ON ASYMPTOTIC STABILITY OF NONCHARACTERISTIC VISCOUS BOUNDARY LAYERS

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Abstract. We extend our recent work with K. Zumbrun on long-time stability of multi-dimensional noncharacteristic viscous boundary layers of a class of symmetrizable hyperbolic-parabolic systems. Our main improvements are (i) to establish the stability for a larger class of systems in dimensions $d \geq 2$, yielding the result for certain magnetohydrodynamics (MHD) layers; (ii) to drop a technical assumption on the so–called glancing set which was used in previous works. We also provide a different proof of low-frequency estimates by employing the method of Kreiss' symmetrizers, replacing the one relying on detailed derivation of pointwise bounds on the resolvent kernel.

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1. Introduction

Boundary layers occur in many physical settings, such as gas dynamics and magnetohydrodynamics (MHD) with inflow or outflow boundary conditions, for example the flow around an airfoil with micro-suction or blowing. Layers satisfying such boundary conditions are called noncharacteristic layers; see, for example, the physical discussion in [S, SGKO]. See also [GMWZ5, YZ, NZ1, NZ2, Z5] for further discussion.

In this paper, we study the stability of boundary layers assuming that the layer is noncharacteristic. Specifically, we consider a boundary layer, or stationary solution, connecting the endstate U_+:

$$\tilde U = \bar U(x_1), \quad \lim_{x_1\to+\infty} \bar U(x_1) = U_+. \tag{1.1}$$

of a general system of viscous conservation laws on the quarter-space

$$\tilde U_t + \sum_j F^j(\tilde U)_{x_j} = \sum_{jk} (B^{jk}(\tilde U)\tilde U_{x_k})_{x_j}, \quad x \in \mathbb{R}^d_+, \quad t > 0, \tag{1.2}$$

$\tilde U, F^j \in \mathbb{R}^n$, $B^{jk} \in \mathbb{R}^{n\times n}$, with initial data $\tilde U(x,0) = \tilde U_0(x)$ and boundary conditions as specified in (B) below.

An fundamental question is to establish *asymptotic stability* of these solutions under perturbation of the initial or boundary data. This question has been investigated in [GR, MeZ1, GMWZ5, GMWZ6, YZ, NZ1, NZ2] for arbitrary-amplitude boundary-layers using Evans function techniques, with the result that linearized and nonlinear stability reduce to a generalized spectral stability, or Evans stability, condition. See also the small-amplitude results of [GG, R3, MN, KNZ, KaK] obtained by energy methods.

In the current paper, as in [N1] for the shock cases, we apply the method of Kreiss' symmetrizers to provide a different proof of estimates on low-frequency part of the solution operator, which allows us to extend the existing stability result in [NZ2] to a larger class of symmetrizable systems including MHD equations, yielding the result for certain MHD layers. We are also able to drop a technical assumption (H4) that was required in previous analysis of [Z2, Z3, Z4, GMWZ1, NZ2].

1.1. **Equations and assumptions.**

We consider the general hyperbolic-parabolic system of conservation laws (1.2) in conserved variable $\tilde U$, with

$$\tilde U = \begin{pmatrix} \tilde u^I \\ \tilde u^{II} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b_1^{jk} & b_2^{jk} \end{pmatrix},$$

$\tilde u^I \in \mathbb{R}^{n-r}$, $\tilde u^{II} \in \mathbb{R}^r$, and

$$\Re\sigma \sum_{jk} b_2^{jk}\xi_j\xi_k \geq \theta|\xi|^2 > 0, \quad \forall \xi \in \mathbb{R}^n\backslash\{0\}.$$

Following [MaZ3, MaZ4, Z3, Z4], we assume that equations (1.2) can be written, alternatively, after a triangular change of coordinates

$$\tilde W := \tilde W(\tilde U) = \begin{pmatrix} \tilde w^I(\tilde u^I) \\ \tilde w^{II}(\tilde u^I, \tilde u^{II}) \end{pmatrix}, \tag{1.3}$$

in *the quasilinear, partially symmetric hyperbolic-parabolic form*

$$\tilde A^0 \tilde W_t + \sum_j \tilde A^j \tilde W_{x_j} = \sum_{jk} (\tilde B^{jk} \tilde W_{x_k})_{x_j} + \tilde G, \tag{1.4}$$

where, defining $\tilde W_+ := \tilde W(U_+)$,

(A1) $\tilde A^j(\tilde W_+), \tilde A^0, \tilde A^1_{11}$ are symmetric, $\tilde A^0$ block diagonal, $\tilde A^0 \geq \theta_0 > 0$,

(A2) for each $\xi \in \mathbb{R}^d \setminus \{0\}$, no eigenvector of $\sum_j \xi_j \tilde A^j (\tilde A^0)^{-1}(\tilde W_+)$ lies in the kernel of $\sum_{jk} \xi_j \xi_k \tilde B^{jk} (\tilde A^0)^{-1}(\tilde W_+)$,

(A3) $\tilde B^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde b^{jk} \end{pmatrix}$, $\sum \tilde b^{jk} \xi_j \xi_k \geq \theta |\xi|^2$, and $\tilde G = \begin{pmatrix} 0 \\ \tilde g \end{pmatrix}$ with $\tilde g(\tilde W_x, \tilde W_x) = \mathcal{O}(|\tilde W_x|^2)$.

Along with the above structural assumptions, we make the following technical hypotheses:

(H0) $F^j, B^{jk}, \tilde A^0, \tilde A^j, \tilde B^{jk}, \tilde W(\cdot), \tilde g(\cdot,\cdot) \in C^{s+1}$, with $s \geq [(d-1)/2] + 4$ in our analysis of linearized stability, and $s \geq s(d) := [(d-1)/2] + 7$ in our analysis of nonlinear stability.

(H1) $\tilde A^{11}_1$ is either strictly positive or strictly negative, that is, either $\tilde A^{11}_1 \geq \theta_1 > 0$, or $\tilde A^{11}_1 \leq -\theta_1 < 0$. (We shall call these cases the *inflow case* or *outflow case*, correspondingly.)

(H2) The eigenvalues of $dF^1(U_+)$ are distinct and nonzero.

(H3) The eigenvalues of $\sum_j \xi_j dF^j(U_+)$ have constant multiplicity with respect to $\xi \in \mathbb{R}^d$, $\xi \neq 0$.

Alternative Hypothesis H3′. The constant multiplicity condition in Hypothesis (H3) holds for the compressible Navier Stokes equations whenever is hyperbolic. We are able to treat symmetric dissipative systems like the equations of viscous MHD, for which the constant multiplicity condition fails, under the following relaxed hypothesis.

(H3′) The eigenvalues of $\sum_j \xi_j dF^j(U_+)$ are either semisimple and of constant multiplicity or totally nonglancing in the sense of [GMWZ6], Definition 4.3.

Additional Hypothesis H4′ (in 3D). In the treatment of the three–dimensional case, the analysis turns out to be quite delicate and we are able to establish the stability under the following additional (generic) hypothesis (see Remark 3.4 and Appendix A for discussions of this condition):

(H4′) In the case the eigenvalue $\lambda_k(\xi)$ of $\sum_j \xi_j dF^j(U_+)$ is semisimple and of constant multiplicity, we assume further that $\nabla_{\tilde\xi} \lambda_k \neq 0$ when $\partial_{\xi_1} \lambda_k = 0$, $\xi \neq 0$.

Remark 1.1. Here we stress that we are able to drop the following structural assumption, which is needed for the earlier analyses of [Z2, Z3, Z4, NZ2].

(H4) The set of branch points of the eigenvalues of $(\tilde A^1)^{-1}(i\tau \tilde A^0 + \sum_{j\neq 1} i\xi_j \tilde A^j)_+$, $\tau \in \mathbb{R}$, $\tilde\xi \in \mathbb{R}^{d-1}$ is the (possibly intersecting) union of finitely many smooth curves $\tau = \eta_q^+(\tilde\xi)$, on which the branching eigenvalue has constant multiplicity s_q (by definition ≥ 2).

We also assume:

(B) Dirichlet boundary conditions in $\tilde{W}$-coordinates:

$$(\tilde{w}^I, \tilde{w}^{II})(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) := (\tilde{h}_1, \tilde{h}_2)(\tilde{x}, t) \tag{1.5}$$

for the inflow case, and

$$\tilde{w}^{II}(0, \tilde{x}, t) = \tilde{h}(\tilde{x}, t) \tag{1.6}$$

for the outflow case, with $x = (x_1, \tilde{x}) \in \mathbb{R}^d$.

This is sufficient for the main physical applications; the situation of more general, Neumann and mixed-type boundary conditions on the parabolic variable $\tilde{w}^{II}$ can be treated as discussed in [GMWZ5, GMWZ6].

1.2. **The Evans condition and strong spectral stability.** A necessary condition for linearized stability is weak spectral stability, defined as nonexistence of unstable spectra $\Re\lambda > 0$ of the linearized operator L about the wave. As described in [Z2, Z3], this is equivalent to nonvanishing for all $\tilde{\xi} \in \mathbb{R}^{d-1}$, $\Re\lambda > 0$ of the *Evans function*

$$D_L(\tilde{\xi}, \lambda),$$

a Wronskian associated with the family of eigenvalue ODE obtained by Fourier transform in the directions $\tilde{x} := (x_2, \dots, x_d)$. See [Z2, Z3, GMWZ5, GMWZ6, NZ2] for further discussion.

Definition 1.2. We define *strong spectral stability* as *uniform Evans stability*:

$$|D_L(\tilde{\xi}, \lambda)| \geq \theta(C) > 0 \tag{D}$$

for $(\tilde{\xi}, \lambda)$ on bounded subsets $C \subset \{\tilde{\xi} \in \mathbb{R}^{d-1}, \Re\lambda \geq 0\} \setminus \{0\}$.

For the class of equations we consider, this is equivalent to the uniform Evans condition of [GMWZ5, GMWZ6], which includes an additional high-frequency condition that for these equations is always satisfied (see Proposition 3.8, [GMWZ5]). A fundamental result proved in [GMWZ5] is that small-amplitude noncharacteristic boundary-layers are always strongly spectrally stable.

Proposition 1.3 ([GMWZ5])**.** *Assuming (A1)-(A3), (H0)-(H2), (H3′), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar{U} - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small satisfy the strong spectral stability condition (D).

As demonstrated in [SZ, Z5], stability of large-amplitude boundary layers may fail for the class of equations considered here, even in a single space dimension, so there is no such general theorem in the large-amplitude case. Stability of large-amplitude boundary-layers may be checked efficiently by numerical Evans computations; see, e.g., [HLZ, CHNZ, HLyZ1, HLyZ2].

1.3. **Main results.** Our main results are as follows.

Theorem 1.4 (Linearized stability)**.** *Assuming (A1)-(A3), (H0)-(H2), (H3′), (H4′), (B), and (D), we obtain the asymptotic* $L^1 \cap H^{[(d-1)/2]+5} \to L^p$ *stability in dimensions* $d \geq 3$, *and any* $2 \leq p \leq \infty$, *with rates of decay*

$$(1.7) \qquad \begin{aligned} |U(t)|_{L^2} &\leq C(1+t)^{-\frac{d-2}{4}-\epsilon}|U_0|_{L^1\cap L^2},\\ |U(t)|_{L^p} &\leq C(1+t)^{-\frac{d-1}{2}(1-1/p)+\frac{1}{2p}-\epsilon}|U_0|_{L^1\cap H^{[(d-1)/2]+5}}, \end{aligned}$$

for some $\epsilon > 0$, *provided that the initial perturbations* U_0 *are in* $L^1 \cap H^{[(d-1)/2]+5}$, *and zero boundary perturbations.*

Theorem 1.5 (Nonlinear stability)**.** *Assuming (A1)-(A3), (H0)-(H2), (H3′), (H4′), (B), and (D), we obtain the asymptotic* $L^1 \cap H^s \to L^p \cap H^s$ *stability in dimensions* $d \geq 3$, *for* $s \geq s(d)$ *as defined in (H0), and any* $2 \leq p \leq \infty$, *with rates of decay*

$$(1.8) \qquad \begin{aligned} |\tilde U(t) - \bar U|_{L^p} &\leq C(1+t)^{-\frac{d-1}{2}(1-1/p)+\frac{1}{2p}-\epsilon}|U_0|_{L^1\cap H^s}\\ |\tilde U(t) - \bar U|_{H^s} &\leq C(1+t)^{-\frac{d-2}{4}-\epsilon}|U_0|_{L^1\cap H^s}, \end{aligned}$$

for some $\epsilon > 0$, *provided that the initial perturbations* $U_0 := \tilde U_0 - \bar U$ *are sufficiently small in* $L^1 \cap H^s$ *and zero boundary perturbations.*

Remark 1.6. As will be seen in the proof, the assumption (H4′) can be dropped in the case $d \geq 4$, though we then lose the factor $t^{-\epsilon}$ in the decay rate.

Our final main result gives the stability for the two–dimensional case that is not covered by the above theorems. We remark here that as shown in [Z2, Z3], Hypothesis (H4) is automatically satisfied in dimensions $d = 1, 2$ and in any dimension for rotationally invariant problems. Thus, in treating the two–dimensional case, we assume this hypothesis without making any further restriction on structure of the systems. Also since the proof does not depend on dimension d, we state the theorem in a general form as follows.

Theorem 1.7 (Two-dimensional case or cases with (H4))**.** *Assume (A1)-(A3), (H0)-(H2), (H3′), (H4), (B), and (D). We obtain asymptotic* $L^1 \cap H^s \to L^p \cap H^s$ *stability of* $\bar U$ *as a solution of* (1.2) *in dimension* $d \geq 2$, *for* $s \geq s(d)$ *as defined in (H0), and any* $2 \leq p \leq \infty$, *with rates of decay*

$$(1.9) \qquad \begin{aligned} |\tilde U(t) - \bar U|_{L^p} &\leq C(1+t)^{-\frac{d}{2}(1-1/p)+1/2p}|U_0|_{L^1\cap H^s}\\ |\tilde U(t) - \bar U|_{H^s} &\leq C(1+t)^{-\frac{d-1}{4}}|U_0|_{L^1\cap H^s}, \end{aligned}$$

provided that the initial perturbations $U_0 := \tilde U_0 - \bar U$ *are sufficiently small in* $L^1 \cap H^s$ *and zero boundary perturbations. Similar statement holds for linearized stability.*

Remark 1.8. The same results can be also obtained for nonzero boundary perturbations as treated in [NZ2]. In fact, in [NZ2], though a bit of tricky, it has been already shown that estimates on solution operator (see Proposition 2.1) for homogenous boundary conditions are enough to treat nonzero boundary perturbations. Thus for sake of simplicity, we only treat zero boundary perturbations in the current paper.

Combining Theorems 1.4, 1.5, 1.7 and Proposition 1.3, we obtain the following small-amplitude stability result.

Corollary 1.9. *Assuming (A1)-(A3), (H0)-(H2), (H3'), (B) for some fixed endstate (or compact set of endstates) U_+, boundary layers with amplitude*

$$\|\bar U - U_+\|_{L^\infty[0,+\infty]}$$

sufficiently small are linearly and nonlinearly stable in the sense of Theorems 1.4, 1.5, and 1.7.

2. Nonlinear stability

The linearized equations of (1.2) about the profile $\bar U$ are

$$U_t = LU := \sum_{j,k} (B^{jk} U_{x_k})_{x_j} - \sum_j (A^j U)_{x_j} \tag{2.1}$$

with initial data $U(0) = U_0$. Then, we obtain the following proposition, extending Proposition 3.5 of [NZ2] under our weaker assumptions.

Proposition 2.1. *Under the hypotheses of Theorem 1.5, the solution operator $\mathcal{S}(t) := e^{Lt}$ of the linearized equations may be decomposed into low frequency and high frequency parts (see below) as $\mathcal{S}(t) = \mathcal{S}_1(t) + \mathcal{S}_2(t)$ satisfying*

$$\begin{aligned} |\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde x}^{\tilde\beta} f|_{L^2_x} &\le C(1+t)^{-(d-2)/4-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-2)/4-\epsilon/2}|f|_{L^{1,\infty}_{\tilde x, x_1}} \\ |\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde x}^{\tilde\beta} f|_{L^{2,\infty}_{\tilde x, x_1}} &\le C(1+t)^{-(d-1)/4-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-1)/4-\epsilon/2}|f|_{L^{1,\infty}_{\tilde x, x_1}} \\ |\mathcal{S}_1(t)\partial_{x_1}^{\beta_1}\partial_{\tilde x}^{\tilde\beta} f|_{L^\infty_x} &\le C(1+t)^{-(d-1)/2-\epsilon/2-|\beta|/2}|f|_{L^1_x} + C(1+t)^{-(d-1)/2-\epsilon/2}|f|_{L^{1,\infty}_{\tilde x, x_1}} \end{aligned} \tag{2.2}$$

for some $\epsilon > 0$ and $\beta = (\beta_1, \tilde\beta)$ with $\beta_1 = 0, 1$, and

$$|\partial_{x_1}^{\gamma_1}\partial_{\tilde x}^{\tilde\gamma}\mathcal{S}_2(t) f|_{L^2} \le Ce^{-\theta_1 t}|f|_{H^{|\gamma_1|+|\tilde\gamma|+3}}, \tag{2.3}$$

for $\gamma = (\gamma_1, \tilde\gamma)$ with $\gamma_1 = 0, 1$.

We shall give a proof of Proposition 2.1 in Section 3. For the rest of this section, we give a rather straightforward proof of the first two main theorems using estimates of the solution operator stated in Proposition 2.1, following nonlinear arguments of [Z3, NZ2].

2.1. **Proof of linearized stability.**

Applying estimates on low- and high-frequency operators $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ obtained in Proposition 2.1, we obtain

$$\begin{aligned} |U(t)|_{L^2} &\le |\mathcal{S}_1(t)U_0|_{L^2} + |\mathcal{S}_2(t)U_0|_{L^2} \\ &\le C(1+t)^{-\frac{d-2}{4}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde x, x_1}}] + Ce^{-\eta t}|U_0|_{H^3} \\ &\le C(1+t)^{-\frac{d-2}{4}-\frac{\epsilon}{2}}|U_0|_{L^1\cap H^3} \end{aligned} \tag{2.4}$$

and (together with Sobolev embedding)

$$
\begin{aligned}
|U(t)|_{L^\infty} &\le |\mathcal{S}_1(t)U_0|_{L^\infty} + |\mathcal{S}_2(t)U_0|_{L^\infty} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde{x},x_1}}] + C|\mathcal{S}_2(t)U_0|_{H^{[(d-1)/2]+2}} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}[|U_0|_{L^1} + |U_0|_{L^{1,\infty}_{\tilde{x},x_1}}] + Ce^{-\eta t}|U_0|_{H^{[(d-1)/2]+5}} \\
&\le C(1+t)^{-\frac{d-1}{2}-\frac{\epsilon}{2}}|U_0|_{L^1\cap H^{[(d-1)/2]+5}}.
\end{aligned}
\tag{2.5}
$$

These prove the bounds as stated in the theorem for $p = 2$ and $p = \infty$. For $2 < p < \infty$, we use the interpolation inequality between L^2 and L^∞.

2.2. **Proof of nonlinear stability.** Defining the perturbation variable $U := \tilde{U} - \bar{U}$, we obtain the nonlinear perturbation equations

$$
U_t - LU = \sum_j Q^j(U, U_x)_{x_j}, \tag{2.6}
$$

where

$$
\begin{aligned}
Q^j(U, U_x) &= \mathcal{O}(|U||U_x| + |U|^2) \\
Q^j(U, U_x)_{x_j} &= \mathcal{O}(|U||U_x| + |U||U_{xx}| + |U_x|^2) \\
Q^j(U, U_x)_{x_j x_k} &= \mathcal{O}(|U||U_{xx}| + |U_x||U_{xx}| + |U_x|^2 + |U||U_{xxx}|)
\end{aligned}
\tag{2.7}
$$

so long as $|U|$ remains bounded.

Applying the Duhamel principle to (2.6), we obtain

$$
U(x,t) = \mathcal{S}(t)U_0 + \int_0^t \mathcal{S}(t-s)\sum_j \partial_{x_j} Q^j(U, U_x)ds \tag{2.8}
$$

where $U(x, 0) = U_0(x)$.

Proof of Theorem 1.5. Define

$$
\begin{aligned}
\zeta(t) := \sup_s \Big(&|U(s)|_{L^2_{\tilde{x}}}(1+s)^{\frac{d-2}{4}+\epsilon} + |U(s)|_{L^\infty_{\tilde{x}}}(1+s)^{\frac{d-1}{2}+\epsilon} \\
&+ (|U(s)| + |U_x(s)|)_{L^{2,\infty}_{\tilde{x},x_1}}(1+s)^{\frac{d-1}{4}+\epsilon}\Big).
\end{aligned}
\tag{2.9}
$$

We shall prove here that for all $t \ge 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$
\zeta(t) \le C(|U_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{2.10}
$$

This bound together with continuity of $\zeta(t)$ implies that

$$
\zeta(t) \le 2C|U_0|_{L^1\cap H^s} \tag{2.11}
$$

for $t \ge 0$, provided that $|U_0|_{L^1\cap H^s} < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s, and the standard principle of continuation, there exists a solution $U \in H^s$ on the open time-interval for which $|U|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well-defined and continuous. Now, let $[0, T)$

be the maximal interval on which $|U|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta > 0$. Recalling the following energy estimate (see Proposition 4.1 of [NZ2]) and the Sobolev embeding inequality $|U|_{W^{2,\infty}} \le C|U|_{H^s}$, we have

$$\begin{aligned}|U(t)|^2_{H^s} &\le Ce^{-\theta t}|U_0|^2_{H^s} + C\int_0^t e^{-\theta(t-\tau)}|U(\tau)|^2_{L^2}d\tau \\ &\le C(|U_0|^2_{H^s} + \zeta(t)^2)(1+t)^{-(d-2)/2-2\epsilon}.\end{aligned} \tag{2.12}$$

and so the solution continues so long as ζ remains small, with bound (2.11), yielding existence and the claimed bounds.

Thus, it remains to prove the claim (2.10). First by (2.8), we obtain

$$\begin{aligned}|U(t)|_{L^2} \le& |\mathcal{S}(t)U_0|_{L^2} + \int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ &+ \int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds\end{aligned} \tag{2.13}$$

where $|\mathcal{S}(t)U_0|_{L^2} \le C(1+t)^{-\frac{d-1}{4}-\epsilon}|U_0|_{L^1\cap H^3}$ and

$$\begin{aligned}&\int_0^t |\mathcal{S}_1(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ &\quad\le C\int_0^t (1+t-s)^{-\frac{d-2}{4}-\frac12-\epsilon}|Q^j(s)|_{L^1} + (1+s)^{-\frac{d-2}{4}-\epsilon}|Q^j(s)|_{L^{1,\infty}_{\tilde x,x_1}}ds \\ &\quad\le C\int_0^t (1+t-s)^{-\frac{d-2}{4}-\frac12-\epsilon}|U|^2_{H^1} + (1+t-s)^{-\frac{d-2}{4}-\epsilon}\Big(|U|^2_{L^{2,\infty}_{\tilde x,x_1}} + |U_x|^2_{L^{2,\infty}_{\tilde x,x_1}}\Big)ds \\ &\quad\le C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t \Big[(1+t-s)^{-\frac{d-2}{4}-\frac12-\epsilon}(1+s)^{-\frac{d-2}{2}-2\epsilon} \\ &\qquad\quad + (1+t-s)^{-\frac{d-2}{4}-\epsilon}(1+s)^{-\frac{d-1}{2}-2\epsilon}\Big]ds \\ &\quad\le C(1+t)^{-\frac{d-2}{4}-\epsilon}(|U_0|^2_{H^s} + \zeta(t)^2)\end{aligned}$$

and

$$\begin{aligned}&\int_0^t |\mathcal{S}_2(t-s)\partial_{x_j}Q^j(s)|_{L^2}ds \\ &\quad\le \int_0^t e^{-\theta(t-s)}|\partial_{x_j}Q^j(s)|_{H^3}ds \\ &\quad\le C\int_0^t e^{-\theta(t-s)}|U|^2_{H^s}ds \\ &\quad\le C(|U_0|^2_{H^s} + \zeta(t)^2)\int_0^t e^{-\theta(t-s)}(1+s)^{-\frac{d-2}{2}-2\epsilon}ds \\ &\quad\le C(1+t)^{-\frac{d-2}{2}-2\epsilon}(|U_0|^2_{H^s} + \zeta(t)^2).\end{aligned}$$

Therefore, combining these above estimates yields

$$|U(t)|_{L^2}(1+t)^{\frac{d-2}{4}+\epsilon} \le C(|U_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{2.14}$$

Similarly, we can obtain estimates for other norms of U in definition of ζ, and finish the proof of claim (2.10) and thus the main theorem. □

Remark 2.2. The decaying factor $t^{-\epsilon}$ is crucial in above analysis when $d = 3$. In fact, the main difficulty here comparing with the shock cases in [N1] is to obtain a refined bound of solutions in L^∞. See further discussion in Section 3 below.

3. Linearized estimates

In this section, we shall give a proof of Proposition 2.1 or bounds on $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$, where we use the same decomposition of solution operator $\mathcal{S}(t) = \mathcal{S}_1(t)+\mathcal{S}_2(t)$ as in [Z2, Z3].

3.1. **High–frequency estimate.** We first observe that our relaxed Hypothesis (H3′) and the dropped Hypothesis (H4) only play a role in low–frequency regimes. Thus, in course of obtaining the high–frequency estimate (2.3), we make here the same assumptions as were made in [NZ2], and therefore the same estimate remains valid as claimed in (2.3) under our current assumptions. We omit to repeat its proof here, and refer the reader to the paper [NZ2], Proposition 3.6.

In the remaining of this section, we shall focus on proving the bounds on low-frequency part $\mathcal{S}_1(t)$ of linearized solution operator.

Taking the Fourier transform in $\tilde{x} := (x_2, \dots, x_d)$ of linearized equation (2.1), we obtain a family of eigenvalue ODE

$$\begin{aligned} \lambda U = L_{\tilde{\xi}} U := \overbrace{(B_{11}U')' - (A_1 U)'}^{L_0 U} - i\sum_{j\neq 1} A_j \xi_j U + i\sum_{j\neq 1} B_{j1}\xi_j U' \\ + i\sum_{k\neq 1} (B_{1k}\xi_k U)' - \sum_{j,k\neq 1} B_{jk}\xi_j\xi_k U. \end{aligned} \tag{3.1}$$

3.2. **The GMWZ's L^2 stability estimate.** Let $U = (u^I, u^{II})^T$ be a solution of resolvent equation $(L_{\tilde{\xi}} - \lambda)U = f$. Following [Z3, GMWZ6], consider the variable W as usual

$$W := \begin{pmatrix} w^I \\ w^{II} \\ w^{II}_{x_1} \end{pmatrix}$$

with $w^I := A_* u^I, w^{II} := b_1^{11} u^I + b_2^{11} u^{II}$, $A_* := A_{11}^1 - A_{12}^1 (b_2^{11})^{-1} b_1^{11}$. Then we can write equations of W as a first order system

$$\begin{aligned} \partial_{x_1} W = \mathcal{G}(x_1, \lambda, \tilde{\xi})W + F \\ \Gamma W = 0 \text{ on } x_1 = 0. \end{aligned} \tag{3.2}$$

For small or bounded frequencies $(\lambda, \tilde{\xi})$, we use the MZ conjugation lemma (see [MeZ1, MeZ3]). That is, given any $(\underline{\lambda}, \underline{\tilde{\xi}}) \in \mathbb{R}^{d+1}$, there is a smooth invertible matrix $\Phi(x_1, \lambda, \tilde{\xi})$ for $x_1 \geq 0$ and $(\lambda, \tilde{\xi})$ in a small neighborhood of $(\underline{\lambda}, \underline{\tilde{\xi}})$, such that (3.2) is equivalent to

$$\partial_{x_1} Y = \mathcal{G}_+(\lambda, \tilde{\xi})Y + \tilde{F}, \quad \tilde{\Gamma}(\lambda, \tilde{\xi})Y = 0 \tag{3.3}$$

where $\mathcal{G}_+(\lambda, \tilde{\xi}) := \tilde{\mathcal{G}}(+\infty, \lambda, \tilde{\xi}), W = \Phi Y, \tilde{F} = \Phi^{-1}F$ and $\tilde{\Gamma} Y := \Gamma \Phi Y$.

Next, there are smooth matrices $V(\lambda,\tilde\xi)$ such that

$$V^{-1}\mathcal{G}_+V = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix} \tag{3.4}$$

with blocks $H(\lambda,\tilde\xi)$ and $P(\lambda,\tilde\xi)$ satisfying the eigenvalues μ of P in $\{|\Re e\mu|\ge c>0\}$ and

$$\begin{aligned} H(\lambda,\tilde\xi) &= H_0(\lambda,\tilde\xi) + \mathcal{O}(\rho^2) \\ H_0(\lambda,\tilde\xi) &:= -(A^1_+)^{-1}\Big((i\tau+\gamma)A^0_+ + \sum_{j=2}^d i\xi_j A^j_+\Big), \end{aligned}$$

with $\lambda = \gamma + i\tau$. We later often use the polar coordinate notation $\zeta = (\tau,\gamma,\tilde\xi), \zeta = \rho\hat\zeta$, where $\hat\zeta = (\hat\tau,\hat\gamma,\hat{\tilde\xi})$ and $\hat\zeta\in S^d$.

Define variables $Z = (u_H,u_P)^T$ as $W = \Phi Y = \Phi V Z$, $\bar\Gamma Z := \Gamma\Phi V Z$, and $(f_H,f_P)^T = V^{-1}\tilde F$. We have

$$\partial_{x_1}\begin{pmatrix} u_H \\ u_P \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & P \end{pmatrix}\begin{pmatrix} u_H \\ u_P \end{pmatrix} + \begin{pmatrix} f_H \\ f_P \end{pmatrix}, \quad \bar\Gamma Z = 0. \tag{3.5}$$

Then the maximal stability estimate for the low frequency regimes in [GMWZ6] states that

$$(\gamma+\rho^2)|u_H|^2_{L^2} + |u_P|^2_{L^2} + |u_H(0)|^2 + |u_P(0)|^2 \lesssim \langle|f_H|,|u_H|\rangle + \langle|f_P|,|u_P|\rangle. \tag{3.6}$$

We note that in the final step there in [GMWZ1], the standard Young's inequality has been used to absorb all terms of (u_H,u_P) into the left-hand side, leaving the L^2 norm of F alone in the right hand side. For our purpose, we shall keep it as stated in (3.6). Here, by $f\lesssim g$, we mean $f\le Cg$, for some C independent of parameter ρ.

We remark also that as shown in [GMWZ1], all of coordinate transformation matrices are uniformly bounded. Thus a bound on $Z = (u_H,u_P)^T$ would yield a corresponding bound on the solution U.

3.3. **L^2 and L^∞ resolvent bounds.** Changing variables as above and taking the inner product of each equation in (3.5) against u_H and u_P, respectively, and integrating the results over $[0,x_1]$, for $x_1>0$, we obtain

$$\begin{aligned} \frac12|u_H(x_1)|^2 &= \frac12|u_H(0)|^2 + \Re e\int_0^{x_1}(H(\lambda,\tilde\xi)u_H\cdot u_H + f_H\cdot u_H)dz, \\ \frac12|u_P(x_1)|^2 &= \frac12|u_P(0)|^2 + \Re e\int_0^{x_1}(P(\lambda,\tilde\xi)u_P\cdot u_P + f_P\cdot u_P)dz. \end{aligned} \tag{3.7}$$

This together with the facts that $|H|\le C\rho$ and $|P|\le C$ yields

$$\begin{aligned} |u_H|^2_{L^\infty(x_1)} &\lesssim |u_H(0)|^2 + \rho|u_H|^2_{L^2} + \langle|f_H|,|u_H|\rangle, \\ |u_P|^2_{L^\infty(x_1)} &\lesssim |u_P(0)|^2 + |u_P|^2_{L^2} + \langle|f_P|,|u_P|\rangle, \end{aligned} \tag{3.8}$$

and thus in view of (3.6) gives

$$(\gamma+\rho^2)|u_H|^2_{L^2} + |u_P|^2_{L^2} + (\hat\gamma+\rho)|u_H|^2_{L^\infty} + |u_P|^2_{L^\infty} \lesssim \langle|f_H|,|u_H|\rangle + \langle|f_P|,|u_P|\rangle. \tag{3.9}$$

Now applying the Young's inequality, we get

$$\langle |f_H|, |u_H|\rangle + \langle |f_P|, |u_P|\rangle \le (\epsilon |u_P|^2_{L^\infty} + C_\epsilon |f_P|^2_{L^1}) + \Big(\epsilon(\hat\gamma+\rho)|u_H|^2_{L^\infty} + \frac{C_\epsilon}{\hat\gamma+\rho}|f_H|^2_{L^1}\Big)$$

and thus for ϵ sufficiently small, together with (3.9),

$$(\gamma+\rho^2)|u_H|^2_{L^2} + |u_P|^2_{L^2} + (\hat\gamma+\rho)|u_H|^2_{L^\infty} + |u_P|^2_{L^\infty} \lesssim \frac{1}{\hat\gamma+\rho}|f_H|^2_{L^1} + |f_P|^2_{L^1}. \tag{3.10}$$

Therefore in term of $Z = (u_H, u_P)^t$,

$$|Z|_{L^\infty(x_1)} \le C(\hat\gamma+\rho)^{-1}|f|_{L^1} \quad \text{and} \quad |Z|_{L^2(x_1)} \le C(\hat\gamma+\rho)^{-3/2}|f|_{L^1}. \tag{3.11}$$

Unfortunately, unlike the shock cases (see [N1]), bounds (3.11) are not enough for our need to close the analysis in dimension $d = 3$. See Remark 2.2. In the following subsection, we shall derive better bounds for Z in both L^∞ and L^2 norms.

3.4. **Refined L^2 and L^∞ resolvent bounds.** With the same notations as above, we prove in this subsection that there hold refined resolvent bounds:

$$\begin{aligned} |Z|_{L^\infty(x_1)} &\lesssim (\hat\gamma+\rho)^{-1+\epsilon}(|f|_{L^1} + |f|_{L^\infty}) \\ |Z|_{L^2(x_1)} &\lesssim (\hat\gamma+\rho)^{-3/2+\epsilon}(|f|_{L^1} + |f|_{L^\infty}) \end{aligned} \tag{3.12}$$

for some small $\epsilon > 0$. We stress here that a refined factor ρ^ϵ in L^∞ is crucial in our analysis for three-dimensional case. See Remark 2.2.

Assumption (H3′) implies the following block structure (see [MeZ3, GMWZ6]).

Proposition 3.1 (Block structure; [GMWZ6])**.** *For all $\underline{\hat\zeta}$ with $\underline{\hat\gamma} \ge 0$ there is a neighborhood ω of $(\underline{\hat\zeta}, 0)$ in $S^d \times \overline{\mathbb{R}}_+$ and there are C^∞ matrices $T(\hat\zeta,\rho)$ on ω such that $T^{-1}H_0T$ has the block diagonal structure*

$$T^{-1}H_0T = H_B(\hat\zeta,\rho) = \rho \hat H_B(\hat\zeta,\rho) \tag{3.13}$$

with

$$\hat H_B(\hat\zeta,\rho) = \begin{bmatrix} Q_1 & 0 & \\ 0 & \ddots & 0 \\ & 0 & Q_p \end{bmatrix}(\hat\zeta,\rho) \tag{3.14}$$

with diagonal blocks Q_k of size $\nu_k \times \nu_k$ such that:

(i) (Elliptic modes) $\Re Q_k$ is either positive definite or negative definite.

(ii) (Hyperbolic modes) $\nu_k = 1$, $\Re Q_k = 0$ when $\hat\gamma = \rho = 0$, and $\partial_{\hat\gamma}(\Re Q_k)\partial_\rho(\Re Q_k) > 0$.

(iii) (Glancing modes) $\nu_k > 1$, Q_k has the following form:

$$Q_k(\hat\zeta,\rho) = i(\underline{\mu}_k \mathrm{Id} + J) + i\sigma Q'_k(\hat{\hat\xi}) + \mathcal{O}(\hat\gamma+\rho), \tag{3.15}$$

where $\sigma := |\hat{\hat\xi} - \underline{\hat{\hat\xi}}|$,

$$J := \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ & & 0 & 0 \end{bmatrix}, \qquad Q'_k(\hat{\hat\xi}) := \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ q_2 & 0 & \cdots & 0 \\ & & \cdots & \\ q_{\nu_k} & 0 & \cdots & 0 \end{bmatrix} \tag{3.16}$$

$q_{\nu_k} \neq 0$, and the lower left hand corner a of Q_k satisfies $\partial_{\hat\gamma}(\Re a)\partial_\rho(\Re a) > 0$.

(iv) (Totally nonglancing modes) $\nu_k > 1$, eigenvalue of Q_k, when $\hat\gamma = \rho = 0$, is totally nonglancing, see Definition 4.3, [GMWZ6].

Proof. For a proof, see for example [Met], Theorem 8.3.1. It is also straightforward to see that for the case (iii),

$$q_{\nu_k}(\hat{\underline{\tilde\xi}}) = |\nabla_{\tilde\xi} D_k(\underline{\zeta}, \underline{\xi}_1)| = c|\nabla_{\tilde\xi}\lambda_k(\xi)|,$$

where c is a nonzero constant, $D_k(\zeta,\xi_1)$ is defined as $\det(iQ_k(\zeta) + \xi_1 Id)$, and $\lambda_k(\xi)$ is the zero of $D_k(\zeta,\xi_1)$ (recalling $\zeta = (\lambda,\tilde\xi)$) satisfying

$$\partial_{\xi_1}\lambda_k = ... = \partial_{\xi_1}^{\nu_k - 1}\lambda_k = 0, \quad \partial_{\xi_1}^{\nu_k}\lambda_k \neq 0 \qquad \text{at } (\underline{\tilde\xi}, \underline{\xi}_1).$$

Thus, assumption (H4′) guarantees the nonvanishing of q_{ν_k}. We skip the proof of other facts. □

We shall treat each mode in turn. The following simple lemma may be found useful.

Lemma 3.2. *Let U be a solution of $\partial_z U = QU + F$ with $U(+\infty) = 0$. Assume that there is a positive [resp., negative] symmetric matrix S such that*

$$\Re SQ := \frac{1}{2}(SQ + Q^* S^*) \geq \theta Id \tag{3.17}$$

for some $\theta > 0$, and $S \geq Id$ [resp., $-S \geq Id$]. Then there holds

$$\begin{aligned} &|U|^2_{L^\infty} + \theta|U|^2_{L^2} \lesssim |F|^2_{L^1} \\ &\text{[resp., } |U|^2_{L^\infty} + \theta|U|^2_{L^2} \lesssim |U(0)|^2 + |F|^2_{L^1} \text{].} \end{aligned} \tag{3.18}$$

Proof. Taking the inner product of the equation of U against SU and integrating the result over $[x_1,\infty]$ for the first case [resp., $[0,x_1]$ for the second case], we easily obtain the lemma. □

Thanks to Proposition 3.1, we can decompose U as follows

$$U = u_P + u_{H_e} + u_{H_h} + u_{H_g} + u_{H_t}, \tag{3.19}$$

corresponding to parabolic, elliptic, hyperbolic, glancing, or totally nonglancing modes.

3.4.1. *Parabolic modes.* Since spectrum of P is away from the imaginary axis, we can assume that

$$P(\lambda,\tilde\xi) = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$$

with $\pm\Re P_\pm \geq c > 0$. Therefore applying Lemma 3.2 with $S = Id$ or $-Id$ yields

$$\begin{aligned} &|u_{P_+}|^2_{L^\infty} + |u_{P_+}|^2_{L^2} \lesssim |F_{P_+}|^2_{L^1}, \\ &|u_{P_-}|^2_{L^\infty} + |u_{P_-}|^2_{L^2} \lesssim |u_{P_-}(0)|^2 + |F_{P_-}|^2_{L^1}. \end{aligned} \tag{3.20}$$

3.4.2. *Elliptic modes.* This is case (i) in Proposition 3.1 when the spectrum of Q_k lies in

$$\{\Re e\mu > \delta\} \qquad [\text{resp., } \{\Re e\mu < -\delta\}].$$

In this case, there are positive symmetric matrices $S^k(\hat\zeta, \rho)$, C^∞ on a neighborhood ω of $(\underline{\zeta}, 0)$ and such that

$$\Re S^k Q^k \geq cId \qquad [\text{resp., } -\Re S^k Q^k \geq cId]$$

for $c > 0$. Thus, Lemma 3.2 again yields

$$\begin{aligned} &|u_{H_{e+}}|^2_{L^\infty} + \rho|u_{H_{e+}}|^2_{L^2} \lesssim |F_{H_{e+}}|^2_{L^1}, \\ &|u_{H_{e-}}|^2_{L^\infty} + \rho|u_{H_{e-}}|^2_{L^2} \lesssim |u_{H_{e-}}(0)|^2 + |F_{H_{e-}}|^2_{L^1}. \end{aligned} \tag{3.21}$$

3.4.3. *Hyperbolic modes.* This is case (ii) in Proposition 3.1. In this case, as shown in [Met] we can write

$$Q^k(\hat\zeta, \rho) = q^k(\hat\zeta)Id + \rho\mathcal{R}^k(\hat\zeta, \rho) \tag{3.22}$$

where q^k is purely imaginary when $\hat\gamma = 0$, $\dot q^k := \partial_{\hat\gamma}\Re e q^k(\hat\zeta)$ does not vanish, and the spectrum of $\dot q^k \mathcal{R}^k(\hat\zeta, 0)$ is contained in the half space $\{\Re e\mu > 0\}$. Therefore, when $\dot q^k > 0$ [resp., $\dot q^k < 0$] and thus for $(\zeta, \hat\gamma)$ sufficiently close to $(\hat\zeta, 0)$

$$\Re e q^k \geq c\hat\gamma, \qquad [\text{resp., } \Re e q^k \leq -c\hat\gamma],$$

we have positive symmetric matrices $S^k(\hat\zeta, \rho)$ satisfying

$$\Re S^k Q^k \geq c(\hat\gamma + \rho)Id \qquad [\text{resp., } -\Re S^k Q^k \geq c(\hat\gamma + \rho)Id]$$

for $c > 0$. Thus, again by Lemma 3.2, we obtain

$$\begin{aligned} &|u_{H_{h+}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{h+}}|^2_{L^2} \lesssim |F_{H_{h+}}|^2_{L^1}, \\ &|u_{H_{h-}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{h-}}|^2_{L^2} \lesssim |u_{H_{h-}}(0)|^2 + |F_{H_{h-}}|^2_{L^1}. \end{aligned} \tag{3.23}$$

3.4.4. *Totally nonglancing modes.* This is case (iv) in Proposition 3.1. As constructed in [GMWZ6], there exist symmetrizers S^k that are positive [resp. negative] definite when the mode is totally incoming [resp. outgoing]. Denote $u_{H_{t+}}$ [resp., $u_{H_{t-}}$] associated with totally incoming [resp. outgoing] modes. Then similarly as in above, we also have

$$\begin{aligned} &|u_{H_{t+}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{t+}}|^2_{L^2} \lesssim |F_{H_{t+}}|^2_{L^1}, \\ &|u_{H_{t-}}|^2_{L^\infty} + (\gamma + \rho^2)|u_{H_{t-}}|^2_{L^2} \lesssim |u_{H_{t-}}(0)|^2 + |F_{H_{t-}}|^2_{L^1}. \end{aligned} \tag{3.24}$$

Thus, putting these estimates together with noting that the stability estimate (3.6) already gives a bound on $|u(0)|$, we easily obtain sharp bounds on u in L^∞ and L^2 for all above cases:

$$|u_k|^2_{L^\infty} + (\gamma + \rho^2)|u_k|^2_{L^2} \lesssim |f|^2_{L^1} + |u_{H_g}|_{L^\infty}|f|_{L^1}, \tag{3.25}$$

for all $k = P, H_e, H_h, H_t$.

3.4.5. *Glancing modes.* Hence, we remain to consider the final case: case (iii) in Proposition 3.1. Recall (3.15)

$$Q_k(\hat{\zeta},\rho) = i(\underline{\mu}_k \mathrm{Id} + J) + i\sigma Q_k'(\hat{\tilde{\xi}}) + \mathcal{O}(\hat{\gamma}+\rho) \tag{3.26}$$

on a neighborhood of $(\underline{\hat{\zeta}},0)$, where $\sigma = |\hat{\tilde{\xi}} - \underline{\hat{\tilde{\xi}}}|$. We consider two cases.

Case a. $\sigma \lesssim (\hat{\gamma}+\rho)^\epsilon$ for some small $\epsilon > 0$. Recall that we consider the reduced system:

$$\partial_{x_1} u_k = \rho Q_k(\hat{\zeta},\rho) u_k + f_k \tag{3.27}$$

with $Q_k(\hat{\zeta},\rho)$ having a form as in (3.26). It is clear that the L^p norm of u_k remains unchanged under the transformation u_k to $u_k e^{-i\underline{\mu}_k x_1}$. Thus, we can assume that $\underline{\mu}_k = 0$. Note that we have the following bounds by (3.11)

$$|u_k|_{L^\infty(x_1)} \lesssim (\hat{\gamma}+\rho)^{-1}|f|_{L^1} \quad \text{and} \quad |u_k|_{L^2(x_1)} \lesssim (\hat{\gamma}+\rho)^{-3/2}|f|_{L^1}. \tag{3.28}$$

To prove the refined bounds (3.12), we first observe that

$$|\partial_{x_1} u_k|_{L^\infty} \lesssim \rho |u_k|_{L^\infty} + |f_k|_{L^\infty} \lesssim |f|_{L^1} + |f|_{L^\infty},$$

where the last inequality is due to (3.28). Now, write $u_k = (u_{k,1},\cdots,u_{k,\nu_k})$. Thanks to the special form of Q_k in (3.26), we have

$$\partial_{x_1} u_{k,\nu_k} = i\rho\sigma Q_k'(\hat{\tilde{\xi}}) u_k + \mathcal{O}(\gamma+\rho^2) u_k + f_k. \tag{3.29}$$

Taking inner product of the equation (3.29) against $\partial_{x_1} u_{k,\nu_k}$, we easily obtain by applying the standard Young's inequality:

$$|\partial_{x_1} u_{k,\nu_k}|_{L^2}^2 \lesssim \rho^2(\hat{\gamma}+\rho)^{2\epsilon}|u_k|_{L^2}^2 + |f_k|_{L^1}|\partial_{x_1} u_{k,\nu_k}|_{L^\infty} \lesssim (\hat{\gamma}+\rho)^{-1+2\epsilon}|f|_{L^1}^2 + |f|_{L^\infty}^2. \tag{3.30}$$

Similarly, for u_{k,ν_k-1} satisfying

$$\partial_{x_1} u_{k,\nu_k-1} = i\rho\sigma Q_k'(\hat{\tilde{\xi}}) u_k + i\rho u_{k,\nu_k} + \mathcal{O}(\gamma+\rho^2) u_k + f_k,$$

we have

$$|\partial_{x_1} u_{k,\nu_k-1}|_{L^2}^2 \lesssim \rho^2(\hat{\gamma}+\rho)^{2\epsilon}|u_k|_{L^2}^2 + \rho|< u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} >| + |f_k|_{L^1}|\partial_{x_1} u_{k,\nu_k}|_{L^\infty}. \tag{3.31}$$

Here, integration by parts and Young's inequality yield

$$\rho|< u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} >| \lesssim \rho |\partial_{x_1} u_{k,\nu_k}|_{L^2}|u_{k,\nu_k-1}|_{L^2} + \rho|u_k(0)|^2.$$

Thus, using the refined bound (3.30) and noting that

$$|u_k(0)|^2 \lesssim |< f, u_k >| \lesssim |f|_{L^1}|u_k|_{L^\infty} \lesssim (\hat{\gamma}+\rho)^{-1}|f|_{L^1}^2,$$

we obtain

$$\rho|< u_{k,\nu_k}, \partial_{x_1} u_{k,\nu_k-1} >| \lesssim \rho(\hat{\gamma}+\rho)^{-2+\epsilon}(|f|_{L^1}^2 + |f|_{L^\infty}^2)$$

Therefore, applying this estimate into (3.31), we get

$$|\partial_{x_1} u_{k,\nu_k-1}|_{L^2}^2 \lesssim (\hat{\gamma}+\rho)^{-1+\epsilon}(|f|_{L^1}^2 + |f|_{L^\infty}^2). \tag{3.32}$$

Using this refined bound, we can estimate the same for u_{k,ν_k-2}, u_{k,ν_k-3}, and so on. Thus, we obtain a refined bound for u_k:

$$|\partial_{x_1} u_k|_{L^2}^2 \lesssim \rho^{-1+\epsilon}(|f|_{L^1}^2 + |f|_{L^\infty}^2) \tag{3.33}$$

where ϵ may be changed in each finite step and smaller than the original one. This and the standard Sobolev imbedding yield

$$|u_k|_{L^\infty}^2 \lesssim |u_k|_{L^2}|\partial_{x_1} u_k|_{L^2} \lesssim (\hat\gamma + \rho)^{-2+\epsilon}(|f|_{L^1}^2 + |f|_{L^\infty}^2) \tag{3.34}$$

which proves the L^∞ refined bound in (3.12) for Z. Using (3.34) into (3.9), we also obtain the refined bound in L^2 as claimed in (3.12):

$$|u_k|_{L^2}^2 \lesssim (\hat\gamma + \rho)^{-3+\epsilon}(|f|_{L^1}^2 + |f|_{L^\infty}^2), \tag{3.35}$$

for some $\epsilon > 0$.

Case b. $\sigma \gtrsim (\hat\gamma + \rho)^\epsilon$ for some small ϵ in $(0, 1/2)$. We shall diagonalize this block. Recall that

$$Q_k(\hat\zeta, \rho) = i\underline{\mu}_k \mathrm{Id} + i \begin{bmatrix} 0 & 1 & 0 & \\ 0 & 0 & \ddots & 0 \\ & \ddots & \ddots & 1 \\ \sigma q_{\nu_k} & & 0 & 0 \end{bmatrix} + \mathcal{O}(\sigma). \tag{3.36}$$

Following [Z2, Z3, GMWZ1], we diagonalize this glancing block by

$$u'_{H_g} := T_{H_g}^{-1} u_{H_g},$$

where $u_{H_g} := u_{H_{g+}} + u_{H_{g-}}$. Here $u_{H_{g\pm}}$ are defined as the projections of u_{H_g} onto the growing (resp. decaying) eigenspaces of $Q_k(\hat\zeta, \rho)$ in (3.36). We recall the following whose proof can be found in [Z2, Z3] or Lemma 12.1, [GMWZ1].

Lemma 3.3 (Lemma 12.1, [GMWZ1])**.** *The diagonalizing transformation T_{H_g} may be chosen so that*

$$|T_{H_g}| \le C, \qquad |T_{H_g}^{-1}| \le C\beta, \qquad |T^{-1}_{H_g|_{H_{g-}}}| \le C\alpha \tag{3.37}$$

where α, β are defined as

$$\beta := \sigma^{-1+1/\nu_k}, \qquad \alpha := \sigma^{(1-[(\nu_k+1)/2])/\nu_k}, \tag{3.38}$$

and $T^{-1}_{H_g|_{H_{g-}}}$ denotes the restriction of $T^{-1}_{H_g}$ to subspace H_{g-}. In particular, $\beta\alpha^{-2} \ge 1$.

Simple calculations show that eigenvalues of Q_k are

$$\alpha_{k,j} = i\underline{\mu_k} + \pi_{k,j} + o(\sigma^{1/\nu_k}), \qquad j = 0, 1, ..., s-1. \tag{3.39}$$

Here, $\pi_{k,j} = \epsilon^j i(q_{\nu_k}\sigma)^{1/\nu_k}$, with $\epsilon = 1^{1/\nu_k}$. We can further change of coordinates if necessary to assume that

$$Q'_k := T_{H_g}^{-1} Q_k T_{H_g} = \mathrm{diag}(\alpha_{k,1}, \cdots, \alpha_{k,l}, \alpha_{k,l+1}, \cdots, \alpha_{k,\nu_k}) \tag{3.40}$$

with

$$-\Re e\ \alpha_{k,j} > 0, \quad j = 1, ..., l,$$
$$\Re e\ \alpha_{k,j} > 0, \quad j = l+1, ..., \nu_k. \tag{3.41}$$

Hence, applying Lemma 3.2 to equations of u'_{H_g} with $S = Id$ or $S = -Id$, we easily obtain

$$\begin{aligned} |u'_{H_{g+}}|^2_{L^\infty} + \rho \min_j |\Re e\ \alpha_{k,j}||u'_{H_{g+}}|^2_{L^2} &\lesssim |F'_{H_{g+}}|^2_{L^1}, \\ |u'_{H_{g-}}|^2_{L^\infty} + \rho \min_j |\Re e\ \alpha_{k,j}||u'_{H_{g-}}|^2_{L^2} &\lesssim |u'_{H_{g-}}(0)|^2 + |F'_{H_{g-}}|^2_{L^1}. \end{aligned} \tag{3.42}$$

The diagonalized boundary condition $\Gamma' := \Gamma_a T_{H_g}$. By computing, we observe that

$$|\Gamma' u'_{H_{g-}}| = |\Gamma u_{H_{g-}}| \geq C^{-1}|u_{H_{g-}}| \geq \frac{C^{-1}|u'_{H_{g-}}|}{|T^{-1}_{H_g|_{H_{g-}}}|} \geq C^{-1}\alpha^{-1}||u'_{H_{g-}}|.$$

Thus,

$$|u'_{H_{g-}}| \leq C\alpha|\Gamma' u'_{H_{g-}}| \leq C\alpha(|\Gamma' u'| + |\Gamma' u'_+|) \leq C\alpha|u'_+|. \tag{3.43}$$

Using this estimate, (3.37), and (3.25), the estimate (3.42) yields

$$\alpha^{-2}|u_{H_g}|^2_{L^\infty} + \rho\alpha^{-2} \min_j |\Re e\ \alpha_{k,j}||u_{H_g}|^2_{L^2} \lesssim \beta^2|f|^2_{L^1}. \tag{3.44}$$

Recalling that α, β are defined as in (3.38) and the fact that we are in the case of $\sigma \geq \rho^\epsilon$ for some small $\epsilon > 0$, we get

$$|u_{H_g}|_{L^\infty} \leq C\alpha\beta|f|_{L^1} \leq C(\hat{\gamma} + \rho)^{-2\epsilon}|f|_{L^1}, \tag{3.45}$$

from which we obtain the refined bounds (3.12) for this case as well.

Remark 3.4. In case b) above, we use the nonvanishing of q_{ν_k} to make sure that σq_{ν_k} is order of σ in the neighborhood ω of $(\underline{\hat{\zeta}}, 0)$ so that the lower left hand entry of Q_k dominates and thus we can be sure to diagonalize the block. Otherwise, the other entries of Q_k in (3.36) may dominate and the behavior is not clear. The nonvanishing of q_{ν_k} is guaranteed by our additional Hypothesis (H4′) as shown in the proof of Proposition 3.1. This is only place in the paper where the assumption (H4′) is used.

3.5. **$L^1 \to L^p$ estimates.** We establish the $L^1 \to L^p$ resolvent bounds for low frequency regime, restricting our attention to the surface

$$\Gamma^{\tilde{\xi}} := \{\lambda\ :\ \Re e\lambda = -\theta_1(|\tilde{\xi}|^2 + |\Im m\lambda|^2)\}, \tag{3.46}$$

for $\theta_1 > 0$. Taking θ_1 to be sufficiently small such that all earlier resolvent estimates are still valid on $\Gamma^{\tilde{\xi}}$, with $\rho := |(\tilde{\xi}, \lambda)|$ being sufficiently small. Thus, we obtain the following:

Proposition 3.5 (Low-frequency bounds)**.** *Under the hypotheses of Theorem 1.5, for $\lambda \in \Gamma^{\tilde{\xi}}$ and $\rho := |(\tilde{\xi}, \lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde{\xi}} - \lambda)^{-1}\partial^\beta_{x_1} f|_{L^p(x_1)} \leq C\rho^{-1-1/p+\epsilon}[\rho^\beta|f|_{L^1(x_1)} + |f|_{L^\infty(x_1)}], \tag{3.47}$$

for all $2 \leq p \leq \infty$, $\beta = 0, 1$, and $\epsilon > 0$.

Proof. Following [Z2, Z3], define the curves

$$(\tilde\xi,\lambda)(\rho,\hat{\tilde\xi},\hat\tau):=(\rho\hat{\tilde\xi},\rho i\hat\tau-\theta_1\rho^2),$$

where $\hat{\tilde\xi}\in\mathbb{R}^{d-1}$, $\hat\tau\in\mathbb{R}$ and $(\hat{\tilde\xi},\hat\tau)\in S^d$: $|\hat{\tilde\xi}|^2+|\hat\tau|^2=1$. As $(\rho,\hat{\tilde\xi},\hat\tau)$ range in the compact set $[0,\delta]\times S^d$, $(\tilde\xi,\lambda)$ traces out the portion of the surface $\Gamma^{\tilde\xi}$ contained in the set $|\tilde\xi|^2+|\lambda|^2\le\delta$. Thus, using L^2 and L^∞ estimates obtained in previous sections with $\hat\gamma=0$ and applying the interpolation inequality between L^2 and L^∞ spaces, we obtain the proposition in the case $\beta=0$.

Now, recalling that $W=\Phi VZ$ and all coordinate transformation matrices are uniformly bounded, the refined bounds of Z therefore imply improved bounds for W and thus U. Bounds for L^p, $2<p<\infty$, are obtained by interpolation inequality between L^2 and L^∞. Hence, we have proved the bounds for $\beta=0$ as claimed.

For $\beta=1$, we expect that $\partial_{x_1}f$ plays a role as "ρf" forcing. Recall that the eigenvalue equations $(L_{\tilde\xi}-\lambda)U=\partial_{x_1}f$ read

$$\begin{aligned}\overbrace{(B^{11}U_{x_1})_{x_1}-(A^1U)_{x_1}}^{L_0U}&-i\sum_{j\neq1}A^j\xi_jU+i\sum_{j\neq1}B^{j1}\xi_jU_{x_1}\\&+i\sum_{k\neq1}(B^{1k}\xi_kU)_{x_1}-\sum_{j,k\neq1}B^{jk}\xi_j\xi_kU-\lambda U=\partial_{x_1}f.\end{aligned}\tag{3.48}$$

Now modifying the nice argument of Kreiss-Kreiss presented in [KK, GMWZ1], we write $U=V+U_1$, where V satisfies

$$(L_0-\lambda)V=\partial_{x_1}f,\qquad x_1\in\mathbb{R}.\tag{3.49}$$

Noting that A^1 and B^{11} depend on x_1 only, we thus obtain by one-dimensional results (see [MaZ3, Z3]) the following pointwise bounds on Green kernel G^0_λ of $\lambda-L_0$,

$$|\partial_{y_1}G^0_\lambda(x_1,y_1)|\le Ce^{-\rho|x_1-y_1|}(\rho+e^{-\theta|y_1|}).\tag{3.50}$$

Hence, employing Hausdorff-Young's inequality, we obtain

$$|V|_{L^p(x_1)}+|V_{x_1}|_{L^p(x_1)}\le C\rho^{-1/p}[\rho|f|_{L^1(x_1)}+|f|_{L^\infty(x_1)}],\tag{3.51}$$

for all $1\le p\le\infty$.

Now from $U_1=U-V$ and equations of U and V, we observe that U_1 satisfies

$$(L_{\tilde\xi}-\lambda)U_1=L(V,V_{x_1}),\tag{3.52}$$

where $L(V,V_{x_1})=\rho\mathcal{O}(|V|+|V_{x_1}|)$.

Therefore applying the result which we just proved for $\beta=0$ to the equations (3.52), we obtain

$$\begin{aligned}|U_1|_{L^p(x_1)}&\le C\rho^{-1-1/p+\epsilon}\Big[|L(V,V_{x_1})|_{L^1(x_1)}+|L(V,V_{x_1})|_{L^\infty(x_1)}\Big]\\&\le C\rho^{-1-1/p+\epsilon}\rho\Big[|V|_{L^q}+|V_{x_1}|_{L^q}\Big]\\&\le C\rho^{-1/p+\epsilon}[|f|_{L^1(x_1)}+\rho^{-1}|f|_{L^\infty(x_1)}].\end{aligned}\tag{3.53}$$

Bounds on V and U_1 clearly give our claimed bounds on U by triangle inequality:

$$|U|_{L^p} \le |V|_{L^p} + |U_1|_{L^p}.$$

We obtain the proposition for the case $\beta = 1$, and thus complete the proof.

□

3.6. **Estimates on the solution operator.** In this subsection, we complete the proof of Proposition 2.1. As mentioned earlier, it suffices to prove the bounds for $\mathcal{S}_1(t)$, where the low frequency solution operator $\mathcal{S}_1(t)$ is defined as

$$\mathcal{S}_1(t) := \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}|\le r} \oint_{\Gamma^{\tilde{\xi}}} e^{\lambda t + i\tilde{\xi}\cdot\tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} d\lambda d\tilde{\xi}. \tag{3.54}$$

Proof of bounds on $\mathcal{S}_1(t)$. Let $\hat{u}(x_1, \tilde{\xi}, \lambda)$ denote the solution of $(L_{\tilde{\xi}} - \lambda)\hat{u} = \hat{f}$, where $\hat{f}(x_1, \tilde{\xi})$ denotes Fourier transform of f, and

$$u(x,t) := \mathcal{S}_1(t) f = \frac{1}{(2\pi i)^d} \int_{|\tilde{\xi}|\le r} \oint_{\Gamma^{\tilde{\xi}}} e^{\lambda t + i\tilde{\xi}\cdot\tilde{x}} (L_{\tilde{\xi}} - \lambda)^{-1} \hat{f}(x_1, \tilde{\xi}) d\lambda d\tilde{\xi}.$$

Using Parseval's identity, Fubini's theorem, the triangle inequality, and Proposition 3.5, we may estimate

$$\begin{aligned} |u|^2_{L^2(x_1,\tilde{x})}(t) &= \frac{1}{(2\pi)^{2d}} \int_{x_1} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\lambda t} \hat{u}(x_1, \tilde{\xi}, \lambda) d\lambda \Big|^2 d\tilde{\xi} dx_1 \\ &\le \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\Re e\lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^2(x_1)} d\lambda \Big|^2 d\tilde{\xi} \\ &\le C[|f|_{L^1(x)} + |f|_{L^{1,\infty}_{\tilde{x},x_1}}]^2 \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\Re e\lambda t} \rho^{-3/2+\epsilon} d\lambda \Big|^2 d\tilde{\xi}. \end{aligned}$$

Specifically, parametrizing $\Gamma^{\tilde{\xi}}$ by

$$\lambda(\tilde{\xi}, k) = ik - \theta_1(k^2 + |\tilde{\xi}|^2), \quad k \in \mathbb{R},$$

we estimate

$$\begin{aligned} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\Re e\lambda t} \rho^{-3/2+\epsilon} d\lambda \Big|^2 d\tilde{\xi} &\le \int_{\tilde{\xi}} \Big| \int_{\mathbb{R}} e^{-\theta_1(k^2+|\tilde{\xi}|^2)t} \rho^{-3/2+\epsilon} dk \Big|^2 d\tilde{\xi} \\ &\le \int_{\tilde{\xi}} e^{-2\theta_1|\tilde{\xi}|^2 t} |\tilde{\xi}|^{-1} \Big| \int_{\mathbb{R}} e^{-\theta_1 k^2 t} |k|^{\epsilon-1} dk \Big|^2 d\tilde{\xi} \\ &\le C t^{-(d-2)/2-\epsilon}, \end{aligned}$$

noting that $\int_{\mathbb{R}^{d-1}} e^{-\theta|x|^2} |x|^{-\alpha} dx$ is finite, provided $\alpha < d - 1$.

Similarly, we estimate

$$\begin{aligned} |u|^2_{L^{2,\infty}_{\tilde{x},x_1}}(t) &\le \frac{1}{(2\pi)^{2d}} \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\Re e\lambda t} |\hat{u}(x_1, \tilde{\xi}, \lambda)|_{L^\infty(x_1)} d\lambda \Big|^2 d\tilde{\xi} \\ &\le C[|f|_{L^1(x)} + |f|_{L^{1,\infty}_{\tilde{x},x_1}}]^2 \int_{\tilde{\xi}} \Big| \oint_{\Gamma^{\tilde{\xi}}} e^{\Re e\lambda t} \rho^{-1+\epsilon} d\lambda \Big|^2 d\tilde{\xi} \end{aligned}$$

where, parametrizing $\Gamma^{\tilde\xi}$ as above, we have

$$\begin{aligned}\int_{\tilde\xi}\Big|\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda\Big|^2 d\tilde\xi &\le \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}\Big|\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk\Big|^2 d\tilde\xi\\ &\le Ct^{-(d-1)/2-\epsilon}.\end{aligned}$$

Finally, we estimate

$$\begin{aligned}|u|_{L^\infty_{\tilde x,x_1}}(t) &\le \frac{1}{(2\pi)^d}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}|\hat u(x_1,\tilde\xi,\lambda)|_{L^\infty(x_1)}d\lambda d\tilde\xi\\ &\le C[|f|_{L^1(x)}+|f|_{L^{1,\infty}_{\tilde x,x_1}}]\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda d\tilde\xi\end{aligned}$$

where, parametrizing $\Gamma^{\tilde\xi}$ as above, we have

$$\begin{aligned}\int_{\tilde\xi}\oint_{\Gamma^{\tilde\xi}} e^{\Re e\lambda t}\rho^{-1+\epsilon}d\lambda d\tilde\xi &\le \int_{\tilde\xi} e^{-\theta_1|\tilde\xi|^2 t}\int_{\mathbb{R}} e^{-\theta_1 k^2 t}|k|^{\epsilon-1}dk d\tilde\xi\\ &\le Ct^{-(d-1)/2-\epsilon/2}.\end{aligned}$$

The x_1-derivative bounds follow similarly by using the version of the $L^1\to L^p$ estimates for $\beta_1=1$. The $\tilde x-$derivative bounds are straightforward by the fact that $\widehat{\partial_{\tilde x}^{\tilde\beta}f}=(i\tilde\xi)^{\tilde\beta}\hat f$. $\square$

4. Two–dimensional case or cases with (H4)

In this section, we give an immediate proof of Theorem 1.7. Notice that the only assumption we make here that differs from those in [NZ2] is the relaxed Hypothesis (H3′), treating the case of totally nonglancing characteristic roots, which is only involved in low–frequency estimates. That is to say, we only need to establish the $L^1\to L^p$ bounds in low-frequency regimes for this new case.

Proposition 4.1 (Low-frequency bounds; [NZ2], Proposition 3.3)**.** *Under the hypotheses of Theorem 1.7, for $\lambda\in\Gamma^{\tilde\xi}$ (see* (3.46)*) and $\rho:=|(\tilde\xi,\lambda)|$, θ_1 sufficiently small, there holds the resolvent bound*

$$|(L_{\tilde\xi}-\lambda)^{-1}\partial_{x_1}^\beta f|_{L^p(x_1)}\le C\gamma_2\rho^{-2/p}\Big[\rho^\beta|\hat f|_{L^1(x_1)}+\beta|\hat f|_{L^\infty(x_1)}\Big], \tag{4.1}$$

for all $2\le p\le\infty$, $\beta=0,1$, and γ_2 is the diagonalization error (see [Z3]*, (5.40)) defined as*

$$\gamma_2:=1+\sum_{j,\pm}\Big[\rho^{-1}|\Im m\lambda-\eta_j^\pm(\tilde\xi)|+\rho\Big]^{1/s_j-1}, \tag{4.2}$$

with $\eta_j^\pm, s_j$ as in (H4).

Proof. We only need to treat the new case: the totally nonglancing blocks Q_t^k. But this is already treated in our previous subsection, Subsection 3.4.4, yielding

$$\begin{aligned}&|u_{H_{t+}}|^2_{L^\infty}+\rho^2|u_{H_{t+}}|^2_{L^2}\lesssim|F_{H_{t+}}|^2_{L^1},\\ &|u_{H_{t-}}|^2_{L^\infty}+\rho^2|u_{H_{t-}}|^2_{L^2}\lesssim|u_{H_{t-}}(0)|^2+|F_{H_{t-}}|^2_{L^1},\end{aligned} \tag{4.3}$$

where the boundary term $|u_{H_{t-}}(0)|^2$ can be treated by applying the L^2 stability estimate (3.6). Thus, together with a use of the standard interpolation inequality, we have obtained

$$|u_{H_t}|_{L^p(x_1)} \le C\gamma_2\rho^{-1}|f|_{L^1(x_1)}, \tag{4.4}$$

for all $2 \le p \le \infty$ and γ_2 defined as in (4.2), yielding (4.1) for $\beta = 0$. For $\beta = 1$, we can follow the Kreiss–Kreiss trick as done in the proof of Proposition 3.5, completing the proof of Proposition 4.1. □

Proof of Theorem 1.7. Proposition 4.1 is Proposition 3.3 in [NZ2] with an extension to the totally nonglancing cases. Thus, we can now follow word by word the proof in [NZ2], yielding the theorem. □

Appendix A. Genericity of (H4′)

Genericity of our additional structural assumption (H4′) is clear. Indeed, violation of the condition would require d equations: $\partial_{\xi_j}\lambda_k(\xi) = 0$ for all $j = 1, \cdots, d$, whereas only $d-1$ parameters in $\xi \in \mathbb{R}^d \setminus \{0\}$ are varied as ξ may be constrained in the unit sphere S^d by homogeneity of $\lambda(\xi)$ in ξ.

Finally, we give the following counterexample of Kevin Zumbrun in the two–dimensional case for which the hypothesis (H4′) fails.

Counterexample A.1. Let

$$A_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.1}$$

Then both A_1 and A_2 are clearly symmetric and do not commute. However, at $\xi_1 = 0$, the matrix $\xi_1 A_1 + \xi_2 A_2$ has an eigenvalue ($\lambda(\xi) \equiv 0$) such that $\nabla\lambda = 0$, violating (H4′).

Counterexamples for higher–dimensional cases can be constructed similarly.

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