done in [15, 31, 16]), we are ignoring the pointwise contribution there, which would also be convected time-decaying delta functions. To see these features, a simple exercise is to do the Fourier transform of the equations about a constant state.

6. Nonlinear analysis

In this section, we shall prove the main nonlinear stability theorem. The proof follows exactly word by word as in the scalar case [16]. We present its sketch here for sake of completeness. Define the nonlinear perturbation

$$\begin{pmatrix} u \\ q \end{pmatrix}(x,t) := \begin{pmatrix} \tilde u \\ \tilde q \end{pmatrix}(x+\alpha(t),t) - \begin{pmatrix} U \\ Q \end{pmatrix}(x), \tag{110}$$

where the shock location $\alpha(t)$ is to be determined later.

Plugging (110) into (1), we obtain the perturbation equation

$$\begin{aligned} u_t + (Au)_x + Lq_x &= N_1(u)_x + \dot\alpha(t)(u_x + U_x), \\ -q_{xx} + q + (Bu)_x &= N_2(u)_x, \end{aligned} \tag{111}$$

where $N_j(u) = O(|u|^2)$ so long as u stays uniformly bounded.

We recall the Green function decomposition

$$G(x,t;y) = G^I(x,t;y) + G^{II}(x,t;y) \tag{112}$$

where $G^I(x,t;y)$ is the low-frequency part. We further define as in Proposition 4.1,

$$\widetilde G^I(x,t;y) = G^I(x,t;y) - E(x,t;y) - R(x,t;y)$$

and

$$\widetilde G^{II}(x,t;y) = G^{II}(x,t;y) + R(x,t;y).$$

Then, we immediately obtain the following from Lemmas 4.2, 4.4 and Proposition 5.2:

Lemma 6.1. *We obtain*

$$\Big|\int_{-\infty}^{+\infty}\partial_y^\beta \widetilde{G}^I(\cdot,t;y)f(y)dy\Big|_{L^p}\le C(1+t)^{-\frac{1}{2}(1/q-1/p)-|\beta|/2}|f|_{L^q}, \tag{113}$$

for all $1\le q\le p, \beta=0,1,$ *and*

$$\Big|\int_{-\infty}^{+\infty}\widetilde{G}^{II}(x,t;y)f(y)dy\Big|_{L^p}\le Ce^{-\eta t}|f|_{H^3}, \tag{114}$$

for all $2\le p\le\infty$.

Proof. (113) is precisely the estimate (89) in Lemma 4.2, recalled here for our convenience. (114) is a straightforward combination of Lemma 4.4 and Proposition 5.2, followed by a use of the interpolation inequality between L^2 and L^∞ and an application of the standard Sobolev imbedding. □

We next show that by Duhamel's principle we have:

Lemma 6.2. *We obtain the reduced integral representation:*

$$\begin{aligned}
u(x,t)=&\int_{-\infty}^{+\infty}(\widetilde{G}^I+\widetilde{G}^{II})(x,t;y)u_0(y)dy\\
&-\int_0^t\int_{-\infty}^{+\infty}\widetilde{G}^I_y(x,t-s;y)\Big(\partial_yL\mathcal{K}N_2(u)+N_1(u)+\dot\alpha(t)u\Big)(y,s)\,dy\,ds\\
&+\int_0^t\int_{-\infty}^{+\infty}\widetilde{G}^{II}(x,t-s;y)\Big(\partial_yL\mathcal{K}N_2(u)+N_1(u)+\dot\alpha(t)u\Big)_y(y,s)\,dy\,ds,\\
q(x,t)=&(\mathcal{K}\partial_x)(N_2(u)-Bu)(x,t),
\end{aligned} \tag{115}$$

and

$$\begin{aligned}
\alpha(t)=&-\int_{-\infty}^{+\infty}e_t(y,t)u_0(y)dy\\
&+\int_0^t\int_{-\infty}^{+\infty}e_y(y,t-s)\Big(\partial_yL\mathcal{K}N_2(u)+N_1(u)+\dot\alpha(t)u\Big)(y,s)\,dy\,ds.
\end{aligned} \tag{116}$$

$$\begin{aligned}
\dot\alpha(t)=&-\int_{-\infty}^{+\infty}e_t(y,t)u_0(y)dy\\
&+\int_0^t\int_{-\infty}^{+\infty}e_{yt}(y,t-s)\Big(\partial_yL\mathcal{K}N_2(u)+N_1(u)+\dot\alpha(t)u\Big)(y,s)\,dy\,ds.
\end{aligned} \tag{117}$$

Proof. By Duhamel's principle and the fact that

$$\int_{-\infty}^{+\infty}G(x,t;y)U_x(y)dy=e^{\mathcal{L}t}U_x(x)=U_x(x),$$

we obtain

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{+\infty} G(x,t;y)u_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\Big(\partial_y L\mathcal{K}N_2(u) + N_1(u) + \dot\alpha(t)u\Big)_y(y,s)\,dy\,ds \\ &+ \alpha(t)U_x. \end{aligned} \tag{118}$$

Thus, by defining the *instantaneous shock location:*

$$\begin{aligned} \alpha(t) = &-\int_{-\infty}^{+\infty} e_t(y,t)u_0(y)dy \\ &+ \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\Big(\partial_y L\mathcal{K}N_2(u) + N_1(u) + \dot\alpha(t)u\Big)(y,s)\,dy\,ds \end{aligned}$$

and using the Green function decomposition (112), we easily obtain the integral representation as claimed in the lemma. □

With these preparations, we are now ready to prove the main theorem, following the standard stability analysis of [25, 39, 40]:

Proof of Theorem 1.4. Define

$$\zeta(t) := \sup_{0\le s\le t, 2\le p\le\infty} \Big[|u(s)|_{L^p}(1+s)^{\frac12(1-1/p)} + |\alpha(s)| + |\dot\alpha(s)|(1+s)^{1/2}\Big]. \tag{119}$$

We shall prove here that for all $t\ge 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \le C(|u_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{120}$$

This bound together with continuity of $\zeta(t)$ implies that

$$\zeta(t) \le 2C|u_0|_{L^1\cap H^s} \tag{121}$$

for $t\ge 0$, provided that $|u_0|_{L^1\cap H^s} < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s, and the standard principle of continuation, there exists a solution $u\in H^s$ on the open time-interval for which $|u|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0,T)$ be the maximal interval on which $|u|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta>0$. By Proposition 5.1, and the Sobolev embeding inequality $|u|_{W^{2,\infty}} \le C|u|_{H^s}$, $s\ge 3$, we have

$$\begin{aligned} |u(t)|_{H^s}^2 &\le Ce^{-\theta t}|u_0|_{H^s}^2 + C\int_0^t e^{-\theta(t-\tau)}\Big(|u(\tau)|_{L^2}^2 + |\dot\alpha|^2\Big)d\tau \\ &\le C(|u_0|_{H^s}^2 + \zeta(t)^2)(1+t)^{-1/2}. \end{aligned} \tag{122}$$

and so the solution continues so long as ζ remains small, with bound (121), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (120). First by representation (115) for u, for any $2 \le p \le \infty$, we obtain

$$\begin{aligned}|u|_{L^p}(t) \le & \Big| \int_{-\infty}^{+\infty} (\widetilde{G}^I + \widetilde{G}^{II})(x,t;y) u_0(y) dy \Big|_{L^p} \\ & + \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^I_y(x,t-s;y) \Big(\partial_y L\mathcal{K} N_2(u) + N_1(u) + \dot\alpha(s) u \Big)(y,s)\, dy \Big|_{L^p} ds \\ & + \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t-s;y) \Big(\partial_y L\mathcal{K} N_2(u) + N_1(u) + \dot\alpha(t) u \Big)_y(y,s)\, dy \Big|_{L^p} ds \\ = & I_1 + I_2 + I_3, \end{aligned} \tag{123}$$

where estimates (113) and (114) yield

$$\begin{aligned} I_1 = & \Big| \int_{-\infty}^{+\infty} (\widetilde{G}^I + \widetilde{G}^{II})(x,t;y) u_0(y) dy \Big|_{L^p} \\ & \le C(1+t)^{-\frac12(1-1/p)} |u_0|_{L^1} + Ce^{-\eta t} |u_0|_{H^3} \\ & \le C(1+t)^{-\frac12(1-1/p)} |u_0|_{L^1 \cap H^3}, \end{aligned}$$

and, with noting that $\partial_y L\mathcal{K}$ is bounded from L^2 to L^2,

$$\begin{aligned} I_2 = & \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^I_y(x,t-s;y) \Big(\partial_y L\mathcal{K} N_2(u) + N_1(u) + \dot\alpha(t) u \Big)(y,s)\, dy \Big|_{L^p} ds \\ & \le C \int_0^t (t-s)^{-\frac12(1/2-1/p)-1/2} (|u|_{L^\infty} + |\dot\alpha|) |u|_{L^2}(s) ds \\ & \le C\zeta(t)^2 \int_0^t (t-s)^{-\frac12(1/2-1/p)-1/2} (1+s)^{-3/4} ds \\ & \le C\zeta(t)^2 (1+t)^{-\frac12(1-1/p)}, \end{aligned}$$

and, together with (122), $s \ge 4$,

$$\begin{aligned} I_3 = & \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t-s;y) \Big(\partial_y L\mathcal{K} N_2(u) + N_1(u) + \dot\alpha(s) u \Big)_y(y,s)\, dy \Big|_{L^p} ds \\ & \le C \int_0^t e^{-\eta(t-s)} |\partial_y L\mathcal{K} N_2(u) + N_1(u) + \dot\alpha(t) u|_{H^4}(s) ds \\ & \le C \int_0^t e^{-\eta(t-s)} (|u|_{H^s} + |\dot\alpha|) |u|_{H^s}(s) ds \\ & \le C(|u_0|_{H^s}^2 + \zeta(t)^2) \int_0^t e^{-\eta(t-s)} (1+s)^{-1} ds \\ & \le C(|u_0|_{H^s}^2 + \zeta(t)^2)(1+t)^{-1}. \end{aligned}$$

Thus, we have proved

$$|u(t)|_{L^p}(1+t)^{\frac{1}{2}(1-1/p)} \le C(|u_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{124}$$

Similarly, using representations (116) and (117) and the estimates in Lemma 4.3 on the kernel $e(y,t)$, we can estimate (see, e.g., [25, 40]),

$$|\dot\alpha(t)|(1+t)^{1/2} + |\alpha(t)| \le C(|u_0|_{L^1} + \zeta(t)^2). \tag{125}$$

This completes the proof of the claim (120), and thus the result for u as claimed. To prove the result for q, we observe that $\mathcal{K}\partial_x$ is bounded from $L^p \to W^{1,p}$ for all $1 \le p \le \infty$, and thus from the representation (115) for q, we estimate

$$\begin{aligned} |q|_{W^{1,p}}(t) &\le C(|N_2(u)|_{L^p} + |u|_{L^p})(t) \\ &\le C|u|_{L^p}(t) \le C|u_0|_{L^1\cap H^s}(1+t)^{-\frac{1}{2}(1-1/p)} \end{aligned} \tag{126}$$

and

$$|q|_{H^{s+1}}(t) \le C|u|_{H^s}(t) \le C|u_0|_{L^1\cap H^s}(1+t)^{-1/4}, \tag{127}$$

which complete the proof of the main theorem. $\square$

Appendix A. Spectral stability in the small-amplitude regime

In this section we verify the spectral stability condition for small-amplitude profiles. Denoting $A = A(U(x))$, $B = B(U(x))$ we have the associated linearized spectral problem

$$\begin{aligned} \lambda u + (Au)_x + Lq_x &= 0, \\ -q_{xx} + q + (Bu)_x &= 0. \end{aligned} \tag{A.1}$$

Using the zero-mass conditions

$$\int u\,dx = 0, \qquad \int q\,dx = 0,$$

we recast system (A.1) in terms of the integrated coordinates, which we denote, again, as u and q. The resulting system reads

$$\lambda u + Au_x + Lq_x \quad = \quad 0, \tag{A.2}$$
$$-q_{xx} + q + Bu_x \quad = \quad 0. \tag{A.3}$$

In what follows we assume that the shocks are weak, that is, $u_\pm \in \mathcal{N}(u_*)$, being $\mathcal{N}$ a neighborhood of a certain state u_*, for which

$$0 < \max_{u\in\mathcal{N}} |u - u_*| \le \epsilon \ll 1,$$

with $\epsilon > 0$ sufficiently small; clearly,

$$|u_* - u_\pm|, |u_- - u_+| = \mathcal{O}(\epsilon)$$

and the shock profile for U is approximately scalar, satisfying,

$$\begin{aligned} U_x &= \mathcal{O}(\epsilon^2)e^{-\eta\epsilon|x|}(r_p(u_*) + \mathcal{O}(\epsilon)), \\ U_{xx} &= \mathcal{O}(\epsilon^3)e^{-\theta\epsilon|x|}, \end{aligned} \tag{A.4}$$

for some $\theta, \eta > 0$. For the principal characteristic field $a_p := a_p(U(x))$ we have

$$\begin{aligned} (a_p)_x &= \mathcal{O}(U_x) < 0, \quad \text{(monotonicity)}, \\ (a_p)_{xx} &= \mathcal{O}(U_{xx}). \end{aligned} \tag{A.5}$$

We shall make use of the following

Lemma Appendix A.1. *Under* (S0) - (S2)*, there exists a scalar function* $\beta = \beta(u) > 0$*, such that*

$$(A_0 L)^\top = \beta B, \tag{A.6}$$

for all $u \in \mathcal{U}$*.*

Proof. Follows by elementary linear algebra facts, since $A_0 LB$ is positive semi-definite with rank one and can be written as $z \otimes w$, for some vectors z and w. It follows the existence of a scalar β, such that $z = \beta w$; it is clearly nonzero and positive because of positive semi-definiteness of $A_0 LB$. □

We start by providing some basic Friedrichs-type energy estimates.

Lemma Appendix A.2. *Assume* u, q *and* $\operatorname{Re}\lambda \geq 0$ *solve* (A.2) - (A.3)*. If* $\epsilon > 0$ *is sufficiently small, then there hold the estimates*

$$(\operatorname{Re}\lambda)|u|_{L^2}^2 + |q|_{L^2}^2 + |q_x|_{L^2}^2 \leq C \int |U_x||u|^2 \, dx \tag{A.7}$$

$$|\operatorname{Im}\lambda| \int |U_x||u|^2 \, dx \leq C \int |U_x|\big(\delta|u|^2 + \delta^{-1}|q|^2\big) \, dx \tag{A.8}$$

for some $C > 0$ *and any* $\delta > 0$*.*

Proof. Multiply (A.2) by $A_0 := A_0(U(x))$ and take the complex L^2 product against u; taking its real part and denoting

$$\bar{A} := (A_0 A((U(x)), \qquad \bar{L} := A_0(U(x))L,$$

we obtain

$$(\operatorname{Re}\lambda)\langle u, A_0 u\rangle + \operatorname{Re}\langle u, \bar{A}u_x\rangle + \operatorname{Re}\langle u, \bar{L}q_x\rangle = 0.$$

Using symmetry of $\bar{A}$ and integrating by parts we get

$$(\operatorname{Re}\lambda)\langle u, A_0 u\rangle - \tfrac{1}{2}\operatorname{Re}\langle u, \bar{A}_x u\rangle + \operatorname{Re}\langle u, \bar{L}q_x\rangle = 0. \tag{A.9}$$

Multiply (A.3) by $\beta := \beta(U(x))$, use (A.6), take the L^2 product against q, integrate by parts and take its real part. This yields

$$c^{-1}|q_x|_{L^2}^2 + c^{-1}|q|_{L^2}^2 + \operatorname{Re}\langle q, \beta_x q\rangle - \operatorname{Re}\langle u, \bar{L}q_x\rangle - \operatorname{Re}\langle \bar{L}_x q, u\rangle = 0, \tag{A.10}$$

because $\beta \geq c^{-1} > 0$. Since the error terms can be absorbed

$$\beta_x, \bar{L}_x = \mathcal{O}(|U_x|) = \mathcal{O}(\epsilon^2),$$

for ϵ sufficiently small, and since A_0 is positive definite, we obtain inequality (A.7). Inequality (A.8) follows in a similar fashion, with the parameter δ arising after application of Young's inequality. □

Corollary Appendix A.3. *There hold the estimates*

$$0 \leq \operatorname{Re}\lambda \leq C\epsilon^2, \tag{A.11}$$

$$|\operatorname{Im}\lambda| \leq C\epsilon, \tag{A.12}$$

for some $C > 0$.

Proof. Estimate (A.11) follows immediately from (A.7). Taking $\delta = \epsilon > 0$ in (A.8), and using (A.7) to control $|q|^2_{L^2}$ we can easily obtain

$$(|\operatorname{Im}\lambda| - C\epsilon) \int |U_x||u|^2 \leq 0,$$

yielding (A.12). □

Appendix A.1. Kawashima-type estimate

Next we carry out an energy estimate for u_x of Kawashima-type (see [8, 27]).

Lemma Appendix A.4. *For each* $\operatorname{Re}\lambda \geq 0$, $\lambda \neq 0$, *there holds*

$$|u_x|^2_{L^2} \leq \bar{C}\big((\operatorname{Re}\lambda)\eta|u|^2_{L^2} + \int |U_x||u|^2\,dx\big), \tag{A.13}$$

for some $\bar{C} > 0$ *and* $\eta > 0$ *with* ϵ^2/η *sufficiently small.*

Proof. Denote $K = K(U(x))$, and take the real part of the L^2 product of Ku_x against (A.2). Since K is skew-symmetric, the result is

$$\operatorname{Re}\langle u_x, KAu_x\rangle = \operatorname{Re}(\lambda\langle Ku_x, u\rangle) + \operatorname{Re}\langle Ku_x, Lq_x\rangle. \tag{A.14}$$

Noticing also that $\operatorname{Im}\langle Ku_x, u\rangle = -\frac{1}{2}\langle K_x u, u\rangle$, we obtain the bound

$$\operatorname{Re}(\lambda\langle Ku_x, u\rangle) \leq C(\operatorname{Re}\lambda)\big(\eta^{-1}|u_x|^2_{L^2} + \eta|u|^2_{L^2}\big) + C|\operatorname{Im}\lambda| \int |U_x||u|^2\,dx, \tag{A.15}$$

for any $\eta > 0$ and some $C > 0$. We also have the estimate

$$\langle Ku_x, Lq_x\rangle \leq C\big(\delta_1|u_x|^2_{L^2} + \delta_1^{-1}|q_x|^2_{L^2}\big), \tag{A.16}$$

for any $\delta_1 > 0$, where we have used Young's inequality in both estimates.

To estimate $\operatorname{Re}\langle u_x, KAu_x\rangle$, observe that from (6), there holds

$$\operatorname{Re}\langle u_x, KAu_x\rangle + \langle u_x, \bar{L}Bu_x\rangle \geq c^{-1}|u_x|^2_{L^2}, \tag{A.17}$$

for some $c > 0$. (Notice that $\langle u_x, \bar{L}Bu_x\rangle \in \mathbb{R}$ because $\bar{L}B$ is symmetric, positive semi-definite.)

Multiply equation (A.3) by $\bar{L}$, take the L^2 product with u_x and integarte by parts. This yields,

$$\langle u_x, \bar{L}Bu_x\rangle = -\langle u_{xx}, \bar{L}q_x\rangle - \langle u_x, \bar{L}_x q_x\rangle - \langle u_x, \bar{L}q\rangle. \tag{A.18}$$

To estimate the first term, take the real part of the L^2 product of u_{xx} against A_0 times (A.2), use $\bar{A}$ symmetric, A_0 positive definite, and integrate by parts to obtain

$$\begin{aligned} -\mathrm{Re}\,\langle u_{xx}, \bar{L}q_x\rangle &\leq -\mathrm{Re}\,(\lambda\langle u_x, (A_0)_x u\rangle) + \tfrac{1}{2}\langle u_x, \bar{A}_x u_x\rangle - \mathrm{Re}\,\langle u_x, \bar{A}_x u\rangle \\ &\leq -(\mathrm{Re}\,\lambda)\mathrm{Re}\,\langle u_x, (A_0)_x u\rangle + (\mathrm{Im}\,\lambda)\mathrm{Im}\,\langle u_x, (A_0)_x u\rangle + \\ &\quad + \tfrac{1}{2}\langle u_x, \bar{A}_x u_x\rangle - \mathrm{Re}\,\langle u_x, \bar{A}_x u\rangle. \end{aligned} \tag{A.19}$$

Using (A.7) and (A.8), and bounding the error terms $(A_0)_x, \bar{A}_x = \mathcal{O}(|U_x|) = \mathcal{O}(\epsilon^2)$, we get

$$-\mathrm{Re}\,\langle u_{xx}, \bar{L}q_x\rangle \leq C\epsilon \int |U_x||u|^2\,dx + C\epsilon|u_x|^2_{L^2}, \tag{A.20}$$

where the term $\langle u_x, \bar{A}_x u\rangle$ has been bounded by

$$\begin{aligned} \int |U_x||u||u_x|\,dx &\leq \frac{C}{2}\Big(\int |U_x|^{3/2}|u|^2\,dx + \int |U_x|^{1/2}|u_x|^2\,dx\Big) \\ &\leq \frac{C}{2}\epsilon \int |U_x||u|^2\,dx + \frac{C}{2}\epsilon|u_x|^2_{L^2}. \end{aligned}$$

We also estimate

$$\mathrm{Re}\,\langle u_x, \bar{L}_x q_x\rangle \leq C\epsilon^2|u_x|^2_{L^2} + C|q_x|^2_{L^2}, \tag{A.21}$$

$$\mathrm{Re}\,\langle u_x, \bar{L}q\rangle \leq C\big(\delta_2|u_x|^2_{L^2} + \delta_2^{-1}|q|^2_{L^2}\big), \tag{A.22}$$

for any $\delta_2 > 0$, using Young's inequality. Putting all together back into (A.18) we get

$$\langle u_x, \bar{L}Bu_x\rangle \leq C\epsilon|u_x|^2_{L^2} + C\int |U_x||u|^2\,dx, \tag{A.23}$$

after using (A.7).

Finally, since $\mathrm{Re}\,\lambda = \mathcal{O}(\epsilon^2)$, taking $\delta_2 = \epsilon$ and ϵ^2/η sufficiently small, we can substitute (A.23), (A.15) and (A.16) back into (A.17), absorb the small terms into the left hand side to obtain (A.13). This proves the result. □

Corollary Appendix A.5. *For all $\epsilon > 0$ sufficiently small and $\mathrm{Re}\,\lambda \geq 0$, there holds the estimate*

$$(\mathrm{Re}\,\lambda)|u|^2_{L^2} + |u_x|^2_{L^2} \leq C\int |U_x||u|^2\,dx, \tag{A.24}$$

for some $C > 0$.

Proof. Take $\bar{C}$ times estimate (A.7) and add to (A.13) to obtain

$$\bar{C}(\operatorname{Re}\lambda)|u|_{L^2}^2 + |u_x|_{L^2}^2 \le \bar{C}(1+C)\int |u_x||u|^2\,dx + \bar{C}(\operatorname{Re}\lambda)\eta|u|_{L^2}^2.$$

Take η sufficiently small, say $\eta = \mathcal{O}(\epsilon)$ so that ϵ^2/η remains small, and after absorbing into the left hand side we obtain the result. $\square$

Appendix A.2. Goodman-type estimate

Finally, we control the term $\int |U_x||u|^2$ by performing a weighted energy estimate in the spirit of Goodman [4, 5] (see also [8, 27]).

Lemma Appendix A.6. *Under* (S0) - (S2), (H0) - (H3), *for all* $\operatorname{Re}\lambda \ge 0$ *there holds the estimate*

$$\operatorname{Re}\lambda\Big(|u|_{L^2}^2 + |u_x|_{L^2}^2\Big) + \hat{C}\int |U_x||u|^2\,dx \le \hat{C}\epsilon|u_x|_{L^2}^2, \tag{A.25}$$

for some $\hat{C} > 0$ *and all* $\epsilon > 0$ *sufficiently small.*

We first recall that there are matrices L_p, R_p diagonalizing matrix A such that

$$\tilde{A} := L_p A R_p = \begin{pmatrix} A_1^- & & 0 \\ & a_p & \\ 0 & & A_2^+ \end{pmatrix} \tag{A.26}$$

where $A_j^\pm$ are symmetric and positive/negative definite, and a_p is scalar satisfying (A.5) and $a_p = \mathcal{O}(\epsilon)$. Defining $v := L_p u$, we rewrite (A.1) as

$$\begin{aligned} \lambda v + \tilde{A} v_x + \tilde{L} q_x &= \tilde{A}(L_p)_x R_p v, \\ -q_{xx} + q + \tilde{B} v_x &= -B(R_p)_x v, \end{aligned} \tag{A.27}$$

where

$$\tilde{A} = L_p A R_p, \qquad \tilde{L} = L_p L, \qquad \tilde{B} = B R_p.$$

Define

$$S := \begin{pmatrix} \phi_- I_{p-1} & & 0 \\ & 1 & \\ 0 & & \phi_+ I_{n-p} \end{pmatrix} \tag{A.28}$$

where block diagonal form is in the same way as of (A.26) and $\phi_\pm$ are scalar functions of $x \in \mathbb{R}$ which are bounded away from zero and satisfying

$$\phi'_\pm = \mp c_* |U_x| \phi_\pm, \qquad \phi_\pm(0) = 1$$

for some sufficiently large constant c_* to be determined later. Once again, we alternatively write $'$ or $(\cdot)_x$ as derivative with respect to x.

In what follows, we shall use $\langle \cdot, \cdot \rangle$ as a weighted norm defined by

$$\langle f, f \rangle := \langle Sf, f \rangle_{L^2}.$$

With this inner product, we note that for any symmetric matrix A,

$$\langle Af_x, f\rangle = -\frac{1}{2}\langle (A_x + (S_x/S)A)f, f\rangle$$

where S_x/S should be understood as $\phi'_\pm/\phi_\pm$ or 0 in each corresponding block.

By our choice of S and $\phi_\pm$, we observe that

$$\begin{aligned}\tilde A_x + (S_x/S)\tilde A &= \begin{pmatrix} (A_1^-)' + (\phi'_-/\phi_-)A_1^- & & 0\\ & a'_p & \\ 0 & & (A_2^+)' + (\phi'_+/\phi_+)A_2^+\end{pmatrix}\\ &\le \begin{pmatrix} -c_* I & & 0\\ & -\theta & \\ 0 & & -c_* I\end{pmatrix}|U_x|\end{aligned} \tag{A.29}$$

Proposition Appendix A.7. *Denoting $v =: (v_-, v_p, v_+)^\top$, we obtain*

$$(\mathrm{Re}\,\lambda)\langle v, v\rangle + \frac{1}{2}c_*\langle |U_x|v_\pm, v_\pm\rangle + \frac{1}{2}\theta\langle |U_x|v_p, v_p\rangle \le -\mathrm{Re}\,\langle \tilde L q_x, v\rangle. \tag{A.30}$$

Proof. We take inner product in the weighted norm of the first equation of (A.27) against v, take the real part of the resulting equation, and make use of integration by parts, yielding

$$(\mathrm{Re}\,\lambda)\langle v, v\rangle - \langle (\tilde A_x + (S_x/S)\tilde A)_x v, v\rangle = -\mathrm{Re}\,\langle \tilde L q_x, v\rangle + \mathrm{Re}\,\langle \tilde A(L'_p R_p v, v\rangle. \tag{A.31}$$

Noting that $L'_p R_p = \mathcal{O}(|U_x|)$ and the fact that $\tilde A$ has the diagonal block (A.26), we estimate

$$|\langle \tilde A L'_p R_p v, v\rangle| \le C\langle |U_x|v_\pm, v_\pm\rangle + C\langle |a_p||U_x|v_p, v_p\rangle.$$

Using this, (A.29) and the fact that $|a_p| = \mathcal{O}(\epsilon)$ is sufficiently small and c_* is sufficiently large, (A.31) immediately yields (A.30). □

Proposition Appendix A.8. *We obtain*

$$(\mathrm{Re}\,\lambda)\langle v_x, v_x\rangle - \mathrm{Re}\,\langle \tilde L q_x, v\rangle \le C\langle \mathcal{O}(|U_x|^2)v, v\rangle + \eta\langle v_x, v_x\rangle \tag{A.32}$$

for sufficiently small $\eta > 0$.

Proof. We now take the inner product of the derivative of the first equation of (A.27) against v_x. We thus obtain

$$\lambda\langle v_x, v_x\rangle + \langle (\tilde A v_x)_x, v_x\rangle + \langle (\tilde L q_x)_x, v_x\rangle = \langle (\tilde A L'_p R_p v)_x, v_x\rangle \tag{A.33}$$

where we estimate by integration by parts,

$$\begin{aligned}\langle (\tilde A v_x)_x, v_x\rangle &= \langle \tilde A_x v_x, v_x\rangle - \frac{1}{2}\langle (\tilde A_x + (S_x/S)\tilde A)v_x, v_x\rangle = \langle \mathcal{O}(|U_x|)v_x, v_x\rangle\\ \langle (\tilde A L'_p R_p v)_x, v_x\rangle &= \langle \mathcal{O}(|U_x|)v_x, v_x\rangle + \langle \mathcal{O}(|U_x|)v, v_x\rangle\end{aligned}$$

and by using the second equation and the semi-definite condition $\tilde{L}\tilde{B} \geq 0$,

$$\begin{aligned}
\langle (\tilde{L}q_x)_x, v_x\rangle &= \langle \tilde{L}q_{xx}, v_x\rangle + \langle \tilde{L}_x q_x, v_x\rangle \\
&= \langle \tilde{L}(q + \tilde{B}v_x + BR_p' v), v_x\rangle + \langle \tilde{L}_x q_x, v_x\rangle \\
&= -\langle \tilde{L}q_x, v\rangle - \langle (\tilde{L}_x + (S_x/S)\tilde{L})q, v\rangle + \langle \tilde{L}\tilde{B}v_x, v_x\rangle + \langle \tilde{L}BR_p' v, v_x\rangle + \langle \tilde{L}_x q_x, v_x\rangle \\
&\geq -\langle \tilde{L}q_x, v\rangle + \langle \tilde{L}BR_p' v, v_x\rangle + \langle \tilde{L}_x q_x, v_x\rangle - \langle (\tilde{L}_x + (S_x/S)\tilde{L})q, v\rangle.
\end{aligned}$$

Thus, (A.33) yields

$$\begin{aligned}
&(\operatorname{Re}\lambda)\langle v_x, v_x\rangle - \operatorname{Re}\langle \tilde{L}q_x, v\rangle \\
&\quad \leq \langle \mathcal{O}(|U_x|^2)v, v\rangle + \eta\langle v_x, v_x\rangle + \langle \tilde{L}_x q_x, v_x\rangle - \langle (\tilde{L}_x + (S_x/S)\tilde{L})q, v\rangle.
\end{aligned} \tag{A.34}$$

By testing the second equation against q, it is easy to see that

$$\langle q_x, q_x\rangle + \langle q, q\rangle \leq C\langle v_x, v_x\rangle.$$

Thus, we have

$$\langle \tilde{L}_x q_x, v_x\rangle - \langle (\tilde{L}_x + (S_x/S)\tilde{L})q, v\rangle \leq C\langle v_x, v_x\rangle^{1/2}\Big(\langle \mathcal{O}(|U_x|^2)v_x, v_x\rangle + \langle \mathcal{O}(|U_x|^2)v, v\rangle\Big)^{1/2}$$

Using the standard Young's inequality and absorbing all necessary terms into the right hand side of (A.34), we thus obtain from (A.34) the important estimate, (A.32), which proves the proposition. □

Combining Propositions Appendix A.7 and Appendix A.8, we are now ready to give:

Proof of Lemma Appendix A.6. Adding (A.32) with (A.30), noting that the "bad" term $\operatorname{Re}\langle \tilde{L}q_x, v\rangle$ gets canceled out, and using the fact that $|U_x| = \mathcal{O}(\epsilon)$ is sufficiently small, we easily obtain

$$\operatorname{Re}\lambda(\langle v, v\rangle + \langle v_x, v_x\rangle) + \theta\langle |U_x| v, v\rangle \leq \eta\langle v_x, v_x\rangle \tag{A.35}$$

which by changing v to the original coordinate u yields the lemma. □

Proof of Theorem 1.6

Add $\hat{C}\epsilon$ times (A.24) to (A.25) to get

$$(\operatorname{Re}\lambda)(1 + \hat{C}\epsilon)|u|_{L^2}^2 + (\hat{C} + C\hat{C}\epsilon)\int |U_x||u|^2\,dx \leq 0,$$

which readily implies $\operatorname{Re}\lambda < 0$, yielding the result. □

Remark Appendix A.9. Theorem 1.6 can be extended to the non-convex case, that is, when the principal characteristic mode is no longer genuinely nonlinear (hypothesis (H2) does not hold). For that purpose, it is possible to modify the Goodman-type weighted energy estimate by means of the Matsumura-Nishihara weight function w [29] (introduced to compensate for the loss of monotonicity), satisfying

$$-\tfrac{1}{2}(wa_p + w_x) = |U_x|,$$

which replaces the 1 in the weight matrix function S in (A.28). This procedure was carried out for the viscous systems case by Fries [2] and it can be done in the present case as well at the expense of further book-keeping. Note that the existence result of [17, 18] includes non-convex systems, a feature that might be useful in applications.

Appendix B. Pointwise reduction lemma

Let us consider the situation of a system of equations of form

$$W_x = \mathbb{A}^\epsilon(x,\lambda)W, \tag{B.1}$$

for which the coefficient $\mathbb{A}^\epsilon$ does not exhibit uniform exponential decay to its asymptotic limits, but instead is *slowly varying* (uniformly on a ϵ-neighborhood $\mathcal{V}$, being $\epsilon > 0$ a parameter). This case occurs in different contexts for rescaled equations, such as (36) in the present analysis.

In this situation, it frequently occurs that not only $\mathbb{A}^\epsilon$ but also certain of its invariant eigenspaces are slowly varying with x, i.e., there exist matrices

$$\mathbb{L}^\epsilon = \begin{pmatrix} L_1^\epsilon \\ L_2^\epsilon \end{pmatrix}(x), \quad \mathbb{R}^\epsilon = \begin{pmatrix} R_1^\epsilon & R_2^\epsilon \end{pmatrix}(x)$$

for which $\mathbb{L}^\epsilon\mathbb{R}^\epsilon(x) \equiv I$ and $|\mathbb{L}\mathbb{R}'| = |\mathbb{L}'\mathbb{R}| \leq C\delta^\epsilon(x)$, uniformly in ϵ, where the pointwise error bound $\delta^\epsilon = \delta^\epsilon(x)$ is small, relative to

$$\mathbb{M}^\epsilon := \mathbb{L}^\epsilon\mathbb{A}^\epsilon\mathbb{R}^\epsilon(x) = \begin{pmatrix} M_1^\epsilon & 0 \\ 0 & M_2^\epsilon \end{pmatrix}(x) \tag{B.2}$$

and "$'$" as usual denotes $\partial/\partial x$. In this case, making the change of coordinates $W^\epsilon = \mathbb{R}^\epsilon Z$, we may reduce (B.1) to the approximately block-diagonal equation

$$Z^{\epsilon\prime} = \mathbb{M}^\epsilon Z^\epsilon + \delta^\epsilon\Theta^\epsilon Z^\epsilon, \tag{B.3}$$

where $\mathbb{M}^\epsilon$ is as in (B.2), $\Theta^\epsilon(x)$ is a uniformly bounded matrix, and $\delta^\epsilon(x)$ is (relatively) small. Assume that such a procedure has been successfully carried out, and, moreover, that there exists an approximate *uniform spectral gap in numerical range*, in the strong sense that

$$\min \sigma(\operatorname{Re} M_1^\epsilon) - \max \sigma(\operatorname{Re} M_2^\epsilon) \geq \eta^\epsilon(x), \qquad \text{for all } x,$$

with pointwise gap $\eta^\epsilon(x) > \eta_0 > 0$ uniformly bounded in x and in ϵ; here and elsewhere $\operatorname{Re} N := \frac{1}{2}(N + N^*)$ denotes the "real", or symmetric part of an operator N. Then, there holds the following *pointwise reduction lemma*, a refinement of the reduction lemma of [24] (see the related "tracking lemma" given in varying degrees of generality in [3, 23, 32, 42, 39]).

Proposition Appendix B.1. *Consider a system* (B.3) *under the gap assumption* (Appendix B)*, with Θ^ϵ uniformly bounded in $\epsilon \in \mathcal{V}$ and for all x. If, for all $\epsilon \in \mathcal{V}$, $\sup_{x\in\mathbb{R}}(\delta^\epsilon/\eta^\epsilon)$ is sufficiently small (i.e., the ratio of pointwise gap $\eta^\epsilon(x)$ and pointwise error bound $\delta^\epsilon(x)$ is uniformly small), then there exist (unique) linear transformations $\Phi_1^\epsilon(x,\lambda)$ and $\Phi_2^\epsilon(x,\lambda)$, possessing the same regularity with respect to the various parameters ϵ, x, λ as do coefficients $\mathbb{M}^\epsilon$ and $\delta^\epsilon(x)\Theta^\epsilon(x)$, for which the graphs $\{(Z_1, \Phi_2^\epsilon(Z_1))\}$ and $\{(\Phi_1^\epsilon(Z_2), Z_2)\}$ are invariant under the flow of* (B.3)*, and satisfying*

$$\sup_{\mathbb{R}} |\Phi_j^\epsilon| \le C \sup_{\mathbb{R}} (\delta^\epsilon/\eta^\epsilon).$$

Moreover, we have the pointwise bounds

$$|\Phi_2^\epsilon(x)| \le C \int_{-\infty}^{x} e^{-\int_y^x \eta^\epsilon(z)dz} \delta^\epsilon(y)dy, \tag{B.4}$$

and symmetrically for Φ_1^ϵ.

Proof. By a change of independent coordinates, we may arrange that $\eta^\epsilon(x) \equiv$ constant, whereupon the first assertion reduces to the conclusion of the tracking/reduction lemma of [24]. Recall that this conclusion was obtained by seeking Φ_2^ϵ as the solution of a fixed-point equation

$$\Phi_2^\epsilon(x) = \mathcal{T}\Phi_2^\epsilon(x) := \int_{-\infty}^{x} \mathcal{F}^{y\to x} \delta^\epsilon(y) Q(\Phi_2^\epsilon)(y)dy.$$

Observe that in the present context we have allowed δ^ϵ to vary with x, but otherwise follow the proof of [24] word for word to obtain the conclusion (see Appendix C of [24], proof of Proposition 3.9). Here, $Q(\Phi_2^\epsilon) = \mathcal{O}(1 + |\Phi_2^\epsilon|^2)$ by construction, and $|\mathcal{F}^{y\to x}| \le Ce^{-\eta(x-y)}$. Thus, using only the fact that $|\Phi_2^\epsilon|$ is bounded, we obtain the bound (B.4) as claimed, in the new coordinates for which η^ϵ is constant. Switching back to the old coordinates, we have instead $|\mathcal{F}^{y\to x}| \le Ce^{-\int_y^x \eta^\epsilon(z)dz}$, yielding the result in the general case. ∎

Remark Appendix B.2. From Proposition Appendix B.1, we obtain reduced flows

$$\begin{cases} Z_1^{\epsilon\prime} = M_1^\epsilon Z_1^\epsilon + \delta^\epsilon(\Theta_{11} + \Theta_{12}^\epsilon \Phi_2^\epsilon) Z_1^\epsilon, \\ Z_2^{\epsilon\prime} = M_2^\epsilon Z_2^\epsilon + \delta^\epsilon(\Theta_{22} + \Theta_{21}^\epsilon \Phi_1^\epsilon) Z_2^\epsilon. \end{cases}$$

on the two invariant manifolds described.

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